Relationship between the fractal dimension of a fractional Brownian motion and its projection

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Aim

- To present the relationship between the fractal dimension of nD fractional Brownian Motion (fBm) and that of its (n-1)D projection.
- To test this relationship on 2D synthesized fBm images.
- To apply it on 3D bone images.

Relationship between box-counting dimension of fBm and its projections

A nD fBm of H index, B_H, is characterized by a self-similarity property, for all scale λ : $\{B_H(\lambda t); t \in R^n\} \stackrel{(d)}{=} \{\lambda^H B_H(t); t \in R^n\}$

Since B^H is a zero mean Gaussian field with stationary increments, its distribution is completely determined by its variogram v :

$$v(t) = \frac{1}{4} E\left((B_H(t) - B_H(0))^2\right) = C_{H,n} \|t\|^{2H}.$$

We are looking for the box-counting dimension of the fBm graph, $\dim_B B_H$. Let $G = \{ (t, B_H(t)) ; t \in [0,1]^n \} \subset R^{n+1}$ be the graph of the fBm.

We call $N_\delta(G)$ the smallest number of sets of diameter at most δ which can cover G. As we have, for small δ,

$$N_\delta(B_H) \approx C\delta^{-D},$$

the box-counting dimension of G is defined by $\dim_B B_H = D$.

The behaviour of the fBm variogram at small scales gives its box-counting dimension :

$$v(t) \le C_{H,n} \|t\|^{2H} \Rightarrow \dim_B B_H \le n+1-H \text{ a.s. and } v(t) \ge C_{H,n} \|t\|^{2H} \Rightarrow \dim_B B_H \ge n+1-H \text{ a.s.},$$

then

$$\dim_B B_H = n+1-H \text{ a.s.}$$

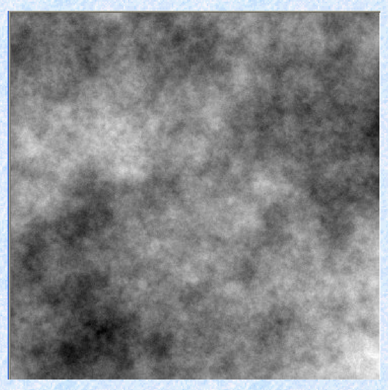
Let now have a look to the windowed X-ray transform of B_H. Fix $\alpha \in S^{n-1}$ a direction in R^n, we can define a Gaussian field with stationary increments $P_\alpha B_H = \{P_\alpha B_H(s) ; s \in <\alpha>^\perp\}$. For $s \in <\alpha>^\perp$, we average B_H with a suitable window φ along the line $\{s+p\alpha ; p \in R\}$ and denote the result $P_\alpha B_H(s)$. There exists two positive constants $C_1(\alpha)$ and $C_2(\alpha)$ such that the variogram v_P of $P_\alpha B_H$ satisfies, for small $s \in <\alpha>^\perp$,

$$C_1(\varphi)\|s\|^{2\min(1,H+1/2)} \le v_P(s) \le C_2(\varphi)\|s\|^{2\min(1,H+1/2)}.$$

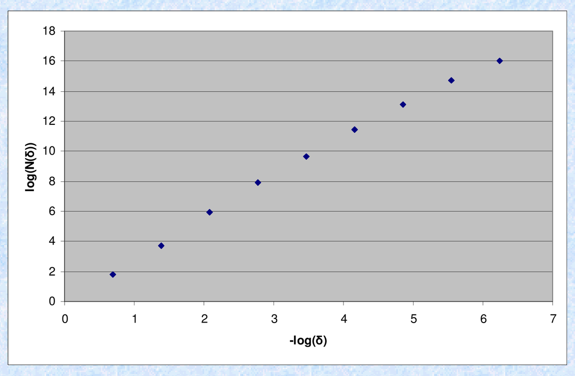
Thus, $\dim_B P_\alpha B_H = \dim <\alpha>^\perp + 1 - \min(1,H+1/2) = n - \min(1,H+1/2)$.

Application to 2D synthesized fBm

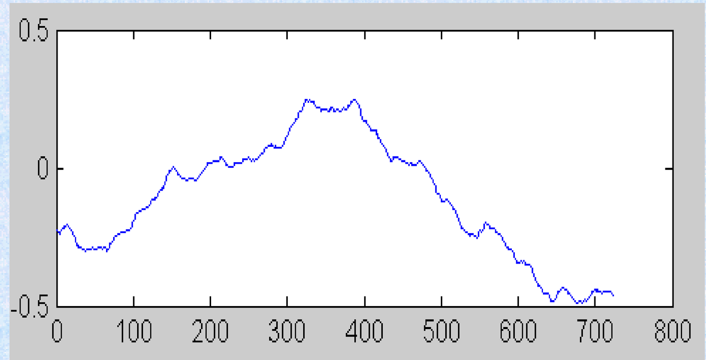
We synthesized 2D fBm images with a size of 725 × 725 pixels and performed their projection following the vertical axis. Points are regularly spaced with a step of 2^{-10}. We then estimated their box-counting dimensions.



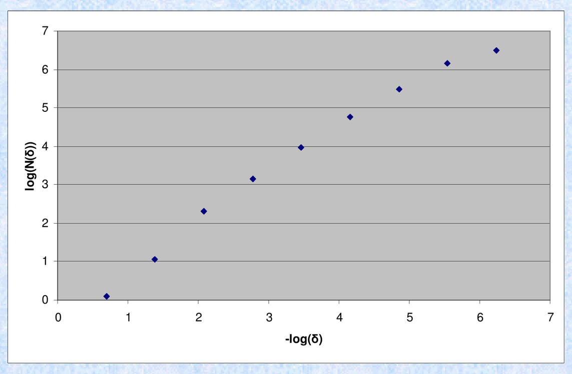
2D fBm (H=0.2).



Plots obtained by the box-counting method.



1D signal of its projection.

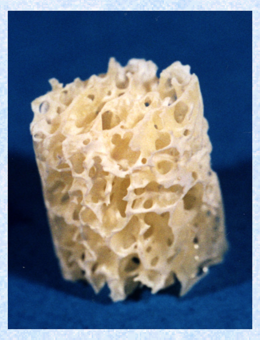


	H=0,2	H=0,2	H=0,3	H=0,3	H=0,4	H=0,4
$\mathrm{Dim}_B B_H = 2+1-H$	2,7605	2,768	2,6967	2,6964	2,622	2,603
$\mathrm{Dim}_B PB_H = 1+1-(H+1/2)$	1,3556	1,2728	1,3427	1,2906	1,1736	1,2155
$\mathrm{Error} = 3/2 - \mathrm{Dim}_B B_H + \mathrm{Dim}_B PB_H$	0,0951	0,0048	0,146	0,0942	0,0516	0,1125

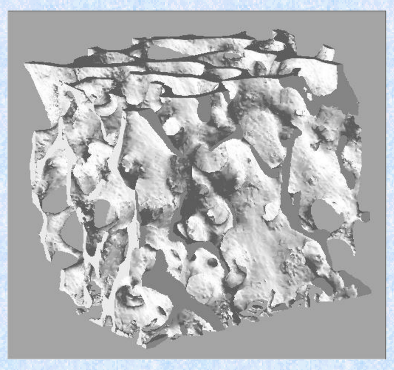
We estimated the dimensions by regression of $\log(N_\delta)$ for $\delta = 2^{-4}, 2^{-5}, 2^{-6}, 2^{-7}$. The fractal dimensions evaluated suffer from excessive bias. It is probably a consequence of the fact that $N_\delta \approx C\delta^{-D}$ for a constant C, rather than $N_\delta \approx \delta^{-D}$.

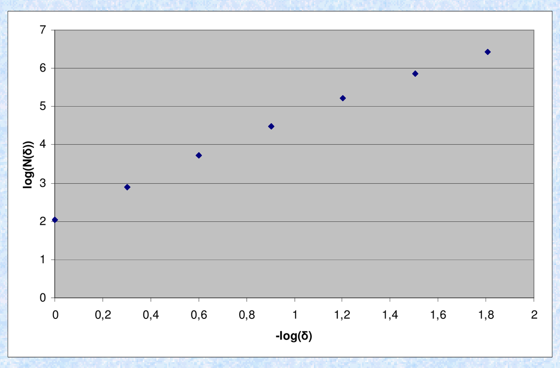
Application to 3D bone images

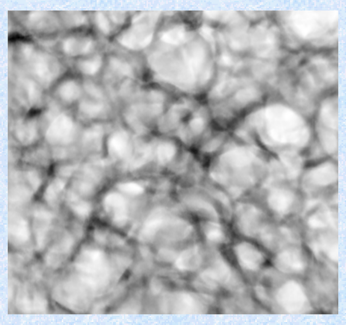
Human bone is composed of trabecular bone whose modifications could lead to osteoporotic fractures. It is a porous material ordered in trabeculae of 100 µm on average thickness and separated of 300 to 500 µm on average. Direct 3D evaluation of trabecular bone structure could not be done in clinical practice due to the complexity, price, low resolution and accessibility of the devices. So the main alternative to the 3D method is the conventional bone radiography. It would be very interesting to link 2D parameters measured on radiographs with 3D ones.

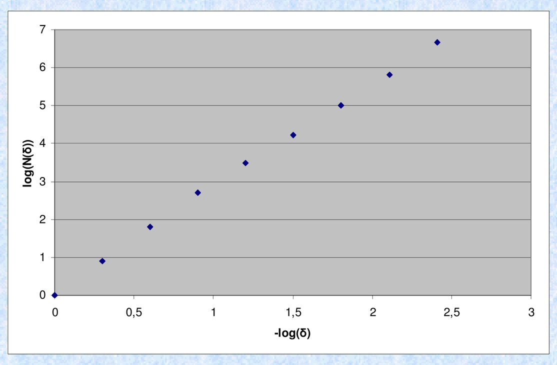


The bone volume is given by a binary random field $B:[0,1]^3 \to \{0,1\}$. We can define the random set $K=\{t \in [0,1]^3 ; B(t)=1\} \subset R^3$. We assume that K is a self-similar set of index $H < 1/2$ such that $\dim_B K = 3-H$. We obtain a radiograph of the bone by performing a windowed X-ray transform in the direction $\alpha \in S^2$ of B denoted $P_\alpha B = \{P_\alpha B(s) ; s \in <\alpha>^\perp\}$. We conjecture the graph of $P_\alpha X$ has a box-counting dimension $\dim_B P_\alpha B = 2+1-(H+1/2)$.









We tested the relation on 8 bone samples. The images were acquired with a high resolution micro tomograph and have a size of 256 pixels³ with a pixel size of 12 µm. We computed the projections by just summing the image following the height axis.

$\mathrm{Dim}_B B = 3-H$	2,6772	2,6538	2,6281	2,8047	2,7057	2,6961	2,5694	2,8398
$\mathrm{Dim}_B PB = 2+1-(H+1/2)$	2,366	2,2994	2,2792	2,2988	2,2676	2,2884	2,2999	2,341
$\mathrm{Error} = 1/2 - \mathrm{Dim}_B B + \mathrm{Dim}_B PB$	0,1888	0,1456	0,1511	-0,0059	0,0619	0,0923	0,2305	0,0012