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Fourier Series Approximation of Linear Fractional Stable Motion

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Abstract An approximation of the linear fractional stable motion by a Fourier sum is presented. In the continuous sample path case precise error bounds are derived. This approximation method is used to develop a simulation method of the sample path of linear fractional stable motions.

Keywords Fractional stable motion · Self-similar process

Mathematics Subject Classification (2000) Primary 60G18 · 60E02 · Secondary 65C50 · 68U20

1 Introduction

Irregular phenomena appear in various fields of scientific research: traffic volume in modern communication and computer networks and mathematical finance, for example. See, e.g., [10, 11, 18] for network traffic and the collection of articles in [4], part B. Important features often discovered are *heavy tails*, statistical *self-similarity* and *long range dependence*. See [5] and [4] for a recent overview. Mathematical models both easy to use and relevant for these applications are *fractional stable motions*, most prominently the fractional Brownian motion. See [14] for a comprehensive introduction to these processes.

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It is a common feature of fractional stable motions that they are self-similar. Moreover, they can exhibit long range dependence as well as continuous sample path even in the heavy tailed (α-stable) case. See, e.g., [5] and the references therein.

Fix any $0 < \alpha \le 2$ and let $Z_\alpha(dz)$ be an independently scattered symmetric α-stable ($S\alpha S$) random measure on $\mathbb{R}$ with Lebesgue control measure ds in the sense of [14]. For $a, b \in \mathbb{R}$ not both equal to zero, $0 < H < 1$ and $t \in \mathbb{R}$ define

$$X(t) = X_{a,b}(t) = \int_{\mathbb{R}} g_{a,b}(t,z)\, Z_\alpha(dz) \tag{1.1}$$

where

$$g_{a,b}(t,z) = a\big((t-z)_+^{H-1/\alpha} - (-z)_+^{H-1/\alpha}\big) + b\big((t-z)_-^{H-1/\alpha} - (-z)_-^{H-1/\alpha}\big).$$

The process $\{X(t)\}_{t\in\mathbb{R}}$ is called a *linear fractional stable motion* (LFSM). Note that the integral is well defined since the integrand is in $L^\alpha(\mathbb{R}, dz)$, see, e.g., [14]. Also note that if $a = b \neq 0$, then (1.1) reduces to

$$X(t) = a\int_{-\infty}^{\infty}\Big(|t-z|^{H-1/\alpha} - |z|^{H-1/\alpha}\Big) Z_\alpha(dz) \tag{1.2}$$

the so-called well balanced linear fractional stable motion.

It follows from basic properties of the random integral definition that $\{X(t)\}_{t\in\mathbb{R}}$ is self-similar with Hurst-index H, that is

$$\{X(ct)\}_{t\in\mathbb{R}} \overset{f.d.}{=} \big\{c^H X(t)\big\}_{t\in\mathbb{R}}$$

and has stationary increments, so that for any $h > 0$

$$\{X(t+h) - X(h)\}_{t\in\mathbb{R}} \overset{f.d.}{=} \{X(t)\}_{t\in\mathbb{R}}.$$

Here $\overset{f.d.}{=}$ denotes equality in distribution of all finite-dimensional marginals.

It follows from Kolmogoroff's Theorem (see [7], Theorem 3.23), using the fact that $\{X(t)\}_{t\in\mathbb{R}}$ has stationary increments and is self-similar with Hurst-coefficient H, that if $H > 1/\alpha$ there exists a modification of $\{X(t)\}_{t\in\mathbb{R}}$ with locally Hölder-continuous sample path of order $0 < \beta < H - 1/\alpha$ see [17] and [9] for an improvement. We will always choose this version. Note that $H > 1/\alpha$ necessarily implies $1 < \alpha \le 2$. Moreover, the case $H > 1/\alpha$ is also considered the so-called *long range dependent* case of LFSM, see, e.g., [14], Remark on p. 345. We will restrict the major part of this article to this most important case.

The purpose of this article is to present a Fourier series approximation of the LSFM $\{X(t)\}_{t\in\mathbb{R}}$ which can be used to efficiently simulate its sample paths. Since there is only a satisfying theory of Fourier series for functions in L^p with $p > 1$ we will restrict ourself to the case $1 < \alpha \le 2$. To our knowledge, in the heavy tailed case $0 < \alpha < 2$, there exist three other methods to simulate the sample path of LFSM. The first method, presented in [15], is an effective implementation of the approximation method presented in [14] using the fast-Fourier-transform algorithm. This simulation method generates the so-called fractional stable noise $Y_n = X(n) - X(n-1)$,

$n = 1, 2, \ldots$, a stationary sequence, by a Riemann sum approximation of the integral representation. Hence, to generate a LFSM sample path of size N one has to compute the cumulative sums $X(n) = \sum_{j=1}^{n} Y_j$, for $n = 1, \ldots, N$, with an error (in terms of the scale parameter of a $S\alpha S$ random variable) growing linearly in n. See Corollary 2.1 in [15]. The second method, presented in [19] is based on a limit theorem for sums of moving averages. It generates an approximation of the LFSM by the normalization of the cumulative sums of a linear process. The error in terms of the scale parameter of a $S\alpha S$ random variable is uniform on any compact set. However, this method always approximates the LFSM $\{X(t)\}_{t\in\mathbb{R}}$ by a stepwise constant function, see Proposition 1 in [19], even in the continuous sample path case $H > 1/\alpha$. The last method is presented in [2] in a much more general context. It is based on the approximation of the LFSM by the sum of a shot noise series and a fractional Brownian motion, a Gaussian process (see Section 6.3 of [2]). A fast and exact synthesis method for one-dimensional fractional Brownian motion is known [12], such that this part can be efficiently simulated. However, the error due to this approximation is only given in terms of Berry-Essen bounds and can not be compared with the rates of convergence of the shot noise series obtained almost surely and in L^r norm.

In contrast to [19], our method will produce directly an approximation of a sample path of LFSM by a continuous function which can be efficiently evaluated at any time point. In particular, we obtain a sample path of a LFSM discretized on a regular grid without additional cost. Moreover, we provide detailed error bounds, in terms of the scale parameter of a $S\alpha S$ random variable, of the approximation, which do not depend on the time point over a compact set, contrary to [15]. Therefore, our approximation can also be used to approximate integral functionals of LFSM. Furthermore our method can be generalized to simulate more general processes than LFSM, obtained through a moving-average integral representation.

This article is organized as follows. In Section 2 we present the general idea behind our method as well as a general convergence result for one fixed time point for any $0 < H < 1$ and $1 < \alpha \le 2$. In Section 3 we consider the continuous sample path case $H > 1/\alpha$. Detailed error estimates in terms of the scale parameter of a $S\alpha S$ random variable (or the standard deviation if $\alpha = 2$) as well as the convergence of all finite-dimensional marginal distributions are obtained in this most important case. We conclude this article with a description of the simulation algorithm in Section 4. The proofs of Proposition 1 and Theorem 2 are postponed to the Appendix.

2 The Approximation Method

In this section we will describe the general idea of our approximation method for LFSM in the case $0 < H < 1$ and $1 < \alpha \le 2$ using Fourier-series. Our method is based on the following basic fact of stable integrals. See [14], Chapter 3 for details.

Property (C): For $f_n, f \in L^\alpha(\mathbb{R}, dz)$ we have

$$\int_{\mathbb{R}} f_n(z)\, Z_\alpha(dz) \to \int_{\mathbb{R}} f(z)\, Z_\alpha(dz) \quad \text{as } n \to \infty \tag{2.1}$$

in probability, if and only if

$$\int_{\mathbb{R}} \left| f_n(z) - f(z) \right|^\alpha dz \to 0 \quad \text{as } n \to \infty. \tag{2.2}$$

Note that since $\alpha > 1$ we know that $\|f\|_\alpha = \left(\int_{\mathbb{R}} |f(z)|^\alpha \, dz\right)^{1/\alpha}$ is a norm on $L^\alpha(\mathbb{R}, dz)$ and hence (2.2) is just $\|f_n - f\|_\alpha \to 0$ as $n \to \infty$.

In the following we will only consider the case $a = 1$ and $b = 0$ in (1.1). The general case as well as the well-balanced case (1.2) can be dealt with similarly. For any $t \in \mathbb{R}$ fixed, note that since $0 < H < 1$ the function $z \mapsto (t - z)_+^{H-1/\alpha} - (-z)_+^{H-1/\alpha}$ belongs to $L^\alpha(\mathbb{R}, dz)$. Then dominated convergence together with Property (C) implies

$$Y_A(t) = \int_{-A}^{A} \left((t - z)_+^{H-1/\alpha} - (-z)_+^{H-1/\alpha}\right) Z_\alpha(dz) \to X(t) \quad \text{as } A \to \infty \tag{2.3}$$

in probability.

Now fix some (large) $A > 0$ and define

$$e_k(z) = \exp\left(i\pi \frac{k}{A} z\right), \ k \in \mathbb{Z}.$$

Then $\{e_k(z) : k \in \mathbb{Z}\}$ is an orthonormal basis in $L^2([-A, A], (2A)^{-1} \, dz)$. For a fixed $t \in \mathbb{R}$ let

$$\rho_t(z) = (t - z)_+^{H-1/\alpha} - (-z)_+^{H-1/\alpha} \text{ and } \varphi(z) = (-z)_+^{H-1/\alpha}$$

for $|z| \le A$ and extend both $\rho_t(z)$ and $\varphi(z)$ periodically to $\mathbb{R}$ with period $2A$, that is $\rho_t(z+2A) = \rho_t(z)$ and $\varphi(z+2A) = \varphi(z)$. Note that $Y_A(t) = \int_{-A}^{A} \rho_t(z) \, Z_\alpha(dz)$. Since $H > 0$ we have $\varphi, \rho_t \in L^\alpha([-A, A], ds) \subset L^1([-A, A], ds)$ since $\alpha > 1$. Hence, we can define the Fourier-coefficients of ρ_t for $k \in \mathbb{Z}$ as

$$\hat{\rho}_t(k) = \frac{1}{2A} \int_{-A}^{A} \rho_t(z) \overline{e_k(z)} \, dz = \left(e^{-i\pi \frac{k}{A} t} - 1\right) \hat{\varphi}(k),$$

where $\hat{\varphi}(k) = \frac{1}{2A} \int_{-A}^{A} \varphi(z) \overline{e_k(z)} \, dz$. Since $1 < \alpha \le 2$, using the fact that the Fourier-series of an $L^\alpha([-A, A], (2A)^{-1} \, dz)$-function f converges to f in $L^\alpha([-A, A], (2A)^{-1} \, dz)$ (see, e.g., [8], Theorem on p. 50), we have

$$\sum_{k=-M}^{M} \hat{\rho}_t(k) e_k(z) \to \rho_t(z) \quad \text{as } M \to \infty \tag{2.4}$$

in $L^\alpha([-A, A], (2A)^{-1} \, dz)$. Therefore, using Property (C) again, we conclude

$$Y_{A,M}(t) = \int_{-A}^{A} \left(\sum_{k=-M}^{M} \hat{\rho}_t(k) e_k(z)\right) Z_\alpha(dz) \to Y_A(t) \quad \text{as } M \to \infty \tag{2.5}$$

in probability. Note that

$$Y_{A,M}(t) = \sum_{k=-M}^{M} \hat{\rho}_t(k) \int_{-A}^{A} e_k(z)\, Z_\alpha(dz).$$

We now approximate $\int_{-A}^{A} e_k(z)\, Z_\alpha(dz)$ by an analogue of a Riemann sum. For integers $L \geq 1$ let $z_j = z_j(A, L) = j(A/L)$ for $j = -L, \ldots, L$ and let $\Delta z_j = [z_j, z_{j+1}[$. Observe that since $e_k(z)$ is uniformly continuous on $[-A, A]$ we have

$$\sum_{j=-L}^{L-1} e_k(z_j) 1_{\Delta z_j}(z) \to e_k(z) \quad \text{as } L \to \infty$$

uniformly on $[-A, A]$ and hence in $L^\alpha([-A, A], (2A)^{-1}\, dz)$. Therefore, in view of Property (C) we get

$$\hat{Z}_{A,L}(k) = \sum_{j=-L}^{L-1} e_k(z_j) Z_\alpha(\Delta z_j) \to \int_{-A}^{A} e_k(z)\, Z_\alpha(dz) \quad \text{as } L \to \infty \tag{2.6}$$

in probability. Note that $Z_\alpha(\Delta z_j)$, $j = -L, \ldots, L-1$ are i.i.d. $S\alpha S$ random variables with scale $(A/L)^{1/\alpha}$, since by normalization we assume that the scale of $Z_\alpha([0, 1])$ equals one.

We now define

$$Y_{A,M,L}(t) = \sum_{k=-M}^{M} \hat{\rho}_t(k) \hat{Z}_{A,L}(k) = \sum_{k=-M}^{M} \left(e^{-i\pi \frac{k}{A} t} - 1\right) \hat{\varphi}(k) \hat{Z}_{A,L}(k) \tag{2.7}$$

to be our approximation of the LFSM $X(t)$ for any $t \in \mathbb{R}$ fixed. In view of (2.3)–(2.6) we haven proven the following.

Theorem 1 *If* $1 < \alpha \leq 2$ *we have, for any fixed* $t \in \mathbb{R}$ *that*

$$\lim_{L \to \infty} Y_{A,M,L}(t) = Y_{A,M}(t),$$

$$\lim_{M \to \infty} Y_{A,M}(t) = Y_A(t)$$

and

$$\lim_{A \to \infty} Y_A(t) = X(t)$$

in probability.

Observe that

$$\hat{Z}_{A,L}(k) = \sum_{j=-L}^{L-1} e^{2i\pi \frac{jk}{2L}} Z_\alpha(\Delta z_j)$$

can be viewed as the discrete Fourier-transform of the sequence $(Z_\alpha(\Delta z_j) : j = -L, \ldots, L-1)$. Hence, we choose $L > M$ a power of 2 in our approximation method in order to apply the Fast-Fourier-Transform algorithm to compute $\widehat{Z}_{A,L}(k)$ for $-M \le k \le M$ from a generated sequence $Z_\alpha(\Delta z_j)$ of i.i.d. $S\alpha S$ random variables. However, the derivation of Theorem 1 does not allow directly to have $M, L \to \infty$ simultaneously. In the next section we will show that if $H > 1/\alpha$ our method also works if $L = M + 1$ or $L = 2M \to \infty$, see Corollary 2 and Remark 2 below. Moreover, we improve Theorem 1 by providing error bounds and show convergence of all finite-dimensional marginal distributions.

3 Error Estimates

In this section we derive error estimates of our approximations $\{Y_{A,M,L}(t)\}_{t\in\mathbb{R}}$ to $\{Y_A(t)\}_{t\in\mathbb{R}}$ and of the latter to the LFSM $\{X(t)\}_{t\in\mathbb{R}}$ in terms of the scale parameter $\|\xi\|_\alpha$ of a $S\alpha S$ random variable ξ, where the notation $\|\cdot\|_\alpha$ is used according to the situation for a function or a $S\alpha S$ random variable. Recall from [14] that a $S\alpha S$ random variable ξ has the characteristic function $\mathbb{E}(e^{ix\xi}) = \exp(-C_\alpha^\alpha |x|^\alpha)$, where C_α denotes the scale parameter of ξ. We also use the notation $C_\alpha = \|\xi\|_\alpha$ which is very illuminating when dealing with stable integrals because the scale parameter of the stable integral $\xi = \int_\mathbb{R} g(z)\, Z_\alpha(dz)$ is $\|\xi\|_\alpha = C_\alpha \big(\int_\mathbb{R} |g(z)|^\alpha\, dz\big)^{1/\alpha} = C_\alpha \|g\|_\alpha$, where C_α is the scale parameter of $Z_\alpha([0,1])$. We will assume without loss of generality that $C_\alpha = 1$.

In the following we only consider the continuous and long range dependence case $H > 1/\alpha$. Note that by (2.3) we can write

$$Y_A(t) = \int_{-A}^{A} \rho_t(z)\, Z_\alpha(dz) \tag{3.1}$$

where ρ_t is a $2A$-periodic function which is now bounded and continuous. In fact, $\rho_t(z) = \varphi(t-z) - \varphi(-z)$ where $\varphi(z) = a(z)_+^{H-1/\alpha} + b(z)_-^{H-1/\alpha}$ for $z \in [-A, A[$. As before we will only consider the case $a = 1$ and $b = 0$ as well as the well balanced case $a = b = 1$ which will allow better estimates because of the symmetry of $\varphi(z) = |z|^{H-1/\alpha}$ in this particular case.

We first analyze the effect of truncation on the integral defining $X(t)$ as in (2.3) above. We fix any $0 < T < A$ and consider the processes only on the finite interval $|t| \le T$. The rate of convergence is obtained using the mean value theorem and the decreasing order of the kernel function φ.

Proposition 1 *Assume $1 < \alpha \le 2$ and $1/\alpha < H < 1$. Then, for all $T > 0$ and $A > 0$ we have for all $|t| \le T$,*

$$\big\| X(t) - Y_A(t) \big\|_\alpha \le C_1(H,\alpha) T (A-T)^{-(1-H)},$$

where $C_1(H,\alpha) = 2(H - 1/\alpha)\big(\alpha(1-H)\big)^{-1/\alpha}$.

Proof See the Appendix. □

We now present the main result of this section. The following theorem provides error bounds in terms of the scale parameter of our Fourier series approximation $Y_{A,M,L}(t)$ of $Y_A(t)$. Note that by (2.6) and (2.7) we can write

$$Y_{A,M,L}(t) = \sum_{j=-L}^{L-1} \left(\sum_{k=-M}^{M} \hat{\rho}_t(k) e_k(z_j) \right) \int_{z_j}^{z_{j+1}} Z_\alpha(dy). \tag{3.2}$$

Recall from [8] that, for $2A$ periodic functions the Dirichlet kernel is given by

$$D_{A,M}(x) = \sum_{k=-M}^{M} e^{i\pi \frac{k}{A} x} = \frac{\sin((2M+1)\pi x/2A)}{\sin(\pi x/2A)}$$

and that for integrable $2A$-periodic functions f we have

$$\sum_{k=-M}^{M} \widehat{f}(k) e_k(x) = D_{A,M} * f(x)$$

where for integrable $2A$-periodic functions f, g the convolution $f * g$ is defined as

$$f * g(x) = \frac{1}{2A} \int_{-A}^{A} f(x-z) g(z)\, dz$$

and

$$\widehat{f}(k) = \frac{1}{2A} \int_{-A}^{A} f(z) \overline{e_k(z)}\, dz.$$

Therefore, we get

$$\sum_{k=-M}^{M} \widehat{\rho_t}(k) e_k(x) = D_{A,M} * \rho_t(x).$$

Hence, we can rewrite (3.2) as

$$Y_{A,M,L}(t) = \sum_{j=-L}^{L-1} \big(D_{A,M} * \rho_t\big)(z_j) \int_{z_j}^{z_{j+1}} Z_\alpha(dy). \tag{3.3}$$

Theorem 2 *Assume* $1 < \alpha \le 2$ *and* $1/\alpha < H < 1$. *Then, for all* $T, A > 0$, $M \ge e^4$ *and* $L > M$, *for any* $|t| \le T$

$$\big\| Y_A(t) - Y_{A,M,L}(t) \big\|_\alpha \le \varepsilon_\varphi(A, M, L),$$

where, if $\varphi(z) = (-z)_+^{H-1/\alpha}$ (*or if* $\varphi(z) = (-z)_-^{H-1/\alpha}$)

$$\varepsilon_\varphi(A, M, L) = A^H \big(C_3(H,\alpha) M^{-1/2} + C_4(H,\alpha) L^{-1/\alpha} \log(M) \big), \tag{3.4}$$

with $C_3(H,\alpha) = 2^{3/2+1/\alpha}(2 + H - 1/\alpha)$ *and, if* $\varphi(z) = |z|^{H-1/\alpha}$

$$\varepsilon_\varphi(A, M, L) = A^H\big(C_3(H,\alpha)M^{-1/2-(H-1/\alpha)} + C_4(H,\alpha)L^{-1/\alpha}\log(M)\big), \quad (3.5)$$

with $C_3(H,\alpha) = 2^{3/2+1/\alpha}(1 + 2H - 2/\alpha)^{-1/2}(1 + H - 1/\alpha)$. *In both cases we have* $C_4(H,\alpha) = 2^3\left(2^{1+H-1/\alpha}(\alpha + 1)^{-1/\alpha} + 1\right)$.

Sketch of the proof Since $\rho_t \in L^\alpha([-A, A], (2A)^{-1}\,dx)$ with $\alpha > 1$, we know from the theorem on p. 50 of [8] that $\rho_t(\cdot) = \sum_{k=-\infty}^{+\infty} \hat\rho_t(k)e_k(\cdot)$, where the series converges in L^α. Furthermore,

$$\rho_t = \sum_{|k|\le M} \hat\rho_t(k)e_k + \sum_{|k|>M} \hat\rho_t(k)e_k = D_{A,M} * \rho_t + \sum_{|k|>M} \hat\rho_t(k)e_k.$$

Since $\alpha > 1$, we can use Minkowski's inequality to get

$$\left\|Y_A(t) - Y_{A,M,L}(t)\right\|_\alpha \le V(A, M) + U(A, M, L),$$

with, on the one hand

$$V(A, M) = \left(\int_{-A}^{A}\left|\sum_{|k|>M} \hat\rho_t(k)e_k(y)\right|^\alpha dy\right)^{1/\alpha},$$

which we bound by $C_3(H,\alpha)A^H M^{-1/2}$, using the rate of convergence of $\sum_{|k|>M} |\hat\rho_t(k)|^2$, Hölder's inequality and Plancherel-identity. On the other hand,

$$U(A, M, L) = \left(\int_{-A}^{A}\left|\sum_{j=-L}^{L-1} 1_{[z_j,z_{j+1}[}(y)\big[D_{A,M} * \rho_t(y) - D_{A,M} * \rho_t(z_j)\big]\right|^\alpha dy\right)^{1/\alpha},$$

is bounded by $C_4(H,\alpha)A^H L^{-1/\alpha}\log(M)$, using the mean value theorem, Hölder regularity of order $H - \frac{1}{\alpha}$ of the function ρ_t and the fact that $\frac{1}{2A}\int_{-A}^{A}\left|D_{A,M}(x)\right| dx \le 2\log(M)$. □

Combining Proposition 1 and Theorem 2 we get the following.

Corollary 1 *Assume* $1 < \alpha \le 2$ *and* $1/\alpha < H < 1$. *Then, for all* $T, A > 0$, $M \ge e^4$ *and* $L > M$, *for any* $|t| \le T$

$$\left\|X(t) - Y_{A,M,L}(t)\right\|_\alpha \le C_1(H,\alpha)T(A - T)^{H-1} + \varepsilon_\varphi(A, M, L),$$

where ε_φ *is given by* (3.4) *or* (3.5) *according to* φ *and* $C_1(H,\alpha) = 2(H - 1/\alpha)\,(\alpha(1 - H))^{-1/\alpha}$.

Observe that by Theorem 2 it follows that we can choose $L = M+1$ in (2.7) and still get convergence.

Remark 1 Let us point out that the error bound estimate does not depend on the time point t. Therefore, one advantage of this method is that integral functionals of the LFSM can be approximated efficiently just as well as values of the process at time points. Actually one can consider the integral $\int_{-T}^{T} g(t)X(t)\,dt$ for a large class of functions g defined on a compact set $[-T,T]$. Replacing $X(t)$ by its approximation $Y_{A,M,L}(t)$ given by (2.7) this integral can be approximated by

$$\int_{-T}^{T} g(t)Y_{A,M,L}(t)\,dt = \sum_{k=-M}^{M}\left(\int_{-T}^{T}\left(e^{-i\pi\frac{k}{A}t}-1\right)g(t)\,dt\right)\hat{\varphi}(k)\hat{Z}_{A,L}(k).$$

Therefore, computing the last deterministic integrals in advance, one can quickly generate approximate realizations of $\int_{-T}^{T} g(t)X(t)\,dt$. Moreover, since $X(t)-Y_{A,M,L}(t)$ is a $S\alpha S$ random variable with $\alpha>1$, there is a constant $c_\alpha(1)$ such that we have $\mathbb{E}\left|X(t)-Y_{A,M,L}(t)\right| \le c_\alpha(1)|\left\|X(t)-Y_{A,M,L}(t)\right\|_\alpha$, according to Property 1.2.17 p. 18 of [14]. Then

$$\mathbb{E}\left|\int_{-T}^{T} g(t)\left(X(t)-Y_{A,M,L}(t)\right)\,dt\right| \le c_\alpha(1)\int_{-T}^{T}|g(t)|\cdot\left\|X(t)-Y_{A,M,L}(t)\right\|_\alpha\,dt,$$

which gives the rate of convergence of the approximations of $\int_{-T}^{T} g(t)X(t)\,dt$ by Corollary 1.

We obtain the convergence of all finite-dimensional marginal distributions from Corollary 1.

Corollary 2 *Assume* $1<\alpha\le 2$ *and* $1/\alpha<H<1$. *Then, for all* $T>0$, *as* $A,M,L\to+\infty$, *such that* $\varepsilon_\varphi(A,M,L)\to 0$ *where* ε_φ *is given by* (3.4) *or* (3.5) *according to* φ, *the approximations* $\{Y_{A,M,L}(t)\}_{t\in[-T,T]}$ *converge in finite-dimensional marginal distributions to the LFSM* $\{X(t)\}_{t\in[-T,T]}$.

Proof Fix any $t_1,\ldots,t_n\in[-T,T]$ and $\lambda_1,\ldots,\lambda_n\in\mathbb{R}$. Then by Corollary 1 and the triangle inequality we obtain

$$\left\|\sum_{j=1}^{n}\lambda_j X(t_j)-\sum_{j=1}^{n}\lambda_j Y_{A,M,L}(t_j)\right\|_\alpha \le \sum_{j=1}^{n}|\lambda_j|\left\|X(t_j)-Y_{A,M,L}(t_j)\right\|_\alpha \to 0$$

as $A,M,L\to\infty$ and the corresponding condition is fulfilled. Since for a sequence ξ_n of $S\alpha S$ random variable $\|\xi_n\|_\alpha\to 0$ implies $\xi_n\to 0$ in probability and the above relation holds for all $\lambda_1,\ldots,\lambda_n\in\mathbb{R}$, the convergence of all finite-dimensional marginal distributions follows. □

Remark 2 If we let $L=M+1$ and $A=A(M)=M^\rho$ for some $0<\rho<1/2$, since $\varepsilon_\varphi(M^\rho,M,M+1)\underset{M\to+\infty}{\longrightarrow}0$, we get from Corollary 2 that $\{Y_{M^\rho,M,M+1}(t)\}_{t\in\mathbb{R}}\Rightarrow\{X(t)\}_{t\in\mathbb{R}}$ as $M\to\infty$ for all finite-dimensional marginal distributions.

Remark 3 In the situation of Corollary 2 it is a challenging open problem to achieve tightness and hence convergence in distribution on some suitable function space.

4 The Simulation Algorithm

In this section we present an effective simulation algorithm of the sample path of the LFSM $\{X(t)\}_{t\in\mathbb{R}}$ defined by (1.1) based on our Fourier approximation given by (2.7). We will only consider the so-called *well balanced* case given by (1.2). Then

$$Y_{A,M,L}(t) = \sum_{k=-M}^{M} \left(e^{-i\pi\frac{k}{A}t} - 1\right)\hat{\varphi}(k)\hat{Z}_{A,L}(k)\,. \tag{4.1}$$

Recall from (2.6) that for $k = -M, \dots, M$ in this case we have

$$\hat{Z}_{A,L}(k) = \sum_{j=-L}^{L-1} e^{2i\pi kj/2L} Z_\alpha(\Delta z_j) \tag{4.2}$$

where $Z_\alpha(\Delta z_j)$, $j = -L, \dots, L-1$ are i.i.d. $S\alpha S$ random variables with scale $(A/L)^{1/\alpha}$. This sequence can easily and exactly be simulated using the algorithm from Chambers et al., see [1]. Note that (4.2) is the discrete Fourier transform of the vector $(Z_\alpha(\Delta z_j) : j = -L, \dots, L-1)$ and can be effectively computed using a variant of the fast Fourier transform algorithm (FFT). Moreover, since $\varphi(z) = |z|^{H-1/\alpha}$ we know from Lemma 1(b), especially (5.4) that

$$\hat{\varphi}(k) = A^{H-1/\alpha} k^{-1-H+1/\alpha} \int_0^k v^{H-1/\alpha} \cos(\pi v)\, dv\,. \tag{4.3}$$

The integral in (4.3) can be effectively computed by decomposing

$$\int_0^k v^{H-1/\alpha}\cos(\pi v)\,dv = \sum_{j=1}^{k} \int_{j-1}^{j} v^{H-1/\alpha}\cos(\pi v)\,dv$$

and approximating each summand by Simpson's rule, see, e.g., [6], Theorem 2.3. Our simulation study presented below shows that the approximation error resulting from applying Simpson's rule has little effect on the accuracy of our simulation method.

Let us denote

$$W_{A,M,L}(t) = \sum_{k=-M}^{M} e^{-i\pi\frac{k}{A}t}\hat{\varphi}(k)\hat{Z}_{A,L}(k) \tag{4.4}$$

the discrete Fourier transform of the vector $(\hat{\varphi}(k)\hat{Z}_{A,L}(k))_{-M\le k\le M}$ such that by (4.1)

$$Y_{A,M,L}(t) = W_{A,M,L}(t) - W_{A,M,L}(0).$$

Let us point out that the fast Fourier transform algorithm allows to compute $W_{A,M,L}$ and $Y_{A,M,L}$ at any point $t_l = \frac{2A}{2M+1}l$ for $-M \le l \le M$ with $O(M\log_2 M)$ operations.

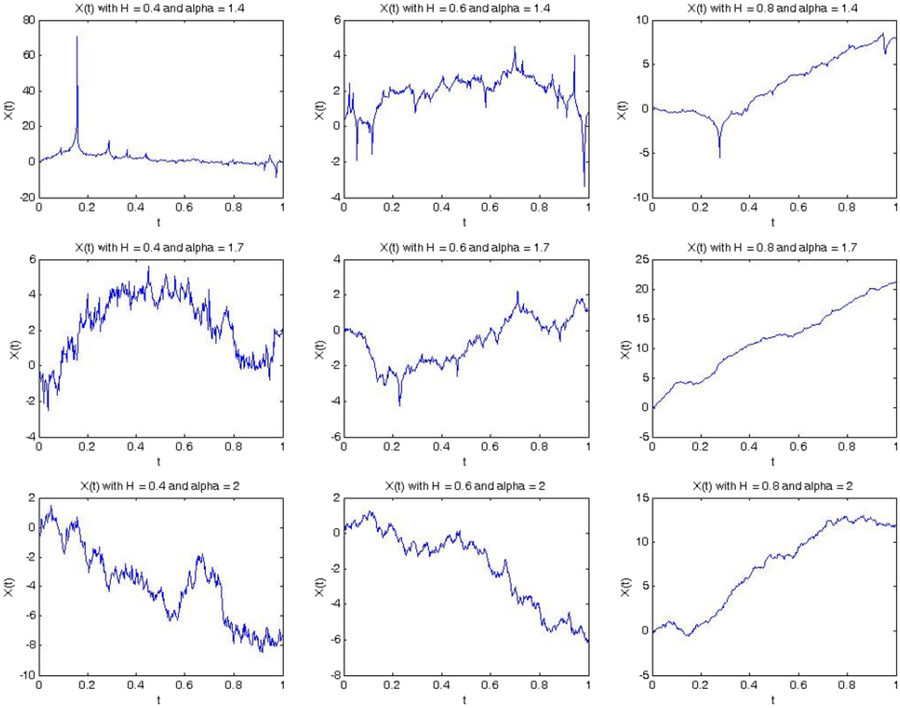

Fig. 1 Approximations of LFSM sample path for various H, α. We used $A = 101$ and $L = 2^{18}$ and $N = M = 2^{16}$

Let us assume that we want to generate an approximate sample path of the LFSM $\{X(t)\}_{t\in\mathbb{R}}$ with Hurst index H over an interval $[a, b]$, given by $N + 1$ equidistant time points. Since the LFSM has stationary increments we may consider $[0, b - a]$ instead of $[a, b]$. Let $\delta_N = \frac{b-a}{N}$ be the step size of the time points over $[0, b - a]$. Using the self-similarity of order H of $\{X(t)\}_{t\in\mathbb{R}}$ we have

$$\{X(0), \dots, X(N\delta_N)\} \overset{d}{=} \left(\delta_N \frac{2M+1}{2A}\right)^H \{X(t_0), \dots, X(t_N)\}.$$

Therefore, we only have to generate $W_{A,M,L}$ at any point $(t_l)_{0\le l\le N-1}$ for some $M \ge N$ which can be done with only $O\left(M \log_2 M\right)$ operations.

We now formulate our simulation algorithm (A MATLAB code can be obtained from the authors upon request). Given a number N of points, a truncation point $A > 0$ and some large integers $M, L \ge N$ (usually one should pick $M = 2^p \ge N$ and $L = 2^q$ with $p > q$ in view of the FFT algorithm) we compute:

Step 1: Compute $\hat{\varphi}(k)$, $k = -M, \dots, M$ by (4.3)
Step 2: Generate $2L$ $S\alpha S$ random variables $Z_{-L}, \dots, Z_{L-1}$ and set $\bar{Z}_j = (A/L)^{1/\alpha} Z_j$
Step 3: Compute the FFT of $(\bar{Z}_{-L}, \dots, \bar{Z}_{L-1})$ to obtain $(\hat{Z}(-M), \dots, \hat{Z}(M))$
Step 4: Set $\hat{W}_k = \hat{\varphi}(k) \cdot \hat{Z}(k)$ for $k = -M, \dots, M$

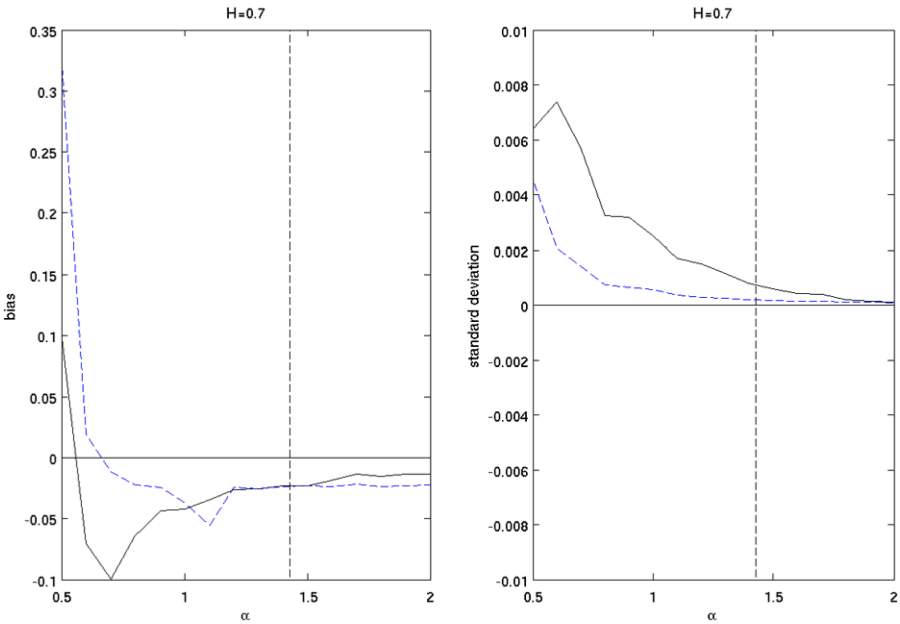

Fig. 2 Bias and standard deviations of the FIRT estimators $\widehat{H}_{FIRT} - H$ (*solid line*) and the wavelet estimators $\widehat{H}_{WT}$ (*dashed line*) studied in [16] as functions of α when $H = 0.7$. The *broken vertical line* located at $\alpha = 1/H$ indicates on the right the long range dependence cases. The bias and standard deviations were computed by using samples of $n = 256$ independent replications of the estimators, obtained from 256 independently simulated paths of the LFSM process for $N = M = 2^{16}$ points, $L = 2^{18}$ and $A = 101$

Step 5: Compute the FFT of $(\hat{W}_{-M}, \dots, \hat{W}_M)$ to get $(W_{A,M,L}(t_{-M}), \dots, W_{A,M,L}(t_M))$

Step 6: Compute $Y_{A,M,L}(t_l) = W_{A,M,L}(t_l) - W_{A,M,L}(0)$ for $0 \le l \le N$.

Note that for any particular sample path, Step 1 of the algorithm have only to be executed once in an initializing step. After the initialization the complexity to compute one approximation sample path is $O(M \log_2(M))$ when choosing M a power of 2. Let us point out that it is about the same cost than in [15] and [19] to get approximations not of the LFSM itself but of the linear fractional stable noise. Therefore, to obtain approximations of the LFSM, the authors have one more step than us which is to compute cumulative sums of the noise.

Sample paths realizations of LFSM for different H and α are given in Figure 1. The computational time (on a dual core PowerMac G5) for $L = 2^{18}$ and $N = M = 2^{16}$ is 46 seconds for the initialization Steps 1 and 0.7 second for each sample path. Let us point out that, even if error bounds of the approximations have only been proved in the cases $\alpha \in (1, 2)$ and $H > 1/\alpha$, one can use the code for any values of $\alpha \in (0, 2)$ and $H \in (0, 1)$. In fact, in view of Theorem 1 we still get an approximation. Actually, note that $H > 1/\alpha$ in Figure 1 only for $\alpha = 1.4$ and $H = 0.8$ or $\alpha \in \{1.7, 2\}$ and $H \in \{0.6, 0.8\}$. Finally, let us mention that when $\alpha = 2$ the LFSM is just the well known fractional Brownian motion for which a lot of numerical methods have been proposed these last years. Moreover, in this particular case, there exists a fast and

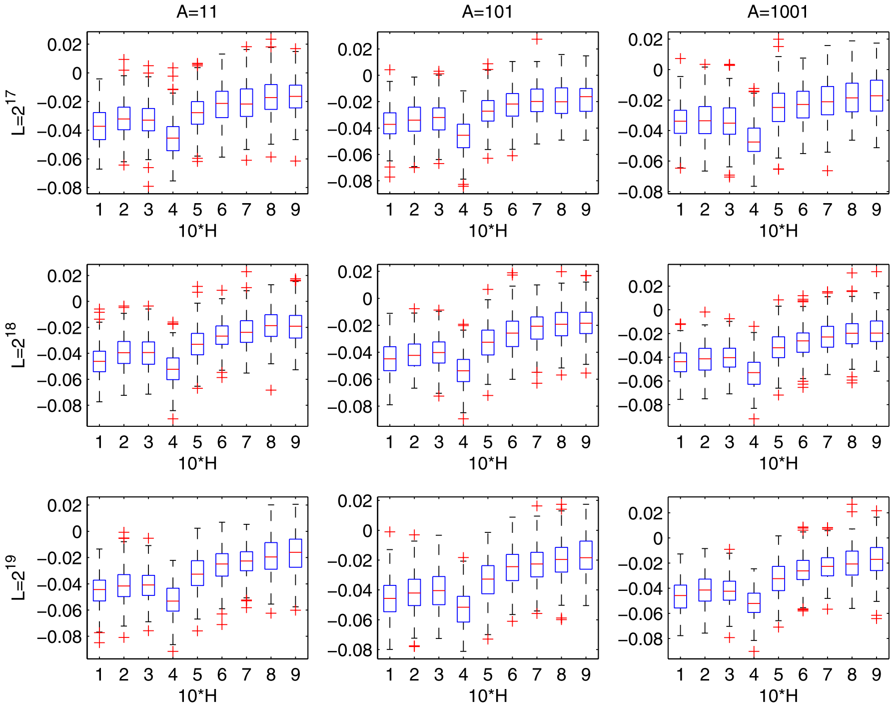

Fig. 3 This figure contains Boxplots for the bias $\widehat{H}_{WT} - H$ for the wavelet estimators $\widehat{H}_{WT}$ studied in [16]. As in Figure 3 in [15] these Boxplots were computed by using samples of $n = 256$ independent replications of the estimators, obtained from 256 independently simulated paths of the LFSM process for an index of stability $\alpha = 1.5$ for $M = 2^{16}$ points. The values of H are $H = 0.1, 0.2, \ldots, 0.9$ (note that $0.6 < 1/\alpha < 0.7$ such that $H = 0.7, 0.8$ and 0.9 correspond to long-range dependence). The values of the discretization parameters A and L, used in the simulation algorithm are indicated in the margins of the figure

exact synthesis method [12], based on the Choleski decomposition of the covariance function and on the stationarity of the increments that allows to apply the embedding circulant matrix method [3]. Then our approximation is not really relevant for this case.

To illustrate the quality of the approximation given by this method we estimate the Hurst parameter H of the LFSM using two type of estimators studied in [16], namely the wavelet $\hat{H}_{WT}$ and the FIRT $\hat{H}_{FIRT}$ estimators based on discrete wavelet transforms of the approximation for the first one, and on discrete differences for the second one. The results obtained for a fixed parameter $H = 0.7$ with respect to the stability index α are given in Figure 2. It can be compared to those of Figure 4 in [16]. Let us point out that both estimators underestimate the real theoretical value of H. However, for $H \geq 1/\alpha$ the bias of the two estimators is relatively low (about 0.02) and the standard deviation is very small (less than 10^{-3}) while in [16] the bias is about 0.01 with standard deviation greater than 0.03.

Figures 3 and 4 contain Boxplots for the bias $\mathbb{E}(\hat{H}_{WT}) - H$ and $\mathbb{E}(\hat{H}_{FIRT}) - H$ for a fixed index of stability $\alpha = 1.5$. In this simulation study a sample of 256 indepen-

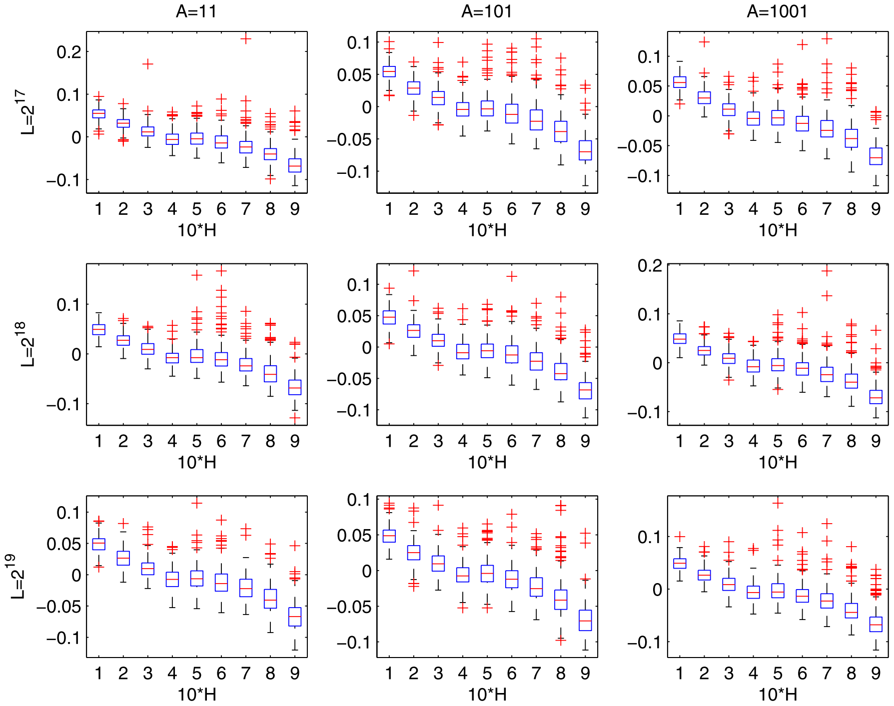

Fig. 4 This figure contains Boxplots for the bias $\widehat{H}_{FIRT} - H$ for the FIRT estimators $\widehat{H}_{FIRT}$ studied in [16]. As in Figure 4 in [15] these Boxplots were computed by using samples of $n = 256$ independent replications of the estimators, obtained from 256 independently simulated paths of the LFSM process for an index of stability $\alpha = 1.5$ for $M = 2^{16}$ points

dent copies of our LFSM approximation of size $N = M = 2^{16}$ points were generated using three different values of the truncation parameter $A = 11, 101$ and 1001, three different values of the step parameter $L = 2^{17}, 2^{18}$ and 2^{19} for the following values of the Hurst parameter $H = 0.1, 0.2, \dots, 0.9$.

As already noticed in Figure 2 for $H = 0.7$, both estimators underestimate the real theoretical value of H for any $H = 0.1, \dots, 0.9$. Let us also point out that the FIRT estimator (Figure 4) exhibits a larger variability than the wavelet one (Figure 3).

Concerning the parameters A and L let us observe that these parameters do not seem to have a significant influence on the estimated results. It is not surprising that L has less influence than M according to the upper bound given by (3.5) obtained in Corollary 1. Actually, since $\alpha = 1.5$ for all $H \neq 0.9$ we have $1/2 - (H - 1/\alpha) < 1/2$. In order to compare our approximation to the ones of [15], we used the same index of stability $\alpha = 1.5$ and the same number $n = 256$ of samples as in Figure 3 (wavelet estimator) and Figure 4 (FIRT estimator) of [15]. Let us point out that their discretization parameters have a greater influence on the wavelet estimator than in our case, which makes the optimal choice for these parameters more difficult. Moreover, as already observed in Figure 2, our standard deviations are really smaller than in [15] and we also obtain better results for the FIRT estimator. For another indication on the quality of our approximation, let us also refer to the bottom panel of Figure 2 in [20] (also obtained for $\alpha = 1.5$) and observe that our results are more accurate.

Appendix

Proof of Proposition 1 Let

$$E_A(t) = X(t) - Y_A(t) = \int_{-\infty}^{-A} \left((t-z)_+^{H-1/\alpha} - (-z)_+^{H-1/\alpha}\right) Z_\alpha(dz)\,.$$

Then

$$\|E_A(t)\|_\alpha^\alpha = \int_A^\infty \left|(t+z)^{H-1/\alpha} - z^{H-1/\alpha}\right|^\alpha dz.$$

Assume first that $0 \le t \le T$. Then, since $1/\alpha < H < 1$, by the mean value theorem $(t+z)^{H-1/\alpha} - z^{H-1/\alpha} \le (H-1/\alpha) z^{H-1-1/\alpha} t$ and hence

$$\|E_A(t)\|_\alpha^\alpha \le (H-1/\alpha)^\alpha t^\alpha \int_A^\infty z^{\alpha H - 1 - \alpha}\, dz \le \frac{(H-1/\alpha)^\alpha T^\alpha}{\alpha(1-H)} A^{\alpha(H-1)}.$$

On the other hand, if $-T \le t < 0$ we have $|(t+z)^{H-1/\alpha} - z^{H-1/\alpha}| = z^{H-1/\alpha} - (t+z)^{H-1/\alpha} \le (H-1/\alpha)(-t)(t+z)^{H-1/\alpha-1}$ and we get

$$\begin{aligned}\|E_A(t)\|_\alpha^\alpha &\le (H-1/\alpha)^\alpha (-t)^\alpha \int_A^\infty (t+z)^{\alpha H - 1 - \alpha}\, dz \\ &\le \frac{(H-1/\alpha)^\alpha T^\alpha}{\alpha(1-H)} (A-T)^{\alpha(H-1)}\end{aligned}$$

and the assertion follows. By a similar computation one gets the same bounds for $a = 0$ and $b = 1$. Since the symmetric case $\varphi(z) = |z|^{H-1/\alpha}$ is the sum of these two cases we obtain the constant $C_1(H,\alpha)$ to hold in either case. □

In order to prove Theorem 2 we need some estimates on the Fourier-transforms of the $2A$-periodic functions ρ_t and φ to control the rate of convergence of their Fourier series.

Lemma 1 *Assume* $1 < \alpha \le 2$ *and* $H > 1/\alpha$.

(a) *If* $\varphi(z) = (-z)_+^{H-1/\alpha}$ *(or if* $\varphi(z) = (-z)_-^{H-1/\alpha}$*) then there exists a constant* $C_2(H,\alpha)$ *such that for all* $k \in \mathbb{Z} \setminus \{0\}$ *and any* $t \in \mathbb{R}$ *we have*

$$\left|\hat\varphi(k)\right| \le C_2(H,\alpha) A^{H-1/\alpha} |k|^{-1},$$
$$\left|\hat\rho_t(k)\right| \le 2C_2(H,\alpha) A^{H-1/\alpha} |k|^{-1},$$

where $C_2(H,\alpha) = 2 + H - 1/\alpha$.

(b) *If* $\varphi(z)=|z|^{H-1/\alpha}$ *then there exists a constant* $C_2(H,\alpha)$ *such that for all* $k\in\mathbb{Z}\setminus\{0\}$ *and any* $t\in\mathbb{R}$ *we have*

$$\big|\hat{\varphi}(k)\big|\le C_2(H,\alpha)A^{H-1/\alpha}|k|^{-1-H+1/\alpha},$$
$$\big|\hat{\rho}_t(k)\big|\le 2C_2(H,\alpha)A^{H-1/\alpha}|k|^{-1-H+1/\alpha},$$

where $C_2(H,\alpha)=1+(H-1/\alpha)$.

Proof Since $\hat{\rho}_t(k)=(e^{-i\pi\frac{k}{A}t}-1)\hat{\varphi}(k)$ we only consider $\hat{\varphi}(k)$. Moreover, since $\hat{\varphi}(-k)=\overline{\hat{\varphi}(k)}$ it suffices to assume $k\ge 1$. We first consider $\varphi(z)=(-z)_+^{H-1/\alpha}$. Then

$$\begin{aligned}\hat{\varphi}(k)&=\frac{1}{2A}\int_0^A z^{H-1/\alpha}e^{i\pi\frac{k}{A}z}\,dz\\&=\frac{1}{2}A^{H-1/\alpha}k^{-1-H+1/\alpha}\bigg[\int_0^k v^{H-1/\alpha}\cos(\pi v)\,dv+i\int_0^k v^{H-1/\alpha}\sin(\pi v)\,dv\bigg].\end{aligned}\tag{5.1}$$

To analyze the first integral on the right-hand side of (5.1) we decompose

$$\begin{aligned}\int_0^k v^{H-1/\alpha}\cos(\pi v)\,dv&=\int_0^1 v^{H-1/\alpha}\cos(\pi v)\,dv+\int_1^k v^{H-1/\alpha}\cos(\pi v)\,dv\\&=C_1+\int_1^k v^{H-1/\alpha}\cos(\pi v)\,dv.\end{aligned}$$

Integrate by parts twice to obtain

$$\begin{aligned}\int_1^k v^{H-1/\alpha}\cos(\pi v)\,dv&=-\frac{H-1/\alpha}{\pi}\int_1^k v^{H-1/\alpha-1}\sin(\pi v)\,dv\\&=\frac{H-1/\alpha}{\pi^2}\big((-1)^k k^{H-1/\alpha-1}-1\big)\\&\quad+\frac{(H-1/\alpha)(1-H+1/\alpha)}{\pi^2}\int_1^k v^{H-1/\alpha-2}\cos(\pi v)\,dv.\end{aligned}$$

Since

$$\frac{(H-1/\alpha)(1-H+1/\alpha)}{\pi^2}\bigg|\int_1^k v^{H-1/\alpha-2}\cos(\pi v)\,dv\bigg|\le\frac{H-1/\alpha}{\pi^2}\big(1-k^{H-1/\alpha-1}\big)$$

we obtain

$$\begin{aligned}\bigg|\int_0^k v^{H-1/\alpha}\cos(\pi v)\,dv\bigg|&\le|C_1|+\bigg|\int_1^k v^{H-1/\alpha}\cos(\pi v)\,dv\bigg|\\&\le 1+2\frac{H-1/\alpha}{\pi^2}.\end{aligned}\tag{5.2}$$

Similarly, we have

$$\int_0^k v^{H-1/\alpha}\sin(\pi v)\,dv = \int_0^1 v^{H-1/\alpha}\sin(\pi v)\,dv + \int_1^k v^{H-1/\alpha}\sin(\pi v)\,dv.$$

Again, using integration by parts we conclude

$$\begin{aligned}&\int_1^k v^{H-1/\alpha}\sin(\pi v)\,dv\\ &\quad= \frac{(-1)^{k+1}k^{H-1/\alpha}-1}{\pi} + \frac{(H-1/\alpha)(1-H+1/\alpha)}{\pi^2}\int_1^k v^{H-1/\alpha-2}\sin(\pi v)\,dv\end{aligned}$$

and hence

$$\begin{aligned}\left|\int_0^k v^{H-1/\alpha}\sin(\pi v)\,dv\right| &\le 1 + \left|\int_1^k v^{H-1/\alpha}\sin(\pi v)\,dv\right|\\ &\le 1 + \frac{1+k^{H-1/\alpha}}{\pi} + \frac{H-1/\alpha}{\pi^2}.\end{aligned} \tag{5.3}$$

Now (5.1) together with (5.2) and (5.3) imply part (a) of the Lemma.

For the proof of part (b) note that if $\varphi(z) = |z|^{H-1/\alpha}$ then

$$\hat{\varphi}(k) = A^{H-1/\alpha}k^{-1-H+1/\alpha}\int_0^k v^{H-1/\alpha}\cos(\pi v)\,dv \tag{5.4}$$

and hence the assertion follows from (5.2). □

Proof of Theorem 2 Let $1 < \alpha \le 2$ and $1/\alpha < H < 1$. Then,

$$\begin{aligned}&\left\|Y_A(t) - Y_{A,M,L}(t)\right\|_\alpha\\ &\quad= \left\|\int_{-A}^{A}\left[\rho_t(y) - \sum_{j=-L}^{L-1} 1_{[z_j,z_{j+1}[}(y)\big(D_{A,M}*\rho_t\big)(z_j)\right] Z_\alpha(dy)\right\|_\alpha\\ &\quad= \left(\int_{-A}^{A}\left|\rho_t(y) - \sum_{j=-L}^{L-1} 1_{[z_j,z_{j+1}[}(y)\big(D_{A,M}*\rho_t\big)(z_j)\right|^\alpha dy\right)^{1/\alpha}.\end{aligned}$$

Since $\alpha > 1$, we can write

$$\rho_t = \sum_{|k|\le M}\hat{\rho}_t(k)e_k + \sum_{|k|>M}\hat{\rho}_t(k)e_k = D_{A,M}*\rho_t + \sum_{|k|>M}\hat{\rho}_t(k)e_k,$$

where the series converges in L^α. Therefore, by Minkowski's inequality we obtain

$$\begin{aligned}&\left\|Y_A(t) - Y_{A,M,L}(t)\right\|_\alpha\\ &\quad\le \left(\int_{-A}^{A}\left|\sum_{|k|>M}\hat{\rho}_t(k)e_k(y)\right|^\alpha dy\right)^{1/\alpha}\end{aligned}$$

$$+\left(\int_{-A}^{A}\left|D_{A,M}*\rho_t(y)-\sum_{j=-L}^{L-1}1_{[z_j,z_{j+1}[}(y)\big(D_{A,M}*\rho_t\big)(z_j)\right|^{\alpha}dy\right)^{1/\alpha}$$

$$=V(A,M)+\left(\int_{-A}^{A}\left|\sum_{j=-L}^{L-1}1_{[z_j,z_{j+1}[}(y)\big[D_{A,M}*\rho_t(y)-D_{A,M}*\rho_t(z_j)\big]\right|^{\alpha}dy\right)^{1/\alpha}$$

$$=V(A,M)+U(A,M,L).$$

We first bound $V(A,M)$. Observe that by Hölder's inequality with $p=2/\alpha$ and $q^{-1}=1-\alpha/2$, for $f\in L^2([-A,A],dy)\subset L^{\alpha}([-A,A],dy)$,

$$\left(\int_{-A}^{A}|f(y)|^{\alpha}dy\right)^{1/\alpha}\le(2A)^{(1/\alpha)-(1/2)}\left(\int_{-A}^{A}|f(y)|^2dy\right)^{1/2}.$$

Hence, combined with the Plancherel-identity we get

$$\begin{aligned}V(A,M)&=\left(\int_{-A}^{A}\left|\sum_{|k|>M}\hat\rho_t(k)e_k(y)\right|^{\alpha}dy\right)^{1/\alpha}\\&\le(2A)^{(1/\alpha)-(1/2)}\left(\int_{-A}^{A}\left|\sum_{|k|>M}\hat\rho_t(k)e_k(y)\right|^{2}dy\right)^{1/2}\\&=(2A)^{1/\alpha}\left(\frac{1}{2A}\int_{-A}^{A}\left|\sum_{|k|>M}\hat\rho_t(k)e_k(y)\right|^{2}dy\right)^{1/2}\\&=(2A)^{1/\alpha}\left(\sum_{|k|>M}\left|\hat\rho_t(k)\right|^2\right)^{1/2}.\end{aligned}$$

In the case $\varphi(z)=(-z)_+^{H-1/\alpha}$, using Lemma 1(a),

$$\begin{aligned}V(A,M)&\le2^{1+1/\alpha}C_2(H,\alpha)A^H\left(\sum_{|k|>M}|k|^{-2}\right)^{1/2}\\&\le2^{3/2+1/\alpha}C_2(H,\alpha)\left(\int_M^{\infty}x^{-2}dx\right)^{1/2}\\&=C_3(H,\alpha)A^HM^{-1/2}\end{aligned}$$

where $C_3(H,\alpha)=2^{3/2+1/\alpha}(2+H-1/\alpha)$. Similarly, if $\varphi(z)=|z|^{H-1/\alpha}$ we get using Lemma 1(b) that

$$V(A,M)\le C_3(H,\alpha)A^H M^{-1/2-(H-1/\alpha)}$$

where now $C_3(H,\alpha)=2^{3/2+1/\alpha}(1+2H-2/\alpha)^{-1/2}(1+H-1/\alpha)$.

Moreover, by Minkowski's inequality

$$U(A,M,L)\le\sum_{j=-L}^{L-1}\left(\int_{z_j}^{z_{j+1}}\left|D_{A,M}*\rho_t(y)-D_{A,M}*\rho_t(z_j)\right|^\alpha dy\right)^{1/\alpha}.$$

Observe further that

$$D_{A,M}*\rho_t(y)-D_{A,M}*\rho_t(z_j)=\frac{1}{2A}\int_{-A}^{A}D_{A,M}(z)\big[\rho_t(y-z)-\rho_t(z_j-z)\big]\,dz.$$

Hence, by Minkowski's integral inequality (see, e.g., [13], p. 177) we obtain

$$\begin{aligned}&\left(\int_{z_j}^{z_{j+1}}\left|\frac{1}{2A}\int_{-A}^{A}D_{A,M}(z)\big[\rho_t(y-z)-\rho_t(z_j-z)\big]\,dz\right|^\alpha dy\right)^{1/\alpha}\\&\quad\le\frac{1}{2A}\int_{-A}^{A}|D_{A,M}(z)|\left(\int_{z_j}^{z_{j+1}}\left|\rho_t(y-z)-\rho_t(z_j-z)\right|^\alpha dy\right)^{1/\alpha}dz.\end{aligned}$$

Therefore, we have to estimate

$$U(A,M,L)\le\frac{1}{2A}\int_{-A}^{A}|D_{A,M}(z)|\sum_{j=-L}^{L-1}\left(\int_{z_j}^{z_{j+1}}\left|\rho_t(y-z)-\rho_t(z_j-z)\right|^\alpha dy\right)^{1/\alpha}dz. \tag{5.5}$$

Note that $\rho_t(y-z)-\rho_t(z_j-z)=\varphi(t-(y-z))-\varphi(t-(z_j-z))-(\varphi(y-z)-\varphi(z_j-z))$. Let us denote $I(\varphi,L,z)=\sum_{j=-L}^{L-1}(\int_{z_j}^{z_{j+1}}|\varphi(y-z)-\varphi(z_j-z)|^\alpha dy)^{1/\alpha}$ and remark that

$$\sum_{j=-L}^{L-1}\left(\int_{z_j}^{z_{j+1}}\left|\rho_t(y-z)-\rho_t(z_j-z)\right|^\alpha dy\right)^{1/\alpha}\le I\left(\check\varphi,L,z+t\right)+I\left(\varphi,L,z\right),$$

where $\check\varphi(y)=\varphi(-y)$.

Note that if $\varphi(z)=(-z)_+^{H-1/\alpha}$ (or $\varphi(z)=(-z)_-^{H-1/\alpha}$) some of the summands in $I(\check\varphi,L,z+t)$ and $I(\varphi,L,z)$ are zero. Moreover, since the estimates in these two cases as well as the symmetric case $\varphi(z)=|z|^{H-1/\alpha}$ are quite similar, we will only consider the symmetric case and leave the details of the other cases to the reader.

Since φ is $2A$ periodic, $I(\varphi,L,\cdot)$ is also $2A$ periodic and we just have to analyze $J(L,z)=I(\varphi,L,z)$ for $-A\le z<A$. Let us denote $j_0=j_0(z)=[(L/A)z]$ where $[v]$ denotes the largest integer less or equal to v. Then $-L\le j_0\le L-1$ and we decompose further $J(L,z)$ (setting $\sum_{j=a}^{b}=0$ if $b<a$) as

$$
\begin{aligned}
J(L,z) = &\sum_{j=-L}^{j_0-1} \left(\int_{z_j}^{z_{j+1}} \left| \varphi(y-z) - \varphi(z_j - z) \right|^\alpha dy \right)^{1/\alpha} \\
&+ \left(\int_{z_{j_0}}^{z_{j_0+1}} \left| \varphi(y-z) - \varphi(z_{j_0} - z) \right|^\alpha dy \right)^{1/\alpha} \\
&+ \left(\int_{z_{j_0+1}}^{z_{j_0+2}} \left| \varphi(y-z) - \varphi(z_{j_0+1} - z) \right|^\alpha dy \right)^{1/\alpha} \\
&+ \sum_{j=j_0+2}^{L-1} \left(\int_{z_j}^{z_{j+1}} \left| \varphi(y-z) - \varphi(z_j - z) \right|^\alpha dy \right)^{1/\alpha} \\
= &\, I_1(z) + I_2(z) + I_3(z) + I_4(z).
\end{aligned}
$$

Observe that if $j \le j_0 - 1$ we have $z_j < z_{j+1} \le z_{j_0} \le z$ and hence for $z_j \le y \le z_{j+1}$ we have $z_j - z \le y - z \le z_{j+1} - z \le 0$. Since $H < 1$ we have $H - 1 - 1/\alpha < 0$ and by the mean value theorem we get in the present case that

$$
\begin{aligned}
|\varphi(y-z) - \varphi(z_j - z)| &= (z - z_j)^{H-1/\alpha} - (z-y)^{H-1/\alpha} \\
&\le (H-1/\alpha)(z - z_{j+1})^{H-1-1/\alpha}(y-z_j).
\end{aligned}
$$

This implies

$$
\begin{aligned}
I_1(z) &= \sum_{j=-L}^{j_0-1} \left(\int_{z_j}^{z_{j+1}} \left| \varphi(z_j - z) - \varphi(y-z) \right|^\alpha dy \right)^{1/\alpha} \\
&\le (H - 1/\alpha) \sum_{j=-L}^{j_0-1} (z - z_{j+1})^{H-1-1/\alpha} \left(\int_{z_j}^{z_{j+1}} (y - z_j)^\alpha \, dy \right)^{1/\alpha} \\
&\le (H - 1/\alpha)(\alpha+1)^{-1/\alpha} \left(\frac{A}{L} \right)^{1+1/\alpha} \sum_{j=-L}^{j_0-1} (z - z_j)^{H-1-1/\alpha} \\
&\le 2(H - 1/\alpha)(\alpha+1)^{-1/\alpha} \left(\frac{A}{L} \right)^{1+1/\alpha} \int_{-L}^{(L/A)z} \big(z - x(A/L) \big)^{H-1-1/\alpha} dx \\
&= 2(\alpha+1)^{-1/\alpha} \left(\frac{A}{L} \right)^{1+1/\alpha} \frac{L}{A} (z+A)^{H-1/\alpha} \le 2 \cdot 2^{H-1/\alpha} (\alpha+1)^{-1/\alpha} A^H L^{-1/\alpha}.
\end{aligned}
$$

Since $H - 1/\alpha \in (0,1)$ the function φ is Hölder of order $H - 1/\alpha$ on $[-A, A)$ with $|\varphi(x) - \varphi(y)| \le |x-y|^{H-1/\alpha}$ for all $-A \le x, y < A$. Since $z \in [z_{j_0}, z_{j_0+1})$

$$
I_2(z) = \left(\int_{z_{j_0}}^{z_{j_0+1}} \left| \varphi(y-z) - \varphi(z_{j_0} - z) \right|^\alpha dy \right)^{1/\alpha} \le (A/L)^H.
$$

Similarly, $I_3(z) \le (A/L)^H$. Moreover, since $H > 1/\alpha$ we get $I_2(z) + I_3(z) \le 2A^H L^{-1/\alpha}$. For $I_4(z)$, observe that if $j \ge j_0 + 1$ we have $z_j \ge z_{j_0+1} > z$ so for $z_j \le y \le z_{j+1}$ we have using the mean value theorem again

$$\begin{aligned}|\varphi(y-z) - \varphi(z_j-z)| &= (y-z)^{H-1/\alpha} - (z_j-z)^{H-1/\alpha}\\ &\le (H-1/\alpha)(z_j-z)^{H-1-1/\alpha}(y-z_j).\end{aligned}$$

This implies

$$\begin{aligned}I_4(z) &= \sum_{j=j_0+2}^{L-1} \left(\int_{z_j}^{z_{j+1}} \left|\varphi(z_j-z) - \varphi(y-z)\right|^\alpha dy\right)^{1/\alpha}\\
&\le (H-1/\alpha) \sum_{j=j_0+2}^{L-1} (z_j-z)^{H-1-1/\alpha} \left(\int_{z_j}^{z_{j+1}} (y-z_j)^\alpha\, dy\right)^{1/\alpha}\\
&\le (H-1/\alpha)(\alpha+1)^{-1/\alpha}\left(\frac{A}{L}\right)^{1+1/\alpha} \sum_{j=j_0+2}^{L-1} (z_j-z)^{H-1-1/\alpha}\\
&\le 2(H-1/\alpha)(\alpha+1)^{-1/\alpha}\left(\frac{A}{L}\right)^{1+1/\alpha} \int_{(L/A)z}^{L-1} \big((A/L)x - z\big)^{H-1-1/\alpha}\, dx\\
&= 2(\alpha+1)^{-1/\alpha}\left(\frac{A}{L}\right)^{1+1/\alpha} \frac{L}{A}(A-z)^{H-1/\alpha}\\
&\le 2\cdot 2^{H-1/\alpha}(\alpha+1)^{-1/\alpha} A^H L^{-1/\alpha}.\end{aligned}$$

Therefore, we have for $-A \le z \le A$ that

$$J(L,z) \le 2\left(2^{1+H-1/\alpha}(\alpha+1)^{-1/\alpha} + 1\right) A^H L^{-1/\alpha} = C_5(H,\alpha) A^H L^{-1/\alpha}.$$

Finally, in view of (5.5) we get

$$U(A,M,L) \le 2C_5(H,\alpha) A^H L^{-1/\alpha} \frac{1}{2A}\int_{-A}^{A} |D_{A,M}(z)|\, dz.$$

Note further that we have

$$\begin{aligned}\frac{1}{2A}\int_{-A}^{A} \left|D_{A,M}(x)\right| dx &= \frac{1}{\pi}\int_{-\frac{2M+1}{2}\pi}^{\frac{2M+1}{2}\pi} \frac{|\sin(x)|}{|(2M+1)\sin(x/(2M+1)|}\, dx\\
&= \frac{1}{\pi}\int_{-\frac{2M+1}{2}\pi}^{\frac{2M+1}{2}\pi} \frac{|\sin(x)|}{|x|}\left|\frac{x/(2M+1)}{\sin(x/(2M+1))}\right| dx\\
&\le \frac{1}{2}\int_{-\frac{2M+1}{2}\pi}^{\frac{2M+1}{2}\pi} \frac{|\sin(x)|}{|x|}\, dx,\end{aligned}$$

using the fact that $|\sin(x)| \geq \frac{2}{\pi}|x|$ for $|x| \leq \frac{\pi}{2}$. Moreover,

$$\frac{1}{2}\int_{-\frac{2M+1}{2}\pi}^{\frac{2M+1}{2}\pi} \frac{|\sin(x)|}{|x|}\,dx \leq \frac{1}{2}\int_{-(M+1)\pi}^{(M+1)\pi} \frac{|\sin(x)|}{|x|}\,dx = \sum_{k=0}^{M}\int_0^{\pi} \frac{\sin(x)}{x+k\pi}\,dx$$

$$\leq \pi + \frac{2}{\pi}\left(1+\log(M)\right).$$

Hence, $\frac{1}{2A}\int_{-A}^{A}\left|D_{A,M}(x)\right|\,dx \leq 2\log(M)$, for $M \geq e^4$, and

$$U(A,M,L) \leq 4C_5(H,\alpha)A^H L^{-1/\alpha}\log(M) = C_4(H,\alpha)A^H L^{-1/\alpha}\log(M),$$

which concludes the proof. □

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