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ON THE PERIMETER OF EXCURSION SETS OF SHOT NOISE RANDOM FIELDS[1]

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In this paper, we use the framework of functions of bounded variation and the coarea formula to give an explicit computation for the expectation of the perimeter of excursion sets of shot noise random fields in dimension $n \geq 1$. This will then allow us to derive the asymptotic behavior of these mean perimeters as the intensity of the underlying homogeneous Poisson point process goes to infinity. In particular, we show that two cases occur: we have a Gaussian asymptotic behavior when the kernel function of the shot noise has no jump part, whereas the asymptotic is non-Gaussian when there are jumps.

1. Introduction. We will consider here a *shot noise random field* which is a real-valued random field given on $\mathbb{R}^n$ by

$$X(x) = \sum_{i \in I} g(x - x_i, m_i), \qquad x \in \mathbb{R}^n, \tag{1.1}$$

where g is a given (deterministic) measurable function (it will be called the *kernel function* of the shot noise), the $\{x_i\}_{i \in I}$ are the points of a homogeneous Poisson point process on $\mathbb{R}^n$ of intensity λ, the $\{m_i\}_{i \in I}$ are called *the marks* and they are independent copies of a random variable m, also all independent of $\{x_i\}_{i \in I}$. Such a random field is a very common model in physics and telecommunications, where it has many applications [5, 6]. It is a natural generalization of shot noise processes ($n = 1$), introduced by Rice [16] to model shot effect noise in electronic devices. More recently, it has also become a widely used model in image processing, mainly for applications in texture synthesis and analysis [13].

Geometric characteristics of random surfaces is an important subject of modern probability research, linked with extremal theory [2, 4] and based on the study of random fields excursion sets. The excursion set of level t of the random field X in an open subset U of $\mathbb{R}^n$ is defined by

$$E_X(t, U) := \{x \in U \text{ such that } X(x) > t\}.$$

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Most of the results are obtained for stationary Gaussian random fields, but recent works have dropped the Gaussian assumption. In particular, in [1], asymptotics for the distribution of critical points and Euler characteristics are obtained for a large class of infinite divisible stationary random fields with suitable regularity assumptions. Central limit theorem for volumes of excursion sets have also been considered in [10] for general stationary random fields, including some shot noise and Gaussian random fields. In this paper, we will be interested in the "perimeter" of excursion sets of shot noise random fields (we will give the precise definition of it in the first section), measured as the $(n-1)$-dimensional Hausdorff measure of its boundary

$$L_X(t,U) := \mathcal{H}^{n-1}\big(\partial E_X(t,U)\cap U\big).$$

In dimension $n=1$, the "perimeter" of an excursion set is the number of crossings of the considered level; in dimension $n=2$ it measures the length of the boundary of the excursion set; in dimension $n=3$ it measures its surface area, and so on.

The shot noise random field is not necessarily smooth or differentiable (this happens, e.g., when the kernel function is an indicator function), and therefore we cannot use the framework of Adler et al. ([2] or [1]) to compute its geometric features (such as the Euler characteristic or intrinsic volumes). Even in the case where the shot noise random field is smooth, it may not satisfy the regularity conditions on the marginal or conditional probability densities of the field and its gradient [7].

As a consequence, we will adopt a different viewpoint, and we will study the excursion sets via the whole function $t \mapsto L_X(t,U)$, and not only its value for a particular fixed t. A convenient functional framework to define and compute these perimeters is the framework of functions of bounded variation. Our main tool will be the so-called *coarea formula* that relates the integral of the perimeter of all excursion sets to the integral of the differential of the function. The coarea formula has been already used in a similar situation by Wschebor [18] and by Zähle in [19] to obtain a general Rice formula for continuous random fields in dimension d. It has also been used by Azaïs and Wschebor in [4] to have a proof of the theorem that computes the expected number of crossings of a random field as a function of some of its marginal and conditional probability densities. See also Adler and Taylor [2], page 283, for a discussion about this theorem. The coarea formula approach is in some sense a weak approach since we will obtain a formula for almost every level t (and not for a specific value of t). But the advantage is that we will be able to make explicit computations in a situation where the methods of [2] or [1] cannot be applied. Let us also mention that in our previous paper [8] we started using the same approach in dimension $n=1$, but we worked under a piecewise regularity assumption more restrictive than functions of bounded variation. We propose here a much more general setting, allowed for any dimension $n \geq 1$. Note also that our results on the coarea formula only rely on a functional assumption and not on a distribution assumption. In particular, they are valid for some processes with jumps and allow to recover partially some recent results in dimension $n = 1$ on Rice formula for the number of crossings for the sum of a smooth process and a pure jump process [11] or for piecewise deterministic Markov processes [9]. However, we focus here on shot noise random fields for which we set convenient assumptions to ensure the bounded variation, derive explicit computation and obtain asymptotic regime as the underlying intensity tends to infinity.

The paper is organized as follows: we start in Section 2 to give some notation and properties of the functions of bounded variation. Then, in Section 3, we define precisely the shot noise random field and give an explicit computation of the perimeter of its excursion sets. We illustrate our results with some examples. Finally in Section 4, we derive two different asymptotic behaviors for the perimeters as the intensity of the underlying homogeneous Poisson point process goes to infinity.

2. Coarea formula.

2.1. *The framework of functions of bounded variation.* For sake of completeness, we first recall here some definitions and properties of functions of bounded variation. We will mainly use the framework and notations of Ambrosio, Fusco and Pallara in [3]. We shall also sometimes refer to Evans and Gariepy [12]. In all the following $\mathcal{L}^n$ will denote the n-dimensional Lebesgue measure in $\mathbb{R}^n$, and $\mathcal{H}^k$ will denote the k-dimensional Hausdorff measure (we will most of the time use it with $k = n-1$ and keep $\mathcal{L}^n$ for $k = n$). When there is no ambiguity we will simply denote dx instead of $\mathcal{L}^n(dx)$ for the Lebesgue measure in integrals. We will also use the notation $\mu \llcorner A$ to denote the restriction of a measure μ to a set A.

Let U be an open subset of $\mathbb{R}^n$. A real-valued function $f \in L^1(U)$ is said to be a *function of bounded variation in* U if

$$V(f,U) := \sup\left\{\int_U f \operatorname{div}\varphi \, dx \,|\, \varphi \in C_c^1(U,\mathbb{R}^n), \|\varphi\|_\infty \le 1\right\} < +\infty,$$

where $C_c^1(U,\mathbb{R}^n)$ denotes the set of continuously differentiable $\mathbb{R}^n$-valued functions with compact support in U. We will denote by $BV(U)$ the space of functions of bounded variation in U. An equivalent definition ([3], pages 117–120) of $f \in BV(U)$ is that $f \in L^1(U)$ is such that its distributional derivative (i.e., its derivative in the sense of distributions) is representable by a *finite Radon measure* in U, that is,

$$\int_U f(x)\frac{\partial \phi}{\partial x_l}(x)\,dx = -\int_U \phi(x) D_l f(dx) \qquad \forall \phi \in C_c^\infty(U,\mathbb{R}), \forall l = 1,\dots,n$$

for some $\mathbb{R}^n$-valued measure $Df = (D_1 f,\dots,D_n f)$. Its total variation is the positive Radon measure denoted by $\|Df\|$ and defined by

$$\|Df\|(E) = \sup_{P \in \mathcal{P}(E)} \sum_{k\in K} \|Df(E_k)\|,$$

for all measurable sets $E \subset U$, with $\mathcal{P}(E)$ the set of finite or countable partitions $P = (E_k)_{k \in K}$ of E into disjoint measurable sets E_k, and where $\|Df(E_k)\|$ denotes the Euclidean norm (in $\mathbb{R}^n$) of $Df(E_k)$. Let us note S^{n-1} the unit sphere of $\mathbb{R}^n$. According to the polar decomposition, which follows from Radon–Nikodym theorem (see Corollary 1.29 of [3]), there exists a unique S^{n-1}-valued function ν_f that is measurable and integrable with respect to the measure $\|Df\|$ and such that $Df = \|Df\|\nu_f$. And we also have

$$V(f, U) = \|Df\|(U).$$

In particular, according to Evans and Gariepy ([12], page 91), when $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is Lipschitz, then it is differentiable almost everywhere, and in this case if we denote $\nabla f(x) = (\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x)) \in \mathbb{R}^n$ the gradient of f at a point x, and $\|\nabla f(x)\|$ its Euclidean norm, then $Df = \nabla f \mathcal{L}^n$, $\|Df\| = \|\nabla f\| \mathcal{L}^n$ and $\nu_f = \nabla f / \|\nabla f\|$, so that

$$V(f, U) = \|Df\|(U) = \int_U \|\nabla f(x)\| \, dx.$$

One can define a norm on $BV(U)$ by

$$\|f\|_{BV(U)} = \|f\|_{L^1(U)} + \|Df\|(U), \tag{2.1}$$

so that $(BV(U), \|\cdot\|_{BV(U)})$ is a Banach space ([3], page 121).

The framework of functions of bounded variation is of special interest to study the perimeter of a set because of the following definition and property. Let E be an $\mathcal{L}^n$-measurable subset of $\mathbb{R}^n$. Then for any open subset $U \subset \mathbb{R}^n$, we say that E is a *set of finite perimeter* in U if its indicator function χ_E is of bounded variation in U. In this case, we define the *perimeter* $L_E(U)$ of E in U as $V(\chi_E, U)$, that is,

$$L_E(U) := \sup\left\{ \int_E \operatorname{div} \varphi \, dx \,|\, \varphi \in C_c^1(U, \mathbb{R}^n), \|\varphi\|_\infty \le 1 \right\}.$$

The term "perimeter" meets here its usual sense in dimension $n = 2$ as "the length of the boundary." Indeed, it can be shown (see [3], page 143) that for all sets E with piecewise C^1 boundary in U and such that $\mathcal{H}^{n-1}(\partial E \cap U)$ is finite, then by Gauss–Green theorem the distributional derivative of χ_E is $D\chi_E = \nu_E \mathcal{H}^{n-1} \angle \partial E$, where ν_E is the inner unit normal to E, and that

$$L_E(U) = \mathcal{H}^{n-1}(\partial E \cap U).$$

Let $f \in BV(U)$. For $t \in \mathbb{R}$, we can consider the excursion set (also sometimes called "upper level set" or "superlevel") of level t of f,

$$E_f(t, U) := \{x \in U \text{ such that } f(x) > t\}.$$

Then for $\mathcal{L}^1$-almost every $t \in \mathbb{R}$, the set $E_f(t, U)$ is of finite perimeter in U. We will denote its perimeter in U by $L_f(t, U)$. Moreover, the function $t \mapsto L_f(t, U)$ belongs to $L^1(\mathbb{R})$, and we have the *coarea formula*,

$$\|Df\|(U) = \int_{\mathbb{R}} L_f(t, U) \, dt. \tag{2.2}$$

The proof of this formula can be found in [3], page 145, or also in [12], page 85.

Now, in order to use this formula, we need to be more explicit about the pointwise properties of f and the decomposition of its distributional derivative Df. Let us recall that f is said *approximately continuous* at $x \in U \subset \mathbb{R}^n$ if

$$\lim_{\rho\to 0} \rho^{-n} \int_{B_\rho(x)} |f(y) - f(x)|\, dy = 0, \tag{2.3}$$

where $B_\rho(x)$ is the Euclidean ball of radius ρ and centered at x. The set S_f of points where this property does not hold is a $\mathcal{L}^n$ negligible Borel set called an *approximate discontinuity set*; see [3], Proposition 3.64, page 160. A point $x \in S_f$ is called an *approximate jump point* of f if there exist $f^+(x), f^-(x) \in \mathbb{R}$ and $\nu_f(x) \in S^{n-1}$ such that $f^+(x) > f^-(x)$ with

$$\lim_{\rho\to 0} \rho^{-n} \int_{B_\rho^+(x,\nu_f(x))} |f(y) - f^+(x)|\, dy = 0 \quad \text{and}$$

$$\lim_{\rho\to 0} \rho^{-n} \int_{B_\rho^-(x,\nu_f(x))} |f(y) - f^-(x)|\, dy = 0,$$

where $B_\rho^+(x,\nu)$ [resp., $B_\rho^-(x,\nu)$] denotes the half-ball determined by $\nu \in S^{n-1}$, that is, $\{y \in B_\rho(x), \langle y - x, \nu\rangle > 0\}$ [resp., $\{y \in B_\rho(x), \langle y - x, \nu\rangle < 0\}$]. The set of approximate jump points is denoted by J_f, and it is a Borel subset of S_f. Moreover, by the Federer–Vol'pert theorem ([3], Theorem 3.78, page 173), since $f \in BV(U)$, the set S_f is a countably $\mathcal{H}^{n-1}$-rectifiable set with $\mathcal{H}^{n-1}(S_f \setminus J_f) = 0$ and

$$Df \angle J_f = (f^+ - f^-)\nu_f \mathcal{H}^{n-1} \angle J_f.$$

By the Radon–Nikodym theorem, the distributional derivative Df can be decomposed into the sum of three terms (see [3], page 184),

$$Df = \nabla f \mathcal{L}^n + (f^+ - f^-)\nu_f \mathcal{H}^{n-1} \angle J_f + D^c f.$$

These three terms are defined in the following way:

- $D^a f := \nabla f \mathcal{L}^n$ is the absolutely continuous part of the Radon measure Df with respect to the Lebesgue measure $\mathcal{L}^n$. And moreover, ∇f is here the *approximate differential* of f; see [3], page 165 and Theorem 3.83, page 176.
- $D^j f := (f^+ - f^-)\nu_f \mathcal{H}^{n-1} \angle J_f$ is the *jump part* of Df.
- The last term $D^c f$ is the so-called *Cantor part* of Df. It has the property to vanish on sets which have a $\mathcal{H}^{n-1}$ finite measure.

2.2. *A general coarea formula.* In our framework, we will be interested in functions that have no Cantor part in their distributional derivative (we will mainly study piecewise $\mathcal{C}^1$ functions). These functions have been introduced by De Giorgi and Ambrosio to study variational problems where both volume and surface energies are involved, and they are called "special functions of bounded variation."

Their set is denoted by $SBV(U)$, and it is a closed subset of $(BV(U), \|\cdot\|_{BV(U)})$; see [3], Corollary 4.3, page 213.

As we already mentioned in the Introduction, our viewpoint will be to study the function $t \to L_f(t,U)$. The coarea formula (2.2) only provides the integral of $t \to L_f(t,U)$ on $\mathbb{R}$, and we will extend it to the integral of $t \to h(t)L_f(t,U)$ on $\mathbb{R}$ for any bounded continuous function h. This is the aim of the following theorem.

THEOREM 1. *Let U be an open subset of $\mathbb{R}^n$, and let $f:U \to \mathbb{R}$ be a function in $SBV(U)$. Using the notation and definitions of the previous section, its distributional derivative is given by*

$$Df = \nabla f \mathcal{L}^n + (f^+ - f^-)\nu_f \mathcal{H}^{n-1} \angle J_f,$$

while

$$\|Df\| = \|\nabla f\| \mathcal{L}^n + (f^+ - f^-)\mathcal{H}^{n-1} \angle J_f.$$

Let $h:\mathbb{R} \to \mathbb{R}$ be a continuous and bounded function. Then

$$\int_{\mathbb{R}} h(t) L_f(t,U)\,dt = \int_U h(f(x)) \|\nabla f(x)\|\,dx + \int_{J_f \cap U} \left(\int_{f^-(y)}^{f^+(y)} h(s)\,ds \right) \mathcal{H}^{n-1}(dy).$$

PROOF. Let $h:\mathbb{R} \to \mathbb{R}$ be a continuous and bounded function. Let us first assume that there exists an $\varepsilon > 0$ such that $h(t) \geq \varepsilon$ for all $t \in \mathbb{R}$. We define for all $t \in \mathbb{R}$,

$$H(t) = \int_0^t h(s)\,ds.$$

Then H is a C^1 diffeomorphism from $\mathbb{R}$ to $\mathbb{R}$. It is strictly increasing, and since $H'(t) = h(t)$ for all $t \in \mathbb{R}$ is bounded, it is also Lipschitz on $\mathbb{R}$.

Now, let u be the function defined on U by $u = H \circ f$. Then by the chain-rule (see [3], page 164 and Theorem 3.96, page 189) we have that $u \in SBV(U)$, its jump set $J_u = J_f$ and its derivative is given by

$$Du = (h \circ f)\nabla f \mathcal{L}^n + (H \circ f^+ - H \circ f^-)\nu_f \mathcal{H}^{n-1} \angle J_f.$$

Then by the coarea formula (2.2), we have that

$$\begin{aligned}
\int_{\mathbb{R}} L_u(s,U)\,ds &= \|Du\|(U) \\
&= \int_U h(f(x))\|\nabla f(x)\|\,dx + \int_{J_f \cap U} (H(f^+(y)) - H(f^-(y)))\mathcal{H}^{n-1}(dy) \\
&= \int_U h(f(x))\|\nabla f(x)\|\,dx + \int_{J_f \cap U} \left(\int_{f^-(y)}^{f^+(y)} h(s)\,ds \right) \mathcal{H}^{n-1}(dy).
\end{aligned}$$

But for all $s \in \mathbb{R}$, we have that $E_u(s, U) = E_f(H^{-1}(s), U)$ where H^{-1} denotes the inverse of the C^1 diffeomorphism H. And thus for all $s \in \mathbb{R}$, we also have that $L_u(s, U) = L_f(H^{-1}(s), U)$. Then by the change of variable $t = H^{-1}(s)$ (see, e.g., [17], page 153), we get

$$\begin{aligned}\int_{\mathbb{R}} h(t) L_f(t, U)\, dt &= \int_{\mathbb{R}} L_u(s, U)\, ds \\ &= \int_U h\big(f(x)\big) \|\nabla f(x)\|\, dx + \int_{J_f} \bigg(\int_{f^-(y)}^{f^+(y)} h(s)\, ds \bigg) \mathcal{H}^{n-1}(dy),\end{aligned}$$

which is the announced formula.

In the general case, when h is not strictly positive, we simply apply the above formula to $h_1 = 1 + \sup(h, 0)$ and to $h_2 = 1 + \sup(-h, 0)$, and then we have it for $h = h_1 - h_2$, which ends the proof of the theorem. □

3. Shot noise random fields. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and $\Phi = \{(x_i, m_i)\}_{i \in I}$ be a Poisson point process on $\mathbb{R}^n \times \mathbb{R}^d$ with intensity $\lambda \mathcal{L}^n \otimes F$, with F a probability measure on $\mathbb{R}^d$. Let $g : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$ be defined such that for F-almost every $m \in \mathbb{R}^d$ the function $g_m := g(\cdot, m)$ belongs to $SBV(\mathbb{R}^n)$. From Section 2, it follows that for such $m \in \mathbb{R}^d$

$$Dg_m = \nabla g_m \mathcal{L}^n + \big(g_m^+ - g_m^-\big) \nu_{g_m} \mathcal{H}^{n-1} \angle J_{g_m}$$

and

$$\begin{aligned}\|g_m\|_{BV(\mathbb{R}^n)} &= \|g_m\|_{L^1(\mathbb{R}^n)} + \|Dg_m\|\big(\mathbb{R}^n\big) \\ &= \|g_m\|_{L^1(\mathbb{R}^n)} + \|\nabla g_m\|_{L^1(\mathbb{R}^n)} + \int_{J_{g_m}} \big(g_m^+(y) - g_m^-(y)\big) \mathcal{H}^{n-1}(dy) \\ &< +\infty.\end{aligned}$$

Under the assumption that

$$\int_{\mathbb{R}^d} \|g_m\|_{L^1(\mathbb{R}^n)} F(dm) < +\infty, \tag{3.1}$$

one can define almost surely the shot noise random field

$$X_\Phi = \sum_{i \in I} \tau_{x_i} g_{m_i}, \tag{3.2}$$

as a random field in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, where $\tau_{x_i} g_{m_i}(x) := g_{m_i}(x - x_i)$.

In the sequel we will also consider $\Phi_j = \Phi \setminus \{(x_j, m_j)\}$ for $j \in I$ and its associated shot noise random field

$$X_{\Phi_j} = \sum_{i \in I; i \neq j} \tau_{x_i} g_{m_i}.$$

Throughout the paper we make the stronger assumption that

$$\int_{\mathbb{R}^d} \|g_m\|_{BV(\mathbb{R}^n)} F(dm) < +\infty. \tag{3.3}$$

3.1. *Regularity of the shot noise random fields*. The aim of the following theorem is to show that the shot noise random field inherits the regularity properties of the kernel functions g_m. The general idea is that if the kernel functions are special functions of bounded variation, then so is locally the shot noise. The theorem also gives the decomposition of its distributional derivative.

In the following, we will need the functional spaces $L^1_{\text{loc}}(\mathbb{R}^n)$ and $SBV_{\text{loc}}(\mathbb{R}^n)$. We recall that they are defined by the following: a function f belongs to $L^1_{\text{loc}}(\mathbb{R}^n)$ [resp., $SBV_{\text{loc}}(\mathbb{R}^n)$] if and only if it belongs to $L^1(U)$ [resp., $SBV(U)$] for all bounded open subset U of $\mathbb{R}^n$.

THEOREM 2. *Under assumption* (3.3), *one can define almost surely* (*a.s.*) *in* $L^1_{\text{loc}}(\mathbb{R}^n)$ *the two shot noise random fields*

$$X_\Phi := \sum_{i\in I} \tau_{x_i} g_{m_i} \quad \text{and} \quad \nabla X_\Phi := \sum_{i\in I} \tau_{x_i} \nabla g_{m_i}.$$

Moreover, a.s. $X_\Phi \in SBV_{\text{loc}}(\mathbb{R}^n)$ *with*

$$DX_\Phi = \nabla X_\Phi \mathcal{L}^n + (X_\Phi^+ - X_\Phi^-)\nu_{X_\Phi} \mathcal{H}^{n-1} \angle_{J_{X_\Phi}},$$

and for $\mathcal{H}^{n-1}$ *almost every* $y \in J_{X_\Phi}$, *there exists a unique* $(x_j, m_j) \in \Phi$ *such that* $y \in J_{\tau_{x_j} g_{m_j}} = x_j + J_{g_{m_j}}$ *and*

$$X_\Phi^+(y) = \tau_{x_j} g_{m_j}^+(y) + X_{\Phi_j}(y) \quad \text{and} \quad X_\Phi^-(y) = \tau_{x_j} g_{m_j}^-(y) + X_{\Phi_j}(y). \tag{3.4}$$

PROOF. Let U be a bounded open set of $\mathbb{R}^n$. First, note that by Campbell's formula,

$$\begin{aligned}\mathbb{E}\Big(\sum_{i\in I} \|\tau_{x_i} g_{m_i}\|_{BV(U)}\Big) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^n} \|\tau_y g_m\|_{BV(U)} \lambda\, dy\, F(dm) \\ &= \lambda \int_{\mathbb{R}^d}\int_{\mathbb{R}^n} \big(\|\tau_y g_m\|_{L^1(U)} + \|\tau_y Dg_m\|(U)\big)\, dy\, F(dm).\end{aligned}$$

By Fubini's theorem and by the translation invariance of $\mathcal{L}^n$, we have

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^n} \|\tau_y g_m\|_{L^1(U)}\, dy\, F(dm) = \mathcal{L}^n(U) \int_{\mathbb{R}^d} \|g_m\|_{L^1(\mathbb{R}^n)} F(dm).$$

Moreover, recalling the notation χ_V to denote the indicator function of a set V, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \|\tau_y Dg_m\|(U)\, dy &= \int_{\mathbb{R}^n} \|Dg_m\|(U-y)\, dy \\ &= \int_{\mathbb{R}^n\times\mathbb{R}^n} \chi_{U-y}(x) \|Dg_m\|(dx)\, dy \\ &= \mathcal{L}^n(U) \|Dg_m\|\big(\mathbb{R}^n\big),\end{aligned}$$

by Fubini's theorem and the translation invariance of $\mathcal{L}^n$. It follows that

$$\mathbb{E}\left(\sum_{i\in I}\|\tau_{x_i}g_{m_i}\|_{BV(U)}\right)=\lambda\mathcal{L}^n(U)\int_{\mathbb{R}^d}\|g_m\|_{BV(\mathbb{R}^n)}F(dm)<+\infty.$$

Hence, almost surely $\sum_{i\in I}\|\tau_{x_i}g_{m_i}\|_{BV(U)}<+\infty$. Since $(SBV(U),\|\cdot\|_{BV(U)})$ is a Banach space, it implies that $X_\Phi=\sum_{i\in I}\tau_{x_i}g_{m_i}$ is almost surely in $(SBV(U),\|\cdot\|_{BV(U)})$. Now, let us identify DX_Φ. First, remark that $\sum_{i\in I}\|\tau_{x_i}g_{m_i}\|_{BV(U)}<+\infty$ implies that

$$\sum_{i\in I}\|\tau_{x_i}\nabla g_{m_i}\|_{L^1(U)}+\sum_{i\in I}\int_{J_{\tau_{x_i}g_{m_i}}\cap U}(\tau_{x_i}g^+_{m_i}(y)-\tau_{x_i}g^-_{m_i}(y))\mathcal{H}^{n-1}(dy)<+\infty,$$

so that the vectorial Radon measure

$$\sum_{i\in I}\tau_{x_i}\nabla g_{m_i}\mathcal{L}^n\angle_U+\sum_{i\in I}(\tau_{x_i}g^+_{m_i}-\tau_{x_i}g^-_{m_i})\nu_{\tau_{x_i}g_{m_i}}\mathcal{H}^{n-1}\angle_{U\cap J_{\tau_{x_i}g_{m_i}}}$$

is well defined. By uniqueness of the Radon–Nikodym decomposition, we get

$$\nabla X_\Phi\mathcal{L}^n\angle_U=\sum_{i\in I}\tau_{x_i}\nabla g_{m_i}\mathcal{L}^n\angle_U, \tag{3.5}$$

$$\begin{aligned}&(X^+_\Phi-X^-_\Phi)\nu_{X_\Phi}\mathcal{H}^{n-1}\angle_{U\cap J_{X_\Phi}}\\&\quad=\sum_{i\in I}(\tau_{x_i}g^+_{m_i}-\tau_{x_i}g^-_{m_i})\nu_{\tau_{x_i}g_{m_i}}\mathcal{H}^{n-1}\angle_{U\cap J_{\tau_{x_i}g_{m_i}}}.\end{aligned} \tag{3.6}$$

Note in particular that the last equality implies that

$$\mathcal{H}^{n-1}\left(U\cap J_{X_\Phi}\cap\left(\bigcup_{i\in I}J_{\tau_{x_i}g_{m_i}}\right)^c\right)=0,$$

where for a set S, S^c denotes the complement of S.

For a fixed point $(x_j,m_j)\in\Phi$, let us remark that since X_Φ and $\tau_{x_j}g_{m_j}$ are both in $SBV(U)$, we also have $X_{\Phi_j}=X_\Phi-\tau_{x_j}g_{m_j}\in SBV(U)$. Analogously, we get

$$\mathcal{H}^{n-1}\left(U\cap J_{X_{\Phi_j}}\cap\left(\bigcup_{i\in I;i\neq j}J_{\tau_{x_i}g_{m_i}}\right)^c\right)=0.$$

Note that when $y\in J_{\tau_{x_j}g_{m_j}}\cap S^c_{X_{\Phi_j}}\cap U$, we obtain that $y\in J_{X_\Phi}$ with (3.4) satisfied. Therefore, it suffices to prove that the set of points in U that belong to $\bigcup_{j\in I}(J_{\tau_{x_j}g_{m_j}}\cap S_{X_{\Phi_j}})$ is $\mathcal{H}^{n-1}$ negligible. We have

$$\begin{aligned}&\mathbb{E}\left(\mathcal{H}^{n-1}\left(\bigcup_{j\in I}(J_{\tau_{x_j}g_{m_j}}\cap S_{X_{\Phi_j}})\cap U\right)\right)\\&\quad\le\mathbb{E}\left(\sum_{j\in I}\mathcal{H}^{n-1}(J_{\tau_{x_j}g_{m_j}}\cap S_{X_{\Phi_j}}\cap U)\right)\\&\quad\le\int_{\mathbb{R}^d}\int_{\mathbb{R}^n}\mathbb{E}(\mathcal{H}^{n-1}(J_{\tau_xg_m}\cap S_{X_\Phi}\cap U))\lambda\,dx\,F(dm),\end{aligned}$$

by Mecke's formula; see [6]. Now, using Fubini's theorem, for F-a.e. m

$$\int_{\mathbb{R}^n} \mathbb{E}\big(\mathcal{H}^{n-1}(J_{\tau_x g_m} \cap S_{X_\Phi} \cap U)\big)\,dx = \mathbb{E}\bigg(\int_{S_{X_\Phi}\cap U} \mathcal{L}^n(y + J_{g_m})\mathcal{H}^{n-1}(dy)\bigg),$$

since $J_{\tau_x g_m} = x + J_{g_m}$ for all $x \in \mathbb{R}^n$. But $\mathcal{L}^n(y + J_{g_m}) = \mathcal{L}^n(J_{g_m}) = 0$ for all $y \in \mathbb{R}^n$, which implies that $\mathcal{H}^{n-1}(\bigcup_{j\in I} J_{\tau_{x_j} g_{m_j}} \cap S_{X_{\Phi_j}} \cap U) = 0$ almost surely. Finally, we complete the proof using the fact that $\mathbb{R}^n$ is covered by a countable union of bounded open sets so that a.s. X_Φ is in $SBV_{\mathrm{loc}}(\mathbb{R}^n)$. □

Under assumption (3.3), X_Φ is well defined a.s. as a function in $SBV(\mathbb{R}^n)$, but we can also consider X_Φ as a real random field indexed by $\mathbb{R}^n$. More precisely, let us define

$$\mathcal{D}_\Phi = \bigg\{x \in \mathbb{R}^n \text{ such that } X_\Phi(x) = \sum_{i\in I} \tau_{x_i} g_{m_i}(x)\bigg\} \cap S^c_{X_\Phi}.$$

Note that the convergence in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ implies that a.s. $\mathcal{L}^n(\mathcal{D}^c_\Phi) = 0$ so that $\mathbb{E}(\mathcal{L}^n(\mathcal{D}^c_\Phi)) = 0$. Moreover, for U a bounded open set in $\mathbb{R}^n$, by Fubini's theorem, one has

$$\mathbb{E}\big(\mathcal{L}^n(\mathcal{D}^c_\Phi \cap U)\big) = \int_U \mathbb{P}\big(x \in \mathcal{D}^c_\Phi\big)\,dx = \mathbb{P}\big(0 \in \mathcal{D}^c_\Phi\big)\mathcal{L}^n(U),$$

where the last equality comes from $\mathbb{P}(x \in \mathcal{D}^c_\Phi) = \mathbb{P}(0 \in \mathcal{D}^c_\Phi)$, by stationarity of the point process $\{x_i\}_{i\in I}$. It follows that for all $x \in \mathbb{R}^n$, $\mathbb{P}(x \in \mathcal{D}^c_\Phi) = 0$ and a.s. $x \in S^c_{X_\Phi}$ and $X_\Phi(x) = \sum_{i\in I} \tau_{x_i} g_{m_i}(x)$. The same remark may be applied to ∇X_Φ since $\mathcal{L}^n(S_{\nabla X_\Phi}) = 0$ as $\nabla X_\Phi \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. This allows to compute the finite dimensional law of the random field $\{(X_\Phi(x), \nabla X_\Phi(x)); x \in \mathbb{R}^n\}$ itself. In particular, the shot noise random fields have the nice property that their characteristic function is explicit; see, for instance, [6], Chapter 2. More precisely, in our framework, the shot noise field $\{(X_\Phi(x), \nabla X_\Phi(x)); x \in \mathbb{R}^n\}$ is stationary, and therefore the joint characteristic function of $X_\Phi(x)$ and $\nabla X_\Phi(x)$ is independent of x and is given for all $u \in \mathbb{R}$ and all $v \in \mathbb{R}^n$ by

$$\begin{aligned}\psi(u, v) &:= \mathbb{E}\big(e^{iuX_\Phi(x) + i\langle v, \nabla X_\Phi(x)\rangle}\big)\\ &= \exp\bigg(\lambda \int_{\mathbb{R}^d}\int_{\mathbb{R}^n} \big(e^{iug_m(y) + i\langle v, \nabla g_m(y)\rangle} - 1\big)\,dy\,F(dm)\bigg).\end{aligned} \tag{3.7}$$

In the following we will also simply denote $\psi(u) = \psi(u, 0) = \mathbb{E}(e^{iuX_\Phi(x)})$ the characteristic function of $X_\Phi(x)$. Let us also mention that the real random variables $X_\Phi(x), \frac{\partial X_\Phi}{\partial x_1}(x), \dots, \frac{\partial X_\Phi}{\partial x_n}(x)$ are integrable with

$$\mathbb{E}\big(X_\Phi(x)\big) = \lambda \int_{\mathbb{R}^d}\int_{\mathbb{R}^n} g_m(y)\,dy\,F(dm) \quad \text{and}$$

$$\mathbb{E}\bigg(\frac{\partial X_\Phi}{\partial x_l}(x)\bigg) = \lambda \int_{\mathbb{R}^d}\int_{\mathbb{R}^n} \frac{\partial g_m}{\partial x_l}(y)\,dy\,F(dm),$$

for all $l = 1, \dots, n$, implying that $\|\nabla X_\Phi\|$ is also integrable. Moreover, under the additional assumption that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^n} g_m(y)^2\,dy\,F(dm) < +\infty \quad \text{and} \tag{3.8}$$
$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^n} \|\nabla g_m(y)\|^2\,dy\,F(dm) < +\infty,$$

the real random variables $X_\Phi(x)$, $\frac{\partial X_\Phi}{\partial x_1}(x), \dots, \frac{\partial X_\Phi}{\partial x_n}(x)$ are also square integrable with

$$\operatorname{Var}\bigl(X_\Phi(x)\bigr) = \lambda\int_{\mathbb{R}^d}\int_{\mathbb{R}^n} g_m(y)^2\,dy\,F(dm) \quad \text{and}$$
$$\operatorname{Var}\left(\frac{\partial X_\Phi}{\partial x_l}(x)\right) = \lambda\int_{\mathbb{R}^d}\int_{\mathbb{R}^n}\left(\frac{\partial g_m}{\partial x_l}(y)\right)^2 dy\,F(dm),$$

for all $l = 1, \dots, n$.

3.2. *Perimeter of the excursion sets*. We consider the excursion set of X_Φ defined as previously by

$$E_{X_\Phi}(t, U) = \bigl\{x \in U \text{ such that } X_\Phi(x) > t\bigr\},$$

as well as $L_{X_\Phi}(t, U)$ its perimeter in U. According to Theorem 1, when $h : \mathbb{R} \to \mathbb{R}$ is a continuous and bounded function, one has a.s. the coarea formula

$$\int_{\mathbb{R}} h(t) L_{X_\Phi}(t, U)\,dt$$
$$= \int_U h\bigl(X_\Phi(x)\bigr)\|\nabla X_\Phi(x)\|\,dx + \int_{J_{X_\Phi}\cap U}\left(\int_{X_\Phi^-(y)}^{X_\Phi^+(y)} h(s)\,ds\right)\mathcal{H}^{n-1}(dy).$$

By (3.4), the jump part rewrites as

$$\begin{aligned}
&\int_{J_{X_\Phi}\cap U}\left(\int_{X_\Phi^-(y)}^{X_\Phi^+(y)} h(s)\,ds\right)\mathcal{H}^{n-1}(dy)\\
&\quad= \sum_{j\in I}\int_{J_{\tau_{x_j} g_{m_j}}\cap U}\left(\int_{\tau_{x_j} g_{m_j}^-(y)+X_{\Phi_j}(y)}^{\tau_{x_j} g_{m_j}^+(y)+X_{\Phi_j}(y)} h(s)\,ds\right)\mathcal{H}^{n-1}(dy)\\
&\quad= \sum_{j\in I}\int_{J_{\tau_{x_j} g_{m_j}}\cap U}\left(\int_{\tau_{x_j} g_{m_j}^-(y)}^{\tau_{x_j} g_{m_j}^+(y)} h\bigl(s + X_{\Phi_j}(y)\bigr)\,ds\right)\mathcal{H}^{n-1}(dy).
\end{aligned} \tag{3.9}$$

We compute the expectation of the jump part of the coarea formula in the next proposition.

PROPOSITION 1. *Let $h:\mathbb{R}\to\mathbb{R}$ be a continuous and bounded function. Then*

$$\mathbb{E}\left(\int_{J_{X_\Phi}\cap U}\left(\int_{X_\Phi^-(y)}^{X_\Phi^+(y)} h(s)\,ds\right)\mathcal{H}^{n-1}(dy)\right)$$
$$=\lambda\mathcal{L}^n(U)\int_{\mathbb{R}^d}\int_{J_{g_m}}\left(\int_{g_m^-(y)}^{g_m^+(y)}\mathbb{E}\big(h\big(s+X_\Phi(0)\big)\big)\,ds\right)\mathcal{H}^{n-1}(dy)F(dm).$$

PROOF. From Mecke's formula (see [6]), taking the expectation in (3.9), we get

$$\mathbb{E}\left(\int_{J_{X_\Phi}\cap U}\left(\int_{X_\Phi^-(y)}^{X_\Phi^+(y)} h(s)\,ds\right)\mathcal{H}^{n-1}(dy)\right)$$
$$=\int_{\mathbb{R}^n\times\mathbb{R}^d}\int_{J_{\tau_x g_m}\cap U}\left(\int_{\tau_x g_m^-(y)}^{\tau_x g_m^+(y)}\mathbb{E}\big(h\big(s+X_\Phi(y)\big)\big)\,ds\right)\mathcal{H}^{n-1}(dy)\lambda\,dx\,F(dm)$$
$$=\int_{\mathbb{R}^n\times\mathbb{R}^d}\int_{J_{g_m}\cap(U-x)}\left(\int_{g_m^-(y)}^{g_m^+(y)}\mathbb{E}\big(h\big(s+X_\Phi(x+y)\big)\big)\,ds\right)\mathcal{H}^{n-1}(dy)\lambda\,dx\,F(dm),$$

by translation invariance of $\mathcal{H}^{n-1}$. Moreover, by stationarity of X_Φ, for all $s\in\mathbb{R}$ and $x,y\in\mathbb{R}^n$,

$$\mathbb{E}\big(h\big(s+X_\Phi(x+y)\big)\big)=\mathbb{E}\big(h\big(s+X_\Phi(0)\big)\big).$$

By Fubini's theorem, integrating with respect to x, this last integral is equal to

$$\lambda\mathcal{L}^n(U)\int_{\mathbb{R}^d}\int_{J_{g_m}}\int_{g_m^-(y)}^{g_m^+(y)}\mathbb{E}\big(h\big(s+X_\Phi(0)\big)\big)\,ds\mathcal{H}^{n-1}(dy)F(dm),$$

which is the announced result. □

We can now give our main result about the mean value of the perimeter of the shot noise random field. It is a direct consequence of the coarea formula of Theorem 1, when taking for h the function $h(t)=e^{iut}$, and of the computation of the expectation of the jump part of the coarea formula given in Proposition 1.

THEOREM 3. *Let X_Φ be a shot noise random field given on $\mathbb{R}^n$ by (3.2) and such that assumption (3.3) is satisfied. For all $t\in\mathbb{R}$, let us denote*

$$C_{X_\Phi}(t)=\mathbb{E}\big(L_{X_\Phi}\big(t,(0,1)^n\big)\big).$$

Then the function $t\mapsto C_{X_\Phi}(t)$ belongs to $L^1(\mathbb{R})$, and its Fourier transform is given for all $u\in\mathbb{R}$, $u\neq 0$ by

$$\widehat{C}_{X_\Phi}(u)=\mathbb{E}\big(e^{iuX_\Phi(0)}\|\nabla X_\Phi(0)\|\big)$$
$$+\mathbb{E}\big(e^{iuX_\Phi(0)}\big)\frac{\lambda}{iu}\int_{\mathbb{R}^d}\int_{J_{g_m}}\big(e^{iug_m^+(y)}-e^{iug_m^-(y)}\big)\mathcal{H}^{n-1}(dy)F(dm),$$

and for $u = 0$, we have

$$\begin{aligned}\widehat{C}_{X_\Phi}(0) &= \mathbb{E}\big(V\big(X_\Phi, (0,1)^n\big)\big)\\ &= \mathbb{E}\big(\|\nabla X_\Phi(0)\|\big) + \lambda \int_{\mathbb{R}^d}\int_{J_{g_m}} \big(g_m^+(y) - g_m^-(y)\big)\mathcal{H}^{n-1}(dy)F(dm).\end{aligned}$$

A direct consequence of Theorem 3 is the following corollary that gives a Rice formula in a weak sense (i.e., for almost every level t) and includes the "jump part."

COROLLARY 1. *Under the assumptions of Theorem 3, and if we moreover assume that the random variable $X_\Phi(0)$ admits a probability density on $\mathbb{R}$, denoted by $t \mapsto p_{X_\Phi(0)}(t)$, then for almost every $t \in \mathbb{R}$ we have*

$$\begin{aligned}C_{X_\Phi}(t) &= \mathbb{E}\big(\|\nabla X_\Phi(0)\| \,|\, X_\Phi(0) = t\big) p_{X_\Phi(0)}(t)\\ &\quad + \lambda \int_{\mathbb{R}^d}\int_{J_{g_m}}\int_{g_m^-(y)}^{g_m^+(y)} p_{X_\Phi(0)}(t-s)\,ds\,\mathcal{H}^{n-1}(dy)F(dm).\end{aligned}$$

PROOF. Note that since $X_\Phi(0)$ admits $p_{X_\Phi(0)}$ for density, we may define the positive measurable function

$$\begin{aligned}C(t) &:= \mathbb{E}\big(\|\nabla X_\Phi(0)\| \,|\, X_\Phi(0) = t\big) p_{X_\Phi(0)}(t)\\ &\quad + \lambda \int_{\mathbb{R}^d}\int_{J_{g_m}}\int_{g_m^-(y)}^{g_m^+(y)} p_{X_\Phi(0)}(t-s)\,ds\,\mathcal{H}^{n-1}(dy)F(dm),\end{aligned}$$

for almost every $t \in \mathbb{R}$. Moreover,

$$\begin{aligned}\int_{\mathbb{R}} C(t)\,dt &= \mathbb{E}\big(\|\nabla X_\Phi(0)\|\big) + \lambda \int_{\mathbb{R}^d}\int_{J_{g_m}} \big(g_m^+(y) - g_m^-(y)\big)\mathcal{H}^{n-1}(dy)F(dm)\\ &\le \mathbb{E}\big(\|\nabla X_\Phi(0)\|\big) + \lambda \int_{\mathbb{R}^d} \|g_m\|_{BV(\mathbb{R}^n)} F(dm) < +\infty,\end{aligned}$$

by assumption (3.3). Then we may compute its Fourier transform and find $\widehat{C} = \widehat{C}_{X_\Phi}$. The result follows from the injectivity of the Fourier transform. □

Let us quote that sufficient conditions for $X_\Phi(0)$ to admit a probability density are given in Section 3.2 of [7].

3.3. *Some particular cases.* In order to have explicit formulas for the mean perimeter $C_{X_\Phi}(t) = \mathbb{E}(L_{X_\Phi}(t, (0,1)^n))$, we need to be able to compute the two terms of $\widehat{C}_{X_\Phi}(u)$ in the formula of Theorem 3. We will give in this section many situations in which the computations are doable. The first case is the one of piecewise constant functions g_m, since in this case the first term vanishes. The second case is when $\mathbb{E}(\|\nabla X_\Phi(0)\|^2)$ is finite, because we are then able to use the joint characteristic function of $X_\Phi(0)$ and $\nabla X_\Phi(0)$ to have an explicit formula for the term $\mathbb{E}(e^{iuX_\Phi(0)}\|\nabla X_\Phi(0)\|)$. This will be the aim of the next section.

Let us start with the piecewise constant case. When the functions g_m are piecewise constant, then $\nabla X_\Phi(0) = 0$ a.s., and therefore we simply have

$$\widehat{C}_{X_\Phi}(u) = \mathbb{E}(e^{iuX_\Phi(0)})\frac{\lambda}{iu}\int_{\mathbb{R}^d}\int_{J_{gm}}(e^{iug_m^+(y)} - e^{iug_m^-(y)})\mathcal{H}^{n-1}(dy)F(dm).$$

EXAMPLE 1. We consider a shot-noise process X_Φ in $\mathbb{R}^2$ made of random shapes; that is, we assume that $n = 2$, that the marks m are given by $m = (\beta, r)$ with $\beta \geq 0$, $r \geq 0$ and with the distribution F given by $F(dm) = F_\beta(d\beta)F_r(dr)$ (having thus β and r independent). We also assume that the functions g_m are of the form $g_m(x) = \beta\chi_{K_r}(x)$ for all $x \in \mathbb{R}^2$, where for each r, K_r is a compact set of $\mathbb{R}^2$ with piecewise smooth boundary and such that the mean perimeter and the mean area, respectively, defined by

$$\overline{p} = \int_{\mathbb{R}_+}\mathcal{H}^1(\partial K_r)F_r(dr) \quad \text{and} \quad \overline{a} = \int_{\mathbb{R}_+}\mathcal{L}^2(K_r)F_r(dr)$$

are both finite. Then in this case, we have

$$\forall u \in \mathbb{R}, u \neq 0, \qquad \widehat{C}_{X_\Phi}(u) = \lambda\overline{p}\frac{\widehat{F}_\beta(u) - 1}{iu}e^{\lambda\overline{a}(\widehat{F}_\beta(u)-1)},$$

where $\widehat{F}_\beta(u) = \int_{\mathbb{R}_+} e^{iu\beta}F_\beta(d\beta)$:

In particular if β follows an exponential distribution of parameter $\mu > 0$, then $\widehat{F}_\beta(u) = \frac{\mu}{\mu - iu}$, and then

$$\widehat{C}_{X_\Phi}(u) = \lambda\overline{p}\frac{1}{\mu - iu}e^{\lambda\overline{a}(iu)/(\mu-iu)}.$$

We recognize here (thanks to tables of Fourier transforms!) the Fourier transform of a noncentral chi-square distribution, and we thus have

$$\text{for a.e. } t \in \mathbb{R}_+, \qquad C_{X_\Phi}(t) = \lambda\mu\overline{p}e^{-\lambda\overline{a}-\mu t}I_0(2\sqrt{\lambda\mu\overline{a}t}),$$

where I_0 is the modified Bessel function of the first kind of order 0 that is given for all $t \in \mathbb{R}$ by $I_0(t) = \frac{1}{\pi}\int_0^\pi e^{t\cos\theta}\,d\theta$.

Another explicit and simple case is when $\beta = 1$ a.s., which implies that $\widehat{F}_\beta(u) = e^{iu}$. Then $\widehat{C}_{X_\Phi}(u)$ is the product of two Fourier transforms: one of a Poisson distribution and one of the indicator function of the interval [0, 1]. Therefore we get

$$\forall k \in \mathbb{N}, \text{ for a.e. } t \in (k, k+1), \qquad C_{X_\Phi}(t) = \lambda\overline{p}\frac{(\lambda\overline{a})^k}{k!}e^{-\lambda\overline{a}}.$$

We illustrate this result on Figure 1 where we show a sample of a shot-noise process made of two indicator functions of squares.

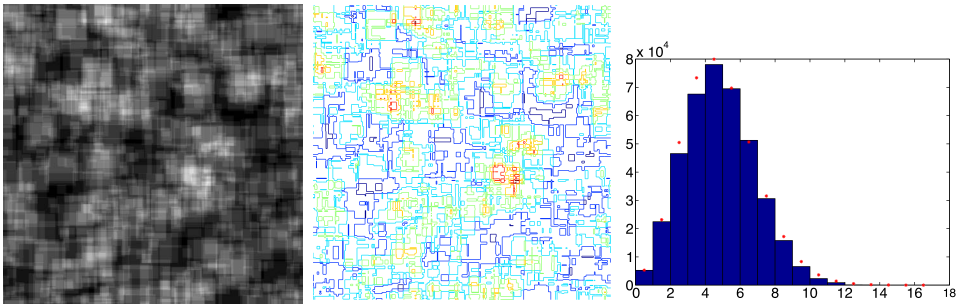

FIG. 1. *We show here on the left a sample* [*in the square domain* $(0, 2000)^2$] *of a shot noise process made of two indicator functions of squares of respective side length* 60 *and* 200, *and with respective probability* 1/2; *with a Poisson point process intensity* $\lambda = 2 \cdot 10^{-4}$. *In the middle we show the boundaries of some of the excursion sets. And on the right we plot the empirical distribution of the perimeter of the excursion sets as a function of the level, together with the expected values of these perimeters* (*red stars*).

3.4. *Link with directional derivatives.* In the general case, when the functions g_m are not piecewise constant, we need to be able to compute the term $\mathbb{E}(e^{iuX_\Phi(0)}\|\nabla X_\Phi(0)\|)$. In order to have an explicit formula for it in terms of the characteristic function of the shot noise [i.e., given by (3.7)], we will first prove the following proposition.

PROPOSITION 2. *Let X and Y be two random variables, such that X is real-valued and Y takes values in $\mathbb{R}^n$, $n \geq 1$. Let ϕ be the joint characteristic function of X and Y given by*

$$\forall u \in \mathbb{R}, \forall v \in \mathbb{R}^n, \qquad \phi(u, v) := \mathbb{E}\big(e^{iuX + i\langle v, Y\rangle}\big).$$

Assume that $\mathbb{E}(\|Y\|^2) < +\infty$. Then

$$\forall u \in \mathbb{R},$$

$$\mathbb{E}\big(e^{iuX}\|Y\|\big) = \frac{-1}{2\pi\omega_{n-1}} \int_{S^{n-1}} \int_0^{+\infty} \frac{1}{t^2}\big(\phi(u, tv) + \phi(u, -tv) - 2\phi(u, 0)\big)\, dt\, \mathcal{H}^{n-1}(dv),$$

where ω_{n-1} is the $\mathcal{L}^{n-1}$ measure of the unit ball of $\mathbb{R}^{n-1}$.

PROOF. We first use the well-known identity

$$\|Y\| = \frac{1}{2\omega_{n-1}} \int_{S^{n-1}} |\langle v, Y\rangle| \mathcal{H}^{n-1}(dv).$$

Now, for any $y \in \mathbb{R}$, we have that

$$\frac{-1}{\pi} \int_0^{+\infty} \frac{1}{t^2}\big(e^{ity} + e^{-ity} - 2\big)\, dt = \frac{2|y|}{\pi} \int_0^{+\infty} \frac{1}{t^2}(1 - \cos(t))\, dt = |y|.$$

We can use this identity for each $\langle v, Y\rangle$, integrate on $v \in S^{n-1}$, multiply by e^{iuX} and finally take the expectation. Then, since for any t and y in $\mathbb{R}$, we have $|e^{ity} + e^{-ity} - 2|/t^2 \le \min(4/t^2, y^2)$, and since $\mathbb{E}(\|Y\|^2) < +\infty$, we obtain by Fubini's theorem that

$$\begin{aligned}
&\mathbb{E}\big(e^{iuX}\|Y\|\big)\\
&\quad = \frac{-1}{2\pi\,\omega_{n-1}} \int_{S^{n-1}} \int_0^{+\infty} \frac{1}{t^2}\big(\mathbb{E}\big(e^{iuX+it\langle v,Y\rangle}\big)\\
&\qquad\qquad + \mathbb{E}\big(e^{iuX-it\langle v,Y\rangle}\big) - 2\mathbb{E}\big(e^{iuX}\big)\big)\,dt\,\mathcal{H}^{n-1}(dv),
\end{aligned}$$

which is the announced result. $\square$

EXAMPLE 2. We give here an example of the use of Proposition 2 in the particular case of radial functions g_m. To have simpler formulas, we will compute only the total variation of the field. We assume here that $n = 2$, that the functions g_m are "cone" functions (with no jumps) given by $g_m(x_1, x_2) = (1 - m\sqrt{x_1^2 + x_2^2})\chi_{B(0,1/m)}(x_1, x_2)$, and that the marks $m \in \mathbb{R}_+$ are distributed with the $\Gamma(3, 1)$ distribution. In this case, the characteristic function of $\nabla X_\Phi(0)$ is given by

$$\begin{aligned}
&\psi(0, v_1, v_2)\\
&\quad = \exp\biggl(\frac{\lambda}{2}\int_{\mathbb{R}_+}\int_{B(0,1/m)} \big(e^{-im(v_1x_1+v_2x_2)/(\sqrt{x_1^2+x_2^2})} - 1\big)m^2e^{-m}\,dx_1\,dx_2\,dm\biggr)\\
&\quad = \exp\biggl(\frac{\lambda}{2}\int_{\mathbb{R}_+}\int_0^{2\pi}\int_0^{1/m} \big(e^{-im(v_1\cos\theta+v_2\sin\theta)} - 1\big)m^2e^{-m}r\,dr\,d\theta\,dm\biggr).
\end{aligned}$$

Therefore, for any $v = (v_1, v_2) \in S^1$ and any $t \in \mathbb{R}$, we have $\psi(0, tv_1, tv_2) = \psi(0, t, 0)$, and we can further compute

$$\begin{aligned}
\psi(0, t, 0) &= \exp\biggl(\frac{\lambda}{2}\int_{\mathbb{R}_+}\int_0^{2\pi}\int_0^{1/m} \big(e^{-imt\cos\theta} - 1\big)m^2e^{-m}r\,dr\,d\theta\,dm\biggr)\\
&= \exp\biggl(\frac{\lambda}{4}\int_{\mathbb{R}_+}\int_0^{2\pi} \big(e^{-imt\cos\theta} - 1\big)e^{-m}\,d\theta\,dm\biggr)\\
&= \exp\biggl(\frac{\lambda}{4}\int_0^{2\pi}\biggl(\frac{1}{1+it\cos\theta} - 1\biggr)d\theta\biggr)\\
&= \exp\biggl(-\frac{\lambda\pi}{2} + \frac{\lambda}{2}\int_0^{\pi}\frac{1}{1+t^2\cos^2\theta}\,d\theta\biggr) = \exp\biggl(-\frac{\lambda\pi}{2} + \frac{\lambda\pi}{2\sqrt{1+t^2}}\biggr).
\end{aligned}$$

Finally, by Theorem 3 and Proposition 2 we can compute the expected total variation of the shot noise random field in $(0, 1)^2$, and we get

$$\begin{aligned}
\mathbb{E}\big(V\big(X_\Phi, (0, 1)^2\big)\big) &= \mathbb{E}\big(\|\nabla X_\phi(0)\|\big)\\
&= \frac{-1}{2}\int_0^{+\infty}\frac{1}{t^2}\big(\psi(0, t, 0) + \psi(0, -t, 0) - 2\psi(0, 0, 0)\big)\,dt
\end{aligned}$$

$$= e^{-(\lambda\pi)/2} \int_0^{+\infty} \frac{e^{(\lambda\pi)/2} - e^{(\lambda\pi)/(2\sqrt{1+t^2})}}{t^2}\,dt$$

$$= \int_0^{\pi/2} \frac{1 - e^{((\lambda\pi)/2)(\cos\alpha - 1)}}{\sin^2\alpha}\,d\alpha.$$

This last integral is related to Bessel functions, and this is not a surprise since Bessel functions are involved in systems that have cylindrical symmetries.

EXAMPLE 3. We consider a shot-noise process in $\mathbb{R}^2$ made of a deterministic function with random amplitude. More precisely, we assume that $n = 2$ and that the marks m are in $\mathbb{R}$ (i.e., $d = 1$) with the distribution $F(dm)$ given by the exponential distribution with parameter 1 so that $\widehat{F}(u) = \frac{1}{1-iu}$. We consider the function $g(x) = g(x_1, x_2) = e^{-x_1}\chi_{\mathbb{R}_+}(x_1)\chi_{[0,1]}(x_2)$ and $g_m = m \times g$. Note that $g \in SBV(\mathbb{R}^2)$ with $J_g = (\{0\} \times [0, 1]) \cup (\mathbb{R}_+ \times \{0\}) \cup (\mathbb{R}_+ \times \{1\})$ and $\nabla g(x_1, x_2) = \binom{-e^{-x_1}}{0}\chi_{\mathbb{R}_+}(x_1)\chi_{[0,1]}(x_2)$. It follows that the joint characteristic function of $X_\Phi(x)$ and $\nabla X_\Phi(x)$ is given by

$$\psi(u, v_1, v_2) = \exp\biggl(\lambda \int_0^{+\infty}\int_0^1 \bigl(\widehat{F}\bigl(e^{-y_1}(u - v_1)\bigr) - 1\bigr)\,dy_1\,dy_2\biggr)$$

$$= \bigl(1 - i(u - v_1)\bigr)^{-\lambda}.$$

Now, we notice that here the gradient ∇g is nonzero only in the x_1 direction, and that moreover it is always nonpositive in that direction. Therefore we have

$$\mathbb{E}\bigl(e^{iuX_\Phi(0)}\|\nabla X_\Phi(0)\|\bigr) = -\mathbb{E}\biggl(e^{iuX_\Phi(0)}\frac{\partial X_\Phi(0)}{\partial x_1}\biggr) = i\frac{\partial\psi}{\partial v_1}(u, 0, 0)$$

$$= \lambda(1 - iu)^{-\lambda-1}.$$

By Theorem 3, it remains to compute the second term corresponding to the jump part to get an explicit expression for $\widehat{C}_{X_\Phi}$. In this example, for $u \neq 0$, we have

$$\frac{1}{iu}\int_{\mathbb{R}^d}\int_{J_{g_m}} \bigl(e^{iug_m^+(y)} - e^{iug_m^-(y)}\bigr)\mathcal{H}^{n-1}(dy)F(dm)$$

$$= \frac{\widehat{F}(u) - 1}{iu} + 2\int_0^{+\infty} \frac{\widehat{F}(ue^{-t}) - 1}{iu}\,dt.$$

Therefore,

$$\widehat{C}_{X_\Phi}(u) = 2\lambda(1 - iu)^{-\lambda-1} - 2\lambda\frac{\log(1 - iu)}{iu}(1 - iu)^{-\lambda}.$$

The second term corresponds to the Fourier transform of the function

$$f(t) = \frac{2\lambda}{\Gamma(\lambda)}\chi_{\mathbb{R}_+}(t)\int_0^t s^{\lambda-1}e^{-s}\bigl(\kappa(\lambda) - \log(s)\bigr)\,ds,$$

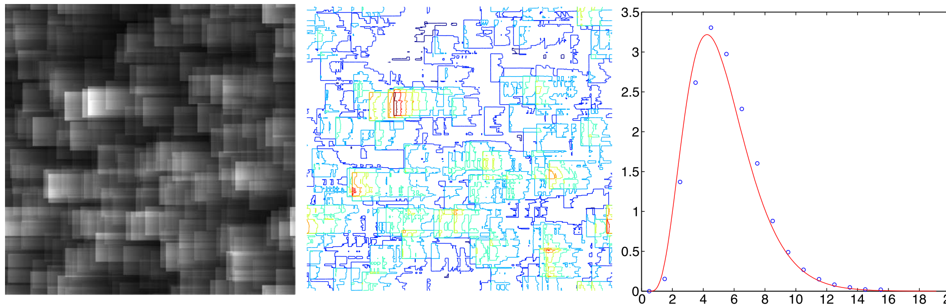

FIG. 2. *In this figure we show a sample of the shot noise random field of example* 3 (*exponential in the horizontal direction*, *with random amplitudes*). *The sample here* (*left image*) *is shown on the square domain* $(0, 10)^2$, *and we have taken* $\lambda = 4$. *In the middle we show the boundaries of some of the excursion sets*, *and on the right we plot the empirical distribution of the perimeter of some excursion sets as a function of the level* (*blue circles*), *together with the expected values of these perimeters* (*red curve*) *given by formula* (3.10).

where κ is the logarithmic derivative of the Γ function, and thus finally we get

$$\text{(3.10)} \qquad \text{for a.e. } t \in \mathbb{R}_+, \qquad C_{X_\Phi}(t) = \frac{2\lambda t^\lambda e^{-t}}{\Gamma(\lambda+1)} + f(t).$$

This example is also illustrated by Figure 2, where we show a sample of such a shot noise random field together with the empirical and the theoretical expected length of its excursion sets.

4. Asymptotic Gaussian behaviour. We assume in this section that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^n} g_m^2(y)\,dy\,F(dm) < +\infty.$$

As the intensity λ of the Poisson point process goes to infinity, it is well known (see, e.g., [14] or [15]) that the shot noise random field converges (after normalization) to a Gaussian random field. More precisely, if we denote by X_λ the shot noise field defined by equation (3.2) with a homogeneous Poisson point process $\{x_i\}_{i\in I}$ of intensity λ, then the random field Z_λ defined by

$$\text{(4.1)} \qquad \forall x \in \mathbb{R}^n, \qquad Z_\lambda(x) = \frac{X_\lambda(x) - \lambda \iint g_m(y)\,dy\,F(dm)}{\sqrt{\lambda}}$$

converges, in the sense of finite dimensional distributions, to a stationary Gaussian random field B of mean 0 and covariance function $\mathbb{E}(B(x)B(0)) = \iint g_m(y-x)g_m(y)\,dy\,F(dm)$. The aim of the following theorem is to give the asymptotic behavior of the mean perimeter of the excursion sets of Z_λ as λ goes to infinity. It shows in particular that when there are no jumps, we have a finite Gaussian asymptotic whereas when there are jumps, the mean perimeters are not bounded anymore and behave like $\sqrt{\lambda}$.

THEOREM 4. *Let* Z_λ *be the normalized shot noise random field defined by equation* (4.1). *Assume that the functions* g_m *satisfying condition* (3.3), *also satisfy the following conditions*:

$$\sigma^2 := \iint g_m^2(y)\, dy\, F(dm) < +\infty, \qquad \sigma_\nabla^2 := \iint \|\nabla g_m(y)\|^2\, dy\, F(dm) < +\infty,$$

and

$$\iint \big(g_m^+(y) - g_m^-(y)\big)\big(|g_m^+(y)| + |g_m^-(y)|\big)\mathcal{H}^{n-1}(dy)\, F(dm) < +\infty.$$

Then we have two different asymptotic behaviors:

(a) *If there are no jumps, that is, if* $\iint (g_m^+(y) - g_m^-(y))\mathcal{H}^{n-1}(dy) F(dm) = 0$, *then as* λ *goes to* $+\infty$, *we have for any fixed* $u \in \mathbb{R}$,

$$\widehat{C}_{Z_\lambda}(u) = \frac{1}{\omega_{n-1}\sqrt{2\pi}} e^{-u^2\sigma^2/2} \int_{S^{n-1}} \sqrt{\iint \langle v, \nabla g_m(y)\rangle^2\, dy\, F(dm)}\, \mathcal{H}^{n-1}(dv) + o(1).$$

(b) *If there are jumps, that is, if* $\iint (g_m^+(y) - g_m^-(y))\mathcal{H}^{n-1}(dy) F(dm) > 0$, *then as* λ *goes to* $+\infty$, *we have for any fixed* $u \in \mathbb{R}$,

$$\widehat{C}_{Z_\lambda}(u) = \sqrt{\lambda} e^{-u^2\sigma^2/2} \left[\left\| \iint \nabla g_m(x)\, dx\, F(dm) \right\| + \iint \big(g_m^+(y) - g_m^-(y)\big)\mathcal{H}^{n-1}(dy)\, F(dm) \right] + o(\sqrt{\lambda}).$$

PROOF. Let us denote $\mu := \iint g_m(y)\, dy\, F(dm)$, and let us recall that

$$\mathbb{E}\big(X_\lambda(0)\big) = \lambda\mu \quad \text{and} \quad \mathrm{Var}\big(X_\lambda(0)\big) = \lambda \iint g_m^2(y)\, dy\, F(dm) = \lambda\sigma^2,$$

and for all $v \in S^{n-1}$,

$$\mathbb{E}\big(\langle v, \nabla X_\lambda(0)\rangle\big) = \lambda \iint \langle v, \nabla g_m(y)\rangle\, dy\, F(dm) =: \lambda\mu_\nabla(v)$$

and

$$\mathrm{Var}\big(\langle v, \nabla X_\lambda(0)\rangle\big) = \lambda \iint \langle v, \nabla g_m(y)\rangle^2\, dy\, F(dm) =: \lambda\sigma_\nabla^2(v).$$

Since $Z_\lambda = (X_\lambda - \lambda\mu)/\sqrt{\lambda}$, the function $\widehat{C}_{Z_\lambda}$ is related to the function $\widehat{C}_{X_\lambda}$ by the relationship

$$\widehat{C}_{Z_\lambda}(u) = \frac{1}{\sqrt{\lambda}} e^{-iu\mu\sqrt{\lambda}} \widehat{C}_{X_\lambda}\left(\frac{u}{\sqrt{\lambda}}\right).$$

And therefore, for $u \neq 0$, we have

$$\begin{aligned}\widehat{C}_{Z_\lambda}(u) &= \frac{1}{\sqrt{\lambda}}\mathbb{E}\big(e^{iu(X_\lambda(0)-\mu\lambda)/\sqrt{\lambda}}\|\nabla X_\lambda(0)\|\big)\\ &\quad + \sqrt{\lambda}\mathbb{E}\big(e^{iu(X_\lambda(0)-\mu\lambda)/\sqrt{\lambda}}\big)\\ &\quad \times \iint \frac{e^{i(u/\sqrt{\lambda})g_m^+(y)} - e^{i(u/\sqrt{\lambda})g_m^-(y)}}{iu/\sqrt{\lambda}}\mathcal{H}^{n-1}(dy)F(dm),\end{aligned}$$

and when $u = 0$, the expected total variation of Z_λ is

$$\begin{aligned}&\mathbb{E}\big(V\big(Z_\lambda, (0,1)^n\big)\big)\\ &\quad = \widehat{C}_{Z_\lambda}(0)\\ &\quad = \frac{1}{\sqrt{\lambda}}\mathbb{E}\big(\|\nabla X_\lambda(0)\|\big) + \sqrt{\lambda}\iint \big(g_m^+(y) - g_m^-(y)\big)\mathcal{H}^{n-1}(dy)F(dm).\end{aligned}$$

We have then two cases, depending on whether there are jumps or not.

We first consider case (a), when there are no jumps, which means that $\iint (g_m^+(y) - g_m^-(y))\mathcal{H}^{n-1}(dy)F(dm) = 0$. Then for F-almost every m, the function g_m is such that its distributional derivative is given by $Dg_m = \nabla g_m \mathcal{L}^n$. And since $g_m \in L^1(\mathbb{R}^n)$, we necessarily have $\int_{\mathbb{R}^n} \nabla g_m(x)\,dx = 0$. [Indeed, a way to see this is to consider the function $t \in \mathbb{R} \mapsto \int_{\mathbb{R}^n} g_m(x + tw)\,dx$ for any vector $w \in \mathbb{R}^n$. This function is constant equal to $\int g_m$, and therefore its derivative at $t = 0$ that is equal to $\int_{\mathbb{R}^n} \langle w, \nabla g_m(x)\rangle\,dx$ is equal to 0.] We then have, using the result of Proposition 2 and the change of variable $t \to t/\sqrt{\lambda}$,

$$\begin{aligned}\widehat{C}_{Z_\lambda}(u) &= \frac{1}{\sqrt{\lambda}}\mathbb{E}\big(e^{iu(X_\lambda(0)-\mu\lambda)/\sqrt{\lambda}}\|\nabla X_\lambda(0)\|\big)\\ &= \frac{-e^{-iu\mu\sqrt{\lambda}}}{2\pi\omega_{n-1}}\int_{S^{n-1}}\int_0^{+\infty}\frac{1}{t^2}\bigg(\psi_\lambda\bigg(\frac{u}{\sqrt{\lambda}}, \frac{tv}{\sqrt{\lambda}}\bigg)\\ &\qquad + \psi_\lambda\bigg(\frac{u}{\sqrt{\lambda}}, -\frac{tv}{\sqrt{\lambda}}\bigg) - 2\psi_\lambda\bigg(\frac{u}{\sqrt{\lambda}}, 0\bigg)\bigg)\,dt\,dv,\end{aligned}$$

where

$$\psi_\lambda\bigg(\frac{u}{\sqrt{\lambda}}, \frac{tv}{\sqrt{\lambda}}\bigg) = \mathbb{E}\big(e^{i(u/\sqrt{\lambda})X_\lambda(0) + i(t/\sqrt{\lambda})\langle v, \nabla X_\lambda(0)\rangle}\big).$$

On the one hand, using formula (3.7) for ψ_λ, we have for any fixed t, u and v,

$$e^{-iu\mu\sqrt{\lambda}}\psi_\lambda\bigg(\frac{u}{\sqrt{\lambda}}, \frac{tv}{\sqrt{\lambda}}\bigg) \underset{\lambda\to+\infty}{\longrightarrow} e^{-u^2\sigma^2/2 - t^2\sigma_\nabla^2(v)/2}.$$

And on the other hand, using the fact that $|e^{ix}+e^{-ix}-2|\le \min(4,x^2)$ for any x real, we get

$$\left|\psi_\lambda\left(\frac{u}{\sqrt{\lambda}},\frac{tv}{\sqrt{\lambda}}\right)+\psi_\lambda\left(\frac{u}{\sqrt{\lambda}},-\frac{tv}{\sqrt{\lambda}}\right)-2\psi_\lambda\left(\frac{u}{\sqrt{\lambda}},0\right)\right| \le \min\left(4,t^2\frac{1}{\lambda}\mathbb{E}\left(\langle v,\nabla X_\lambda(0)\rangle^2\right)\right).$$

Now since $\mathbb{E}(\langle v,\nabla X_\lambda(0)\rangle)=0$, we have $\mathbb{E}(\langle v,\nabla X_\lambda(0)\rangle^2)=\mathrm{Var}(\langle v,\nabla X_\lambda(0)\rangle)=\lambda\sigma_\nabla^2(v)\le\lambda\sigma_\nabla^2$. Therefore, we can use the dominated convergence theorem and obtain

$$\widehat{C}_{Z_\lambda}(u)\underset{\lambda\to+\infty}{\longrightarrow}\frac{-e^{-u^2\sigma^2/2}}{2\pi\omega_{n-1}}\int_{S^{n-1}}\int_0^{+\infty}\frac{1}{t^2}\left(2e^{-t^2\sigma_\nabla^2(v)/2}-2\right)dt\,dv = \frac{e^{-u^2\sigma^2/2}}{\omega_{n-1}\sqrt{2\pi}}\int_{S^{n-1}}\sqrt{\sigma_\nabla^2(v)}\,dv,$$

which completes the proof of case (a).

For case (b), when there are jumps, which means that $\iint(g_m^+(y)-g_m^-(y))\times\mathcal{H}^{n-1}(dy)F(dm)$ is strictly positive, then $\iint\nabla g_m(x)\,dx\,F(dm)$ is not necessarily equal to 0 anymore. In this case, we will consider $\widehat{C}_{Z_\lambda}(u)/\sqrt{\lambda}$, and we will show that it converges to a finite strictly positive limit. To begin with, we use again Proposition 2 and the change of variable $t\mapsto t/\lambda$, to have

$$\frac{1}{\lambda}\mathbb{E}\left(e^{iu(X_\lambda(0)-\mu\lambda)/\sqrt{\lambda}}\|\nabla X_\lambda(0)\|\right) = \frac{-e^{-iu\mu\sqrt{\lambda}}}{2\pi\omega_{n-1}}\int_{S^{n-1}}\int_0^{+\infty}\frac{1}{t^2}\left(\psi_\lambda\left(\frac{u}{\sqrt{\lambda}},\frac{tv}{\lambda}\right)+\psi_\lambda\left(\frac{u}{\sqrt{\lambda}},-\frac{tv}{\lambda}\right)-2\psi_\lambda\left(\frac{u}{\sqrt{\lambda}},0\right)\right)dt\,dv.$$

Now, on the one hand, for any fixed t, u and v we have

$$e^{-iu\mu\sqrt{\lambda}}\psi_\lambda\left(\frac{u}{\sqrt{\lambda}},\frac{tv}{\lambda}\right)\underset{\lambda\to+\infty}{\longrightarrow}e^{-u^2\sigma^2/2+it\mu_\nabla(v)}.$$

On the other hand, we have

$$\left|\psi_\lambda\left(\frac{u}{\sqrt{\lambda}},\frac{tv}{\lambda}\right)+\psi_\lambda\left(\frac{u}{\sqrt{\lambda}},-\frac{tv}{\lambda}\right)-2\psi_\lambda\left(\frac{u}{\sqrt{\lambda}},0\right)\right| \le \min\left(4,t^2\frac{1}{\lambda^2}\mathbb{E}\left(\langle v,\nabla X_\lambda(0)\rangle^2\right)\right).$$

Now, here $\mathbb{E}(\langle v,\nabla X_\lambda(0)\rangle^2)=\mathrm{Var}(\langle v,\nabla X_\lambda(0)\rangle)+\mathbb{E}(\langle v,\nabla X_\lambda(0)\rangle)^2=\lambda\sigma_\nabla^2(v)+\lambda^2\mu_\nabla(v)^2$. Therefore, we can again use the dominated convergence theorem and get

$$
\begin{aligned}
&\frac{1}{\lambda}\mathbb{E}\big(e^{iu(X_\lambda(0)-\mu\lambda)/\sqrt{\lambda}}\|\nabla X_\lambda(0)\|\big)\\
&\quad = \frac{-e^{-u^2\sigma^2/2}}{2\pi\,\omega_{n-1}}\int_{S^{n-1}}\int_0^{+\infty}\frac{1}{t^2}\big(e^{it\mu_\nabla(v)}+e^{-it\mu_\nabla(v)}-2\big)\,dt\,dv+o(1)\\
&\quad = \frac{e^{-u^2\sigma^2/2}}{2\omega_{n-1}}\int_{S^{n-1}}\big|\mu_\nabla(v)\big|\,dv+o(1)\\
&\quad = e^{-u^2\sigma^2/2}\bigg\|\iint \nabla g_m(x)\,dx\,F(dm)\bigg\|+o(1).
\end{aligned}
$$

For the jump part, we use the inequality

$$
\begin{aligned}
&\bigg|\frac{e^{i(u/\sqrt{\lambda})g_m^+(y)}-e^{i(u/\sqrt{\lambda})g_m^-(y)}}{iu/\sqrt{\lambda}}-\big(g_m^+(y)-g_m^-(y)\big)\bigg|\\
&\qquad \le \frac{|u|}{2\sqrt{\lambda}}\big(g_m^+(y)-g_m^-(y)\big)\big(\big|g_m^+(y)\big|+\big|g_m^-(y)\big|\big)
\end{aligned}
$$

and the fact that

$$
\mathbb{E}\big(e^{iu(X_\lambda(0)-\mu\lambda)/\sqrt{\lambda}}\big)\underset{\lambda\to+\infty}{\longrightarrow} e^{-u^2\sigma^2/2}
$$

to obtain, thanks to the hypothesis in the statement of the theorem, that

$$
\begin{aligned}
&\mathbb{E}\big(e^{iu(X_\lambda(0)-\mu\lambda)/\sqrt{\lambda}}\big)\iint\frac{e^{i(u/\sqrt{\lambda})g_m^+(y)}-e^{i(u/\sqrt{\lambda})g_m^-(y)}}{iu/\sqrt{\lambda}}\mathcal{H}^{n-1}(dy)F(dm)\\
&\quad = e^{-u^2\sigma^2/2}\iint\big(g_m^+(y)-g_m^-(y)\big)\mathcal{H}^{n-1}(dy)F(dm)+o(1).
\end{aligned}
\tag{4.2}
$$

□

Using additional assumptions on the order-three moments of X_λ and ∇X_λ, it is possible to obtain bounds of convergence for $\widehat{C}_{Z_\lambda}(u)$ in a way similar to the technical result of our previous paper [7] in the framework of smooth functions g_m.

EXAMPLE 4. Assume we are considering a shot noise random field on the plane (i.e., $n=2$) and such that there are no jumps. Then if we denote for $i, j = 1$ or 2,

$$
\gamma_{ij}=\int_{\mathbb{R}^d}\int_{\mathbb{R}^2}\frac{\partial g_m}{\partial x_i}(x)\frac{\partial g_m}{\partial x_j}(x)\,dx\,F(dm),
$$

then

$$
\widehat{C}_{Z_\lambda}(u)\underset{\lambda\to+\infty}{\longrightarrow}\frac{1}{2\sqrt{2\pi}}e^{-u^2\sigma^2/2}\int_0^{2\pi}\sqrt{\gamma_{11}\cos^2\theta+\gamma_{22}\sin^2\theta+2\gamma_{12}\cos\theta\sin\theta}\,d\theta.
$$

This shows that we have a weak convergence of $C_{Z_\lambda}(t)$ to the formula for the length of level curves in the Gaussian case (i.e., exactly the formula obtained through Rice formula and probability density functions of Gaussian fields in [4]).

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