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Self-similar Random Fields and Rescaled Random Balls Models

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Abstract We study generalized random fields which arise as rescaling limits of spatial configurations of uniformly scattered random balls as the mean radius of the balls tends to 0 or infinity. Assuming that the radius distribution has a power-law behavior, we prove that the centered and renormalized random balls field admits a limit with self-similarity properties. Our main result states that all self-similar, translation- and rotation-invariant Gaussian fields can be obtained through a unified zooming procedure starting from a random balls model. This approach has to be understood as a microscopic description of macroscopic properties. Under specific assumptions, we also get a Poisson-type asymptotic field. In addition to investigating stationarity and self-similarity properties, we give L^2-representations of the asymptotic generalized random fields viewed as continuous random linear functionals.

Keywords Self-similarity · Generalized random field · Poisson point process · Fractional Poisson field · Fractional Brownian field

Mathematics Subject Classification (2000) Primary 60G60 · 60G78 · Secondary 60G20 · 60D05 · 60G55 · 60G10 · 60F05

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Introduction

In this work we construct essentially all Gaussian, translation- and rotation-invariant, H-self-similar generalized random fields on $\mathbb{R}^d$ in a unified manner as scaling limits of a random balls model. The self-similarity index H ranges over all of $\mathbb{R} \setminus \mathbb{Z}$ and the random balls model is of germ-grain type. It arises by aggregation of spherical grains attached to uniformly scattered germs given by a Poisson point process in d-dimensional space. By a similar scaling procedure, we obtain also non-Gaussian random fields with interesting properties, in particular a model of the type "fractional Poisson field." Its covariance functional coincides with that of the Gaussian H-self-similar field, so that it fulfills a second-order self-similarity property. Although not self-similar in law, this Poisson field presents a property of "aggregate similarity" which takes into account both Poisson structure and self-similarity.

We observe two distinctly separate behaviors depending on whether the self-similarity index H belongs to an interval of type $(m, m+1/2)$ or of type $[m-1/2, m)$ for some integer m. In the first case, the scaling limit applies to random balls models with balls of arbitrarily small radii. In the opposite case, the corresponding germ-grain models have arbitrarily large spherical grains.

The scaling procedure which acts on the random balls model is based on the assumption that the grains have random radius, independent and identically distributed, with a distribution having a power-law behavior either in zero or at infinity. The resulting configuration of mass, obtained by counting the number of balls that cover any given point in space, suitably centered and normalized, exhibits limit distributions under scaling. For the case of the random balls radius distribution being heavy-tailed at infinity, the corresponding scaling operation amounts to zooming out over larger areas of space while renormalizing the mass. In the opposite case, when the radius of balls is given by an intensity with prescribed power-law behavior close to zero, the scaling which is applied entails zooming in successively smaller regions of space. Infinitesimally small microballs will emerge and eventually shape the resulting limit fields. In particular, our results unify and extend in some directions the previous works on similar topics in Kaj et al. [15] (case $H \in (-d/2, 0)$) and Biermé and Estrade [4] (case $H \in (0, 1/2)$). Preliminary and less general versions of some of the results presented here have appeared in Biermé et al. [5] (case $H \in (-d/2, 0) \cup (0, 1/2)$). Let us emphasize that the main novelty of this paper is the extension to any noninteger values of H and the complete description of the asymptotic fields.

Dobrushin [9] characterized the stationary self-similar Gaussian generalized random fields in their spectral form. In this work we obtain the subclass of such random fields that are isotropic, since the random balls models under consideration are rotationally symmetric. In order to obtain the whole range of self-similarity behavior, it is necessary to work not only with stationary random fields but with the wider class of generalized random fields with stationary increments or stationary nth increments. In this sense our approach also links to the line of work initiated by Matheron [18].

The paper is organized as follows. After having introduced the modeling framework and the setting of the investigation, we discuss in Sect. 2 some principles for scaling limit analysis and state two main results, which cover a Gaussian limit regime and a Poisson limit regime. Section 3 is devoted to the properties of the limiting random fields: stationarity and self- or aggregate-similarity. The main results, in particular Theorem 4.7, are presented in Sect. 4 with the study of all self-similar isotropic stationary generalized random fields. In particular we prove that all such Gaussian fields arise as scaling limits of the random balls model. In Sect. 5 we give a pointwise representation of the generalized self-similar fields with positive self-similarity index $H > 0$ and discuss a few explicit examples.

1 Setting

We present first a unified framework which includes and extends both of the distinct modeling scenarios studied in [15] and [4], respectively. Let $B(x, r)$ denote the ball in $\mathbb{R}^d$ with center at x and radius r and consider a family of grains $X_j + B(0, R_j)$ in $\mathbb{R}^d$ generated by a Poisson point process $(X_j, R_j)_j$ in $\mathbb{R}^d \times \mathbb{R}^+$. Equivalently, we let $N(\mathrm{d}x, \mathrm{d}r)$ be a Poisson random measure on $\mathbb{R}^d \times \mathbb{R}^+$ and associate with each random point $(x, r) \in \mathbb{R}^d \times \mathbb{R}^+$ the random ball $B(x, r)$. We assume that the intensity measure of N is given by $\kappa\,\mathrm{d}x\,F(\mathrm{d}r)$, where κ is a positive constant and F is a nonnegative measure on $\mathbb{R}^+$, σ-finite on $(0, +\infty)$. Moreover, we assume throughout the paper that the ball radius intensity $F(\mathrm{d}r)$ is such that

$$\int_{\mathbb{R}^+} r^d F(\mathrm{d}r) < +\infty. \tag{1}$$

Note that if F is a probability measure, this assumption implies that the expected volume of a ball is finite.

For measurable sets $A \subset \mathbb{R}^d \times \mathbb{R}^+$, we let $N(A) = \int_A N(\mathrm{d}x, \mathrm{d}r)$ denote the number of balls with random location and radius (x, r) contained in A and view the values of $A \mapsto N(A)$ as integer-valued random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We recall the basic facts (see [17], Chap. 10 for instance) that $N(A)$ is Poisson distributed with mean $\int_A \kappa\,\mathrm{d}x\,F(\mathrm{d}r)$ (if the integral diverges, then $N(A)$ is countably infinite with probability one) and that if $A_1, \dots, A_n$ are disjoint, then $N(A_1), \dots, N(A_n)$ are independent. We also recall that for measurable functions $k : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$, the stochastic integral $\int k(x, r)\,N(\mathrm{d}x, \mathrm{d}r)$ of k with respect to N exists $\mathbb{P}$-a.s. if and only if

$$\int_{\mathbb{R}^d \times \mathbb{R}^+} \min(|k(x, r)|, 1)\,\mathrm{d}x\,F(\mathrm{d}r) < \infty. \tag{2}$$

1.1 Power-law Assumption

For $\beta \neq d$, we introduce the following asymptotic power-law assumption for the behavior of F near 0 or at infinity:

$$\mathbf{A}(\beta)\colon \quad F(\mathrm{d}r) = f(r)\mathrm{d}r \quad \text{with } f(r) \sim r^{-\beta-1} \text{ as } r \to 0^{d-\beta},$$

where by convention $0^\alpha = 0$ if $\alpha > 0$ and $0^\alpha = +\infty$ if $\alpha < 0$.

The range of parameter values under consideration will be $d - 1 < \beta < 2d$. Then, according to (1), under assumption $\mathbf{A}(\beta)$, it is natural to consider the asymptotic behavior of F near 0 for $d - 1 < \beta < d$ and at infinity for $d < \beta < 2d$.

1.2 Random Field

We consider random fields defined on a space of measures, in the same spirit as the random functionals of [15] or the generalized random fields of [3]. Let $\mathcal{M}$ denote the space of signed measures μ on $\mathbb{R}^d$ with finite total variation

$$\|\mu\| := |\mu|\big(\mathbb{R}^d\big) < \infty, \tag{3}$$

where $|\mu|$ is the total variation measure of μ, and $\|\cdot\|$ is a norm on $\mathcal{M}$ (see, e.g., [21], p. 161). For any $\mu \in \mathcal{M}$, $\mu(B(x,r))$ is a measurable function on $\mathbb{R}^d \times \mathbb{R}^+$ for which

$$\int_{\mathbb{R}^d\times\mathbb{R}^+} |\mu(B(x,r))|\,\mathrm{d}x\, F(\mathrm{d}r) \le v_d|\mu|\big(\mathbb{R}^d\big)\int_{\mathbb{R}^+} r^d F(\mathrm{d}r) < +\infty, \tag{4}$$

in view of (1), where v_d is the Lebesgue measure of the unit ball in $\mathbb{R}^d$. In particular, (2) applies with $k(x,r) = \mu(B(x,r))$. We may hence introduce a generalized random field X defined on $\mathcal{M}$ by

$$X(\mu) = \int_{\mathbb{R}^d\times\mathbb{R}^+} \mu(B(x,r))N(\mathrm{d}x,\mathrm{d}r), \quad \mu \in \mathcal{M}. \tag{5}$$

Condition (4) is even sufficient and necessary for $X(\mu)$ to have finite expected value, and in this case

$$\mathbb{E}X(\mu) = \int_{\mathbb{R}^d\times\mathbb{R}^+} \mu(B(x,r))\kappa\,\mathrm{d}x\, F(\mathrm{d}r) = \kappa v_d \mu\big(\mathbb{R}^d\big)\int_{\mathbb{R}^+} r^d F(\mathrm{d}r).$$

Let us also note that the random field X is linear on each vectorial subspace of $\mathcal{M}$ in the sense that for all $\mu_1,\dots,\mu_n \in \mathcal{M}$ and $a_1,\dots,a_n \in \mathbb{R}$, almost surely,

$$X(a_1\mu_1 + \cdots + a_n\mu_n) = a_1X(\mu_1) + \cdots + a_nX(\mu_n).$$

Furthermore the characteristic function of $X(\mu)$ is given by (see [17], Lemma 10.2)

$$\mathbb{E}\big(e^{itX(\mu)}\big) = \exp\bigg(\int_{\mathbb{R}^d\times\mathbb{R}^+} \big(e^{it\mu(B(x,r))} - 1\big)\kappa\,\mathrm{d}x\, F(\mathrm{d}r)\bigg), \quad t \in \mathbb{R}. \tag{6}$$

Our first proposition adds to this a simple topological structure.

Proposition 1.1 *The random field* $X : (\mathcal{M}, \|\cdot\|) \to (L^2(\Omega,\mathcal{A},\mathbb{P}), \|\cdot\|_2)$ *is a continuous random linear functional*, *where* $\|\cdot\|$ *is given by* (3), *and* $\|\cdot\|_2$ *is the usual norm on* $L^2(\Omega,\mathcal{A},\mathbb{P})$.

Proof Let $\mu \in \mathcal{M}$. The random variable $X(\mu)$ is in $L^2(\Omega,\mathcal{A},\mathbb{P})$, and so X can be considered as a linear functional $X : \mathcal{M} \to L^2(\Omega,\mathcal{A},\mathbb{P})$. Moreover, for any $\mu \in \mathcal{M}$, by Fubini's theorem,

$$\mathrm{Var}(X(\mu)) = \int_{\mathbb{R}^d\times\mathbb{R}^+} \mu(B(x,r))^2\,\kappa\,\mathrm{d}x\, F(\mathrm{d}r)$$

$$\le \kappa\,\|\mu\|\int_{\mathbb{R}^d\times\mathbb{R}^d}|\mu(B(x,r))|\,\mathrm{d}x\,F(\mathrm{d}r) \tag{7}$$
$$\le \kappa\,v_d\left(\int_{\mathbb{R}^+} r^d\,F(\mathrm{d}r)\right)\|\mu\|^2<\infty.$$

Similarly, $|\mathbb{E}(X(\mu))|\le\kappa\,v_d\,(\int_0^{+\infty}r^dF(\mathrm{d}r))\|\mu\|$. Therefore, according to (1), one can find a positive constant $c_d>0$ such that

$$\|X(\mu)\|_2=\sqrt{\mathrm{Var}(X(\mu))+\mathbb{E}(X(\mu))^2}\le c_d\|\mu\|,$$

which shows the continuity of X. □

The random linear functional $X-\mathbb{E}(X)$ is also a continuous linear functional from $(\mathcal{M},\|\cdot\|)$ to $(L^2(\Omega,\mathcal{A},\mathbb{P}),\|\cdot\|_2)$. The corresponding subordinated norm of $X-\mathbb{E}(X)$ is given by

$$|\!|\!|X-\mathbb{E}(X)|\!|\!|=\sup_{\|\mu\|\le1}\|X(\mu)-\mathbb{E}(X(\mu))\|_2=\sup_{\|\mu\|\le1}\sqrt{\mathrm{Var}(X(\mu))}.$$

For $\mu=\delta_0$, the Dirac mass at the origin of $\mathbb{R}^d$, we get $\mathrm{Var}(X(\delta_0))=\kappa\,v_d(\int_{\mathbb{R}^+}r^dF(\mathrm{d}r))$ and may conclude in view of (7) that

$$|\!|\!|X-\mathbb{E}(X)|\!|\!|=\sqrt{\left(\kappa\,v_d\int_{\mathbb{R}^+}r^dF(\mathrm{d}r)\right)}. \tag{8}$$

2 Scaling Limit

2.1 Scaled Random Fields

Let us introduce now the notion of "scaling," by which we indicate an action: a change of scale acts on the size of the grains. The scaling procedure performed in [15] acts on grains of volume v changed by shrinking into grains of volume $\rho\,v$ with small parameter ρ ("small scaling" behavior). The same is performed in [4] in the context of a homogenization, but the scaling acts in the opposite way: the radii r of grains are changed into r/ε (which is a "large scaling" behavior). To cover both mechanisms we introduce the random field which is obtained by applying the rescaling of measures $\mu\mapsto\mu^\rho$, where $\mu^\rho(B)=\mu(\rho B)$ for $\rho>0$ and measurable subsets B of $\mathbb{R}^d$. Let us denote by $F_\rho(\mathrm{d}r)$ the image measure of $F(\mathrm{d}r)$ by the change of scale $r\mapsto\rho r$ and remark that

$$X\big(\mu^\rho\big)=\int_{\mathbb{R}^d\times\mathbb{R}^+}\mu(B(x,r))N\big(\mathrm{d}\rho^{-1}x,\mathrm{d}\rho^{-1}r\big),\quad\forall\mu\in\mathcal{M},$$

where the intensity measure of $N(\mathrm{d}\rho^{-1}x,\mathrm{d}\rho^{-1}r)$ is $\kappa\,\rho^{-d}\mathrm{d}x\,F_\rho(\mathrm{d}r)$. It is natural from this viewpoint to have μ representing an observation window and interpret limits $\rho\to0$ as *zoom-out* and limits $\rho\to\infty$ as *zoom-in* of the random configurations of balls in space.

Let us multiply the intensity measure by λ/κ ($\lambda > 0$) and consider the associated random field on $\mathcal{M}$ given by

$$\int_{\mathbb{R}^d\times\mathbb{R}^+} \mu(B(x,r))N_{\lambda,\rho}(\mathrm{d}x,\mathrm{d}r),$$

where $N_{\lambda,\rho}(\mathrm{d}x,\mathrm{d}r)$ is the Poisson random measure with intensity measure $\lambda\,\mathrm{d}x\,F_\rho(\mathrm{d}r)$ and $\mu\in\mathcal{M}$. Choosing $\lambda=\kappa\rho^{-d}$, this random field has the same law as $\{X(\mu^\rho);\ \mu\in\mathcal{M}\}$. Results are expected concerning the asymptotic behavior of this scaled random balls model under hypothesis $\mathbf{A}(\beta)$ as $\rho\to 0$ or $\rho\to+\infty$. We choose ρ as the basic model parameter, consider $\lambda=\lambda(\rho)$ as a function of ρ, and define on $\mathcal{M}$ the random field

$$X_\rho(\mu)=\int_{\mathbb{R}^d\times\mathbb{R}^+} \mu(B(x,r))N_{\lambda(\rho),\rho}(\mathrm{d}x,\mathrm{d}r). \tag{9}$$

Then, we are looking for a normalization term $n(\rho)$ such that the centered field converges in distribution,

$$\frac{X_\rho(.)-\mathbb{E}(X_\rho(.))}{n(\rho)}\overset{\text{fdd}}{\to} W(.), \tag{10}$$

and we are interested in the nature of the limit field W. The convergence (10) holds whenever

$$\mathbb{E}\left(\exp\left(i\frac{X_\rho(\mu)-\mathbb{E}(X_\rho(\mu))}{n(\rho)}\right)\right)\to\mathbb{E}(\exp(iW(\mu)))$$

for all μ in a convenient subspace of $\mathcal{M}$. A scaling analysis of power law tails reveals that under $\mathbf{A}(\beta)$ we expect

$$\mathrm{Var}(X_\rho(\mu))\sim\lambda(\rho)\rho^\beta\mathrm{Var}(X(\mu)),\qquad \rho\to 0^{\beta-d},$$

which suggests the asymptotic relation $n(\rho)^2\sim\lambda(\rho)\,\rho^\beta$ to obtain the convergence of (10) in $(L^2(\Omega,\mathcal{A},\mathbb{P}),\|\cdot\|_2)$. However, in view of (8), the norm of $(X_\rho-\mathbb{E}(X_\rho))/n(\rho)$ as a continuous linear functional from $(\mathcal{M},\|\cdot\|)$ to $(L^2(\Omega,\mathcal{A},\mathbb{P}),\|\cdot\|_2)$ is given by

$$\left\|\left|\frac{X_\rho-\mathbb{E}(X_\rho)}{n(\rho)}\right\|\right|=\sqrt{\left(v_d\int_{\mathbb{R}^+}r^dF(\mathrm{d}r)\right)}\sqrt{\frac{\lambda(\rho)\rho^d}{n(\rho)^2}}. \tag{11}$$

In particular, (11) is not bounded for $n(\rho)^2=\lambda(\rho)\rho^\beta$ as $\rho\to 0^{\beta-d}$, and the Banach–Steinhaus theorem states that there exists a dense subset of $\mathcal{M}$ on which the rescaled process $(X_\rho(\mu)-\mathbb{E}(X_\rho(\mu)))/\sqrt{\lambda(\rho)\rho^\beta}$ cannot converge in $(L^2(\Omega,\mathcal{A},\mathbb{P}),\|\cdot\|_2)$. Therefore, we study in the sequel the convergence (10) on strict subspaces of $\mathcal{M}$. This will allow us to get in the limit a continuous linear functional taking values in $(L^2(\Omega,\mathcal{A},\mathbb{P}),\|\cdot\|_2)$, despite the fact that the convergence holds only for finite-dimensional distributions.

2.2 Gaussian Limit Regime

For $\beta \neq d$, let us define the space of measures

$$\mathcal{M}^{\beta} = \Big\{ \mu \in \mathcal{M} :\ \exists \alpha \text{ s.t. } \alpha < \beta < d \text{ or } d < \beta < \alpha \\ \text{and } \int_{\mathbb{R}^d \times \mathbb{R}^d} |z - z'|^{d-\alpha} |\mu|(\mathrm{d}z)|\mu|(\mathrm{d}z') < +\infty \Big\},$$

where $|z|$ denotes the Euclidean norm of $z \in \mathbb{R}^d$, and $|\mu|$ is the total variation measure of $\mu \in \mathcal{M}$. We remark that the integral assumption is a finite Riesz energy assumption for $\beta > d$ and that $\mathcal{M}^{\beta} = \{0\}$ when $\beta \geq 2d$. In both cases $d - 1 < \beta < d$ and $d < \beta < 2d$, if $\mu \in \mathcal{M}$ satisfies

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} |z - z'|^{d-\alpha} |\mu|(\mathrm{d}z)|\mu|(\mathrm{d}z') < +\infty$$

for some α ($\alpha < \beta < d$ and $d < \beta < \alpha$, respectively), then the same holds for any γ between β and α. In particular, for any $\mu \in \mathcal{M}^{\beta}$,

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} |z - z'|^{d-\beta} |\mu|(\mathrm{d}z)|\mu|(\mathrm{d}z') < +\infty.$$

We also introduce the subspace of finite signed measures of vanishing total mass,

$$\mathcal{M}_1 = \Big\{ \mu \in \mathcal{M} : \int_{\mathbb{R}^d} \mu(\mathrm{d}z) = 0 \Big\},$$

and consider the subspaces

$$\widetilde{\mathcal{M}}_{\beta} = \begin{cases} \mathcal{M}^{\beta} & \text{for } d < \beta < 2d, \\ \mathcal{M}^{\beta} \cap \mathcal{M}_1 & \text{for } d - 1 < \beta < d. \end{cases} \tag{12}$$

Theorem 2.1 *Let $d - 1 < \beta < 2d$ with $\beta \neq d$. Let F be a nonnegative measure on $\mathbb{R}^+$ which satisfies $\mathbf{A}(\beta)$. For all positive functions λ such that $\lambda(\rho)\rho^{\beta} \underset{\rho \to 0^{\beta-d}}{\longrightarrow} +\infty$, the limit*

$$\frac{X_{\rho}(\mu) - \mathbb{E}(X_{\rho}(\mu))}{\sqrt{\lambda(\rho)\rho^{\beta}}} \underset{\rho \to 0^{\beta-d}}{\overset{\text{fdd}}{\longrightarrow}} W_{\beta}(\mu)$$

holds for all $\mu \in \widetilde{\mathcal{M}}_{\beta}$, in the sense of finite-dimensional distributions of the random functionals. Here W_{β} is the centered Gaussian random linear functional on $\widetilde{\mathcal{M}}_{\beta}$ with covariance functional

$$\operatorname{Cov}(W_{\beta}(\mu), W_{\beta}(\nu)) = \mathbb{E}(W_{\beta}(\mu)W_{\beta}(\nu)) = c_{\beta} \int_{\mathbb{R}^d \times \mathbb{R}^d} |z - z'|^{d-\beta} \mu(\mathrm{d}z)\nu(\mathrm{d}z') \tag{13}$$

for a constant c_{β} only depending on β.

Remark 2.2 Equation (13) defines a covariance function, called generalized covariance function in [18]. The value of the constant c_β is given by (19) below.

Proof We begin with two lemmas. The first lemma describes the covariance function and is based on some technical estimates for the intersection volume of two balls. The second one, inspired by Lemma 1 of [15], stands for Lebesgue's theorem with assumptions that are well adapted to the present setting.

Lemma 2.3 *Let $d-1<\beta<2d$ with $\beta \neq d$. There exists a real constant c_β such that for all $\mu \in \widetilde{\mathcal{M}}_\beta$,*

$$0 < \int_{\mathbb{R}^d \times \mathbb{R}^+} \mu(B(x,r))^2 r^{-\beta-1}\,\mathrm{d}r\,\mathrm{d}x = c_\beta \int_{\mathbb{R}^d \times \mathbb{R}^d} |z-z'|^{d-\beta} \mu(\mathrm{d}z)\mu(\mathrm{d}z') < +\infty.$$

Proof Let us introduce the function γ defined on $[0,\infty)$ by

$$\gamma(u) = \text{Lebesgue measure of } B(0,1) \cap B(u\mathbf{e},1) \tag{14}$$

for any unit vector $\mathbf{e} \in \mathbb{R}^d$. The function γ is decreasing, supported on $[0,2]$, bounded by $\gamma(0)=v_d$, continuous on $[0,2]$, and smooth on $(0,2)$. Define γ_β as

$$\gamma_\beta(u) = \begin{cases} \gamma(u)-\gamma(0), & d-1<\beta<d, \\ \gamma(u), & d<\beta<2d. \end{cases}$$

We notice that for $d-1<\beta<d$, $|\gamma_\beta(u)| \le \gamma(0)$ and $|\gamma_\beta(u)| \le \sup_{v>0}|\gamma'(v)|\,u$. Hence, for some constant $C>0$, $|\gamma_\beta(u)| \le C\,u^{d-\alpha}$ for any $0 \le d-\alpha \le 1$, that is, any α in $[d-1,d]$. For $d<\beta<2d$, one can find $C>0$ such that $|\gamma_\beta(u)| \le C\,u^{d-\alpha}$ for any $\alpha \ge \beta$. In particular, we may take α such that $d-1<\alpha<\beta$ for the case $d-1<\beta<d$ and α such that $\beta<\alpha<2d$ for $d<\beta<2d$, and for both cases, we have a $C>0$ with

$$\forall u>0, \quad |\gamma_\beta(u)| \le C u^{d-\alpha}. \tag{15}$$

Step 1. For $\mu \in \widetilde{\mathcal{M}}_\beta$, let us prove that $\int_{\mathbb{R}^d \times \mathbb{R}^+} \mu(B(x,r))^2 r^{-\beta-1}\,\mathrm{d}r\,\mathrm{d}x < +\infty$. We introduce the function φ defined by

$$\varphi(r) = \int_{\mathbb{R}^d} \mu(B(x,r))^2\,\mathrm{d}x, \quad r>0. \tag{16}$$

Using successively Fubini's theorem, homogeneity, and (14), we get

$$\begin{aligned} \varphi(r) &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \left(\int_{\mathbb{R}^d} \mathbf{1}_{B(z,r)}(x) \mathbf{1}_{B(z',r)}(x)\,\mathrm{d}x \right) \mu(\mathrm{d}z)\mu(\mathrm{d}z') \\ &= r^d \int_{\mathbb{R}^d \times \mathbb{R}^d} \gamma(|z-z'|/r) \mu(\mathrm{d}z)\mu(\mathrm{d}z'). \end{aligned}$$

Therefore $\varphi(r) \le \gamma(0)\,|\mu|(\mathbb{R}^d)^2\,r^d$. Moreover, since $\mu \in \widetilde{\mathcal{M}}_\beta$,

$$\varphi(r) = r^d \int_{\mathbb{R}^d \times \mathbb{R}^d} \gamma_\beta(|z-z'|/r) \mu(\mathrm{d}z)\mu(\mathrm{d}z'), \tag{17}$$

and we can choose α such that $\int_{\mathbb{R}^d\times\mathbb{R}^d}|z-z'|^{d-\alpha}|\mu|(\mathrm{d}z)|\mu|(\mathrm{d}z')<+\infty$ and (15) holds. Then

$$\varphi(r)\le Cr^{\alpha}\int_{\mathbb{R}^d\times\mathbb{R}^d}|z-z'|^{d-\alpha}|\mu|(\mathrm{d}z)|\mu|(\mathrm{d}z').$$

Finally, one can find $C>0$ such that

$$\varphi(r)\le C\min\bigl(r^d,r^{\alpha}\bigr) \tag{18}$$

and

$$\int_0^{+\infty}\varphi(r)r^{-\beta-1}\,\mathrm{d}r=\int_{\mathbb{R}^d\times\mathbb{R}^+}\mu(B(x,r))^2r^{-\beta-1}\,\mathrm{d}r\,\mathrm{d}x<+\infty.$$

Step 2. We prove the equality stated in the lemma, which is

$$\int_0^{+\infty}\varphi(r)r^{-\beta-1}\,\mathrm{d}r=c_\beta\int_{\mathbb{R}^d\times\mathbb{R}^d}|z-z'|^{d-\beta}\mu(\mathrm{d}z)\mu(\mathrm{d}z'),$$

using the previous notation. To this end we wish to replace φ by (17) in the left-hand side integral. Using estimates (15) on $|\gamma_\beta|$, one can show that the integral

$$I_\beta(u):=\int_{\mathbb{R}^+}\gamma_\beta(u/r)r^{d-\beta-1}\,\mathrm{d}r$$

is well defined for all $u\in\mathbb{R}_+$. Furthermore, I_β is homogeneous of order $d-\beta$ so that

$$\forall u>0,\quad I_\beta(u)=I_\beta(1)u^{d-\beta}.$$

This proves that

$$\int_0^{+\infty}\varphi(r)r^{-\beta-1}\,\mathrm{d}r=I_\beta(1)\int_{\mathbb{R}^d\times\mathbb{R}^d}|z-z'|^{d-\beta}\mu(\mathrm{d}z)\mu(\mathrm{d}z'),$$

which completes the proof of the lemma with

$$c_\beta=I_\beta(1)=\int_{\mathbb{R}^+}\gamma_\beta(1/r)r^{d-\beta-1}\,\mathrm{d}r. \tag{19}$$

□

Now let us state a second lemma, which is the main tool to establish our scaling limit results.

Lemma 2.4 *Let F be a nonnegative measure on $\mathbb{R}^+$ satisfying $\mathbf{A}(\beta)$ for $\beta\neq d$.*

(i) *Assume that g is a continuous function on $\mathbb{R}^+$ such that for some $0<p<\beta<q$, there exists $C>0$ such that*

$$|g(r)|\le C\min\bigl(r^q,r^p\bigr).$$

Then

$$\int_{\mathbb{R}^+} g(r) F_\rho(\mathrm{d}r) \sim \rho^\beta \int_{\mathbb{R}^+} g(r) r^{-\beta-1}\,\mathrm{d}r \quad \text{as } \rho \to 0^{\beta-d}.$$

(ii) *Let g_ρ be a family of continuous functions on $\mathbb{R}^+$. Assume that*

$$\lim_{\rho\to 0^{\beta-d}} \rho^\beta g_\rho(r) = 0 \quad \text{and} \quad \rho^\beta |g_\rho(r)| \le C \min\bigl(r^p, r^q\bigr)$$

for some $0 < p < \beta < q$ and $C > 0$. Then

$$\lim_{\rho\to 0^{\beta-d}} \int_{\mathbb{R}^+} g_\rho(r) F_\rho(\mathrm{d}r) = 0.$$

Proof (i) Let us assume, for instance, that $\beta < d$ (the proof of the case $\beta > d$ is similar and can be found in [15]). Let $\varepsilon > 0$. Since F satisfies $\mathbf{A}(\beta)$, there exists $\delta > 0$ such that

$$r < \delta \quad \Rightarrow \quad \bigl|f(r) - r^{-\beta-1}\bigr| \le \varepsilon r^{-\beta-1}. \tag{20}$$

Let us remark that the assumptions on g ensure that

$$\int_0^{+\infty} |g(r)| r^{-\beta-1}\,\mathrm{d}r < +\infty.$$

On the one hand, since $\int_0^{\delta\rho} g(r) F_\rho(\mathrm{d}r) = \int_0^{\delta\rho} g(r) f(\frac{r}{\rho}) \frac{\mathrm{d}r}{\rho}$, we get by (20)

$$\left|\int_0^{\delta\rho} g(r) F_\rho(\mathrm{d}r) - \rho^\beta \int_0^{\delta\rho} g(r) r^{-\beta-1}\,\mathrm{d}r\right| \le \varepsilon \rho^\beta \int_{\mathbb{R}^+} |g(r)| r^{-\beta-1}\,\mathrm{d}r.$$

On the other hand, for $\delta\rho > 1$, since $|g(r)| \le C r^p$,

$$\left|\int_{\delta\rho}^{\infty} g(r) F_\rho(\mathrm{d}r) - \rho^\beta \int_{\delta\rho}^{\infty} g(r) r^{-\beta-1}\mathrm{d}r\right| \le C C_1(\delta) \rho^p + \frac{C}{\beta - p} \delta^{p-\beta} \rho^p,$$

where $C_1(\delta) = \int_\delta^{+\infty} r^p F(\mathrm{d}r) \le \delta^{p-d} \int_{\mathbb{R}^+} r^d F(\mathrm{d}r) < \infty$. Since $p < \beta$, we obtain (i).

(ii) We follow the same lines as for (i) and can assume similarly that $\beta < d$. Since F satisfies $\mathbf{A}(\beta)$, there exists $\delta > 0$ such that

$$r < \delta \quad \Rightarrow \quad |f(r)| \le 2 r^{-\beta-1}. \tag{21}$$

The assumptions on g_ρ ensure that for all $\rho > 0$,

$$\int_0^{+\infty} \rho^\beta |g_\rho(r)| r^{-\beta-1}\,\mathrm{d}r < +\infty \quad \text{with } \lim_{\rho\to+\infty} \int_0^\infty \rho^\beta |g_\rho(r)| r^{-\beta-1}\,\mathrm{d}r = 0,$$

by Lebesgue's theorem. Since $\int_0^{\delta\rho} g_\rho(r) F_\rho(\mathrm{d}r) = \int_0^{\delta\rho} g_\rho(r) f(\frac{r}{\rho}) \frac{\mathrm{d}r}{\rho}$, we get by (21)

$$\left|\int_0^{\delta\rho} g_\rho(r) F_\rho(\mathrm{d}r)\right| \le 2 \int_0^\infty \rho^\beta |g_\rho(r)| r^{-\beta-1}\,\mathrm{d}r.$$

Therefore,

$$\lim_{\rho\to+\infty}\int_0^{\delta\rho} g_\rho(r)F_\rho(\mathrm{d}r)=0. \tag{22}$$

Moreover, for $\delta\rho>1$, since $C_1(\delta)=\int_\delta^{+\infty} r^p F(\mathrm{d}r)<+\infty$ and $|g_\rho(r)|\le C\rho^{-\beta}r^p$,

$$\left|\int_{\delta\rho}^\infty g_\rho(r)F_\rho(\mathrm{d}r)\right|\le C\rho^{-\beta}\int_{\delta\rho}^\infty r^pF_\rho(\mathrm{d}r)\le CC_1(\delta)\rho^{-(\beta-p)}. \tag{23}$$

We conclude the proof using (22) and (23), since $p<\beta$. □

We start now with the proof of Theorem 2.1. Let us denote

$$n(\rho):=\sqrt{\lambda(\rho)\rho^\beta}$$

and define the function φ_ρ on $\mathbb{R}^+$ by

$$\varphi_\rho(r)=\int_{\mathbb{R}^d}\Psi\left(\frac{\mu(B(x,r))}{n(\rho)}\right)\mathrm{d}x,$$

where

$$\Psi(v)=e^{iv}-1-iv. \tag{24}$$

According to (6), the characteristic function of the normalized field $(X_\rho(.)-E(X_\rho(.)))/n(\rho)$ is given by

$$\mathbb{E}\left(\exp\left(i\frac{X_\rho(\mu)-\mathbb{E}(X_\rho(\mu))}{n(\rho)}\right)\right)=\exp\left(\int_{\mathbb{R}^+}\lambda(\rho)\varphi_\rho(r)F_\rho(\mathrm{d}r)\right).$$

By assumption, $n(\rho)$ tends to $+\infty$ as $\rho\to 0^{\beta-d}$ so that $\Psi(\frac{\mu(B(x,r))}{n(\rho)})$ behaves like $-\frac{1}{2}(\frac{\mu(B(x,r))}{n(\rho)})^2$. Therefore, we write

$$\int_{\mathbb{R}^+}\lambda(\rho)\varphi_\rho(r)F_\rho(\mathrm{d}r)=-\frac{1}{2}\int_{\mathbb{R}^+}\varphi(r)\lambda(\rho)n(\rho)^{-2}F_\rho(\mathrm{d}r)+\int_{\mathbb{R}^+}\Delta_\rho(r)F_\rho(\mathrm{d}r), \tag{25}$$

where the function φ is introduced in (16), and

$$\begin{aligned}\Delta_\rho(r)&=\lambda(\rho)\varphi_\rho(r)+\frac{1}{2}\lambda(\rho)n(\rho)^{-2}\varphi(r) \qquad (26)\\ &=\lambda(\rho)\int_{\mathbb{R}^d}\left(\Psi\left(\frac{\mu(B(x,r))}{n(\rho)}\right)+\frac{1}{2}\left(\frac{\mu(B(x,r)}{n(\rho)}\right)^2\right)\mathrm{d}x.\end{aligned}$$

Since $\mu\in\widetilde{\mathcal{M}}_\beta$, the function φ is continuous on $\mathbb{R}^+$ and satisfies (18). Thus, by Lemma 2.4(i), the first term of the right-hand side of (25) converges to $-\frac{1}{2}\int_{\mathbb{R}^+}\varphi(r)r^{-\beta-1}\,\mathrm{d}r$. Moreover, by Lemma 2.3, we obtain

$$\lim_{\rho\to 0^{\beta-d}}\int_{\mathbb{R}^+}\varphi(r)\lambda(\rho)n(\rho)^{-2}F_\rho(\mathrm{d}r)=c_\beta\int_{\mathbb{R}^d\times\mathbb{R}^d}|z-z'|^{d-\beta}\mu(\mathrm{d}z)\mu(\mathrm{d}z').$$

For the second term, let us verify that Δ_ρ given by (26) satisfies the assumptions of Lemma 2.4(ii). First, let us remark that the function Δ_ρ is continuous on $\mathbb{R}^+$ since $\mu \in \mathcal{M}$. Because $|\Psi(v) - (-\frac{v^2}{2})| \le \frac{|v|^3}{6}$ and

$$\int_{\mathbb{R}^d} |\mu(B(x,r))|^3 \, \mathrm{d}x \le \|\mu\|^2 \int_{\mathbb{R}^d} |\mu(B(x,r))| \, \mathrm{d}x \le v_d \, \|\mu\|^3 r^d,$$

we also check that

$$\left|\lambda(\rho)^{-1} n(\rho)^2 \Delta_\rho(r)\right| \le \frac{1}{6} v_d \, \|\mu\|^3 \, n(\rho)^{-1} r^d.$$

Finally, since $|\Psi(v)| \le \frac{|v|^2}{2}$, by (18) there exists $C > 0$ such that

$$\left|\lambda(\rho)^{-1} n(\rho)^2 \Delta_\rho(r)\right| \le C r^\alpha$$

for some α with $(\alpha - \beta)(\beta - d) > 0$. Therefore, $\int_{\mathbb{R}^+} \Delta_\rho(r) F_\rho(\mathrm{d}r)$ tends to 0 according to Lemma 2.4(ii), and so

$$\begin{aligned}\lim_{\rho \to 0^{\beta - d}} \mathbb{E}\left(\exp\left(i \frac{X_\rho(\mu) - \mathbb{E}(X_\rho(\mu))}{n(\rho)}\right)\right) \\ = \exp\left(-\frac{1}{2} c_\beta \int_{\mathbb{R}^d \times \mathbb{R}^d} |z - z'|^{d-\beta} \mu(\mathrm{d}z) \mu(\mathrm{d}z')\right).\end{aligned}$$

Hence $(X_\rho(\mu) - \mathbb{E}(X_\rho(\mu)))/n(\rho)$ converges in distribution to the centered Gaussian random variable $W(\mu)$ whose variance is equal to

$$\mathbb{E}\big(W(\mu)^2\big) = c_\beta \int_{\mathbb{R}^d \times \mathbb{R}^d} |z - z'|^{d-\beta} \mu(\mathrm{d}z) \mu(\mathrm{d}z').$$

By linearity, the covariance of W satisfies (13). □

With similar arguments, we can state a further scaling result leading to a non-Gaussian limit.

2.3 Poisson Limit Regime

In this section we keep the notation introduced in Sect. 2.2 for the Gaussian limit regime.

Theorem 2.5 *Let $d - 1 < \beta < 2d$ with $\beta \neq d$. Let F be a nonnegative measure on $\mathbb{R}^+$ satisfying $\mathbf{A}(\beta)$. For all positive functions λ such that $\lambda(\rho)\rho^\beta \underset{\rho \to 0^{\beta - d}}{\longrightarrow} a^{d-\beta}$ for some $a > 0$, we have, in the sense of finite-dimensional distributions of random functionals, the scaling limit*

$$X_\rho(\mu) - \mathbb{E}(X_\rho(\mu)) \overset{\text{fdd}}{\to} J_\beta(\mu_a)$$

for all $\mu \in \widetilde{\mathcal{M}}_\beta$. *Here* J_β *is the centered random linear functional on* $\widetilde{\mathcal{M}}_\beta$ *defined as*

$$J_\beta(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}^+} \mu(B(x,r)) \widetilde{N}_\beta(\mathrm{d}x, \mathrm{d}r),$$

where $\widetilde{N}_\beta$ *is a compensated Poisson random measure with intensity* $\mathrm{d}x\, r^{-\beta-1}\mathrm{d}r$, *and* μ_a *is defined by* $\mu_a(A) = \mu(a^{-1}A)$.

Proof Let us recall that a compensated Poisson measure $\widetilde{N}$ of intensity n is such that $\widetilde{N} + n$ is a Poisson measure of intensity n. Therefore, the stochastic integral $\int k(x,r)\, \widetilde{N}(\mathrm{d}x, \mathrm{d}r)$ of a measurable function $k : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$ with respect to a compensated Poisson measure $\widetilde{N}$ of intensity n exists $\mathbb{P}$-a.s. if and only if

$$\int_{\mathbb{R}^d \times \mathbb{R}^+} \min\bigl(|k(x,r)|, k(x,r)^2\bigr)\, n(\mathrm{d}x, \mathrm{d}r) < \infty \tag{27}$$

(see [17], Theorem 10.15 for instance).

By Lemma 2.3, using once again the function φ introduced in (16), for all $\mu \in \widetilde{\mathcal{M}}_\beta$, we have

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^+} \mu(B(x,r))^2 r^{-\beta-1}\, \mathrm{d}r\, \mathrm{d}x = \int_{\mathbb{R}^+} \varphi(r) r^{-\beta-1}\, \mathrm{d}r < +\infty.$$

Hence, in view of (27) with $n(\mathrm{d}x, \mathrm{d}r) = \mathrm{d}x\; r^{-\beta-1}\mathrm{d}r$ and $k(x,r) = \mu(B(x,r))$, the random field J_β is well defined on $\widetilde{\mathcal{M}}_\beta$, with characteristic function

$$\mathbb{E}(\exp(i J_\beta(\mu))) = \exp\biggl(\int_{\mathbb{R}^+ \times \mathbb{R}^d} \Psi(\mu(B(x,r)))\, \mathrm{d}x\, r^{-\beta-1}\mathrm{d}r\biggr), \tag{28}$$

where Ψ is given by (24).

On the other hand, the characteristic function for the centered Poisson random balls model equals

$$\mathbb{E}(\exp(i(X_\rho(\mu) - \mathbb{E}(X_\rho(\mu))))) = \exp\biggl(\int_{\mathbb{R}^+ \times \mathbb{R}^d} \Psi(\mu(B(x,r)))\, \mathrm{d}x\, \lambda(\rho) F_\rho(\mathrm{d}r)\biggr).$$

Define for $r > 0$,

$$\widetilde{\varphi}(r) = \int_{\mathbb{R}^d} \Psi(\mu(B(x,r)))\, \mathrm{d}x.$$

For $\mu \in \widetilde{\mathcal{M}}_\beta$, using $|\Psi(v)| \le |v|^2/2$ and (18), we get that there exists $C > 0$ such that

$$|\widetilde{\varphi}(r)| \le C \min\bigl(r^d, r^\alpha\bigr)$$

for some α with $(\alpha - \beta)(\beta - d) > 0$. Thus, by Lemma 2.4(i),

$$\int_{\mathbb{R}^+} \lambda(\rho) \widetilde{\varphi}(r) F_\rho(\mathrm{d}r) \underset{\rho \to 0^{\beta-d}}{\sim} a^{d-\beta} \int_0^\infty \widetilde{\varphi}(r) r^{-\beta-1}\, \mathrm{d}r,$$

and hence,

$$\lim_{\rho\to 0^{\beta-d}} \mathbb{E}(\exp(i(X_\rho(\mu)-\mathbb{E}(X_\rho(\mu)))) = \exp\left(a^{d-\beta}\int_{\mathbb{R}^+}\widetilde{\varphi}(r)\, r^{-\beta-1}\,\mathrm{d}r\right).$$

Finally, it is sufficient to remark that

$$a^{d-\beta}\int_{\mathbb{R}^+}\widetilde{\varphi}(r)r^{-\beta-1}\,\mathrm{d}r = a^d\int_{\mathbb{R}^+}\widetilde{\varphi}(a^{-1}r)r^{-\beta-1}\,\mathrm{d}r$$

with

$$a^d\,\widetilde{\varphi}\big(a^{-1}r\big) = a^d\int_{\mathbb{R}^d}\Psi\big(\mu\big(B\big(x,a^{-1}r\big)\big)\big)\,\mathrm{d}x = \int_{\mathbb{R}^d}\Psi(\mu_a(B(x,r)))\,\mathrm{d}x,$$

to obtain

$$\lim_{\rho\to 0^{\beta-d}} \mathbb{E}(\exp(i(X_\rho(\mu)-\mathbb{E}(X_\rho(\mu))))) = \mathbb{E}(\exp(iJ_\beta(\mu_a))). \qquad \square$$

Lemma 2.3 and (13) yield the following remark.

Remark 2.6 The covariance function of J_β is given for all $\mu,\nu\in\widetilde{\mathcal{M}}_\beta$ by

$$\begin{aligned}\mathrm{Cov}(J_\beta(\mu),J_\beta(\nu)) &= \int_{\mathbb{R}^d\times\mathbb{R}^+}\mu(B(x,r))\nu(B(x,r))\,\mathrm{d}x\,r^{-\beta-1}\mathrm{d}r\\ &= c_\beta\int_{\mathbb{R}^d\times\mathbb{R}^d}|z-z'|^{d-\beta}\mu(\mathrm{d}z)\nu(\mathrm{d}z'),\end{aligned}$$

and so J_β and W_β have the same covariance function on $\widetilde{\mathcal{M}}_\beta$.

3 Properties of the Limiting Random Generalized Fields

In this section we discuss some of the main properties of the fields we obtain as scaling limits. The limits inherit from the random balls model a stationarity property and acquire, due to the nature of the performed scaling, certain self-similarity properties.

3.1 Stationarity

Following the same ideas as in [9] or [18], we define a notion of stationarity which characterizes the translation invariance of a random linear functional over a subset of signed measures. We say as usual that a subspace $\mathcal{S}\subset\mathcal{M}$ is closed for translations if, for any $\mu\in\mathcal{S}$ and any $s\in\mathbb{R}^d$, we have $\tau_s\mu\in\mathcal{S}$, where $\tau_s\mu$ is defined by $\tau_s\mu(A)=\mu(A-s)$, for any Borel set A. To provide a more general framework for stationary random fields, we introduce the following subspaces of measures with vanishing moments. For any $n \in \mathbb{N} \setminus \{0\}$, denote by $\mathcal{M}_n$ the subspace of measures $\mu \in \mathcal{M}$ such that $\int_{\mathbb{R}^d} |z|^{n-1} |\mu|(\mathrm{d}z) < +\infty$ which satisfy

$$\int_{\mathbb{R}^d} z^j \mu(\mathrm{d}z) = \int_{\mathbb{R}^d} z_1^{j_1} \cdots z_d^{j_d} \mu(\mathrm{d}z) = 0 \tag{29}$$

for all $j = (j_1, \ldots, j_d) \in \mathbb{N}^d$ with $0 \le j_1 + \cdots + j_d < n$ (see [18], where similar spaces of measures are introduced). Here, the class $\mathcal{M}_1$ was already used for the setting of Theorem 2.1. For convenience, we also put $\mathcal{M}_0 = \mathcal{M}$. A simple but tedious computation shows that when $\mu \in \mathcal{M}_n$ satisfies $\int_{\mathbb{R}^d} |z|^{2n-2} |\mu|(\mathrm{d}z) < +\infty$ for $n \ge 1$, then

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} |z - z'|^{2k} \mu(\mathrm{d}z) \mu(\mathrm{d}z') = 0, \quad 0 \le k < n.$$

In particular, the subspaces $\mathcal{M}_n$ defined by (29) are closed under translations for any $n \in \mathbb{N}$.

Definition 3.1 Let $n \in \mathbb{N}$. Let X be a random field defined on a subspace $\mathcal{S} \subset \mathcal{M}_n$ closed for translations. The field X is translation invariant if

$$\forall \mu \in \mathcal{S},\ \forall s \in \mathbb{R}^d, \quad X(\tau_s \mu) \overset{\text{fdd}}{=} X(\mu). \tag{30}$$

More precisely, one says that X is stationary when $n = 0$ and has stationary nth increments when $n > 0$.

It follows that if X has stationary nth increments on a subspace $\mathcal{S} \subset \mathcal{M}_n$, then its restriction on $\mathcal{S} \cap \mathcal{M}_{n+1} \subset \mathcal{M}_{n+1}$ has stationary $(n + 1)$th increments. This terminology comes from [9], where $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$ is the Schwartz space. In this setting the generalized field X has stationary nth increments if all its partial derivatives of order n are stationary.

By the translation invariance of the Lebesgue measure, for any $\rho > 0$, the random field X_ρ defined by (9) is stationary on $\mathcal{M}$. The fields W_β and J_β obtained as limit fields on $\widetilde{\mathcal{M}}_\beta$ in Theorem 2.1 and Theorem 2.5 are not defined on the full space $\mathcal{M}$. But $\widetilde{\mathcal{M}}_\beta$ is closed for translations. Therefore, when considering the limiting random fields on $\widetilde{\mathcal{M}}_\beta$, one has the following property.

Proposition 3.2 *Let $d - 1 < \beta < 2d$ with $\beta \neq d$. Then W_β and J_β are translation invariant on $\widetilde{\mathcal{M}}_\beta$.*

In other words, from (12), W_β and J_β defined on $\widetilde{\mathcal{M}}_\beta$ are both stationary if $d < \beta < 2d$, and they have stationary first increments if $d - 1 < \beta < d$.

3.2 Self-similarity

Let $a > 0$ and denote by μ_a the dilated measure defined by $\mu_a(A) = \mu(a^{-1}A)$ for any Borel set A. A subspace $\mathcal{S} \subset \mathcal{M}$ is said to be closed for dilations if, for any $\mu \in \mathcal{S}$ and any $a > 0$, we have $\mu_a \in \mathcal{S}$. The following definition extends the standard definition of self-similarity for pointwise defined random fields.

Definition 3.3 Let $H \in \mathbb{R}$. A random field X, defined on a subspace $\mathcal{S}$ of $\mathcal{M}$ which is closed for dilations, is said to be self-similar with index H if

$$\forall \mu \in \mathcal{S},\ \forall a > 0, \quad X(\mu_a) \overset{\text{fdd}}{=} a^H X(\mu).$$

Once noticed that $\widetilde{\mathcal{M}}_\beta$ is closed for dilations and observing the consequence of dilation on the covariance of W_β, the following property is straightforward.

Proposition 3.4 *The field W_β, defined on $\widetilde{\mathcal{M}}_\beta$, is self-similar with index $H = \frac{d-\beta}{2}$ that runs over $(-d/2, 1/2) \setminus \{0\}$.*

In contrast to the Gaussian field W_β, the Poisson limit field J_β is not self-similar. A similarity property which applies in great generality to long-range dependent processes is discussed in [14]. The following is a version for spatial random fields.

Definition 3.5 A random field X with $\mathbb{E}X = 0$, defined on a subspace $\mathcal{S}$ of $\mathcal{M}$ which is closed for dilations, is said to be aggregate-similar if there exists a sequence of positive real numbers $(a_m)_{m \geq 1}$ such that

$$\forall \mu \in \mathcal{S},\ \forall m \geq 1, \quad X(\mu_{a_m}) \overset{\text{fdd}}{=} \sum_{i=1}^{m} X^i(\mu),$$

where $(X^i)_{i \geq 1}$ are i.i.d. copies of X.

Thus, a random field is aggregate-similar if the path $\mu_{a_m} \mapsto X(\mu_{a_m})$, as we trace along the sequence of dilations given by a_m, passes all aggregates $\sum_{i=1}^m X^i$ of X, in the distributional sense. We may write, equivalently,

$$\forall \mu \in \mathcal{S},\ \forall m \geq 1, \quad X(\mu) \overset{\text{fdd}}{=} \sum_{i=1}^{m} X^i(\mu_{a_m^{-1}}),$$

which immediately shows that an aggregate-similar random field is infinitely divisible.

Any self-similar zero-mean Gaussian random field is aggregate-similar. Indeed, if X_H is Gaussian with $\mathbb{E}X_H = 0$ and self-similar with index H, then letting $a_m = m^{1/2H}$, we have

$$X_H(\mu_{a_m}) \overset{\text{fdd}}{=} m^{1/2} X_H(\mu) \overset{\text{fdd}}{=} \sum_{i=1}^{m} X_H^i(\mu), \quad m \geq 1. \tag{31}$$

In particular, W_β is aggregate-similar on $\widetilde{\mathcal{M}}_\beta$ with respect to the sequence $a_m = m^{1/(d-\beta)}$. For $d-1 < \beta < d$, we have $a_m^{-1} \to 0$, and hence μ_{a_m} represents a zoom-in of W_β as $m \to \infty$. This is in contrast to the case $d < \beta < 2d$, for which $a_m^{-1} \to \infty$. Consequently, the succession of aggregates $\sum_{i=1}^m W_\beta^i(\mu)$ of $W_\beta(\mu)$ appears as the sequence of measures μ_{a_m} performs a zoom-out, in the limit $m \to \infty$.

Turning next to the non-Gaussian field J_β, by (28),

$$\log \mathbb{E}(\exp(i J_\beta(\mu_a))) = a^{d-\beta} \log \mathbb{E}(\exp(i J_\beta(\mu))).$$

Thus, J_β is aggregate-similar with respect to a_m given by $a_m^{d-\beta} = m$. This property provides an interpretation of the dilation parameter a in Theorem 2.5. If we assume in the theorem that $\lambda(\rho)\rho^\beta \to a_m^{d-\beta} = m$ as $\rho^{\beta-d} \to 0$ for arbitrary $m \geq 1$, then

$$X_\rho(\mu) - \mathbb{E}(X_\rho(\mu)) \overset{\text{fdd}}{\to} J_\beta(\mu_{a_m}) \overset{\text{fdd}}{=} \sum_{i=1}^{m} J_\beta^i(\mu).$$

The guiding asymptotic quantity $\lambda\rho^\beta$ may be interpreted as the expected number of very large ($\beta > d$) or very small ($\beta < d$) balls which cover a point asymptotically. Thus, the more of such extreme grains are allowed asymptotically, the larger number of i.i.d. copies of the basic field J_β appears in the limit.

We may continue this line of reasoning by providing a limit result for $J_\beta(\mu_{a_m})$ as $m \to \infty$. In view of Theorems 2.5 and 2.1, this result is not at all surprising.

Proposition 3.6 *As $a^{d-\beta} \to \infty$, for all μ in $\widetilde{\mathcal{M}}_\beta$,*

$$\frac{1}{a^{(d-\beta)/2}} J_\beta(\mu_a) \overset{\text{fdd}}{\to} W_\beta(\mu).$$

Proof Consider the subsequence $a_m = m^{1/(d-\beta)}$. It follows immediately from aggregate-similarity and the central limit theorem that

$$\frac{1}{a_m^{(d-\beta)/2}} J_\beta(\mu_{a_m}) \overset{\text{fdd}}{=} \frac{1}{\sqrt{m}} \sum_{i=1}^{m} J_\beta^i(\mu) \overset{\text{fdd}}{\to} W_\beta(\mu), \quad m \to \infty,$$

since $J_\beta(\mu)$ and $W_\beta(\mu)$ have the same variance. A standard argument completes the proof of convergence in distribution along an arbitrary sequence. □

4 Self-similar Random Fields of Arbitrary Order

We consider in this section an extension of our methods in order to obtain random fields with the self-similarity property for any index $H \in \mathbb{R} \setminus \mathbb{Z}$. To state our main results, Theorems 4.7 and 4.8, a preliminary study of self-similar random fields of arbitrary order is required.

4.1 Dobrushin's Characterization of Self-similar Random Fields

Dobrushin [9] gives a complete description of Gaussian translation-invariant self-similar generalized random fields on $\mathbb{R}^d$. For this purpose, he considers continuous random linear functionals of $\mathcal{S}(\mathbb{R}^d)'$, where $\mathcal{S}(\mathbb{R}^d)'$ is the topological dual of the

Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of all infinitely differentiable rapidly decreasing real functions on $\mathbb{R}^d$ (see, e.g., [10, 11]). As usual, $\mathcal{S}(\mathbb{R}^d)$ is equipped with the topology that corresponds to the following notion of convergence: $\varphi_n \to \varphi$ if and only if for all $N \in \mathbb{N}$ and $j \in \mathbb{N}^d$,

$$\sup_{z\in\mathbb{R}^d} (1+|z|)^N \left|D^j(\varphi_n - \varphi)(z)\right| \to 0,$$

where $D^j\varphi(z) = \frac{\partial^{j_1}\cdots\partial^{j_d}}{\partial z_1^{j_1}\cdots\partial z_d^{j_d}}\varphi(z)$ denotes the partial derivative of order $j = (j_1, \dots, j_d)$. Then, a linear functional $X : \mathcal{S}(\mathbb{R}^d) \to L^2(\Omega, \mathcal{A}, \mathbb{P})$ is continuous if and only if $\varphi_n \to 0$ in $\mathcal{S}(\mathbb{R}^d)$ implies that

$$\mathbb{E}\big(X(\varphi_n)^2\big) \to 0.$$

To each function $\varphi \in \mathcal{S}(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$, one can uniquely associate a signed measure $\widetilde{\varphi} \in \mathcal{M}$ defined by $\widetilde{\varphi}(\mathrm{d}z) = \varphi(z)\,\mathrm{d}z$. For notational simplicity, we identify any function $\varphi \in L^1(\mathbb{R}^d)$ with its image $\widetilde{\varphi}$ in $\mathcal{M}$, so that $L^1(\mathbb{R}^d) \subset \mathcal{M}$. Therefore any random linear functional on $\mathcal{M}$, when restricted to $\mathcal{S}(\mathbb{R}^d)$, can be viewed as a linear functional on $\mathcal{S}(\mathbb{R}^d)$.

Proposition 4.1 *Let $\rho > 0$. The random field X_ρ induces a continuous random linear functional on $\mathcal{S}(\mathbb{R}^d)$.*

Proof By (11), the random field X_ρ is a continuous random linear functional on $(\mathcal{M}, \|\cdot\|)$. Then, to prove the continuity of X_ρ on $\mathcal{S}(\mathbb{R}^d)$, it is sufficient, using Lebesgue's theorem, to notice that the previous identification implies that if $\mu_n = \widetilde{\varphi_n} \to 0$ in $\mathcal{S}(\mathbb{R}^d)$, then $\|\mu_n\| = \int_{\mathbb{R}^d} |\varphi_n(z)|\,\mathrm{d}z \to 0$. □

Now, put

$$\mathcal{S}_n\big(\mathbb{R}^d\big) = \mathcal{S}\big(\mathbb{R}^d\big) \cap \mathcal{M}_n, \quad n \geq 0.$$

In particular, $\mathcal{S}_0(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d)$. We obtain the continuity properties of W_β and J_β by observing that $\mathcal{S}(\mathbb{R}^d) \cap \widetilde{\mathcal{M}}_\beta = \mathcal{S}(\mathbb{R}^d)$ when $d < \beta < 2d$, while $\mathcal{S}(\mathbb{R}^d) \cap \widetilde{\mathcal{M}}_\beta = \mathcal{S}(\mathbb{R}^d) \cap \mathcal{M}_1 = \mathcal{S}_1(\mathbb{R}^d)$ for $d-1 < \beta < d$.

Proposition 4.2 *Let $d-1 < \beta < 2d$ with $\beta \neq d$. The random fields W_β and J_β induce continuous random linear functionals on $\mathcal{S}_n(\mathbb{R}^d)$ for any $n \geq 1$ if $d-1 < \beta < d$ and any $n \geq 0$ if $d < \beta < 2d$.*

Proof Note that by (13) and Remark 2.6, for any $\mu \in \widetilde{\mathcal{M}}_\beta$,

$$\mathbb{E}\big(W_\beta(\mu)^2\big) = \mathbb{E}\big(J_\beta(\mu)^2\big) \leq |c_\beta| \int_{\mathbb{R}^d\times\mathbb{R}^d} |z-z'|^{d-\beta} |\mu|(\mathrm{d}z)|\mu|(\mathrm{d}z'). \tag{32}$$

A straightforward use of Lebesgue's theorem concludes the proof. □

Then, restricted to $\mathcal{S}_n(\mathbb{R}^d)$, the Gaussian field W_β is a translation-invariant self-similar generalized field. We refer to [19] for a synthesis using orthonormal basis of $L^2(\mathbb{R}^d)$ in the case $d < \beta < 2d$ and to [6] for other examples of self-similar generalized fields obtained by random wavelet expansions in the general case. In [9] Dobrushin focuses on the spectral representation of such Gaussian fields. Since the law of a centered Gaussian field is characterized by its covariance function, let us introduce a second-order self-similarity property. For $H \in \mathbb{R}$, we say that a random linear functional X on $\mathcal{S}_n(\mathbb{R}^d)$ is a second-order self-similar field of order H if, for all $a > 0$ and $\varphi, \psi \in \mathcal{S}_n(\mathbb{R}^d)$,

$$\operatorname{Cov}(X(\varphi_a), X(\psi_a)) = a^{2H}\operatorname{Cov}(X(\varphi), X(\psi)), \quad \text{where } \varphi_a(x) = a^{-d}\varphi\big(a^{-1}x\big). \tag{33}$$

We denote by $\widehat{\varphi}(\xi) = \int_{\mathbb{R}^d} e^{-iz\cdot\xi}\varphi(z)\,\mathrm{d}z$ the Fourier transform of $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and recall that $\widehat{\varphi}$ is infinitely differentiable rapidly decreasing on $\mathbb{R}^d$ with complex values. Moreover, for $n \geq 1$, the spaces $\mathcal{S}_n(\mathbb{R}^d)$ are obtained as

$$\mathcal{S}_n\big(\mathbb{R}^d\big) = \big\{\varphi \in \mathcal{S}\big(\mathbb{R}^d\big);\, D^j\widehat{\varphi}(0) = 0, |j| < n\big\}. \tag{34}$$

Then Theorem 3.2 of [9] can be reformulated as follows.

Theorem 4.3 *Let $n \geq 0$, and let X be a continuous random linear functional on $\mathcal{S}_n(\mathbb{R}^d)$. Then X is translation-invariant and second-order self-similar field of order $H \in \mathbb{R}$ if and only if for all $\varphi, \psi \in \mathcal{S}_n(\mathbb{R}^d)$,*

$$\begin{aligned}\operatorname{Cov}(X(\varphi), X(\psi)) = {} & \int_{S^{d-1}}\int_{\mathbb{R}^+} \widehat{\varphi}(r\theta)\overline{\widehat{\psi}(r\theta)}r^{-2H-1}\,\mathrm{d}r\,\mathrm{d}\sigma(\theta) \\ & + \sum_{|j|=|k|=n} A_{j,k}\alpha_j(\varphi)\overline{\alpha_k(\psi)},\end{aligned} \tag{35}$$

where σ is a finite positive measure on the unit sphere S^{d-1}, $\alpha_j(\varphi) = \int_{\mathbb{R}^d}\varphi(x)x^j\,\mathrm{d}x = i^{|j|}D^j\widehat{\varphi}(0)$ for $j = (j_1, \ldots, j_d) \in \mathbb{N}^d$ with $|j| = j_1 + \cdots + j_d = n$, and $A = (A_{j,k})_{|j|=|k|=n}$ is a symmetric positive definite real matrix. Moreover, if $H < n$, then $A = 0$; if $H = n$, then $\sigma = 0$; and if $H > n$, then $A = 0$ and $\sigma = 0$.

We make the further comment that generalized random fields defined on $\mathcal{S}_n(\mathbb{R}^d)$ for some $n > 0$ roughly correspond to suitable derivatives of random fields defined on $\mathcal{S}(\mathbb{R}^d)$. More precisely, since the Schwartz class is closed under differentiation, if X is a continuous random linear functional on $\mathcal{S}(\mathbb{R}^d)$, one can define for any $j \in \mathbb{N}^d$ the partial derivative of X of order j as the continuous random linear functional defined by

$$\forall\varphi \in \mathcal{S}\big(\mathbb{R}^d\big), \quad D^jX(\varphi) = (-1)^{|j|}X(D^j\varphi).$$

Moreover, [9] states the following property (see Lemma 1.2.1 on p. 23 of [3] for a proof).

Proposition 4.4 *For any $n \in \mathbb{N}$, $\mathcal{S}_n(\mathbb{R}^d) = \operatorname{Span}\{D^j\varphi : \varphi \in \mathcal{S}(\mathbb{R}^d), j \in \mathbb{N}^d, |j| = n\}$.*

Therefore, the knowledge of a generalized random field X on $\mathcal{S}_n(\mathbb{R}^d)$ is equivalent to the knowledge of all its partial derivatives $D^j X$ of order j with $|j| = n$. Furthermore, X has stationary nth increments if and only if its partial derivatives $D^j X$ of order j with $|j| = n$ are stationary.

Note that W_β and J_β share the same covariance function by Remark 2.6, so that they are both second-order self-similar fields of order $\frac{d-\beta}{2}$. Moreover, due to the isotropy of balls and the rotation invariance of Lebesgue measure, it is straightforward to conclude that W_β and J_β are isotropic random fields. We obtain the following result, which is of Plancherel's type and gives the covariance function of W_β and J_β in spectral form.

Proposition 4.5 *Fix $d - 1 < \beta < 2d$ with $\beta \neq d$. There exists $k_\beta > 0$ such that, for any $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$ if $d < \beta < 2d$ and for any $\varphi, \psi \in \mathcal{S}_1(\mathbb{R}^d)$ if $d - 1 < \beta < d$, we have*

$$\begin{aligned}\operatorname{Cov}(W_\beta(\varphi), W_\beta(\psi)) &= \operatorname{Cov}(J_\beta(\varphi), J_\beta(\psi)) \\ &= c_\beta \int_{\mathbb{R}^d \times \mathbb{R}^d} |z - z'|^{d-\beta} \varphi(z)\psi(z')\, \mathrm{d}z\, \mathrm{d}z' \\ &= k_\beta \int_{\mathbb{R}^d} \widehat{\varphi}(\xi)\overline{\widehat{\psi}}(\xi)\, |\xi|^{\beta-2d}\, \mathrm{d}\xi.\end{aligned}$$

Proof By combining Propositions 3.2, 3.4, and 4.2 it follows that W_β is a continuous random linear functional on $\mathcal{S}(\mathbb{R}^d)$ if $d < \beta < 2d$ and on $\mathcal{S}_1(\mathbb{R}^d)$ if $d - 1 < \beta < d$, which is translation-invariant and second-order self-similar of order $H = \frac{d-\beta}{2}$. By Theorem 4.3 its covariance function is given by (35). The measure σ is invariant under rotation by isotropy of W_β and hence, up to a constant, equals to the Lebesgue measure on the sphere. □

4.2 Arbitrary-order Self-similar Random Fields as Scaling Limits

To exploit Dobrushin's characterization theorem (Theorem 4.3) further, we next consider a general class of Gaussian random fields which are self-similar with arbitrary index $H \in \mathbb{R} \setminus \mathbb{Z}$. For such an index H, let us introduce the parameter

$$\beta_H = d - 2\left(H - \left[H + \frac{1}{2}\right]\right) \in (d - 1, d + 1] \setminus \{d\} \tag{36}$$

and write

$$\lceil H \rceil_+ = \begin{cases} [H] + 1, & H > 0, \\ 0, & H < 0, \end{cases}$$

where $[H]$ is the integer part of H. Let B_H be a continuous random field defined on $\mathcal{S}_{\lceil H \rceil_+}$, which is centered, Gaussian and isotropic, and whose covariance functional is given by

$$\mathrm{Cov}(B_H(\varphi), B_H(\psi)) = k_{\beta_H} \int_{\mathbb{R}^d} \widehat{\varphi}(\xi)\overline{\widehat{\psi}}(\xi)\,|\xi|^{-2H-d}\,\mathrm{d}\xi, \quad \varphi, \psi \in \mathcal{S}_{\lceil H\rceil_+}(\mathbb{R}^d), \tag{37}$$

where the constant k_{β_H} corresponds to the constant k_β introduced in Proposition 4.5 with $\beta = \beta_H$ as in (36).

In what follows we will see that for H such that $[H + \frac{1}{2}] < H$ or equivalently such that $\beta_H < d$, the field B_H may be explicitly constructed as the scaling limit of a random germ-grain model where the radius of grains accumulates at zero. In the opposite case where $[H + \frac{1}{2}] > H$ or equivalently $\beta_H > d$, the field B_H may be explicitly constructed as the scaling limit of a random germ-grain model where grains have a heavy-tailed radius distribution at infinity. This is the purpose of Theorem 4.7 below.

In the case $d = 1$ and $0 < H < 1$ with $H \neq \frac{1}{2}$, then either $\beta_H < 1$ or $\beta_H > 1$, corresponding to $0 < H < \frac{1}{2}$ or $\frac{1}{2} < H < 1$, and the Gaussian field B_H is obtained either as a zoom-in or as a zoom-out procedure. These two different microscopic descriptions lead to two different macroscopic dependence behaviors. It has to be compared with the usual fractional Brownian motion, which is negatively correlated for $0 < H < \frac{1}{2}$ and positively correlated for $\frac{1}{2} < H < 1$. In [7, 8] similar ideas are developed using the vocabulary of antipersistent and persistent fractional Brownian motion.

In order to link the Dobrushin fields B_H and the limit fields W_β we obtained in the previous section, we will use fractional integration and differentiation. In [19] a similar procedure is used to synthesize Gaussian self-similar random fields with $H \in (-d/2, 0)$. To introduce the method, we consider for $\varphi \in \mathcal{S}(\mathbb{R}^d)$ the usual Laplacian operator

$$\Delta\varphi = \sum_{j=1}^{d} \frac{\partial^2 \varphi}{\partial z_j^2}$$

and recall that for any $\xi \in \mathbb{R}^d$,

$$\widehat{\Delta\varphi}(\xi) = -|\xi|^2 \widehat{\varphi}(\xi).$$

Next, for any $m \in \mathbb{Z}$, we may define formally the operator $(-\Delta)^{-\frac{m}{2}}$ by the relation

$$\widehat{(-\Delta)^{-m/2}\varphi}(\xi) = |\xi|^{-m} \widehat{\varphi}(\xi), \quad \xi \in \mathbb{R}^d.$$

In order to give a precise meaning to this operator, let us denote by $\mathcal{F}$ the Fourier transform on $\mathcal{S}(\mathbb{R}^d)$ and recall that $\mathcal{F}$ is injective on $\mathcal{S}(\mathbb{R}^d)$. We introduce the intersection space

$$\mathcal{S}_\infty(\mathbb{R}^d) = \bigcap_{n\geq 0} \mathcal{S}_n(\mathbb{R}^d).$$

Thus, $\mathcal{S}_\infty(\mathbb{R}^d) \neq \emptyset$ since this space contains any function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ such that $\widehat{\varphi}$ vanishes in a neighborhood of 0. Then, let us consider $\mathcal{F}(\mathcal{S}_\infty(\mathbb{R}^d)) = \{\widehat{\varphi};\, \varphi \in \mathcal{S}_\infty(\mathbb{R}^d)\}$, equipped with the usual topology of the Schwartz space of complex-valued functions. Therefore, $\mathcal{F}$ is a linear homeomorphism from $\mathcal{S}_\infty(\mathbb{R}^d)$ to $\mathcal{F}(\mathcal{S}_\infty(\mathbb{R}^d))$. We can define on $\mathcal{F}(\mathcal{S}_\infty(\mathbb{R}^d))$ the operator T_m by

$$T_m\psi(\xi) = |\xi|^{-m}\psi(\xi), \quad \xi \in \mathbb{R}^d,\ \psi \in \mathcal{F}\big(\mathcal{S}_\infty\big(\mathbb{R}^d\big)\big).$$

Proposition 4.6 *For any $m \in \mathbb{Z}$, the operator T_m is a linear homeomorphism on $\mathcal{F}(\mathcal{S}_\infty(\mathbb{R}^d))$. Moreover, $(-\Delta)^{-m/2} := \mathcal{F}^{-1} \circ T_m \circ \mathcal{F}$ is a linear homeomorphism on $\mathcal{S}_\infty(\mathbb{R}^d)$.*

Proof Let $m \in \mathbb{Z}$. For any $n \geq 1$,

$$\mathcal{S}_n\big(\mathbb{R}^d\big) = \big\{\varphi \in \mathcal{S}\big(\mathbb{R}^d\big);\ D^j\widehat{\varphi}(0) = 0, |j| < n\big\}.$$

Therefore, if $\psi \in \mathcal{F}(\mathcal{S}_\infty(\mathbb{R}^d))$, $T_m\psi$ is a smooth function, rapidly decreasing, with partial derivatives of any order vanishing at 0. Moreover, $\overline{\psi(\xi)} = \psi(-\xi)$ such that $\overline{T_m\psi(\xi)} = T_m\psi(-\xi)$, for any $\xi \in \mathbb{R}^d$. Hence $T_m\psi \in \mathcal{F}(\mathcal{S}_\infty(\mathbb{R}^d))$. It is then clear that T_m is a linear homeomorphism on $\mathcal{F}(\mathcal{S}_\infty(\mathbb{R}^d))$. The proof is completed by using the fact that $\mathcal{F}$ is a linear homeomorphism from $\mathcal{S}_\infty(\mathbb{R}^d)$ onto $\mathcal{F}(\mathcal{S}_\infty(\mathbb{R}^d))$. ☐

Theorem 4.7 *Let $H \in \mathbb{R}$ with $H \notin \frac{1}{2}\mathbb{Z}$ for $d = 1$ and $H \notin \mathbb{Z}$ for $d \geq 2$. Set $m = [H + \frac{1}{2}]$ and $\beta_H = d - 2(H - m)$. Then*

$$B_H(\varphi) \overset{\text{fdd}}{=} W_{\beta_H}\big((-\Delta)^{-m/2}\varphi\big), \quad \varphi \in \mathcal{S}_\infty\big(\mathbb{R}^d\big).$$

Moreover, let F be a σ-finite nonnegative measure on $\mathbb{R}^+$ satisfying $\mathbf{A}(\beta_H)$. For all positive functions λ such that $\lambda(\rho)\rho^{\beta_H} \underset{\rho \to 0^{m-H}}{\longrightarrow} +\infty$, the limit

$$\frac{X_\rho((-\Delta)^{-\frac{m}{2}}\varphi) - \mathbb{E}(X_\rho((-\Delta)^{-\frac{m}{2}}\varphi))}{\sqrt{\lambda(\rho)\rho^{\beta_H}}} \underset{\rho \to 0^{m-H}}{\overset{\text{fdd}}{\longrightarrow}} B_H(\varphi)$$

holds for all $\varphi \in \mathcal{S}_\infty(\mathbb{R}^d)$, in the sense of finite-dimensional distributions of the random functionals.
For the case $H > -d/2$, the covariance functional of B_H has the representation

$$\operatorname{Cov}(B_H(\varphi), B_H(\psi)) = C(H) \int_{\mathbb{R}^d \times \mathbb{R}^d} |z - z'|^{2H} \varphi(z)\psi(z')\,\mathrm{d}z\,\mathrm{d}z',$$
$$\varphi, \psi \in \mathcal{S}_\infty\big(\mathbb{R}^d\big),$$

with a constant $C(H)$ prescribed by (40) below.

Proof According to Proposition 4.5, since $\beta_H \in (d-1, d+1) \subset (d-1, 2d)$ for $d = 1$ and $\beta_H \in (d-1, d+1] \subset (d-1, 2d)$ for $d \geq 2$ with $\beta_H \neq d$, the random field W_{β_H} is well defined on $\mathcal{S}_\infty(\mathbb{R}^d)$. Moreover, for any $\varphi, \psi \in \mathcal{S}_\infty(\mathbb{R}^d)$, we have

$$\operatorname{Cov}\big(W_{\beta_H}\big((-\Delta)^{-m/2}\varphi\big), W_{\beta_H}\big((-\Delta)^{-m/2}\psi\big)\big)$$

$$= c_{\beta_H} \int_{\mathbb{R}^d \times \mathbb{R}^d} |z - z'|^{d-\beta_H} (-\Delta)^{-m/2} \varphi(z) (-\Delta)^{-m/2} \psi(z') \, \mathrm{d}z \, \mathrm{d}z' \tag{38}$$

$$= k_{\beta_H} \int_{\mathbb{R}^d} \widehat{(-\Delta)^{-m/2}\varphi}(\xi) \overline{\widehat{(-\Delta)^{-m/2}\psi}(\xi)} \, |\xi|^{\beta_H - 2d} \, \mathrm{d}\xi. \tag{39}$$

By (39) and (37), we get

$$\begin{aligned}\mathrm{Cov}\big(W_{\beta_H}\big((-\Delta)^{-m/2}\varphi\big), W_{\beta_H}\big((-\Delta)^{-m/2}\psi\big)\big) &= k_{\beta_H} \int_{\mathbb{R}^d} \widehat{\varphi}(\xi) \overline{\widehat{\psi}}(\xi) \, |\xi|^{\beta_H - 2d - 2m} \, \mathrm{d}\xi \\ &= \mathrm{Cov}(B_H(\varphi), B_H(\psi)).\end{aligned}$$

Since the two random fields W_{β_H} and B_H are Gaussian, it is enough to conclude that

$$B_H(\varphi) \overset{\mathrm{fdd}}{=} W_{\beta_H}\big((-\Delta)^{-m/2}\varphi\big).$$

Then, Theorem 2.1 provides the finite-dimensional-distribution limit.
Next, let us consider the covariance functional for $H > -d/2$. By rewriting (38),

$$\mathrm{Cov}(B_H(\varphi), B_H(\psi)) = c_{\beta_H} \int_{\mathbb{R}^d} |z|^{d-\beta_H} \big((-\Delta)^{-m/2}\varphi * (-\Delta)^{-m/2}\psi\big)(z) \, \mathrm{d}z$$

with

$$(-\Delta)^{-m/2}\varphi * (-\Delta)^{-m/2}\psi(z) = \int_{\mathbb{R}^d} (-\Delta)^{-m/2}\varphi(z - z') (-\Delta)^{-m/2}\psi(z') \, \mathrm{d}z'.$$

Using Fourier transforms,

$$(-\Delta)^{-m/2}\varphi * (-\Delta)^{-m/2}\psi(z) = (-\Delta)^{-m}(\varphi * \psi(z)),$$

so that

$$\mathrm{Cov}(B_H(\varphi), B_H(\psi)) = c_{\beta_H} \int_{\mathbb{R}^d} |z|^{d-\beta_H} (-\Delta)^{-m} (\varphi * \psi)(z) \, \mathrm{d}z.$$

Here, since $\Delta|z|^{2H} = 2H(2(H-1)+d)|z|^{2H-2}$ for $z \neq 0$, one can find a constant $c_{H,m}$ such that $|z|^{d-\beta_H} = |z|^{2H-2m} = c_{H,m}\Delta^m |z|^{2H}$ for any $m \geq 0$ and $z \neq 0$. Then, since $H > -d/2$, integrating by parts, we obtain

$$\int_{\mathbb{R}^d} |z|^{d-\beta_H} (-\Delta)^{-m} (\varphi * \psi(z)) \, \mathrm{d}z = c_{H,m} \int_{\mathbb{R}^d} |z|^{2H} \Delta^m \big((-\Delta)^{-m} (\varphi * \psi(z))\big) \, \mathrm{d}z.$$

Thus,

$$\mathrm{Cov}(B_H(\varphi), B_H(\psi)) = C(H) \int_{\mathbb{R}^d \times \mathbb{R}^d} |z - z'|^{2H} \varphi(z) \psi(z') \, \mathrm{d}z \, \mathrm{d}z'$$

with

$$C(H) = (-1)^m \, c_{H,m} \, c_{\beta_H}. \tag{40}$$

□

Under the same parameter assumptions, as in the previous theorem, we may define analogously a continuous generalized random field P_H on $\mathcal{S}_\infty(\mathbb{R}^d)$ by

$$P_H(\varphi) = J_{\beta_H}\big((-\Delta)^{-m/2}\varphi\big), \quad \varphi \in \mathcal{S}_\infty\big(\mathbb{R}^d\big). \tag{41}$$

The effect of a dilation by $a > 0$ is given by

$$J_{\beta_H}\big(\big((-\Delta)^{-m/2}\varphi\big)_a\big) = J_{\beta_H}\big(a^m(-\Delta)^{-m/2}(\varphi_a)\big) = a^m P_H(\varphi_a).$$

This allows us to extend Theorem 2.5 to the case of a general index H. By Proposition 4.5, the covariance functional of P_H coincides with that of B_H, so that P_H can be extended to a continuous linear functional on $\mathcal{S}_{\lceil H\rceil_+}(\mathbb{R}^d)$.

Theorem 4.8 *Take a real number H, $H \notin \frac{1}{2}\mathbb{Z}$ for $d = 1$, $H \notin \mathbb{Z}$ for $d \geq 2$. As above, let $m = [H + \frac{1}{2}]$ and $\beta_H = d - 2(H - m)$. Let F be a nonnegative measure on $\mathbb{R}^+$ which satisfies $\mathbf{A}(\beta_H)$. For all positive functions λ such that $\lambda(\rho)\rho^{\beta_H} \underset{\rho\to 0^{m-H}}{\longrightarrow} a^{2(H-m)}$ for some $a > 0$, we have in the sense of finite-dimensional distributions of random functionals the scaling limit*

$$X_\rho\big((-\Delta)^{-\frac{m}{2}}\varphi\big) - \mathbb{E}\big(X_\rho\big((-\Delta)^{-\frac{m}{2}}\varphi\big)\big) \underset{\rho\to 0^{m-H}}{\overset{\text{fdd}}{\longrightarrow}} a^m P_H(\varphi_a)$$

for all $\varphi \in \mathcal{S}_\infty(\mathbb{R}^d)$.

5 Pointwise Representation of the Random Fields B_H and P_H

In this section we discuss the case of a positive self-similarity index and assume henceforth $H > 0$. For $H \notin \mathbb{N}$, note that $\lceil H\rceil_+ = \lceil H\rceil$, where $\lceil H\rceil = [H] + 1$, and recall that the Gaussian field B_H is defined on $\mathcal{S}_{\lceil H\rceil}(\mathbb{R}^d)$. By Proposition 4.4,

$$\mathcal{S}_{\lceil H\rceil}\big(\mathbb{R}^d\big) = \operatorname{Span}\big\{D^j\varphi : \varphi \in \mathcal{S}\big(\mathbb{R}^d\big),\ j \in \mathbb{N}^d,\ |j| = \lceil H\rceil\big\}.$$

A natural question that arises in this context is whether it is possible to find a continuous random linear functional Y on $\mathcal{S}(\mathbb{R}^d)$ such that

$$\forall\varphi \in \mathcal{S}\big(\mathbb{R}^d\big), \quad D^jY(\varphi) = (-1)^{|j|}B_H\big(D^j\varphi\big), \quad j \in \mathbb{N}^d \text{ with } |j| = \lceil H\rceil.$$

The same question applies to the Poisson field P_H defined by (41). We will use the representation of generalized random fields as defined by Matheron [18], to provide an answer (see also the links between “generalized random fields” and “punctual random fields” in [3]). This will allow us to extend B_H and P_H as continuous random linear functionals on the whole space $\mathcal{S}(\mathbb{R}^d)$.

5.1 Representation of Generalized Random Fields

Let X be a continuous random linear functional on a subset $\mathcal{S}$ of $\mathcal{S}(\mathbb{R}^d)$. We say that a continuous function $\widetilde{X}:\mathbb{R}^d\to L^2(\Omega,\mathcal{A},\mathbb{P})$ is a representation of X if, for any $\varphi\in\mathcal{S}$,

$$X(\varphi)\overset{L^2(\Omega,\mathcal{A},\mathbb{P})}{=}\int_{\mathbb{R}^d}\widetilde{X}(t)\varphi(t)\,\mathrm{d}t.$$

In order to obtain representations $\widetilde{B}_H(t)$ of B_H and $\widetilde{P}_H(t)$ of P_H, for any $t\in\mathbb{R}^d$, we will consider an approximation in $\mathcal{S}_{\lceil H\rceil}(\mathbb{R}^d)$ of the Dirac mass δ_t at t.

Following the ideas of [18], let $\theta\in\mathcal{S}(\mathbb{R}^d)$ be a positive even function such that its Fourier transform $\widehat{\theta}$ satisfies $\widehat{\theta}(0)=\int_{\mathbb{R}^d}\theta(z)\,\mathrm{d}z=1$. Let $n\in\mathbb{N}$ with $n\neq 0$ and set $\theta_n(z)=n^d\theta(nz)$. For $t\in\mathbb{R}^d$, let $\tau_t\theta_n=\theta_n(z-t)$. Write $l!=l_1!\cdots l_d!$ for $l=(l_1,\dots,l_d)\in\mathbb{N}^d$. Then, consider the functions defined by

$$\Theta_t^n=\tau_t\theta_n-\sum_{|l|<\lceil H\rceil}\frac{(-1)^{|l|}}{l!}t^lD^l\theta_n,\quad t\in\mathbb{R}^d.$$

On the one hand, since $\theta\in\mathcal{S}(\mathbb{R}^d)$, which is closed under dilations and differentiations, $\Theta_t^n\in\mathcal{S}(\mathbb{R}^d)$. On the other hand, let us remark that, for $\xi\in\mathbb{R}^d$,

$$\widehat{\Theta_t^n}(\xi)=\widehat{\theta}_n(\xi)\left(e^{-it\cdot\xi}-\sum_{|l|<\lceil H\rceil}\frac{1}{l!}t^l(-i\xi)^l\right)=\widehat{\theta}\left(\frac{\xi}{n}\right)\left(e^{-it\cdot\xi}-\sum_{k=0}^{\lceil H\rceil-1}\frac{(-it\cdot\xi)^k}{k!}\right),\tag{42}$$

using the fact

$$\sum_{|l|=k}\frac{1}{l!}t^l(-i\xi)^l=\frac{(-it\cdot\xi)^k}{k!},\quad k\in\mathbb{N},$$

which is a generalization of the binomial theorem. But for any $k\in\mathbb{N}$ and $j\in\mathbb{N}^d$, we get

$$D^j\left(\frac{(-it\cdot\xi)^k}{k!}\right)\Big/_{\xi=0}=\begin{cases}(-i)^{|j|}t^j & \text{if } |j|=k,\\ 0 & \text{else.}\end{cases}$$

Then by Leibnitz formula we obtain that $D^j\widehat{\Theta_t^n}(0)=0$ for any $j\in\mathbb{N}^d$ such that $|j|<\lceil H\rceil$. According to (34), Θ_t^n belongs to $\mathcal{S}_{\lceil H\rceil}(\mathbb{R}^d)$. Therefore we can consider the sequences of random functions defined by $(B_H(\Theta_\cdot^n))_{n\geq 1}$ and $(P_H(\Theta_\cdot^n))_{n\geq 1}$, where $B_H(\Theta_\cdot^n):t\mapsto B_H(\Theta_t^n)$ for all $n\geq 1$ and similarly for $P_H(\Theta_\cdot^n)$.

Theorem 5.1 *Let $H>0$ with $H\notin\frac{1}{2}\mathbb{N}$ for $d=1$ and $H\notin\mathbb{N}$ for $d\geq 2$. The finite-dimensional distributions of $(B_H(\Theta_\cdot^n))_{n\geq 1}$ converge in $L^2(\Omega,\mathcal{A},\mathbb{P})$ to a representation $\widetilde{B_H}$ of B_H on $\mathcal{S}_{\lceil H\rceil}(\mathbb{R}^d)$ with the covariance function*

$$\Gamma_H(t,s)=k_{\beta_H}\int_{\mathbb{R}^d}\left(e^{-it\cdot\xi}-\sum_{0\leq k<\lceil H\rceil}\frac{(-it\cdot\xi)^k}{k!}\right)$$

$$\times \overline{\left(e^{-is\cdot\xi} - \sum_{0\le k<\lceil H\rceil} \frac{(-is\cdot\xi)^k}{k!}\right)} |\xi|^{-d-2H}\,\mathrm{d}\xi$$

$$= C(H)\left(|t-s|^{2H} - \sum_{|l|<\lceil H\rceil} \frac{(-1)^{|l|}}{l!}\left(s^l D^l |t|^{2H} + t^l D^l |s|^{2H}\right)\right), \quad (43)$$

where the constants k_{β_H} and $C(H)$ have been introduced in Proposition 4.5 and Theorem 4.7, respectively.

Similarly, the finite-dimensional distributions of $(P_H(\Theta_\cdot^n))_{n\ge1}$ converge in $L^2(\Omega,\mathcal{A},\mathbb{P})$ to a representation $\widetilde{P_H}$ of P_H on $\mathcal{S}_{\lceil H\rceil}(\mathbb{R}^d)$ with the same covariance function as $\widetilde{B_H}$.

Proof Let $n\in\mathbb{N}\setminus\{0\}$ and $t\in\mathbb{R}^d$. By the choice of θ we have $\Theta_t^n\in\mathcal{S}_{\lceil H\rceil}(\mathbb{R}^d)$. Let $n,m\in\mathbb{N}\setminus\{0\}$ and define the covariance

$$\Gamma_{n,m}(t,s) := \mathrm{Cov}\big(B_H(\Theta_t^n), B_H(\Theta_s^m)\big) = \mathrm{Cov}\big(P_H(\Theta_t^n), P_H(\Theta_s^m)\big), \quad t,s\in\mathbb{R}^d.$$

By (37) this covariance can be written as

$$\Gamma_{n,m}(t,s) = k_{\beta_H}\int_{\mathbb{R}^d} \widehat{\Theta_t^n}(\xi)\overline{\widehat{\Theta_s^m}}(\xi)\,|\xi|^{-2H-d}\,\mathrm{d}\xi.$$

Then, according to (42), Lebesgue's theorem implies that the limit in $\Gamma_{n,m}(t,s) \underset{n,m\to+\infty}{\longrightarrow} \Gamma_H(t,s)$ is given by

$$\Gamma_H(t,s) := k_{\beta_H}\int_{\mathbb{R}^d}\left(e^{-it\cdot\xi} - \sum_{k<\lceil H\rceil}\frac{(-it\cdot\xi)^k}{k!}\right)$$

$$\times \overline{\left(e^{-is\cdot\xi} - \sum_{k<\lceil H\rceil}\frac{(-is\cdot\xi)^k}{k!}\right)} |\xi|^{-2H-d}\,\mathrm{d}\xi.$$

Therefore, the finite-dimensional distributions of $(B_H(\Theta_\cdot^n))_{n\ge1}$ converge in $L^2(\Omega,\mathcal{A},\mathbb{P})$ to a centered random field $\widetilde{B_H}$. The finite-dimensional distributions of $(P_H(\Theta_\cdot^n))_{n\ge1}$ converge similarly to a limit $\widetilde{P_H}$. Both limit fields have the covariance function Γ_H.

Let us prove that $\widetilde{B_H}$ is a representation of B_H on $\mathcal{S}_{\lceil H\rceil}(\mathbb{R}^d)$. The covariance function Γ_H of $\widetilde{B_H}$ is continuous with respect to each variable, and so $\widetilde{B_H}:\mathbb{R}^d\to L^2(\Omega,\mathcal{A},\mathbb{P})$ is continuous. Then, the random linear functional $X:\varphi\in\mathcal{S}(\mathbb{R}^d)\mapsto \int_{\mathbb{R}^d}\widetilde{B_H}(t)\varphi(t)(\mathrm{d}t)$ is well defined since

$$\mathrm{Var}(X(\varphi)) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathrm{Cov}\big(\widetilde{B_H}(t),\widetilde{B_H}(s)\big)\varphi(t)\varphi(s)\,\mathrm{d}t\,\mathrm{d}s < +\infty,$$

using the fact that $\mathrm{Var}(\widetilde{B_H})(t)\le C|t|^{2H}$. Finally, for any $\varphi\in\mathcal{S}_{\lceil H\rceil}(\mathbb{R}^d)$, we have $\mathrm{Var}(X(\varphi)) = \mathrm{Var}(B_H(\varphi))$ by (37), since $\int_{\mathbb{R}^d} t^l\varphi(t)(\mathrm{d}t) = 0$ for $|l|<\lceil H\rceil$, which proves that $\widetilde{B_H}$ is a representation of B_H on $\mathcal{S}_{\lceil H\rceil}(\mathbb{R}^d)$. The same arguments hold to prove that $\widetilde{P_H}$ is a representation of P_H on $\mathcal{S}_{\lceil H\rceil}(\mathbb{R}^d)$.

It remains to establish (43). By Theorem 4.7, for all $n, m \in \mathbb{N} \setminus \{0\}$,

$$\Gamma_{n,m}(t,s) = C(H)\int_{\mathbb{R}^d\times\mathbb{R}^d} |z-z'|^{2H}\Theta_t^n(z)\Theta_s^m(z')\,\mathrm{d}z\,\mathrm{d}z'.$$

For any $z' \in \mathbb{R}^d$, the function $f_{z'}(z) = |z-z'|^{2H}$ admits continuous derivatives of order l on $\mathbb{R}^d$ for any $|l| < \lceil H\rceil$. Therefore, for any $z' \in \mathbb{R}^d$,

$$\begin{aligned}\int_{\mathbb{R}^d} |z-z'|^{2H}\Theta_t^n(z)\,\mathrm{d}z &= f_{z'} * \theta_n(t) - \sum_{|l|<\lceil H\rceil}\frac{(-1)^{|l|}}{l!}t^l D^l f_{z'} * \theta_n(0) \\ &\xrightarrow[n\to+\infty]{} |t-z'|^{2H} - \sum_{|l|<\lceil H\rceil}\frac{(-1)^{|l|}}{l!}t^l D^l |z'|^{2H}.\end{aligned}$$

By Lebesgue's theorem, as $n \to +\infty$,

$$\lim_{n\to+\infty}\Gamma_{n,m}(t,s) = C(H)\int_{\mathbb{R}^d}\left(|t-z'|^{2H} - \sum_{|l|<\lceil H\rceil}\frac{(-1)^{|l|}}{l!}t^l D^l|z'|^{2H}\right)\Theta_s^m(z')\,\mathrm{d}z'.$$

As previously, we obtain

$$\int_{\mathbb{R}^d}|t-z'|^{2H}\Theta_s^m(z')\,\mathrm{d}z' \xrightarrow[m\to+\infty]{} |t-s|^{2H} - \sum_{|l|<\lceil H\rceil}\frac{(-1)^{|l|}}{l!}s^l D^l|t|^{2H},$$

while

$$\int_{\mathbb{R}^d} D^l|z'|^{2H}\Theta_s^m(z')\,\mathrm{d}z' \xrightarrow[m\to+\infty]{} D^l|s|^{2H}.$$

Therefore $\Gamma_H(t,s) = \lim_{n,m\to+\infty}\Gamma_{n,m}(t,s)$ is also equal to (43). □

Remark 5.2 In the case $H < 0$, one cannot find any representation of either B_H or P_H on $\mathcal{S}(\mathbb{R}^d)$. This is due to the fact that the variance of a random field which is second-order self-similar of order $H < 0$ is not bounded around 0.

Since B_H is Gaussian, $\widetilde{B_H}$ is also Gaussian as a limit in $L^2(\Omega, \mathcal{A}, \mathbb{P})$ of a Gaussian functional. The spectral representation of $\widetilde{B_H}$ is given by

$$\widetilde{B_H}(t) \overset{\text{fdd}}{=} \sqrt{k_{\beta_H}}\int_{\mathbb{R}^d}\left(e^{-it\cdot\xi} - \sum_{k<\lceil H\rceil}\frac{(-it\cdot\xi)^k}{k!}\right)|\xi|^{-H-d/2}W(\mathrm{d}\xi), \qquad (44)$$

where W is the complex Brownian measure. This field is called elliptic Gaussian self-similar random field in [2].

Specializing to the case $d=1$, the covariance function Γ_H in (43) equals

$$C(H)\left(|t-s|^{2H}-\sum_{l<\lceil H\rceil}(-1)^l\binom{2H}{l}\left(\left(\frac{s}{t}\right)^l|t|^{2H}+\left(\frac{t}{s}\right)^l|s|^{2H}\right)\right),$$

where $\binom{2H}{l}=(2H)\cdots(2H-(l-1))/l!$. Therefore, $\widetilde{B_H}$ is up to a multiplicative constant an $\lceil H\rceil$th-order fractional Brownian motion as defined in [20].

5.2 Properties of the Pointwise Representation

One can define the $\lceil H\rceil$th increments of $\widetilde{B_H}$ with lag $h\in\mathbb{R}^d$, which correspond to the discrete differentiation of order $\lceil H\rceil$, by

$$\Delta_h^{\lceil H\rceil}\widetilde{B_H}(t)=\sum_{p=0}^{\lceil H\rceil}\binom{\lceil H\rceil}{p}(-1)^{\lceil H\rceil-p}\,\widetilde{B_H}(t+ph).$$

Then

$$\Delta_h^{\lceil H\rceil}\widetilde{B_H}(t)=\lim_{n\to+\infty}B_H\left(\sum_{p=0}^{\lceil H\rceil}\binom{\lceil H\rceil}{p}(-1)^{\lceil H\rceil-p}\,\tau_{t+ph}\theta_n\right),$$

and the stationarity of B_H implies that $\widetilde{B_H}$ has stationary $\lceil H\rceil$th increments in the wide sense: for all $t,s,h,h'\in\mathbb{R}^d$, the covariances $\operatorname{Cov}(\Delta_h^{\lceil H\rceil}\widetilde{B_H}(s),\Delta_{h'}^{\lceil H\rceil}\widetilde{B_H}(s+t))$ do not depend on s (see [24] or [12] for instance).

Proposition 5.3 *Let $H>0$ with $H\notin\mathbb{N}$. Then the Gaussian random field $\widetilde{B_H}$ has stationary $\lceil H\rceil$th increments. Moreover, this field admits continuous partial derivatives of order $l\in\mathbb{N}^d$ in mean square for any $|l|<\lceil H\rceil$ such that $D^l\widetilde{B_H}$ has stationary $(\lceil H\rceil-|l|)$ increments, is self-similar of order $H-|l|$, and satisfies $D^l\widetilde{B_H}(0)=0$ almost surely.*

Proof Recall that Γ_H denotes the covariance function of $\widetilde{B_H}$. Since $\lceil H\rceil\geq 1$, it is straightforward to see that Γ_H admits symmetric partial derivatives of order $l\in\mathbb{N}^d$ for any $|l|<\lceil H\rceil$, with $\frac{\partial^{2l}\Gamma_H}{\partial s^l\partial t^l}(s,t)$ given by

$$k_H\int_{\mathbb{R}^d}\left(e^{-it\cdot\xi}-\sum_{k<\lceil H\rceil-|l|}\frac{(it\cdot\xi)^k}{k!}\right)\overline{\left(e^{-is\cdot\xi}-\sum_{k<\lceil H\rceil-|l|}\frac{(is\cdot\xi)^k}{k!}\right)}\xi^{2l}|\xi|^{-d-2H}\,\mathrm{d}\xi.$$

By Theorem 2.2.2 of [1], this means that $\widetilde{B_H}$ admits a continuous partial derivative of order l in mean square, $D^l\widetilde{B_H}$, which is a Gaussian random field with covariance given by $\operatorname{Cov}(D^l\widetilde{B_H}(t),D^l\widetilde{B_H}(s))=\frac{\partial^{2l}\Gamma_H}{\partial s^l\partial t^l}(s,t)$. A straightforward change of variables yields, for all $a>0$,

$$\operatorname{Cov}\big(D^l\widetilde{B_H}(at),D^l\widetilde{B_H}(as)\big)=a^{2(H-|l|)}\operatorname{Cov}\big(D^l\widetilde{B_H}(t),D^l\widetilde{B_H}(s)\big).$$

Since $D^l\widetilde{B_H}$ is Gaussian, this implies that $D^l\widetilde{B_H}$ is self-similar of order $H-|l|$, that is,

$$\left\{D^l\widetilde{B_H}(at), t\in\mathbb{R}^d\right\}\overset{\text{fdd}}{=}a^{H-|l|}\left\{D^l\widetilde{B_H}(t), t\in\mathbb{R}^d\right\}\quad\text{for all } a>0.$$

Moreover, for all $t,s,h,h'\in\mathbb{R}^d$,

$$\begin{aligned}&\operatorname{Cov}\left(\Delta_h^{\lceil H\rceil-|l|}D^l\widetilde{B_H}(s),\Delta_{h'}^{\lceil H\rceil-|l|}D^l\widetilde{B_H}(s+t)\right)\\&\quad=k_{\beta_H}\int_{\mathbb{R}^d}e^{-it\cdot\xi}(e^{-ih\cdot\xi}-1)^{\lceil H\rceil-|l|}(e^{ih'\cdot\xi}-1)^{\lceil H\rceil-|l|}\xi^{2l}|\xi|^{-2H-d}\,\mathrm{d}\xi,\end{aligned}$$

and $D^l\widetilde{B_H}$ has stationary $(\lceil H\rceil-|l|)$th increments. Finally, $\operatorname{Var}(D^l\widetilde{B_H}(0))=0$ implies that $D^l\widetilde{B_H}(0)=0$ almost surely. □

Remark 5.4

(a) One can prove that $\widetilde{B_H}$ is the only Gaussian random field with stationary $\lceil H\rceil$th increments, which is self-similar of order H and isotropic.

(b) The representation $\widetilde{P_H}$ of P_H obtained in Theorem 5.1 is not Gaussian but shares the same covariance function as $\widetilde{B_H}$. Therefore it satisfies the same second-order properties: stationary $\lceil H\rceil$th increments, self-similarity of order H, and isotropy.

5.3 Fractional Brownian Field and Fractional Poisson Field

For $0<H<1$, the random field $\widetilde{B_H}$ corresponds to the well-known fractional Brownian field with Hurst parameter equal to H, and (44) is known as the harmonizable representation of the fractional Brownian field (see [13] for a review).

We consider the special case $0<H<1/2$ for which $d-1<\beta_H=d-2H<d$. For this range of parameters, $\lceil H\rceil=1$, and

$$\widetilde{\mathcal{M}}_{\beta_H}=\mathcal{M}^{\beta_H}\cap\mathcal{M}_1,\quad \mathcal{M}_1=\left\{\mu\in\mathcal{M}:\int_{\mathbb{R}^d}\mu(dz)=0\right\}.$$

It follows that all pointwise increment measures $\delta_x-\delta_0$, $x\in\mathbb{R}^d$, belong to $\widetilde{\mathcal{M}}_{\beta_H}$ and are hence admissible for evaluating the limit fields W_{β_H} and J_{β_H}. Using the representations $\widetilde{B_H}$ and $\widetilde{P_H}$ in Theorem 5.1, it is verified that $\widetilde{B_H}(x)\overset{\text{fdd}}{=}W_{\beta_H}(\delta_x-\delta_0)$ and $\widetilde{P_H}(x)\overset{\text{fdd}}{=}J_{\beta_H}(\delta_x-\delta_0)$.

To analyze the properties of $\widetilde{P_H}$, we observe, using (28),

$$\log\mathbb{E}(\exp(i\,\widetilde{P_H}(x)))=\int_{\mathbb{R}^+\times\mathbb{R}^d}\Psi(\delta_x(B(y,r))-\delta_0(B(y,r)))\,\mathrm{d}y\,r^{-\beta_H-1}\,\mathrm{d}r,\tag{45}$$

where Ψ is given by (24). Here,

$$\delta_x(B(y,r))-\delta_0(B(y,r))=\begin{cases}1, & |x-y|<r<|y|,\\-1, & |y|<r<|x-y|,\\0, & \text{otherwise}\end{cases}$$

and hence we may recast (45) into

$$\begin{aligned}\log \mathbb{E}(\exp(i\,\theta \widetilde{P_H}(x))) &= \Psi(\theta) \int_{\mathbb{R}^+\times\mathbb{R}^d} \mathbf{1}_{\{|x-y|<r<|y|\}}\,\mathrm{d}y\, r^{-\beta_H-1}\,\mathrm{d}r \\ &\quad + \Psi(-\theta)) \int_{\mathbb{R}^+\times\mathbb{R}^d} \mathbf{1}_{\{|y|<r<|x-y|\}}\,\mathrm{d}y\, r^{-\beta_H-1}\,\mathrm{d}r \\ &= (-c_{\beta_H})|x|^{2H}(\Psi(\theta)+\Psi(-\theta)).\end{aligned}$$

This is the logarithmic characteristic functional of the difference of two independent random variables, both having a Poisson distribution with intensity $(-c_{\beta_H})\,|x|^{2H}$. Hence, $\widetilde{P}_H(x)$, $x \in \mathbb{R}^d$, defines a mean zero integer-valued symmetrized Poisson-distributed random field such that for any $x, x' \in \mathbb{R}^d$,

$$\mathrm{Cov}(\widetilde{P_H}(x), \widetilde{P_H}(x')) = (-c_{\beta_H})\big(|x|^{2H} + |x'|^{2H} - |x-x'|^{2H}\big).$$

By analogy with fractional Brownian field, this makes it natural to view $\widetilde{P_H}$ as a fractional Poisson field.

By adding random weights to the model we obtain a relation between $\widetilde{P_H}$ and so-called Chentsov random fields, in particular Takenaka fields, see [23], [22], Chap. 8. By (45),

$$\widetilde{P_H}(x) \overset{\mathrm{fdd}}{=} \int_{\mathbb{R}^d\times\mathbb{R}^+} (\mathbf{1}_{B(x,r)}(y) - \mathbf{1}_{B(0,r)}(y))\widetilde{N}_{\beta_H}(\mathrm{d}y, \mathrm{d}r),$$

where $\widetilde{N}_{\beta_H}$ is a compensated Poisson random measure with intensity $r^{-\beta_H-1}\,\mathrm{d}r\,\mathrm{d}y$. Fix a parameter $1<\alpha<2$ and consider the Poisson measure $\widetilde{N}_{\beta_H}(\mathrm{d}y, \mathrm{d}r, \mathrm{d}w)$ with intensity measure $|w|^{-(1+\alpha)} r^{-\beta_H-1}\,\mathrm{d}r\,\mathrm{d}y$. The random field

$$Y(x) = \int_{\mathbb{R}^d\times\mathbb{R}^+\times\mathbb{R}} (\mathbf{1}_{B(x,r)}(y) - \mathbf{1}_{B(0,r)}(y))\, w \widetilde{N}_{\beta_H}(\mathrm{d}y, \mathrm{d}r, \mathrm{d}w)$$

is a variation of $\widetilde{P_H}$ where random weights w are applied symmetrically with intensity $|w|^{-(1+\alpha)}$ to the original Poisson points (y, r). Consequently,

$$Y(x) \overset{\mathrm{fdd}}{=} \int_{\mathbb{R}^d\times\mathbb{R}^+} (\mathbf{1}_{B(x,r)}(y) - \mathbf{1}_{B(0,r)}(y)) M_\alpha(\mathrm{d}y, \mathrm{d}r),$$

where M_α is a symmetric α-stable random measure with associated measure proportional to $r^{-\beta_H-1}\,\mathrm{d}r\,\mathrm{d}y$ [22, Theorem 3.12.2]. By properties of stochastic integrals with respect to symmetric α-stable measures we have, for some positive constant C,

$$\begin{aligned}\log \mathbb{E}(\exp(i, \theta Y(x))) &= -C \int_{\mathbb{R}^+\times\mathbb{R}^+} |\theta|^\alpha\, |\mathbf{1}_{B(x,r)}(y) - \mathbf{1}_{B(0,r)}(y)|^\alpha\,\mathrm{d}y\, r^{-\beta_H-1}\mathrm{d}r \\ &= -C|\theta|^\alpha \int_{\mathbb{R}^+\times\mathbb{R}^+} \mathbf{1}_{B(x,r)\Delta B(0,r)}(y)\,\mathrm{d}y\; r^{-\beta_H-1}\,\mathrm{d}r,\end{aligned}$$

where Δ denotes the symmetric set difference. Hence,

$$Y(x) \overset{\mathrm{fdd}}{=} \int_{\mathbb{R}^d\times\mathbb{R}^+} \mathbf{1}_{B(x,r)\Delta B(0,r)}(y)\, M_\alpha(\mathrm{d}y, \mathrm{d}r),$$

which defines a symmetric α-stable random field which is self-similar with index $H' = (d - \beta_H)/\alpha \in (0, 1/\alpha)$, known as an (α, H')-Takenaka field, see [22], Definition 8.4.1 (the parameter β of the reference corresponds to $d - \beta_H$ in our notation). It is noticed in [22] that, moreover, $\widetilde{B_H}$ is a $(2, H)$-Takenaka field. Randomly weighted random balls models also arise in applications such as teletraffic modeling. For the one-dimensional case with parameter values $d = 1 < \beta_H < \alpha < 2$ and M_α as above, the process

$$Z(t) = \int_{\mathbb{R}\times\mathbb{R}^+} |(0, t) \cap (y, y + r)|\, M_\alpha(\mathrm{d}y, \mathrm{d}r), \quad t \geq 0,$$

has been called a Telecom process. It arises as a scaling limit of a random intervals model with one-sided weights, see Kaj and Taqqu [16].

The fractional Poisson field $\widetilde{P_H}$ shares with $\widetilde{B_H}$ and with (α, H)-Takenaka fields [22, Theorem 8.6.3] the well-known interesting invariance property under restriction to lower-dimensional hyperplanes. For example, any cut along a line through a planar fractional field in $\mathbb{R}^2$ generates a one-dimensional fractional process of the same kind. To see this, let H_k be a k-dimensional hyperplane in $\mathbb{R}^d$. We consider $\mathbb{R}^d = H_k \oplus H_k^\perp$ and write $\bar{x}_k$ for the restriction to H_k of $x = \bar{x}_k + (x - \bar{x}_k) \in \mathbb{R}^d$. To emphasize the dimensional dependence, we write here $\widetilde{B_{H,d}}(x)$ and $\widetilde{P_{H,d}}(x)$, respectively, if the fractional fields are defined on $\mathbb{R}^d$.

Proposition 5.5 *Given* $H \in (0, 1/2)$, *let* $\beta'_H = \beta_H - d + k \in (k - 1, k)$. *Then the measure* $\delta_{\bar{x}_k} - \delta_0$ *belongs to* $\widetilde{\mathcal{M}}_{\beta'_H}$, *and we have*

$$\widetilde{B_{H,d}}(\bar{x}_k) \overset{\text{fdd}}{=} \widetilde{B_{H',k}}(\bar{x}_k)$$

and

$$\widetilde{P_{H,d}}(\bar{x}_k) \overset{\text{fdd}}{=} \widetilde{P_{H',k}}(\bar{x}_k)$$

for $H' = \frac{k-\beta'_H}{2} = \frac{d-\beta_H}{2} = H$.

Proof It is enough to consider hyperplanes of the form $x = (x_1, \ldots, x_k, 0, \ldots, 0)$. Then, clearly, $|\bar{x}_k|^{d-\beta_H} = |\bar{x}_k|^{k-\beta'_H}$, which carries over to showing that the covariances of the pair of relevant random fields coincide. ☐

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