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Hölder regularity for operator scaling stable random fields

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Abstract

We investigate the sample path regularity of operator scaling α-stable random fields. Such fields were introduced in [H. Biermé, M.M. Meerschaert, H.P. Scheffler, Operator scaling stable random fields, Stochastic Process. Appl. 117 (3) (2007) 312–332.] as anisotropic generalizations of self-similar fields and satisfy the scaling property $\{X(c^E x); x \in \mathbb{R}^d\} \stackrel{(fdd)}{=} \{c^H X(x); x \in \mathbb{R}^d\}$ where E is a $d \times d$ real matrix and $H > 0$. In the case of harmonizable operator scaling random fields, the sample paths are locally Hölderian and their Hölder regularity is characterized by the eigen decomposition of $\mathbb{R}^d$ with respect to E. In particular, the directional Hölder regularity may vary and is given by the eigenvalues of E. In the case of moving average operator scaling α-stable random fields, with $\alpha \in (0, 2)$ and $d \geq 2$, the sample paths are almost surely discontinuous.

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1. Introduction

In this paper we consider operator scaling stable random fields as introduced in [1]. More precisely, if E is a real $d \times d$ matrix whose eigenvalues have positive real parts, a scalar-valued

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random field $(X(x))_{x\in\mathbb{R}^d}$ is called *operator scaling* for E and $H > 0$ if

$$\forall c > 0, \quad \{X(c^E x); x \in \mathbb{R}^d\} \overset{(fdd)}{=} \{c^H X(x); x \in \mathbb{R}^d\}, \tag{1}$$

where $\overset{(fdd)}{=}$ means equality of finite dimensional distributions and as usual $c^E = \exp(E \log c)$ with $\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}$ the matrix exponential. These fields can be seen as anisotropic generalizations of self-similar random fields. Let us recall that a scalar-valued random field $(X(x))_{x\in\mathbb{R}^d}$ is said to be H-*self-similar* with $H > 0$ if

$$\forall c > 0, \quad \{X(cx); x \in \mathbb{R}^d\} \overset{(fdd)}{=} \{c^H X(x); x \in \mathbb{R}^d\}.$$

Then a H-self-similar field is also an operator scaling field for the identity matrix $E = I_d$ of size $d \times d$. Numerous natural phenomena have been shown to be self-similar. For instance, self-similar random fields are required to model persistent phenomena in internet traffic, hydrology, geophysics or financial markets, e.g. [2–5]. A very important class of such fields are given by Gaussian random fields and especially by fractional Brownian fields. The fractional Brownian field B_H, where $H \in (0, 1)$ is the so-called Hurst parameter, is H-self-similar and has stationary increments, i.e. $\{B_H(x + h) - B_H(h); x \in \mathbb{R}^d\} \overset{(fdd)}{=} \{B_H(x); x \in \mathbb{R}^d\}$ for any $h \in \mathbb{R}^d$. It is an isotropic generalization of the famous fractional Brownian motion, implicitly introduced in [6] and defined in [7].

However, the isotropy property is a serious drawback for many applications in medicine [8,9], in geophysics [10] and in hydrology [11], just to mention a few. In particular, [8,9] introduce two classes of anisotropic Gaussian random fields for X-ray pictures of bones modeling, to help for diagnosis of osteoporosis. More precisely, [8] proposes to use the fractional Brownian sheet which exhibits different scaling properties in the d orthogonal directions that characterize its anisotropy. The fractional Brownian sheet, first introduced in [12], is operator scaling for a diagonal matrix but it does not have stationary increments, which is a natural assumption since bones can be considered as homogeneous materials. Therefore, [9] introduces anisotropic Brownian fields which have stationary increments. Proposition 5 of [9] shows that the directional regularity of any Gaussian random field with stationary increments is constant except maybe on a hyperplane of dimension at most $d - 1$. However, the sample path regularity of the Brownian fields studied in [9] does not depend on any direction. The Gaussian random fields introduced in [1] illustrate Proposition 5 of [9]. These random fields were introduced to model sedimentary aquifers, which exhibit different scaling properties in different directions and not necessarily orthogonal ones (see [11]). They have stationary increments, satisfy the operator scaling property (1) and their anisotropic behavior is driven by a $d \times d$ matrix E, not necessarily diagonal. Moreover, both Gaussian and α-stable operator scaling random fields are defined in [1]. Actually, Gaussian random fields are not convenient for some heavy tails phenomena modeling. For this purpose, α-stable random fields have been introduced. Let us recall that a scalar-valued random field $\{X(x); x \in \mathbb{R}^d\}$ is symmetric α-stable (SαS), for $\alpha \in (0, 2)$, if any linear combination $\sum_{k=1}^{n} a_k X(x_k)$ is SαS. We address to [4] for a well understanding of such fields. Self-similar isotropic α-stable fields with stationary increments have been extensively used to propose alternative to Gaussian modeling (see [13,5] for instance). Then, operator scaling stable random fields, as defined by [1], are well fitted to mimic persistent, heavy-tailed and anisotropic phenomena.

Two different classes of such fields are defined in [1], using a moving average representation as well as an harmonizable one. In the Gaussian case $\alpha = 2$, according to [1] there exist modifications of these fields which are almost surely Hölder-continuous of certain indices. We give similar results here in the stable case $\alpha \in (0, 2)$ for harmonizable operator scaling stable random fields. Actually, we obtain their critical global and directional Hölder exponents, which are given by the eigenvalues of E. In general, such fields are anisotropic and their sample path properties varies with the direction. In particular, in the case where E is diagonalizable, for any eigenvector θ_j associated with the real eigenvalue λ_j, harmonizable operator scaling stable random fields admit $H_j = 1/\lambda_j$ as critical Hölder exponent in direction θ_j. Let us point out that we establish an accurate upper bound for the modulus of continuity. Such upper bound has already been given in the case of real harmonizable fractional stable motions ($d = 1$) in [14] and in the case of some Gaussian random processes in [15]. Then, in this paper, we generalize these results to any dimension d and any harmonizable operator scaling stable fields. We also obtain such an upper bound in the case of Gaussian operator scaling random fields, which improves the sample path properties established in [1].

Furthermore, whereas in the Gaussian case $\alpha = 2$, moving average and harmonizable fields have the same kind of sample path regularity properties, this is no more true in the case $\alpha \in (0, 2)$. In particular, we show that for $d \geq 2$, a moving average operator scaling stable random field does not admit any continuous modification. Remark that if $d = 1$, the sample path regularity properties are already known since the processes studied are self-similar moving average stable processes, see for example [4,14,16].

One of the main tools for the study of sample paths of operator scaling random fields is the change of polar coordinates with respect to the matrix E introduced in [17]. If X is a Gaussian operator scaling random field with stationary increments, using (1), we can write its variogram as

$$v^2(x) = \mathbb{E}\left(X^2(x)\right) = \tau_E(x)^{2H}\,\mathbb{E}\left(X^2(\ell_E(x))\right),$$

where $\tau_E(x)$ is the radial part of x with respect to E and $\ell_E(x)$ is its polar part. Therefore, in the Gaussian case, the sample path regularity depends on the behavior of the polar coordinates $(\tau_E(x), \ell_E(x))$ around $x = 0$. Such property also holds in the stable case $\alpha \in (0, 2)$. The Hölder sample path regularity properties follow from estimates of $\tau_E(x)$ compared to $\|x\|$. These estimates are given in Section 3 and their proofs are postponed to the Appendix.

Furthermore, the other main tool we use to study the sample paths of harmonizable operator scaling α-stable random fields is a series representation. Representations in series of infinitely divisible laws have been studied in [18–21]. As in [14], our study is based on a LePage series representation. Actually, the main idea is to choose a representation which is a conditionally Gaussian series.

In Section 2, we recall the definition of harmonizable operator scaling random fields. Then, Sections 3 and 4 are devoted to the main tools we need for the study of their sample path regularity. More precisely, Section 3 deals with the polar coordinates with respect to a matrix E and Section 4 gives the LePage series representation. In Section 5, the sample path properties of harmonizable operator scaling random fields and the Hausdorff dimension of their graph are given. Section 6 is concerned with moving average operator scaling random fields.

2. Harmonizable representation

Let us recall the definition of harmonizable operator scaling stable random fields, introduced by [1]. Let us stress that the parametrization used in this paper is not the same one as in [1], see Remark 2.1.

Let E be a real $d \times d$ matrix. Let $\lambda_1, \ldots, \lambda_d$ be the complex eigenvalues of E and $a_j = \Re\left(\lambda_j\right)$ for each $j = 1, \ldots, d$. We assume that

$$\min_{1 \le j \le d} a_j > 1. \tag{2}$$

Let $\psi : \mathbb{R}^d \to [0, \infty)$ be a continuous, E^t-homogeneous function, which means according to Definition 2.6 of [1] that

$$\psi(c^{E^t} x) = c\psi(x) \quad \text{for all } c > 0 \text{ and } x \in \mathbb{R}^d.$$

Moreover, we assume that $\psi(x) \neq 0$ for $x \neq 0$. Such functions were studied in detail in [17], Chapter 5 and various examples are given in Theorem 2.11 and Corollary 2.12 of [1].

Let $0 < \alpha \le 2$ and $W_\alpha(\mathrm{d}\xi)$ be a complex isotropic α-stable random measure on $\mathbb{R}^d$ with Lebesgue control measure (see [4] p.281). If $\alpha = 2$, $W_\alpha(\mathrm{d}\xi)$ is a complex isotropic Gaussian random measure. Let $q = \operatorname{trace}(E)$.

Definition 2.1. The random field

$$X_\alpha(x) = \Re \int_{\mathbb{R}^d} \left(\mathrm{e}^{\mathrm{i}\langle x, \xi\rangle} - 1\right) \psi(\xi)^{-1-q/\alpha} W_\alpha(\mathrm{d}\xi), \quad x \in \mathbb{R}^d, \tag{3}$$

is called harmonizable operator scaling stable random field.

Remark 2.1. As already mentioned, (3) is not exactly the representation used in [1] to define an harmonizable operator scaling stable random field. However, the class of random fields defined by (3) and the class of harmonizable operator scaling stable random fields defined in [1] are the same. More precisely, let $\tilde{E}$ be a real $d \times d$ matrix, $\tilde{q} = \operatorname{tr}\tilde{E}$ and $\tilde{\psi} : \mathbb{R}^d \to [0, +\infty[$ a $\tilde{E}^t$-homogeneous function. For some convenient $H > 0$, let us consider

$$X_{\tilde{\psi}}(x) = \Re \int_{\mathbb{R}^d} \left(\mathrm{e}^{\mathrm{i}\langle x, \xi\rangle} - 1\right) \tilde{\psi}(\xi)^{-H-\tilde{q}/\alpha} W_\alpha(\mathrm{d}\xi), \quad x \in \mathbb{R}^d, \tag{4}$$

an harmonizable operator scaling stable random field as defined in [1]. Let $\psi = \tilde{\psi}^H$ and $E = \tilde{E}/H$. Then, ψ is a E^t-homogeneous function and

$$X_{\tilde{\psi}}(x) = \Re \int_{\mathbb{R}^d} \left(\mathrm{e}^{\mathrm{i}\langle x, \xi\rangle} - 1\right) \psi(\xi)^{-1-q/\alpha} W_\alpha(\mathrm{d}\xi) = X_\alpha(x)$$

with X_α defined by (3). Furthermore, $\tilde{\psi}$, $\tilde{E}$ and H satisfy the assumptions of Theorem 4.1 of [1] if and only if ψ and E satisfy our assumptions.

Remark 2.2. For notational sake of simplicity we denote the kernel function by

$$f(x, \xi) = \left(\mathrm{e}^{\mathrm{i}\langle x, \xi\rangle} - 1\right) \psi(\xi)^{-1-q/\alpha}, \quad \xi \in \mathbb{R}^d \setminus \{0\}. \tag{5}$$

Let us remark that, since (2) is fulfilled, $f(x, \cdot) \in L^\alpha(\mathbb{R}^d)$ for any $x \in \mathbb{R}^d$, which is a necessary and sufficient condition for X_α to be well-defined by (3). Moreover, from Corollary 4.2 of [1], X_α has stationary increments and satisfies the following operator scaling property

$$\forall \varepsilon > 0, \quad \left\{X_\alpha(\varepsilon^E x); x \in \mathbb{R}^d\right\} \stackrel{(fdd)}{=} \left\{\varepsilon X_\alpha(x); x \in \mathbb{R}^d\right\}. \tag{6}$$

Note that, for any $H > 0$, since $X_\alpha = X_{\psi^{1/H}}$ is also defined by (4),

$$\forall c > 0, \quad \left\{X_\alpha(c^{\tilde{E}} x); x \in \mathbb{R}^d\right\} \overset{(fdd)}{=} \left\{c^H X_\alpha(x); x \in \mathbb{R}^d\right\}, \tag{7}$$

with $\tilde{E} = HE$, according to Corollary 4.2 of [1]. Actually, (7) is simply obtained from (6) choosing $c = \varepsilon^{1/H}$. This is related Remark 4.4 of [1].

Now, let us give some examples of operator scaling harmonizable stable random fields.

Example 2.1. Let I_d be the identity matrix of size $d \times d$, $H \in (0, 1)$, $E = I_d/H$ and $\psi(x) = \|x\|^H$ with $\|\cdot\|$ the Euclidean norm. Then the random field X_α defined by (3) is a real harmonizable stable random field (see [4] for details on such fields). In this case, X_α satisfies (6) for $E = I_d/H$ and (7) for $\tilde{E} = I_d$, which means that

$$\forall c > 0, \quad \{X_\alpha(cx); x \in \mathbb{R}^d\} \overset{(fdd)}{=} \{c^H X_\alpha(x); x \in \mathbb{R}^d\},$$

i.e. X_α is self-similar with exponent H. Let us quote that, if $\alpha = 2$, X_α is a fractional Brownian field and its critical Hölder exponent is given by its Hurst index H (see Theorem 8.3.2 of [22] for instance).

Example 2.2. Assume that E is diagonalizable and that all its eigenvalues are real, denoted by $\left(a_j\right)_{1\le j\le d}$. Let $(\theta_j)_{1\le j\le d}$ be a basis of some corresponding eigenvectors and consider the function ψ defined by

$$\psi(x) = \left(\sum_{j=1}^{d} \left|\langle x, \theta_j\rangle\right|^{2/a_j}\right)^{1/2}, \quad x \in \mathbb{R}^d.$$

The function ψ is clearly continuous and non-negative on $\mathbb{R}^d$. Moreover, since

$$\langle c^{E^t} x, \theta_j\rangle = \langle x, c^E \theta_j\rangle = c^{a_j}\langle x, \theta_j\rangle,$$

ψ is also E^t-homogeneous. Finally, $\psi(x) = 0$ if and only if $x = 0$, since $(\theta_j)_{1\le j\le d}$ is a basis of $\mathbb{R}^d$. Then we can define X_α by (3). For all $j = 1, \dots, d$, the operator scaling property (6), applied with $\varepsilon = c^{1/a_j}$, implies that

$$\forall c > 0, \quad \left\{X_\alpha\left(ct\theta_j\right); t \in \mathbb{R}\right\} \overset{(fdd)}{=} \left\{c^{1/a_j} X_\alpha\left(t\theta_j\right); t \in \mathbb{R}\right\},$$

since $\left(c^{1/a_j}\right)^E \theta_j = \left(c^{1/a_j}\right)^{a_j} \theta_j = c\theta_j$. Therefore, the random field X_α is self-similar with exponent $H_j = 1/a_j$ in the direction θ_j. In particular, in the Gaussian case ($\alpha = 2$), the process $\left(X_2\left(t\theta_j\right)\right)_{t\in\mathbb{R}}$ is a fractional Brownian motion with Hurst index H_j and its critical Hölder exponent is equal to H_j.

One of the main tool in the study of operator scaling random fields is the change of coordinates in a kind of polar coordinates with respect to the matrix E. Then, before we study the sample path regularity of X_α, we recall in the next section the definition of these coordinates and give some estimates of the radial part.

3. Polar coordinates

According to Chapter 6 of [17], since E is a real $d \times d$ matrix whose eigenvalues have positive real parts, there exists a norm $\|\cdot\|_E$ on $\mathbb{R}^d$ such that the map

$$\begin{aligned} \Psi_E : \quad (0,\infty) \times S_E &\longrightarrow \mathbb{R}^d \backslash \{0\} \\ (r, \theta) &\longmapsto r^E \theta \end{aligned}$$

is a homeomorphism, where

$$S_E = \{x \in \mathbb{R}^d : \|x\|_E = 1\} \tag{8}$$

is the unit sphere for $\|\cdot\|_E$. Hence we can write any $x \in \mathbb{R}^d \backslash \{0\}$ uniquely as

$$x = \tau_E(x)^E \ell_E(x) \tag{9}$$

with $\tau_E(x) > 0$ and $\ell_E(x) \in S_E$. Here, for any $x \in \mathbb{R}^d \backslash \{0\}$, $\tau_E(x)$ should be interpreted as the *radial part* of x and $\ell_E(x) \in S_E$ as its *directional part*. Moreover, $x \mapsto \tau_E(x)$ and $x \mapsto \ell_E(x)$ are continuous maps. We also know that $\tau_E(x) \to \infty$ as $x \to \infty$ and $\tau_E(x) \to 0$ as $x \to 0$. Hence we can extend τ_E continuously by setting $\tau_E(0) = 0$. Finally, we can observe that $S_E = \{x \in \mathbb{R}^d : \tau_E(x) = 1\}$ is a compact set and define

$$m_E = \min_{S_E} \|x\| \quad \text{and} \quad M_E = \max_{S_E} \|x\|. \tag{10}$$

Let us now recall the formula of integration in *polar coordinates* established in [1].

Proposition 3.1. *There exists a unique finite Radon measure σ_E on the unit sphere S_E defined by (8) such that for all $f \in L^1(\mathbb{R}^d, \mathrm{d}x)$,*

$$\int_{\mathbb{R}^d} f(x)\mathrm{d}x = \int_0^\infty \int_{S_E} f(r^E\theta)\sigma_E(\mathrm{d}\theta)r^{q-1}\mathrm{d}r.$$

As already mentioned in the introduction, the Hölder sample path regularity properties of X_α follow from estimates of $\tau_E(x)$ compared to $\|x\|$ around $x = 0$, i.e. from the Hölder regularity of τ_E around 0, see [1]. Then, in order to get an upper bound for the modulus of continuity (for any α), we need some precise estimates of $\tau_E(x)$.

As done in [23] for the study of operator-self-similar Gaussian random fields we use the Jordan decomposition of the matrix E to get estimates of τ_E. From the Jordan decomposition's theorem (see [24] p. 129 for instance), there exists a real invertible $d \times d$ matrix P such that $D = P^{-1}EP$ is of the real canonical form, which means that D is composed of diagonal blocks which are either Jordan cell matrix of the form

$$\begin{pmatrix} \lambda & 0 & \dots & 0 \\ 1 & \lambda & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 1 & \lambda \end{pmatrix}$$

with λ a real eigenvalue of E or blocks of the form

$$\begin{pmatrix} \Lambda & 0 & \dots & \dots & 0 \\ I_2 & \Lambda & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & I_2 & \Lambda \end{pmatrix} \quad \text{with } \Lambda = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \text{ and } I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{11}$$

where the complex numbers $a \pm \mathrm{i}b$ ($b \neq 0$) are complex conjugated eigenvalues of E.

Let us denote by $\|\cdot\|$ the subordinated norm of the Euclidean norm on the matrix space. Precise estimates of τ_E follow from the next lemma.

Lemma 3.2. *Let J be either a Jordan cell matrix of size l or a block of the form (11) of size $2l$ associated with the eigenvalue $\lambda \in \mathbb{C}$. Then, for any $t \in (0, e^{-1}] \cup [e, +\infty)$*

$$t^a \leq \|t^J\| \leq \sqrt{2l} e t^a \left|\log t\right|^{l-1}$$

with $a = \Re(\lambda)$.

Proof. See the Appendix. $\square$

Let us be more precise on the Jordan decomposition of E.

Notation. Let us recall that the eigenvalues of E are denoted by λ_j, $j = 1, \ldots, d$ and that $a_j = \Re\left(\lambda_j\right) > 1$ for $j = 1, \ldots, d$. There exist $J_1, \ldots, J_p$, where each J_j is either a Jordan cell matrix or a block of the form (11), and P a real $d \times d$ invertible matrix such that

$$E = P \begin{pmatrix} J_1 & 0 & \ldots & 0 \\ 0 & J_2 & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & J_p \end{pmatrix} P^{-1}.$$

We can assume that each J_j is associated with the eigenvalue λ_j of E and that

$$1 < a_1 \leq \cdots \leq a_p.$$

We also set $H_j = a_j^{-1}$ and have

$$0 < H_p \leq \cdots \leq H_1 < 1. \tag{12}$$

If $\lambda_j \in \mathbb{R}$, J_j is a Jordan cell matrix of size $\tilde{l_j} = l_j \in \mathbb{N}\backslash\{0\}$. If $\lambda_j \in \mathbb{C}\backslash\mathbb{R}$, J_j is a block of the form (11) of size $\tilde{l_j} = 2l_j \in 2\mathbb{N}\backslash\{0\}$. Then for any $t > 0$,

$$t^E = P \begin{pmatrix} t^{J_1} & 0 & \ldots & 0 \\ 0 & t^{J_2} & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & t^{J_p} \end{pmatrix} P^{-1}.$$

We denote by $(e_1, \ldots, e_d)$ the canonical basis of $\mathbb{R}^d$ and set $f_j = Pe_j$ for every $j = 1, \ldots, d$. Hence, $(f_1, \ldots, f_d)$ is a basis of $\mathbb{R}^d$. For all $j = 1, \ldots, p$, let

$$W_j = \operatorname{span}\left(f_k; \sum_{i=1}^{j-1} \tilde{l_i} + 1 \leq k \leq \sum_{i=1}^{j} \tilde{l_j}\right). \tag{13}$$

Then, each W_j is a E-invariant set and $\mathbb{R}^d = \bigoplus_{j=1}^{p} W_j$.

The following result gives bounds on the growth rate of $\tau_E(x)$ in terms of the real parts of the eigenvalues of E.

Proposition 3.3. *For any $1 \leq k \leq p$, let W_k be the E-invariant subspace of dimension l_k or $2l_k$ associated with H_k^{-1} by (13). Then, for any $r \in (0,1)$ there exist some finite positive constant $c_1, c_2 > 0$ such that for every $1 \leq j_0 \leq j \leq p$,*

$$c_1 \|x\|^{H_{j_0}} |\log \|x\||^{-(p_{j_0,j}-1)H_{j_0}} \leq \tau_E(x) \leq c_2 \|x\|^{H_j} |\log \|x\||^{(p_{j_0,j}-1)H_j}$$

holds for any $x \in \oplus_{k=j_0}^{j} W_k \backslash \{0\}$ with $\|x\| \leq r$ and $p_{j_0,j} = \max_{j_0 \leq k \leq j} l_k$.

Proof. See the Appendix. □

Then, we easily deduce the following corollary.

Corollary 3.4. *For any $1 \leq k \leq p$, let W_k be the E-invariant subspace of dimension l_k or $2l_k$ associated with H_k^{-1} by (13). Then, for any $r \in (0,1)$ there exist some finite positive constant $c_1, c_2 > 0$ such that for any $x \in W_j \setminus \{0\}$, $1 \leq j \leq p$, with $\|x\| \leq r$*

$$c_1 \|x\|^{H_j} |\log \|x\||^{-(l_j-1)H_j} \leq \tau_E(x) \leq c_2 \|x\|^{H_j} |\log \|x\||^{(l_j-1)H_j}$$

and for any $x \in \mathbb{R}^d \backslash \{0\}$ with $\|x\| \leq r$

$$c_1 \|x\|^{H_1} |\log \|x\||^{-(l-1)H_1} \leq \tau_E(x) \leq c_2 \|x\|^{H_p} |\log \|x\||^{(l-1)H_p},$$

where $l = \max_{1 \leq j \leq p} l_j$.

Therefore we have precise estimates for the Hölder regularity of the radial part. Let us remark that we improve the first statement of Lemma 2.1 of [1] and that the second one can also be improved in a similar way. From these estimates we deduce the Hölder regularity of X_α in Section 5. As already mentioned, the study of the sample paths is based on a series representation. Then, before we state regularity properties, it remains to give the LePage series representation of harmonizable operator scaling stable random fields.

4. Representation as a LePage series

An overview on series representations of infinitely divisible distributions without Gaussian part can be found for example in [21,25] and references therein. In particular, LePage series representation [18,19] have been used in [26,14] to study the sample path regularity of some self-similar α-stable random motions with $\alpha \in (0,2)$. Here, this representation is also the main representation we use in the case $\alpha \in (0,2)$. Actually, in the Gaussian case $\alpha = 2$, such representation does not hold.

Let us now introduce some notations we will use throughout the paper. Let μ be an arbitrary probability measure absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$ and let m be its Radon–Nikodym derivative that is $\mu(\mathrm{d}\xi) = m(\xi)\,\mathrm{d}\xi$.

Notation. Let $(T_n)_{n \geq 1}$, $(g_n)_{n \geq 1}$ and $(\xi_n)_{n \geq 1}$ be independent sequences of random variables.

- T_n is the nth arrival time of a Poisson process with intensity 1.
- $(g_n)_{n \geq 1}$ is a sequence of i.i.d. isotropic complex random variables so that $g_n \overset{(d)}{=} \mathrm{e}^{\mathrm{i}\theta} g_n$ for any $\theta \in \mathbb{R}$. We also assume that $0 < \mathbb{E}(|g_n|^\alpha) < +\infty$.
- $(\xi_n)_{n \geq 1}$ is a sequence of i.i.d. random variables with common law $\mu(\mathrm{d}\xi) = m(\xi)\,\mathrm{d}\xi$.

According to Chapter 3 and Chapter 4 of [4], stochastic integrals with respect to an α-stable random measure Λ can be represented as a LePage series as soon as the control measure of Λ is a finite measure. The next proposition generalizes this representation to a complex isotropic

α-stable random measure W_α with Lebesgue control measure. It is a consequence of Lemma 4.1 of [26], which is a correction of Lemma 1.4 of [27]. This proposition can also be deduced from [21,20], which are concerned with series representations of stochastic integrals with respect to infinitely divisible random measures.

Proposition 4.1. *Let $\alpha \in (0,2)$. Then, for every complex-valued function $h \in L^\alpha\left(\mathbb{R}^d\right)$, the series*

$$Y^h = \sum_{n=1}^{+\infty} T_n^{-1/\alpha} m\left(\xi_n\right)^{-1/\alpha} h\left(\xi_n\right) g_n$$

converges almost surely. Furthermore,

$$C_\alpha Y^h \overset{(d)}{=} \int_{\mathbb{R}^d} h\left(\xi\right) W_\alpha\left(\mathrm{d}\xi\right)$$

with $W_\alpha\left(\mathrm{d}\xi\right)$ a complex isotropic α-stable random measure on $\mathbb{R}^d$ with Lebesgue control measure and

$$C_\alpha = \mathbb{E}\left(\left|\Re\left(g_1\right)\right|^\alpha\right)^{-1/\alpha} \left(\frac{1}{2\pi}\int_0^\pi \left|\cos\left(x\right)\right|^\alpha \mathrm{d}x\right)^{1/\alpha} \left(\int_0^{+\infty} \frac{\sin\left(x\right)}{x^\alpha}\mathrm{d}x\right)^{-1/\alpha}. \tag{14}$$

Remark 4.1. According to Proposition 4.1, taking $\alpha \in (0,2)$, the random measure

$$\Lambda_\alpha\left(d\xi\right) = C_\alpha \sum_{n=1}^{+\infty} T_n^{-1/\alpha} m\left(\xi_n\right)^{-1/\alpha} g_n \delta_{\xi_n}\left(\mathrm{d}\xi\right)$$

is a complex isotropic α-stable random measure with Lebesgue control measure.

Proof. Let $V_n = m\left(\xi_n\right)^{-1/\alpha} h\left(\xi_n\right) g_n$. Then, V_n, $n \geq 1$, are i.i.d. isotropic complex random variables. By Lemma 4.1 in [26], the series Y^h converges almost surely and

$$\forall z \in \mathbb{C}, \quad \mathbb{E}\left(\exp\left(\mathrm{i}\Re\left(\bar{z}Y^h\right)\right)\right) = \exp\left(-\sigma^\alpha \left|z\right|^\alpha\right)$$

with

$$\sigma^\alpha = \mathbb{E}\left(\left|\Re\left(V_1\right)\right|^\alpha\right) \int_0^{+\infty} \frac{\sin\left(x\right)}{x^\alpha}\mathrm{d}x.$$

Since g_1 is invariant by rotation and independent with ξ_1,

$$\mathbb{E}\left(\left|\Re\left(V_1\right)\right|^\alpha\right) = \mathbb{E}\left(m\left(\xi_1\right)^{-1}\left|h\left(\xi_1\right)\right|^\alpha\right)\mathbb{E}\left(\left|\Re\left(g_1\right)\right|^\alpha\right) = \mathbb{E}\left(\left|\Re\left(g_1\right)\right|^\alpha\right)\int_{\mathbb{R}^d}\left|h\left(\xi\right)\right|^\alpha \mathrm{d}\xi.$$

Moreover, by definition of an isotropic α-stable random measure (see [4]),

$$\forall z \in \mathbb{C}, \mathbb{E}\left(\exp\left(\mathrm{i}\Re\left(\bar{z}\int_{\mathbb{R}^d} h\left(\xi\right) W_\alpha\left(\mathrm{d}\xi\right)\right)\right)\right) = \exp\left(-c_\alpha^\alpha\left(h\right)\left|z\right|^\alpha\right)$$

with $c_\alpha^\alpha\left(h\right) = \left(\frac{1}{2\pi}\int_0^\pi \left|\cos\left(x\right)\right|^\alpha \mathrm{d}x\right)\int_{\mathbb{R}^d}\left|h\left(\xi\right)\right|^\alpha \mathrm{d}\xi$. Then,

$$\begin{aligned}\forall z \in \mathbb{C}, \quad \mathbb{E}\left(\exp\left(\mathrm{i}\Re\left(C_\alpha \bar{z} Y^h\right)\right)\right) &= \exp\left(-c_\alpha^\alpha\left(h\right)\left|z\right|^\alpha\right)\\ &= \mathbb{E}\left(\exp\left(\mathrm{i}\Re\left(\bar{z}\int_{\mathbb{R}^d} h\left(\xi\right) W_\alpha\left(\mathrm{d}\xi\right)\right)\right)\right)\end{aligned}$$

with C_α defined by (14). This implies that

$$C_\alpha Y^h \overset{(d)}{=} \int_{\mathbb{R}^d} h(\xi)\, W_\alpha(\mathrm{d}\xi),$$

which concludes the proof. □

From the previous proposition, we deduce the following statement which is the main series representation we use in our investigation.

Proposition 4.2. *Let $\alpha \in (0,2)$. For every $x \in \mathbb{R}^d$, the series*

$$Y_\alpha(x) = C_\alpha \Re\left(\sum_{n=1}^{+\infty} T_n^{-1/\alpha} m(\xi_n)^{-1/\alpha} f(x,\xi_n)\, g_n\right), \tag{15}$$

where f is defined by (5) and C_α by (14), converges almost surely. Furthermore,

$$\left\{Y_\alpha(x);\, x \in \mathbb{R}^d\right\} \overset{(fdd)}{=} \left\{X_\alpha(x);\, x \in \mathbb{R}^d\right\},$$

where X_α is defined by (3).

Proof. From Proposition 4.1, for any $x \in \mathbb{R}^d$, the convergence of the series follows from the fact that $f(x,\cdot) \in L^\alpha\left(\mathbb{R}^d\right)$. The equality of finite dimensional distributions between X_α and Y_α is obtained by linearity of the integral and the sum, which define the fields, and Proposition 4.1. □

Using LePage representation (15) of X_α and the estimates given in Section 3, we give an upper bound for the modulus of continuity of X_α and obtain the critical Hölder regularity of its sample paths in the next section.

5. Hölder regularity and Hausdorff dimension

Throughout this section we fix K a non-empty compact set of $\mathbb{R}^d$ and investigate the Hölder regularity on K of the harmonizable operator scaling stable random field X_α defined by (3).

Let us recall that for the Gaussian case $\alpha = 2$, according to Theorem 5.4 of [1], the Hölder regularity of X_2 depends on the subspaces $\left(W_j\right)_{1\le j\le p}$ defined by (13) and associated with the eigenvalues of E. More precisely, Theorem 5.4 of [1] implies that, when restricted to the subspace W_j, the Gaussian random field $\left\{X_2(x);\, x \in W_j\right\}$ admits H_j as critical Hölder exponent. This follows from the fact that the regularity of X_2 on W_j is determined by the regularity of τ_E around 0 on W_j. Here, we give an upper bound for the modulus of continuity of X_α in the general case $\alpha \in (0,2]$. Then we prove that the critical Hölder exponents are the same as in the Gaussian case $\alpha = 2$. Let us state our main result when $\alpha \in (0,2)$.

Theorem 5.1. *Let $\alpha \in (0,2)$ and X_α be defined by (3). Then, there exists a modification X_α^* of X_α on K such that, with τ_E defined by (9), for any $\varepsilon > 0$*

$$\lim_{\delta\downarrow 0} \sup_{\substack{x,y\in K\\ 0<\|x-y\|\le\delta}} \frac{\left|X_\alpha^*(x) - X_\alpha^*(y)\right|}{\tau_E(x-y)\left|\log \tau_E(x-y)\right|^{1/\alpha+1/2+\varepsilon}} = 0 \quad \textit{almost surely.} \tag{16}$$

This result was proved in the case of harmonizable self-similar stable processes in [14], i.e. in the case of Example 2.1 with $d = 1$. The main idea is to use the LePage series representation (15) where g_n, $n \geq 1$, are Gaussian complex isotropic random variables. It remains to choose the density distribution m of ξ_n. In [14], the authors choose

$$m(\xi) = \frac{c_\eta}{|\xi| \, |\log|\xi||^{1+\eta}}, \quad \xi \in \mathbb{R}\backslash\{0\}$$

where $c_\eta > 0$. A straightforward generalization in higher dimension d leads to choose

$$m(\xi) = \frac{c_\eta}{\|\xi\|^d \, |\log\|\xi\||^{1+\eta}}, \quad \xi \in \mathbb{R}^d\backslash\{0\}.$$

Remark that in this case (i.e. Example 2.1) the matrix $E = I_d/H = E^t$ and that we can choose $\|\cdot\|_{E^t} = \|\cdot\|$. Then, using classical polar coordinates, we obtain that for all $x \neq 0$,

$$(\tau_{E^t}(x), \ell_{E^t}(x)) = \left(\|x\|^H, \frac{x}{\|x\|}\right)$$

and therefore that

$$m(\xi) = \frac{c_\eta}{\tau_{E^t}(\xi)^q \, |\log \tau_{E^t}(\xi)|^{1+\eta}}$$

since $q = \operatorname{trace}(E) = d/H$. Note that by this way m only depends on the radial part τ_{E^t}.

Proof of Theorem 5.1. We can assume without loss of generality that $K = [0, 1]^d$. According to Proposition 4.2, for every $x \in \mathbb{R}^d$

$$Y_\alpha(x) = C_\alpha \Re\left(\sum_{n=1}^{+\infty} T_n^{-1/\alpha} m(\xi_n)^{-1/\alpha} f(x, \xi_n) g_n\right)$$

converges almost surely and $Y_\alpha \overset{(fdd)}{=} X_\alpha$. As already mentioned, we assume that $g_n, n \geq 1$ are Gaussian complex isotropic random variables. Moreover, we choose as density distribution of ξ_n

$$m(\xi) = \frac{c_\eta}{\tau_{E^t}(\xi)^q \, |\log \tau_{E^t}(\xi)|^{1+\eta}}, \quad \xi \in \mathbb{R}^d\backslash\{0\}, \tag{17}$$

where $\tau_{E^t}(\xi)$ is given by (9), $\eta > 0$ and $c_\eta > 0$ is such that $\int_{\mathbb{R}^d} m(\xi)\, d\xi = 1$.

As in the proof of the Kolmogorov–Centsov Theorem (see [28]), we exhibit $(x_k)_k$ a countable dense sequence of elements of K and a finite positive constant C such that for $\tau_E(x_k - x_{k'})$ small enough,

$$|Y_\alpha(x_k) - Y_\alpha(x_{k'})| \leq C \tau_E(x_k - x_{k'}) \, |\log \tau_E(x_k - x_{k'})|^{1/\alpha+1/2+\varepsilon}$$

almost surely. Then, X_α satisfies the same property. Finally, we give a modification X_α^* of X_α for which (16) holds. In the first step, we construct the sequence $(x_k)_k$ and state some useful properties of this sequence.

Step 1. Let $r \in (0, 1)$. By Corollary 3.4, there exist a finite positive constant c_2 and $l \in \mathbb{N}\backslash\{0\}$ such that

$$\tau_E(x) \leq c_2 \|x\|^{H_p} \, |\log\|x\||^{(l-1)H_p}, \tag{18}$$

for any $x \in \mathbb{R}^d\backslash\{0\}$ with $\|x\| \leq r$. Up to change c_2 in (18), we can assume that

$$c_2 d^{H_p/2} 2^{-H_p} (\log 2)^{(l-1)H_p} > 1. \tag{19}$$

For any $k \in \mathbb{N}\backslash\{0\}$, let us choose $\nu_k \in \mathbb{N}\backslash\{0\}$ the smallest positive integer such that

$$c_2 d^{H_p/2} 2^{-\nu_k H_p} (\nu_k \log 2)^{(l-1)H_p} \leq 2^{-k}. \tag{20}$$

This implies that $\lim_{k\to+\infty} \nu_k = +\infty$. Moreover, the definition of ν_k and (19) imply that $\nu_k > 1$ for every $k \in \mathbb{N}\backslash\{0\}$. Therefore, for every $k \in \mathbb{N}\backslash\{0\}$, since $1 \leq \nu_k - 1 < \nu_k$, the definition of ν_k leads to

$$c_2 d^{H_p/2} 2^{-(\nu_k-1)H_p} ((\nu_k - 1)\log 2)^{(l-1)H_p} > 2^{-k}.$$

Hence,

$$2^{-k} \left(2\sqrt{d}\right)^{-H_p} c_2^{-1} < \left(2^{-\nu_k} (\nu_k \log 2)^{l-1}\right)^{H_p} \leq 2^{-k} \left(\sqrt{d}\right)^{-H_p} c_2^{-1}. \tag{21}$$

Then, since $\lim_{k\to+\infty} \nu_k = +\infty$, considering the logarithm of each member of (21), one easily proves that

$$\lim_{k\to+\infty} \frac{k}{\nu_k} = H_p.$$

Hence, there exist two positive finite constants c_3, c_4 such that

$$\forall k \in \mathbb{N}\backslash\{0\}, \quad c_3 2^{k/H_p} k^{l-1} \leq 2^{\nu_k} \leq c_4 2^{k/H_p} k^{l-1}. \tag{22}$$

For every $k \in \mathbb{N}\backslash\{0\}$ and $j = (j_1, \ldots, j_d) \in \mathbb{Z}^d$ we set

$$x_{k,j} = \frac{j}{2^{\nu_k}}, \quad \text{and} \quad \mathcal{D}_k = \left\{x_{k,j} : j \in \mathbb{Z}^d \cap \left[0, 2^{\nu_k}\right]^d\right\}.$$

Let us remark that the sequence $(\mathcal{D}_k)_k$ is increasing and set $\mathcal{D} = \bigcup_{k=1}^{+\infty} \mathcal{D}_k$.

Let us now prove that for k large enough, $\mathcal{D}_k$ is a 2^{-k} net of K for τ_E in the sense that for any $x \in K$ one can find $x_{k,j} \in \mathcal{D}_k$ such that $\tau_E(x - x_{k,j}) \leq 2^{-k}$. Let us fix $x \in K$ and choose j_i such that $j_i \leq 2^{\nu_k} x_i < j_i + 1$ for $1 \leq i \leq d$. Without loss of generality, we can assume that $x \notin \mathcal{D}_k$. Then, $0 < \left\|x - x_{k,j}\right\| \leq 2^{-\nu_k}\sqrt{d}$ and since $\lim_{k\to+\infty} \nu_k = +\infty$, for k large enough,

$$\left\|x - x_{k,j}\right\| \leq 2^{-\nu_k}\sqrt{d} \leq r.$$

Hence, since $t \mapsto t^{H_p} |\log(t)|^{(l-1)H_p}$ is an increasing function on $(0, e^{1-l}]$, (18) and (20) imply that

$$\tau_E\left(x - x_{k,j}\right) \leq 2^{-k}$$

for k large enough. Then, for k large enough, $\mathcal{D}_k$ is a 2^{-k} net of K for τ_E.

Step 2. Almost surely, for any $x, y \in \mathcal{D}$

$$Y_\alpha(x) - Y_\alpha(y) = C_\alpha \Re\left(\sum_{n=1}^{+\infty} T_n^{-1/\alpha} m(\xi_n)^{-1/\alpha} \left(f(x,\xi_n) - f(y,\xi_n)\right) g_n\right),$$

where C_α is defined by (14) and f by (5). Let us consider the random variable

$$R(x,y) = \sum_{n=1}^{+\infty} T_n^{-1/\alpha} m(\xi_n)^{-1/\alpha} \left(f(x,\xi_n) - f(y,\xi_n)\right) g_n.$$

Since the sequences $(T_n)_n$, $(\xi_n)_n$ and $(g_n)_n$ are independent and since $(g_n)_n$ is a sequence of i.i.d. Gaussian complex isotropic random variables, $R(x, y)$ is a Gaussian isotropic complex random variable conditionally to $(T_n, \xi_n)_n$. Remark that

$$Y_\alpha(x) - Y_\alpha(y) = C_\alpha \Re(R(x, y)) \quad \text{almost surely.}$$

Therefore, conditionally to $(T_n, \xi_n)_n$, $Y_\alpha(x) - Y_\alpha(y)$ is a real centered Gaussian random variable with variance

$$\begin{aligned} v^2\left((x, y) \mid (T_n, \xi_n)_n\right) &= \frac{C_\alpha^2}{2} \mathbb{E}\left(|R(x, y)|^2 \mid (T_n, \xi_n)_n\right) \\ &= \frac{C_\alpha^2}{2} \mathbb{E}\left(|g_1|^2\right) \sum_{n=1}^{+\infty} T_n^{-2/\alpha} m(\xi_n)^{-2/\alpha} |f(x - y, \xi_n)|^2, \end{aligned} \tag{23}$$

since $|f(x, \xi_n) - f(y, \xi_n)| = |f(x - y, \xi_n)|$.

As in [15], let

$$\varphi(t) = \sqrt{2Ad \log \frac{1}{t}}, \quad 0 < t < 1 \tag{24}$$

where A is a finite positive constant such that $A > 2/H_p - 1/H_1$.

For $k \in \mathbb{N}\backslash\{0\}$, we consider

$$E_{i,j}^k = \left\{\omega : \left|Y_\alpha\left(x_{k,i}\right) - Y_\alpha\left(x_{k,j}\right)\right| > v\left(\left(x_{k,i}, x_{k,j}\right) \mid (T_n, \xi_n)_n\right) \varphi\left(\tau_E\left(x_{k,i} - x_{k,j}\right)\right)\right\}$$

for any $(i, j) \in \mathbb{Z}^d \times \mathbb{Z}^d$, $i \neq j$, such that $\tau_E\left(x_{k,i} - x_{k,j}\right) < 1$. Then,

$$\mathbb{P}\left(E_{i,j}^k\right) = \mathbb{E}\left(\mathbb{E}\left(\mathbf{1}_{E_{i,j}^k} \mid (T_n, \xi_n)_n\right)\right). \tag{25}$$

Let us give an upper bound of this probability for a well chosen (k, i, j). Let Z be a real centered Gaussian random variable with variance 1. Then, (25) implies that

$$\mathbb{P}\left(E_{i,j}^k\right) = \mathbb{P}\left(|Z| > \varphi\left(\tau_E\left(x_{k,i} - x_{k,j}\right)\right)\right).$$

Let us choose $\delta \in (0, 1)$ and set for $k \in \mathbb{N}\backslash\{0\}$,

$$\delta_k = 2^{-(1-\delta)k} \quad \text{and} \quad I_k = \left\{(i, j) \in \left(\mathbb{Z}^d \cap \left[0, 2^{\nu_k}\right]^d\right)^2 : 0 < \tau_E\left(x_{k,i} - x_{k,j}\right) \le \delta_k\right\}. \tag{26}$$

For every $(i, j) \in I_k$, since φ is a decreasing function

$$\mathbb{P}\left(|Z| > \varphi\left(\tau_E\left(x_{k,i} - x_{k,j}\right)\right)\right) \le \mathbb{P}\left(|Z| > \varphi(\delta_k)\right).$$

We recall that

$$\forall u \ge 0, \quad \mathbb{P}(Z > u) \le \frac{e^{-u^2/2}}{\sqrt{2\pi} u}.$$

Therefore, for every $k \in \mathbb{N}\backslash\{0\}$ and $(i, j) \in I_k$,

$$\mathbb{P}\left(E_{i,j}^k\right) \le \sqrt{\frac{2}{\pi}} \frac{e^{-\varphi^2(\delta_k)/2}}{\varphi(\delta_k)} = \frac{2^{-(1-\delta)kAd}}{\sqrt{\pi Ad(1-\delta) k \log 2}}$$

since $\delta_k = 2^{-(1-\delta)k}$. Hence,

$$\sum_{k=1}^{\infty} \sum_{(i,j)\in I_k} \mathbb{P}\left(E_{i,j}^k\right) \leq \frac{1}{\sqrt{\pi Ad\,(1-\delta)\log 2}} \sum_{k=1}^{+\infty} 2^{-(1-\delta)kAd}\,\mathrm{card}(I_k).$$

Let us give an upper bound of $\mathrm{card}(I_k)$. First, let us remark that one can find a finite positive constant c_5 such that, for any $k \in \mathbb{N}\backslash\{0\}$ and any $x \in \mathbb{R}^d\backslash\{0\}$ satisfying $\tau_E(x) \leq \delta_k$,

$$\|x\| \leq c_5 \tau_E(x)^{1/H_1} \left|\log \tau_E(x)\right|^{l-1}.$$

This inequality is established in the proof of Proposition 3.3, see Eq. (35). Then, since $t \mapsto t^{1/H_1} |\log t|^{l-1}$ is an increasing function on $(0, r_0)$, for a well chosen r_0, there exists a finite positive constant c_6 such that for any $k \in \mathbb{N}\backslash\{0\}$ and any $x \in \mathbb{R}^d\backslash\{0\}$ satisfying $\tau_E(x) \leq \delta_k$,

$$\|x\| \leq c_6 \delta_k^{1/H_1} |\log \delta_k|^{l-1} = ((1-\delta)\log 2)^{l-1} c_6 \delta_k^{1/H_1} k^{l-1}.$$

Hence, one can find a finite positive constant $C > 0$ such that for any $k \in \mathbb{N}\backslash\{0\}$ and any $i \in \mathbb{Z}^d \cap [0, 2^{\nu_k}]^d$,

$$\mathrm{card}\left\{j \in \mathbb{Z}^d \cap \left[0, 2^{\nu_k}\right]^d : (i,j) \in I_k\right\} \leq C\left(\delta_k^{1/H_1} 2^{\nu_k} k^{l-1}\right)^d.$$

By definition of I_k, for every $k \in \mathbb{N}\backslash\{0\}$,

$$\mathrm{card}\, I_k \leq C\left(2^{\nu_k}+1\right)^d \delta_k^{d/H_1} 2^{d\nu_k} k^{d(l-1)}.$$

Then, by (22), there exists a finite constant $C > 0$ such that for all $k \in \mathbb{N}\backslash\{0\}$,

$$\mathrm{card}\, I_k \leq C \delta_k^{d/H_1} 2^{2kd/H_p} k^{3d(l-1)}.$$

Hence, since $\delta_k = 2^{-(1-\delta)k}$

$$\sum_{k=1}^{\infty} \sum_{(i,j)\in I_k} \mathbb{P}\left(E_{i,j}^k\right) \leq \frac{C}{\sqrt{A\,(1-\delta)}} \sum_{k=1}^{\infty} k^{3d(l-1)} 2^{-kd\left(-\frac{2}{H_p}+\frac{1-\delta}{H_1}+(1-\delta)A\right)}$$

with $C > 0$ a finite positive constant. Since $A > \frac{2}{H_p} - \frac{1}{H_1}$, choosing δ small enough, the last inequality implies that

$$\sum_{k=1}^{\infty} \sum_{(i,j)\in I_k} \mathbb{P}\left(E_{i,j}^k\right) < +\infty.$$

By the Borel–Cantelli lemma, almost surely there exists an integer $k^*(\omega)$ such that for every $k \geq k^*(\omega)$,

$$|Y_\alpha(x) - Y_\alpha(y)| \leq v\left((x,y) \mid (T_n, \xi_n)_n\right) \varphi\left(\tau_E(x-y)\right) \tag{27}$$

for all $x, y \in \mathcal{D}_k$, $x \neq y$, with $\tau_E(x-y) \leq \delta_k$.

Step 3. As in [14] let us give an upper bound of the conditional variance $v^2\left((x,y) \mid (T_n,\xi_n)_n\right)$, defined by (23), with respect to $\tau_E(x-y)$. Since f is defined by (5)

$$v^2\left((x,y) \mid (T_n,\xi_n)_n\right) \leq \frac{C_\alpha^2}{2} \mathbb{E}\left(|g_1|^2\right) \sigma^2(\tau_E(x-y)),$$

where, for all $h \geq 0$,

$$\sigma^2(h) = \sum_{n=1}^{+\infty} T_n^{-2/\alpha} m(\xi_n)^{-2/\alpha} \min\left(M_E \left\|h^{E^t}\xi_n\right\|, 2\right)^2 \psi(\xi_n)^{-2-2q/\alpha}, \tag{28}$$

with M_E defined by (10) and m by (17). For the sake of clearness we postpone the proof of the control of $\sigma^2(h)$ in Appendix and state it in the following lemma.

Lemma 5.2. *Let $\eta > 0$, m be the density probability associated with η by (17) and σ^2 be defined by (28) with M_E given by (10). For any $\gamma \in (0,1)$ there exists a finite constant $c > 0$ such that for all $h \in (0, 1-\gamma)$,*

$$\mathbb{E}\left(\sigma^2(h) \mid (T_n)_n\right) \le c \sum_{n=1}^{+\infty} T_n^{-2/\alpha} h^2 \left|\log h\right|^{(1+\eta)(2/\alpha-1)} \quad \textit{almost surely.}$$

Following [14] let us denote

$$b(h) = h|\log h|^{(1+\eta)/\alpha}.$$

Then by Lemma 5.2,

$$\mathbb{E}\left(\sum_{k=1}^{+\infty} \frac{\sigma^2(2^{-k})}{b^2(2^{-k})} \middle| (T_n)_n\right) < +\infty \quad \text{almost surely.}$$

Therefore by independence of $(T_n)_n$ and $(\xi_n)_n$, almost surely

$$\lim_{k\to+\infty} \frac{\sigma^2(2^{-k})}{b^2(2^{-k})} = 0.$$

Up to change the Euclidean norm $\|\cdot\|$ by the equivalent norm $\|\cdot\|_{E^t}$ defined in Lemma 6.1.5 of [17] the map $h \mapsto \left\|h^{E^t}\xi\right\|$ is increasing and so is $h \mapsto \sigma^2(h)$. Also, one can conclude, as in [14], that almost surely

$$\lim_{h\to 0} \frac{\sigma^2(h)}{b^2(h)} = 0.$$

Therefore, up to change k^*, (24) and (27) imply that for every $k \ge k^*(\omega)$,

$$|Y_\alpha(x) - Y_\alpha(y)| \le \sqrt{2dA}\,\tau_E(x-y)\left|\log \tau_E(x-y)\right|^{(1+\eta)/\alpha+1/2} \tag{29}$$

for all $x, y \in \mathcal{D}_k$ such that $0 < \tau_E(x-y) \le \delta_k$, with δ_k defined by (26). Let

$$\Omega^* = \bigcup_{n=1}^{+\infty} \bigcap_{k\ge n} \bigcap_{\substack{x,y\in\mathcal{D}_k \\ 0<\tau_E(x-y)\le\delta_k}} \Big\{|X_\alpha(x) - X_\alpha(y)| \le \sqrt{2dA}\,\tau_E(x-y)\left|\log\tau_E(x-y)\right|^{(1+\eta)/\alpha+1/2}\Big\}.$$

Since X_α and Y_α have the same finite dimensional distributions, $\mathbb{P}(\Omega^*) = 1$.

Step 4. Let us now define a modification X_α^* of X_α that satisfies (29) for all $x, y \in K$ with $\tau_E(x-y)$ small enough and some constant $C > 0$ instead of $\sqrt{2dA}$. Let $\omega \in \Omega^*$, by Step 3 there exists $k^*(\omega) \ge 1$ such that X_α satisfies (29) for $k \ge k^*(\omega)$, $x, y \in \mathcal{D}_k$ and $0 < \tau_E(x-y) \le \delta_k$ with δ_k defined by (26).

Let us recall that by Lemma 2.2 of [1], there exists a finite constant $K_E \ge 1$ such that for all $x, y \in \mathbb{R}^d$

$$\tau_E(x+y) \le K_E\left(\tau_E(x) + \tau_E(y)\right).$$

Let

$$F(h)=\sqrt{2dA}h\,|\log h|^{(1+\eta)/\alpha+1/2}, \quad h>0 \tag{30}$$

and $k_0\in\mathbb{N}\setminus\{0\}$ such that $2^{k_0}\delta_{k_0+1}>3K_E^2$ and F is increasing on $(0,\delta_{k_0}]$. Up to change $k^*(\omega)$, we can assume that $k^*(\omega)\geq k_0$.

Let $x,y\in\mathcal{D}$ such that $x\neq y$ and $3K_E^2\tau_E(x-y)\leq\delta_{k^*(\omega)}$. Then, there exists a unique $k\geq k^*(\omega)$ such that $\delta_{k+1}<3K_E^2\tau_E(x-y)\leq\delta_k$. Furthermore, since $x,y\in\mathcal{D}$, one can find $n\geq k+1$ such that $x,y\in\mathcal{D}_n$. Moreover, by Step 1, up to change $k^*(\omega)$, for $j=k,\ldots,n-1$, we can choose $x^{(j)},y^{(j)}\in\mathcal{D}_j$ such that

$$\tau_E\left(x-x^{(j)}\right)\leq 2^{-j}\quad\text{and}\quad\tau_E\left(y-y^{(j)}\right)\leq 2^{-j}.$$

By construction $\tau_E\left(x^{(k)}-y^{(k)}\right)\leq K_E^2\left(\tau_E(x-y)+2^{1-k}\right)$. Let us point out that since $k\geq k_0$, $2^k\delta_{k+1}\geq 2^{k_0}\delta_{k_0+1}>3K_E^2$. Therefore, one easily sees that $2^{-k}<\frac{\delta_{k+1}}{3K_E^2}<\tau_E(x-y)$ and gets

$$\tau_E\left(x^{(k)}-y^{(k)}\right)\leq 3K_E^2\tau_E(x-y).$$

On the one hand, by Step 3, $3K_E^2\tau_E(x-y)\leq\delta_k$ implies that

$$\left|X_\alpha\left(x^{(k)}\right)-X_\alpha\left(y^{(k)}\right)\right|\leq F\left(\tau_E\left(x^{(k)}-y^{(k)}\right)\right).$$

On the other hand we can write

$$X_\alpha(x)-X_\alpha\left(x^{(k)}\right)=\sum_{j=k}^{n-1}\left(X_\alpha\left(x^{(j+1)}\right)-X_\alpha\left(x^{(j)}\right)\right)$$

with $\tau_E\left(x^{(j+1)}-x^{(j)}\right)\leq 3K_E^2 2^{-(j+1)}\leq\delta_{j+1}$ since $j\geq k_0$. Moreover, note that $x^{(j)}\in\mathcal{D}_j\subset\mathcal{D}_{j+1}$ and Step 3 again implies that

$$\left|X_\alpha(x)-X_\alpha\left(x^{(k)}\right)\right|\leq\sum_{j=k}^{n-1}F\left(\tau_E\left(x^{(j+1)}-x^{(j)}\right)\right)\leq CF(\delta_{k+1}),$$

where $C=\sum_{j=0}^{+\infty}(j+1)^{(1+\eta)/\alpha+1/2}\delta_j<+\infty$. With similar computations for $X_\alpha(y)-X_\alpha\left(y^{(k)}\right)$, we get

$$\begin{aligned}|X_\alpha(x)-X_\alpha(y)|&\leq F\left(\tau_E\left(x^{(k)}-y^{(k)}\right)\right)+2CF(\delta_{k+1})\\&\leq(1+2C)F\left(3K_E^2\tau_E(x-y)\right).\end{aligned}$$

Therefore, since F is defined by (30), one can find a finite constant $C>0$ such that for all $x,y\in\mathcal{D}$ satisfying $0<3K_E^2\tau_E(x-y)\leq\delta_{k^*(\omega)}$,

$$|X_\alpha(x)-X_\alpha(y)|\leq C\tau_E(x-y)\,|\log\tau_E(x-y)|^{(1+\eta)/\alpha+1/2}. \tag{31}$$

We now give a modification of X_α. For $x\in\mathcal{D}$, we set

$$X_\alpha^*(x)(\omega)=X_\alpha(x)(\omega).$$

For $x \in K$, let $x^{(n)} \in \mathcal{D}$ be such that $\lim_{n\to+\infty} x^{(n)} = x$. In view of (31), $\left(X_\alpha^*\left(x^{(n)}\right)(\omega)\right)_n$ is a Cauchy sequence and then converges. We set

$$X_\alpha^*(x)(\omega) = \lim_{n\to+\infty} X_\alpha^*\left(x^{(n)}\right)(\omega).$$

Remark that this limit does not depend on the choice of $\left(x^{(n)}\right)$. Moreover, since X_α is stochastically continuous, X_α^* is a modification of X_α. Finally, by continuity of τ_E, we easily see that

$$\left|X_\alpha^*(x)(\omega) - X_\alpha^*(y)(\omega)\right| \le C\tau_E(x-y)\left|\log \tau_E(x-y)\right|^{(1+\eta)/\alpha+1/2}$$

for all $x, y \in K$ such that $0 < 3K_E^2\tau_E(x-y) < \delta_{k^*(\omega)}$, which concludes the proof. □

Following the same lines as the proof of Theorem 5.1 we obtain a similar result for a class of Gaussian random fields including the operator scaling ones defined in [1] ($\alpha = 2$). Let us remark that Y_α is not defined for $\alpha = 2$. However, in Step 2 of the proof, let us replace Y_α by X a centered Gaussian random field and $v^2\left((x,y) \mid (T_n,\xi_n)_n\right)$ by the variance of $X(x) - X(y)$

$$v^2((x,y)) = \mathbb{E}\left((X(x) - X(y))^2\right).$$

Furthermore let us replace Step 3 by the assumption that for some $\beta \in \mathbb{R}$ and $\delta_0 > 0$ there exists a finite constant $C > 0$ such that for $x, y \in K$ with $0 < \tau_E(x-y) \le \delta_0$,

$$\mathbb{E}\left((X(x) - X(y))^2\right) \le C\tau_E(x-y)^2\left|\log \tau_E(x-y)\right|^\beta. \tag{32}$$

Then Step 1, Step 2 and Step 4 yield the following proposition.

Proposition 5.3. *Let $X = (X(x))_{x\in\mathbb{R}^d}$ be a centered Gaussian random field satisfying (32) for some $\beta \in \mathbb{R}$. There exists a modification X^* of X on K such that*

$$\lim_{\delta\downarrow 0} \sup_{\substack{x,y\in K\\ 0<\|x-y\|\le\delta}} \frac{|X^*(x) - X^*(y)|}{\tau_E(x-y)\left|\log \tau_E(x-y)\right|^{1/2+\beta+\varepsilon}} = 0 \quad \textit{almost surely}$$

for any $\varepsilon > 0$ and with τ_E defined by (9).

Remark 5.1. Let us point out that if X_2 is an operator scaling Gaussian random field as defined in [1], then

$$\mathbb{E}\left((X_2(x) - X_2(y))^2\right) = \tau_E(x-y)^2\,\mathbb{E}\left(X_2(\ell_E(x-y))^2\right)$$

and X_2 satisfies (32) with $\beta = 0$ according to Eq. (5.2) of [1]. Therefore Proposition 5.3 is more precise than one could expect from Theorem 5.1, replacing α by 2.

Let us also mention that [29] gives a different proof of a similar result for some Gaussian operator scaling random fields with stationary increments.

For all $j = 1, \ldots, p$ we set $K_j = K \cap \bigoplus_{k=1}^j W_k$, where W_k is the E-invariant subspace of dimension l_k or $2l_k$ associated with H_k^{-1} by (13). Note that $K_p = K$.

Corollary 5.4. *Let $\alpha \in (0, 2]$ and X_α be defined by (3). There exists a modification X_α^* of X_α on K such that for all $j = 1, \ldots, p$ and any $\varepsilon > 0$,*

$$\lim_{\delta \downarrow 0} \sup_{\substack{x,y \in K_j \\ 0<\|x-y\|\leq\delta}} \frac{\left|X_\alpha^*(x) - X_\alpha^*(y)\right|}{\|x-y\|^{H_j} \left|\log \|x-y\|\right|^{H_j(p_j-1)+\beta+1/2+\varepsilon}} = 0 \quad \textit{almost surely,}$$

with $p_j = \max_{1\leq k\leq j} l_k$, $\beta = 1/\alpha$ if $\alpha \neq 2$ and $\beta = 0$ if $\alpha = 2$.

Proof. It follows from Theorem 5.1, Propositions 5.3 and 3.3, since $a_j \leq a_p$ for any $j = 1 \ldots p$. □

Corollary 5.5. *Let $\alpha \in (0, 2]$ and X_α be defined by (3). There exists a modification X_α^* of X_α which has locally H-Hölder sample paths on $\mathbb{R}^d$ for every $H \in (0, H_p)$.*

Proof. It is a simple consequence of Corollary 5.4. □

Now, as in [1], we are looking for global and directional Hölder critical exponents of the harmonizable stable random field X_α. These exponents have been introduced in [9] in the Gaussian realm but can be defined for any random field, see [1]. Let us first recall Definition 5.1 of [1] which introduces the global Hölder critical exponent of a random field.

Definition 5.1. Let $H \in (0, 1)$. A real-valued random field $(X(x))_{x\in\mathbb{R}^d}$ is said to have Hölder critical exponent H if there exists a modification X^* of X that satisfies the following two properties:

(i) for any $s \in (0, H)$, the sample paths of X^* satisfy almost surely a uniform Hölder condition of order s on any compact set, i.e. for any compact set $K' \subset \mathbb{R}^d$, there exists a finite positive random variable A such that almost surely

$$\left|X^*(x) - X^*(y)\right| \leq A\|x-y\|^s \quad \text{for all } x, y \in K' \tag{33}$$

(ii) for any $s \in (H, 1)$, almost surely the sample paths of X^* fail to satisfy any uniform Hölder condition of order s.

Remark 5.2. Note that the Hölder critical exponent, if it exists, is well-defined since any continuous modification of X and X^* are indistinguishable.

Moreover, according to Definition 5.3 of [1], the directional regularities of a random field X are defined as follows.

Definition 5.2. Let S^{d-1} be the Euclidean unit sphere. A real-valued random field $X = (X(x))_{x\in\mathbb{R}^d}$ admits $H(u)$ as directional regularity in direction $u \in S^{d-1}$ if the process $(X(tu))_{t\in\mathbb{R}}$ admits $H(u)$ as Hölder critical exponent.

Now let us give the directional and global Hölder critical exponents of X_α.

Proposition 5.6. *The random field X_α admits H_p as Hölder critical exponent. Moreover, for any $j = 1, \ldots, p$, if $u \in W_j \cap S^{d-1}$, with W_j defined by (13) and S^{d-1} the Euclidean unit sphere of $\mathbb{R}^d$, the random field X_α admits H_j as directional regularity in the direction u.*

Proof. For Z a real SαS random variable we let

$$\|Z\|_\alpha = \left(-\log\left(\mathbb{E}\left(\exp\left(\mathrm{i}Z\right)\right)\right)\right)^{1/\alpha}.$$

Let τ_E and l_E be defined by (9). Then, for any $x, y \in \mathbb{R}^d$, $x \neq y$,

$$\|X_\alpha^*(x) - X_\alpha^*(y)\|_\alpha = D_\alpha\left(\ell_E(x-y)\right)\tau_E(x-y)$$

where for all $\theta \in S_E$,

$$D_\alpha(\theta) = \left(d_\alpha \int_{\mathbb{R}^d} \left|\mathrm{e}^{\mathrm{i}\langle\theta,\xi\rangle} - 1\right|^\alpha \psi(\xi)^{-\alpha-q}\mathrm{d}\xi\right)^{1/\alpha} \quad \text{with } d_\alpha = \frac{1}{2\pi}\int_0^\pi |\cos(t)|^\alpha\,\mathrm{d}t.$$

From Lebesgue's Theorem, the function D_α is continuous on the compact set S_E, with positive values. Let us denote $m_\alpha = \min_{\theta\in S_E} D_\alpha(\theta) > 0$.

Let $u \in W_j \cap S^{d-1}$ with $1 \le j \le p$. According to Corollary 3.4, for $|t-t'|$ small enough,

$$\|X_\alpha^*(tu) - X_\alpha^*(t'u)\|_\alpha \ge m_\alpha c_1 \left|t-t'\right|^{H_j} \left|\log\left|t-t'\right|\right|^{-(l_j-1)H_j}.$$

Therefore, for any $s > H_j$, it implies that $\frac{X_\alpha^*(tu)-X_\alpha^*(t'u)}{|t-t'|^s}$ is almost surely unbounded as $|t-t'| \downarrow 0$. Then, almost surely $\left(X_\alpha^*(tu)\right)_{t\in\mathbb{R}}$ does not satisfy (33) on [0, 1].

Moreover, Corollary 5.4 implies that $\left(X_\alpha^*(tu)\right)_{t\in\mathbb{R}}$ satisfies (33) on any non-empty compact set $K' \subset \mathbb{R}^d$ for any $s < H_j$. Thus H_j is the directional regularity of X_α in the direction u.

Hence, one can find a direction $u \in S^{d-1}$ in which almost surely $\left(X_\alpha^*(tu)\right)_{t\in\mathbb{R}}$ does not satisfy (33) on [0, 1] for any $s > H_p$. Therefore, almost surely $\left(X_\alpha^*(x)\right)_{x\in\mathbb{R}^d}$ cannot satisfy (33) for any $s > H_p$. Then, by Corollary 5.5, X_α admits H_p as Hölder critical exponent. □

Remark 5.3. In the diagonalizable case (see Example 2.2), the W_j, $j = 1\dots p$, are the eigenspaces associated with the eigenvalues of E. In particular, for θ_j an eigenvector related to the eigenvalue $\lambda_j = a_j$, the critical Hölder exponent in direction θ_j is $H_j = 1/a_j$.

Proposition 5.6, compared to Theorem 5.4 of [1], shows that operator scaling stable fields, defined through an harmonizable representation share the same sample path properties as the Gaussian ones. Therefore it is natural that the box and the Hausdorff dimensions of their graphs on a compact set are the same as the Gaussian ones, obtained in Theorem 5.6 of, [1]. We also refer to Falconer [30] for the definitions and properties of box and the Hausdorff dimension. We denote by $\dim_{\mathcal{H}} \mathcal{A}$, respectively $\dim_{\mathcal{B}} \mathcal{A}$, the Hausdorff dimension and the box dimension of the set A, respectively.

Proposition 5.7. *Let $\alpha \in (0,2)$ and X_α^* be as in Theorem 5.1. For $a, b \in \mathbb{R}^d$ with $a_i < b_i$ $(i = 1,\dots,d)$, let $K = \prod_{i=1}^d [a_i, b_i]$ and*

$$\mathcal{G}(X_\alpha^*)(\omega) = \{(x, X_\alpha^*(x)(\omega));\, x \in K\}$$

be the graph of a realization of the field X_α^* over the compact K. Then, almost surely,

$$\dim_{\mathcal{H}}\mathcal{G}(X_\alpha^*) = \dim_{\mathcal{B}}\mathcal{G}(X_\alpha^*) = d + 1 - H_p.$$

Proof. The proof is very similar to those of Theorem 5.6 [1]. It also uses the same kinds of arguments as in [31]. For sake of completeness we recall the main ideas. Corollary 5.5 allows as usual to state the upper bound

$$\dim_{\mathcal{H}}\mathcal{G}(X_\alpha^*) \le \overline{\dim}_{\mathcal{B}}\mathcal{G}(X_\alpha^*) \le d + 1 - H_p, \quad \text{almost surely}$$

where $\overline{\dim}_{\mathcal{B}}$ denotes the upper box dimension. The lower bound will also follow from the Frostman criterion (Theorem 4.13(a) in [30]). One has to prove that the integral

$$I_s = \int_{K\times K} \mathbb{E}\left[\left((X_\alpha^*(x) - X_\alpha^*(y))^2 + \|x-y\|^2\right)^{-s/2}\right] \mathrm{d}x\mathrm{d}y,$$

is finite to get that almost surely $\dim_{\mathcal{H}} \mathcal{G}(X_\alpha^*) \geq s$. In our case, the fundamental lemma of [32] allows us to write this integral using the characteristic function of the SαS field X_α^*. Actually, when one remarks that, using Fourier-inversion, $(\xi^2+1)^{-s/2} = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\xi t} f_s(t)\mathrm{d}t$, where $f_s \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, one gets

$$I_s = \int_{K\times K} \left(\frac{1}{2\pi}\|x-y\|^{-s} \int_{\mathbb{R}} \mathrm{e}^{-|t|^\alpha \frac{\|X_\alpha^*(x)-X_\alpha^*(y)\|_\alpha^\alpha}{\|x-y\|^\alpha}} f_s(t)\mathrm{d}t\right) \mathrm{d}x\mathrm{d}y.$$

By a change of variables, as $f_s \in L^\infty(\mathbb{R})$, one can find a finite positive constant $C > 0$ such that

$$\begin{aligned} I_s &\leq C \int_{K\times K} \|x-y\|^{1-s} \left\|X_\alpha^*(x) - X_\alpha^*(y)\right\|_\alpha^{-1} \mathrm{d}x\mathrm{d}y \\ &\leq C m_\alpha^{-1} \int_{K\times K} \|x-y\|^{1-s} \tau_E(x-y)^{-1} \mathrm{d}x\mathrm{d}y, \end{aligned}$$

where $m_\alpha = \min_{\theta\in S_E}\left(d_\alpha \int_{\mathbb{R}^d} \left|\mathrm{e}^{\mathrm{i}\langle\theta,\xi\rangle} - 1\right|^\alpha \psi(\xi)^{-\alpha-q}\mathrm{d}\xi\right)^{1/\alpha}$. Since $\int_{K\times K}\|x-y\|^{1-s}\tau_E(x-y)^{-1}\mathrm{d}x\mathrm{d}y < +\infty$ as long as $s < d+1-H_p$ (see [1]),

$$\underline{\dim}_{\mathcal{B}} \mathcal{G}(X_\alpha^*) \geq \dim_{\mathcal{H}} \mathcal{G}(X_\alpha^*) \geq d+1-H_p \quad \text{almost surely},$$

where $\underline{\dim}_{\mathcal{B}}$ denotes the lower box dimension. The proof is then complete. □

6. Moving average representation

We proved in the previous section that harmonizable operator scaling stable random fields share many properties with Gaussian operator random fields. In particular, they have locally Hölder sample paths and critical directional Hölder exponent depending on the directions. In the Gaussian case ($\alpha = 2$), [1] establishes such properties in the framework of both harmonizable and moving average Gaussian operator scaling random fields. However, for stable laws, harmonizable and moving average representations do not have the same behavior as we see in this section.

Let us recall the definition of moving average operator scaling stable random fields introduced in [1]. Let $0 < \alpha \leq 2$. We consider $M_\alpha(dy)$ an independently scattered $S\alpha S$ random measure on $\mathbb{R}^d$ with Lebesgue control measure, see [4] for details on such random measures. As before, $q = \mathrm{trace}(E)$. Let $\varphi : \mathbb{R}^d \to [0,\infty)$ be a continuous E-homogeneous function. We assume moreover that there exists $s > 1$ such that φ is (s, E)-*admissible*. According to Definition 2.7 of [1] it means that $\varphi(x) \neq 0$ for $x \neq 0$ and that for any $0 < A < B$ there exists a finite positive constant $C > 0$ such that for $A \leq \|y\| \leq B$,

$$|\varphi(x+y) - \varphi(y)| \leq C\tau_E(x)^s$$

holds for any $\tau_E(x) \leq 1$.

Definition 6.1. The random field

$$Z_\alpha(x) = \int_{\mathbb{R}^d} \left(\varphi(x-y)^{1-q/\alpha} - \varphi(-y)^{1-q/\alpha}\right) M_\alpha(\mathrm{d}y), \quad x \in \mathbb{R}^d \tag{34}$$

is called moving average operator scaling stable random field.

Remark 6.1. As in the harmonizable case, the representation we use is not exactly the same one as in [1]. However, as in Remark 2.1, up to change the parametrization, the class defined by (34) and the class of moving average random fields defined in [1] are the same.

From Theorem 3.1 and Corollary 3.2 of [1], since φ is (s, E)-admissible with $s > 1$, the random field Z_α is well-defined, stochastically continuous, has stationary increments and satisfies the following operator scaling property

$$\forall c > 0, \quad \left\{Z_\alpha(c^E x); x \in \mathbb{R}^d\right\} \overset{(fdd)}{=} \left\{c Z_\alpha(x); x \in \mathbb{R}^d\right\},$$

as the harmonizable field X_α.

In the Gaussian case ($\alpha = 2$), the variograms of Z_2 and X_2, respectively defined by (34) and (3), are similar. Then, [1] proves that Z_2 and X_2 admit the same critical global and directional Hölder exponents. Moreover, Z_2 satisfies (32) for $\beta = 0$ and then the conclusions of Proposition 5.3 and Corollary 5.4 hold for $\beta = 0$. However, when $\alpha \in (0, 2)$, let us recall that moving average self-similar α-stable random motions does not have in general continuous sample paths (see [4]). The next proposition states the same property for Z_α.

Proposition 6.1. *Let $\alpha \in (0, 2)$ and Z_α be defined by (34). Assume $d \geq 2$. Then, any modification of the random field Z_α is almost surely unbounded on every open ball.*

Proof. Let us remark that $\varphi(0) = 0$ by continuity and E-homogeneity and $q = \sum_{j=1}^{d} a_j > d > \alpha$ since $d \geq 2$. Then, for any open ball U, since $U^* = U \cap \mathbb{Q}^d$ is a dense sequence in U, for any $y \in U$

$$f^*\left(U^*, y\right) := \sup_{x \in U^*} \left|\varphi(x - y)^{1-q/\alpha} - \varphi(-y)^{1-q/\alpha}\right| = +\infty.$$

Then $\int_{\mathbb{R}^d} f^*(U^*, y)^\alpha \, \mathrm{d}y = +\infty$ and the necessary condition for sample boundedness (10.2.14) of Theorem 0.2.3 p. 450 of [4] fails. Theorem 10.2.3 and Corollary 9.5.5 of [4] give the conclusion. □

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Appendix

Proof of Lemma 3.2. The lower bound is straightforward. Actually, for any $t > 0$, t^λ is an eigenvalue of the matrix t^J and therefore $t^a = \left|t^\lambda\right| \leq \|t^J\|$.

Let us prove the upper bound. Let us recall that the norm defined for a matrix $A = (a_{ij})_{1 \leq i,j, \leq d'}$ by $\|A\|_\infty = \max_{1 \leq i \leq d'} \sum_{j=1}^{d'} |a_{ij}|$ is the subordinated norm of the infinite norm $\|x\|_\infty = \max_{1 \leq i \leq d'} |x_i|$ for $x \in \mathbb{R}^{d'}$.

Let us first assume that J is a Jordan cell matrix of size l. In this case $\lambda = a \in \mathbb{R}$ and

$$t^J = t^a \begin{pmatrix} 1 & 0 & \dots & 0 \\ \log t & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \dfrac{(\log t)^{l-1}}{(l-1)!} & \dots & \log t & 1 \end{pmatrix}.$$

From the definition of the subordinated norm $\|\cdot\|_\infty$, we can deduce that $\|t^J\|_\infty = t^a \sum_{j=0}^{l-1} \frac{|\log t|^j}{j!}$. Then, for any $t \in (0, e^{-1}] \cup [e, +\infty)$ we get $|\log(t)| \geq 1$ and

$$\left\|t^J\right\| \leq \sqrt{l}\left\|t^J\right\|_\infty \leq \sqrt{l}t^a\,|\log(t)|^{l-1}\sum_{j=0}^{l-1}\frac{1}{j!}.$$

Therefore, for any $t \in (0, e^{-1}] \cup [e, +\infty)$,

$$\|t^J\| \leq \sqrt{l}et^a\,|\log t|^{l-1}.$$

Let us now assume that J is a block of the form (11) of size $2l$ associated with the eigenvalue $\lambda = a + ib$ for $b \neq 0$. Then $t^J = t^a R(t)N(t)$ where

$$R(t) = \begin{pmatrix} R_b(t) & 0 & \dots & 0 \\ 0 & R_b(t) & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & R_b(t) \end{pmatrix} \quad \text{with } R_b(t) = \begin{pmatrix} \cos(b\log t) & -\sin(b\log t) \\ \sin(b\log t) & \cos(b\log t) \end{pmatrix},$$

and

$$N(t) = \begin{pmatrix} I_2 & 0 & \dots & 0 \\ N_1(t) & I_2 & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ N_{l-1}(t) & \dots & N_1(t) & I_2 \end{pmatrix} \quad \text{with } N_j(t) = \begin{pmatrix} \dfrac{|\log t|^j}{j!} & 0 \\ 0 & \dfrac{|\log t|^j}{j!} \end{pmatrix}.$$

Hence,

$$\left\|t^J\right\| \leq t^a\,\|R(t)\|\,\|N(t)\|.$$

Since $R(t)$ is an orthogonal matrix, $\|R(t)\| = 1$. Furthermore, $N(t)$ is a $(2l) \times (2l)$ matrix and

$$\|N(t)\| \leq \sqrt{2l}\,\|N(t)\|_\infty = \sqrt{2l}\sum_{j=0}^{l-1}\frac{|\log t|^j}{j!}.$$

Therefore, we also obtain that

$$\|t^J\| \leq \sqrt{2l}et^a\,|\log t|^{l-1}$$

for any $t \in (0, e^{-1}] \cup [e, +\infty)$. $\square$

Proof of Proposition 3.3. Let $r \in (0, 1)$ and $x \in \bigoplus_{k=j_0}^{j} W_k \backslash \{0\}$ such that $\|x\| \leq r$.

We first establish the lower bound of Proposition 3.3. Since for any $r_0 \in (0, r)$ the function

$$y \mapsto \|y\|^{H_{j_0}}\,|\log\|y\||^{-H_{j_0}\left(p_{j_0,j}-1\right)}\,\tau_E(y)^{-1}$$

is continuous on the compact set $\left\{y \in \mathbb{R}^d \,/\, r_0 \leq \|y\| \leq r\right\}$, the key issue here is to prove the lower bound around the origin. Moreover, let us remark that one can find $r_0 \in (0, r)$ such that for any $y \in \mathbb{R}^d$ with $\|y\| \leq r_0$, we have $\tau_E(y) \leq e^{-1}$. Then, without loss of generality, we can assume that $\|x\| \leq r_0$. Hence, $x = \tau_E(x)^E \ell_E(x)$ with $\tau_E(x) \leq e^{-1}$. Since $\mathbb{R}^d = \bigoplus_{k=1}^{p} W_k$,

$\ell_E(x) = \sum_{k=1}^{p} \ell_k(x)$ with $\ell_k(x) \in W_k$, $k = 1, \ldots, p$. For any $k = 1, \ldots, p$, let L_k be the coordinates of $\ell_k(x)$ in the basis $\left(f_{\sum_{i=1}^{k-1}\tilde{l}_i+1}, \ldots, f_{\sum_{i=1}^{k}\tilde{l}_i}\right)$ of W_k. Hence, by definition of P,

$$P^{-1}\ell_E(x) = \begin{pmatrix} L_1 \\ \vdots \\ L_p \end{pmatrix} \quad \text{and} \quad x = \tau_E(x)^E \ell_E(x) = P \begin{pmatrix} \tau_E(x)^{J_1} L_1 \\ \vdots \\ \tau_E(x)^{J_p} L_p \end{pmatrix}.$$

Since $x \in \bigoplus_{k=j_0}^{j} W_k$, $\ell_E(x) \in \bigoplus_{k=j_0}^{j} W_k$ and then $L_k = 0$ for $k \notin \{j_0, \ldots, j\}$. Then,

$$\|x\| \le \|P\| \left(\sum_{k=j_0}^{j} \left\| \tau_E(x)^{J_k} L_k \right\|^2 \right)^{1/2} \le \|P\| \left(\sum_{k=j_0}^{j} \left\| \tau_E(x)^{J_k} \right\|^2 \|L_k\|^2 \right)^{1/2}.$$

By Lemma 3.2, since $\tau_E(x) \le e^{-1}$,

$$\|x\| \le \sqrt{2}e \|P\| \left(\sum_{k=j_0}^{j} l_k \tau_E(x)^{2a_k} |\log \tau_E(x)|^{2(l_k-1)} \|L_k\|^2 \right)^{1/2}.$$

Since $\tau_E(x) \le e^{-1}$, $a_k \ge a_{j_0}$ and $l_k \le p_{j_0,j} = \max_{j_0 \le i \le j} l_i \le d$,

$$\begin{aligned} \|x\| &\le \sqrt{2d}e \|P\| \tau_E(x)^{a_{j_0}} |\log \tau_E(x)|^{\left(p_{j_0,j}-1\right)} \left(\sum_{k=j_0}^{j} \|L_k\|^2 \right)^{1/2} \\ &\le \sqrt{2d}e \|P\| \tau_E(x)^{a_{j_0}} |\log \tau_E(x)|^{\left(p_{j_0,j}-1\right)} \left\| P^{-1} l_E(x) \right\|. \end{aligned}$$

Then,

$$\|x\| \le \sqrt{2d}e M_E \|P\| \left\| P^{-1} \right\| \tau_E(x)^{a_{j_0}} |\log \tau_E(x)|^{\left(p_{j_0,j}-1\right)}, \tag{35}$$

where M_E is defined by (10).

Consider the logarithm of both sides of Eq. (35). Then, since $a_{j_0} > 0$, one can find two finite positive constants c_1 and c_2 such that for $\tau_E(x)$ small enough,

$$\log \|x\| \le c_1 \log \tau_E(x) + c_2.$$

Hence, choosing r_0 small enough, one can find a finite constant $C > 0$ such that

$$|\log \tau_E(x)| \le C |\log \|x\||. \tag{36}$$

Using (36) in (35), we obtain that there exists a finite constant $C > 0$ such that for $\|x\| \le r_0$

$$\|x\|^{H_{j_0}} |\log \|x\||^{-H_{j_0}\left(p_{j_0,j}-1\right)} \le C\tau_E(x),$$

which gives the lower bound of Proposition 3.3, up to change C.

Let us now establish the upper bound of Proposition 3.3. Since for any $r_0 \in (0, r)$ the function

$$y \mapsto \frac{\tau_E(y)}{\|y\|^{H_j} |\log \|y\||^{H_j\left(p_{j_0,j}-1\right)}}$$

is continuous on the compact set $\left\{y \in \mathbb{R}^d / r_0 \le \|y\| \le r\right\}$, the key issue here is also to prove the upper bound around the origin. Therefore, we can also assume that $\|x\| \le r_0$, with r_0 chosen as previously, such that $\tau_E(x) \le e^{-1}$.

Let us write $x = \sum_{k=1}^{p} x_k$ with each $x_k \in W_k$. We then denote by X_k the coordinates of x_k in the basis $\left(f_{\sum_{i=1}^{k-1}\tilde{l}_i+1}, \dots, f_{\sum_{i=1}^{k}\tilde{l}_i}\right)$ of W_k. Since $x \in \bigoplus_{k=j_0}^{j} W_k$, $X_k = 0$ for every $k \notin \{j_0, \dots, j\}$. Hence, by definition of P,

$$P^{-1}x = \begin{pmatrix} X_1 \\ \vdots \\ X_p \end{pmatrix} \quad \text{and} \quad \ell_E(x) = \tau_E(x)^{-E} x = P \begin{pmatrix} \tau_E(x)^{-J_1} X_1 \\ \vdots \\ \tau_E(x)^{-J_p} X_p \end{pmatrix}.$$

Then, $\|\ell_E(x)\| \le \|P\| \left(\sum_{k=j_0}^{j} \left\|\tau_E(x)^{-J_k} X_k\right\|^2\right)^{1/2}$ and, since $\tau_E(x)^{-1} \ge e$, Lemma 3.2 yields that

$$\|\ell_E(x)\| \le \sqrt{2}e \|P\| \left(\sum_{j=j_0}^{j} l_k \tau_E(x)^{-2a_k} |\log \tau_E(x)|^{2(l_k-1)} \|X_k\|^2\right)^{1/2}.$$

Hence, using the facts that $\tau_E(x)^{-1} \ge e > 1$, $a_k \le a_j$ and $l_k \le p_{j_0,j}$,

$$\begin{aligned} 0 < m_E = \min_{S_E} \|y\| &\le \sqrt{2d}e \|P\| \left\|P^{-1}\right\| \tau_E(x)^{-a_j} |\log \tau_E(x)|^{p_{j_0,j}-1} \sqrt{\sum_{k=j_0}^{j} \|X_k\|^2} \\ &\le \sqrt{2d}e \|P\| \left\|P^{-1}\right\| \tau_E(x)^{-a_j} |\log \tau_E(x)|^{p_{j_0,j}-1} \left\|P^{-1}x\right\|. \end{aligned}$$

Then, by (36) and since $\left\|P^{-1}x\right\| \le \left\|P^{-1}\right\| \|x\|$, there exists a finite constant $C > 0$ such that for $\|x\| \le r_0$,

$$\tau_E(x) < C \|x\|^{H_j} |\log \|x\||^{H_j\left(p_{j_0,j}-1\right)},$$

which, up to change C, gives the upper bound of Proposition 3.3 and concludes the proof. □

Proof of Lemma 5.2. It is sufficient to consider

$$I(h) = \mathbb{E}\left(m(\xi_n)^{-2/\alpha} \min\left(M_E \left\|h^{E^t}\xi_n\right\|, 2\right)^2 \psi(\xi_n)^{-2-2q/\alpha}\right)$$

with M_E defined by (10) and where the density distribution m of ξ_n is associated with $\eta > 0$ by (17). By definition,

$$I(h) = \int_{\mathbb{R}^d} m(\xi)^{1-2/\alpha} \psi(\xi)^{-2-2q/\alpha} \min\left(M_E \left\|h^{E^t}\xi\right\|, 2\right)^2 \mathrm{d}\xi.$$

Using the formula of integration in *polar coordinates* with respect to E^t, see Proposition 3.1,

$$\begin{aligned} I(h) = \int_{S_{E^t}} \int_0^{+\infty} & m\left(r^{E^t}\theta\right)^{1-2/\alpha} \psi\left(r^{E^t}\theta\right)^{-2-2q/\alpha} \\ & \times \min\left(M_E \left\|(hr)^{E^t}\theta\right\|, 2\right)^2 r^{q-1} \mathrm{d}r \sigma_{E^t}(\mathrm{d}\theta). \end{aligned}$$

Since ψ is E^t-homogeneous,

$$I(h) = \int_{S_{E^t}} \int_0^{+\infty} m\left(r^{E^t}\theta\right)^{1-2/\alpha} \psi(\theta)^{-2-2q/\alpha}$$
$$\times \min\left(M_E \left\|(hr)^{E^t}\theta\right\|, 2\right)^2 r^{-2+q-1-2q/\alpha} \mathrm{d}r\sigma_{E^t}(\mathrm{d}\theta)$$
$$= c_\eta^{1-2/\alpha} \int_{S_{E^t}} \int_0^{+\infty} \psi(\theta)^{-2-2q/\alpha}$$
$$\times \min\left(M_E \left\|(hr)^{E^t}\theta\right\|, 2\right)^2 r^{-3} |\log(r)|^{(1+\eta)(2/\alpha-1)} \mathrm{d}r\sigma_{E^t}(\mathrm{d}\theta).$$

By the change of variable $\rho = hr$, $I(h)$ is equal to

$$c_\eta^{1-2/\alpha} h^2 \int_{S_{E^t}} \int_0^{+\infty} \psi(\theta)^{-2-2q/\alpha}$$
$$\times \min\left(M_E \left\|\rho^{E^t}\theta\right\|, 2\right)^2 \rho^{-3} \left|\log\left(\frac{\rho}{h}\right)\right|^{(1+\eta)(2/\alpha-1)} \mathrm{d}r\sigma_{E^t}(\mathrm{d}\theta).$$

For any $\gamma \in (0,1)$, there exists A_γ such that for every $\rho > 0$ and every $h \leq 1-\gamma$,

$$\left|\log\left(\frac{\rho}{h}\right)\right| = |\log(\rho) - \log(h)| \leq A_\gamma \, ||\log(\rho)| + 1| \, |\log(h)|.$$

Since $2/\alpha > 1$,

$$I(h) \leq A_\gamma^{2/\alpha-1} c_\eta^{1-2/\alpha} h^2 |\log(h)|^{(1+\eta)(2/\alpha-1)} (I_1 + I_2)$$

with

$$I_1 = 4 \int_{S_{E^t}} \psi(\theta)^{-2-2q/\alpha} \sigma_{E^t}(\mathrm{d}\theta) \int_1^{+\infty} \rho^{-3} ||\log(\rho)| + 1|^{(1+\eta)(2/\alpha-1)} \mathrm{d}\rho$$

and

$$I_2 = M_E^2 M_{E^t}^2 \int_{S_{E^t}} \psi(\theta)^{-2-2q/\alpha} \sigma_{E^t}(\mathrm{d}\theta) \int_0^1 \left\|\rho^{E^t}\right\|^2 \rho^{-3} ||\log(\rho)| + 1|^{(1+\eta)(2/\alpha-1)} \mathrm{d}\rho,$$

where M_E and M_{E^t} are defined by (10). Since ψ is continuous with positive value on the compact set S_{E^t},

$$\int_{S_{E^t}} \psi(\theta)^{-2-2q/\alpha} \sigma(\mathrm{d}\theta) < +\infty.$$

Hence $I_1 < +\infty$.

It follows from Lemma 3.2 that for any $\delta' \in (0,1)$, there exists a constant $c'_\delta > 0$ such that

$$\|\rho^{E^t}\| \leq C\rho^{a_1} |\log|\rho||^{l-1}$$

for all $\rho \leq \delta'$. Hence, since $a_1 > 1$,

$$\int_0^1 \left\|\rho^{E^t}\right\|^2 \rho^{-3} ||\log(\rho)| + 1|^{(1+\eta)(2/\alpha-1)} d\rho < +\infty$$

such that $I_2 < +\infty$, which concludes the proof. $\square$

References

[1] H. Biermé, M.M. Meerschaert, H.P. Scheffler, Operator scaling stable random fields, Stochastic Process. Appl. 117 (3) (2007) 312–332.

[2] J. Lévy Véhel, Fractals in Engineering: From Theory to Industrial Applications, Springer, New York, 1997.

[3] P. Abry, P. Goncalves, J. Lévy Véhel, Lois d'échelle, fractales et ondelettes, vol. 1, Hermes, 2002.

[4] G. Samorodnitsky, M.S. Taqqu, Stochastic models with infinite variance, in: Stable non-Gaussian Random Processes, in: Stochastic Modeling, Chapman & Hall, New York, 1994.

[5] W. Willinger, V. Paxson, M.S. Taqqu, Self-similarity and heavy tails: Structural modeling of network traffic, in: A Practical Guide to Heavy Tails (Santa Barbara, CA, 1995), Birkhäuser Boston, Boston, MA, 1998, pp. 27–53.

[6] A.N. Kolmogorov, Wienersche Spiralen und einige andere interessante Kurven in Hilbertsche Raum, C. R. (Dokl.) Acad. Sci. URSS 26 (1940) 115–118.

[7] B.B. Mandelbrot, J. Van Ness, Fractional Brownian motion, fractional noises and applications, SIAM Rev. 10 (1968) 422–437.

[8] S. Leger, Analyse stochastique de signaux multi-fractaux et estimations de paramètres, Ph.D. Thesis, Université d'Orléans. http://www.univ-orleans.fr/mapmo/publications/leger, 2000.

[9] A. Bonami, A. Estrade, Anisotropic analysis of some Gaussian models, J. Fourier Anal. Appl. 9 (3) (2003) 215–236.

[10] S. Pecknold, S. Lovejoy, D. Schertzer, C. Hooge, J.F. Malouin, The simulation of universal multifractals, in: J.M. Perdang, A. Lejeune (Eds.), Cellular Automata: Prospects in Astrophysical Applications, World Scientific, Singapore, 1993, pp. 228–267.

[11] D. Benson, M.M. Meerschaert, B. Bäumer, H.P. Scheffler, Aquifer operator-scaling and the effect on solute mixing and dispersion, Water Res. Res. 42 (2006) 1–18.

[12] A. Kamont, On the fractional anisotropic wiener field, Probab. Math. Statist. 16 (1996) 85–98.

[13] T. Mikosch, S. Resnick, H. Rootzén, A. Stegeman, Is network traffic approximated by stable Lévy motion or fractional Brownian motion? Ann. Appl. Probab. 12 (1) (2002) 23–68.

[14] N. Kôno, M. Maejima, Hölder continuity of sample paths of some self-similar stable processes, Tokyo J. Math. 14 (1) (1991).

[15] N. Kôno, On the modulus of continuity of sample functions of gaussian processes, J. Math. Kyoto Univ. 10 (1970) 493–536.

[16] K. Takashima, Sample path properties of ergodic self-similar processes, Osaka J. Math. 26 (1) (1989) 159–189.

[17] M.M. Meerschaert, H.P. Scheffler, Heavy tails in theory and practice, in: Limit Distributions for Sums of Independent Random Vectors, in: Wiley Series in Probability and Statistics, John Wiley & Sons Inc., New York, 2001.

[18] R. LePage, Conditional moments for coordinates of stable vectors, in: Probability Theory on Vector Spaces, IV (Łańcut, 1987), in: Lecture Notes in Math., vol. 1391, Springer, Berlin, 1989, pp. 148–163.

[19] R. LePage, Multidimensional infinitely divisible variables and processes. II, in: Probability in Banach Spaces, III (Medford, MA, 1980), in: Lecture Notes in Math., vol. 860, Springer, Berlin, 1981, pp. 279–284.

[20] J. Rosiński, On series representations of infinitely divisible random vectors, Ann. Probab. 18 (1) (1990) 405–430.

[21] J. Rosiński, On path properties of certain infinitely divisible processes, Stochastic Process. Appl. 33 (1) (1989) 73–87.

[22] R.J. Adler, The Geometry of Random Field, John Wiley & Sons, 1981.

[23] J.D. Mason, Y. Xiao, Sample path properties of operator-self-similar Gaussian random fields, Teor. Veroyatnost. Primenen. 46 (1) (2001) 94–116.

[24] M.W. Hirsch, S. Smale, Differential Equations, Dynamical Systems, and Linear Algebra, in: Pure and Applied Mathematics, vol. 60, Academic Press, New York, London, 1974.

[25] J. Rosiński, Series representations of Lévy processes from the perspective of point processes, in: Lévy Processes, Birkhäuser Boston, Boston MA, 2001, pp. 401–415.

[26] N. Kôno, M. Maejima, Self-similar stable processes with stationary increments, in: Stable Processes and Related Topics (Ithaca, NY, 1990), in: Progr. Probab., vol. 25, Birkhäuser Boston, Boston, MA, 1991, pp. 275–295.

[27] M.B. Marcus, G. Pisier, Characterizations of almost surely continuous p-stable random Fourier series and strongly stationary processes, Acta Math. 152 (3–4) (1984) 245–301.

[28] I. Karatzas, E. Shreve, Brownian Motion and Stochastic Calculus, Springer-Verlag, 1998.

[29] M. Clausel, Quelques notions d'irrégularité uniforme et ponctuelles: le point de vue ondelettes. Ph.D. Thesis, Université Paris XII, 2008.
[30] K. Falconer, Fractal Geometry, 2nd edn, in: Mathematical Foundations and Applications, John Wiley & Sons Inc., Hoboken, NJ, 2003.
[31] A. Benassi, S. Cohen, J. Istas, Local self-similarity and the Hausdorff dimension, C.R. Acad Sci. Paris Ser.I 336 (3) (2003) 267–272.
[32] A. Ayache, F. Roueff, A Fourier formulation of the Frostman criterion for random graphs and its applications to wavelet series, Appl. Comput. Harmon. Anal. 14 (2003) 75–82.