Fractional Poisson fields

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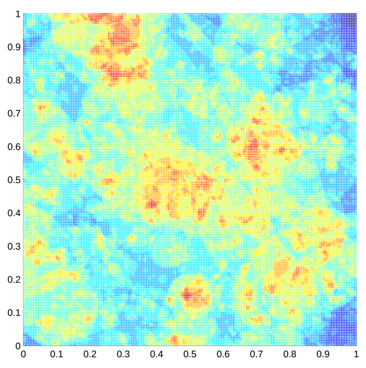
Random balls models

Slice by slice construction

Slice number $j \in \mathbb{Z}$:
a family of grains $X_j^n + B(0, R_j^n)$ in $\mathbb{R}^d$ built up from a Poisson point process $(X_j^n, R_j^n)_n$ in $\mathbb{R}^d \times (2^{-j-1}, 2^{-j})$.

EQUIV: a Poisson random measure N_j on $\mathbb{R}^d \times \mathbb{R}^+$ with intensity measure $\mathrm{d}x \otimes r^{-\beta-1}\mathbf{1}_{(2^{-j-1},2^{-j})}(r)\mathrm{d}r$.

Random point $(x, r) \in \mathbb{R}^d \times \mathbb{R}^+ \curvearrowright$ random ball $B(x, r)$



Associated shot-noise random field

We define the slice random field T_j on $\mathbb{R}^d$ by

$$\begin{aligned} T_j(y) &= \sum_n \mathbf{1}_{B(X_j^n, R_j^n)}(y) \\ &= \# \text{ grains containing } y \in \mathbb{R}^d \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^+} \mathbf{1}_{B(x,r)}(y) N_j(dx, dr). \end{aligned}$$

d $= 1 \rightarrow$ numbers of connections to a server at time y
d $= 2 \rightarrow$ discretized gray level at point y in a picture
d $= 3 \rightarrow$ mass density of a 3D granular media in y

We also define the centered piling up random field

$$F_{j_{min},j_{max}}(y) = \sum_{j=j_{min}}^{j_{max}} \left(T_j(y) - T_j(0)\right) \;,\; (j_{min}, j_{max}) \in \mathbb{Z}^2 .$$

Simulation on $[0,1]^d$

We propose to simulate our field on $[0,1]^d$. Note that for $j \in \mathbb{Z}$ balls with centers not in $[-1-2^{-j}, 2+2^{-j}]^d$ do not contribute to $\left\{T_j(y); y \in [0,1]^d\right\}$. Moreover,

$$\#\left\{(X_j^n, R_j^n); X_j^n \in [-1-2^{-j}, 2+2^{-j}]^d\right\} \sim \mathcal{P}(2^{-j\beta} c_\beta d_j),$$

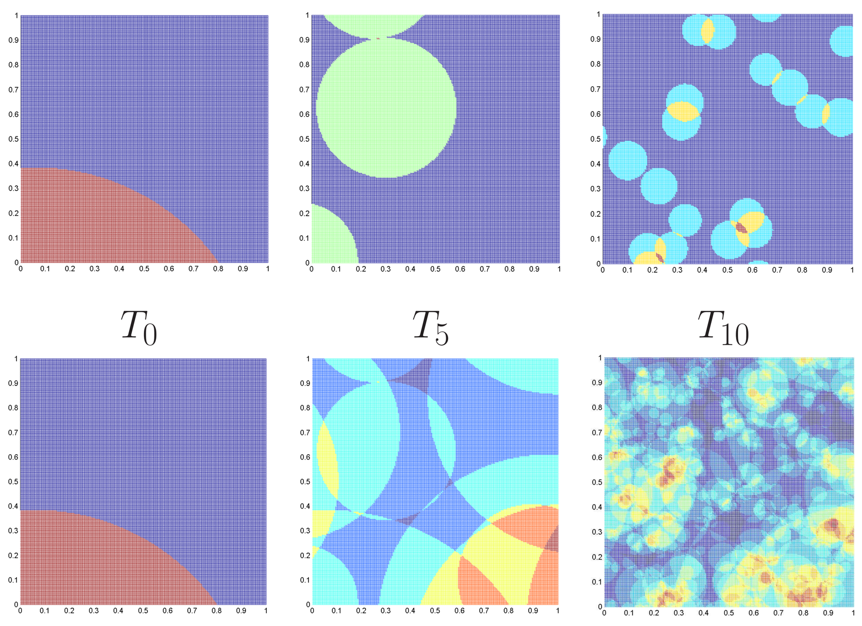
with $c_\beta = (2^\beta - 1)\beta^{-1}$ and $d_j = (3 + 2^{-j+1})^d$.

We consider

- (ν_j) a family of independent Poisson rv with parameter $2^{-j\beta} c_\beta d_j$;
- (X_j^n) a family of independent rv with uniform law on $[-1-2^{-j}, 2+2^{-j}]^d$;
- (R_j^n) a family of independent rv with law $2^{j\beta} c_\beta^{-1} r^{-\beta-1}\mathbf{1}_{(\alpha^{j+1},\alpha^j]}(r)\,dr$, simulated with the pseudo-inverse method: $R_j^n = 2^{-j}(2^\beta - (2^\beta - 1)U[0,1])^{-1/\beta}$).

Then

$$\left\{F_{j_{min},j_{max}}(y) \,;\, y \in [0,1]^d\right\} \overset{fdd}{=} \left\{\sum_{j=j_{min}}^{j_{max}} \left(\sum_{n=1}^{\nu_j} \left((\mathbf{1}_{B(X_j^n, R_j^n)}(y) - \mathbf{1}_{B(X_j^n, R_j^n)}(0)\right)\right) \,;\, y \in [0,1]^d\right\}.$$



T_0 T_5 T_{10}

$F_{0,0}$ $F_{0,5}$ $F_{0,10}$

Fractional Poisson Field

Letting $jmin, jmax$ going to $-\infty, +\infty$ we get

$$F_{j_{min},j_{max}}(y) \longrightarrow F(y) = \sum_{j \in \mathbb{Z}} \left(T_j(y) - T_j(0)\right)$$

where the field $\{F(y); y \in \mathbb{R}^d\}$ may be expressed as

$$F(y) = \int_{\mathbb{R}^d \times \mathbb{R}^+} \left(\mathbf{1}_{B(x,r)}(y) - \mathbf{1}_{B(x,r)}(0)\right) N(\mathrm{d}x, \mathrm{d}r) \tag{1}$$

with N a Poisson random measure of intensity $\mathrm{d}x \otimes r^{-\beta-1}\,\mathrm{d}r$.

F is clearly not Gaussian but shares the same covariance function as the Fractional Brownian Field

$$\mathrm{cov}(F(y), F(y')) = 1/2 \left(||y||^{d-\beta} + ||y'||^{d-\beta} - ||y - y'||^{d-\beta}\right) \tag{2}$$

(1) and (2) yield calling F as **Fractional Poisson Field**

Normal convergence

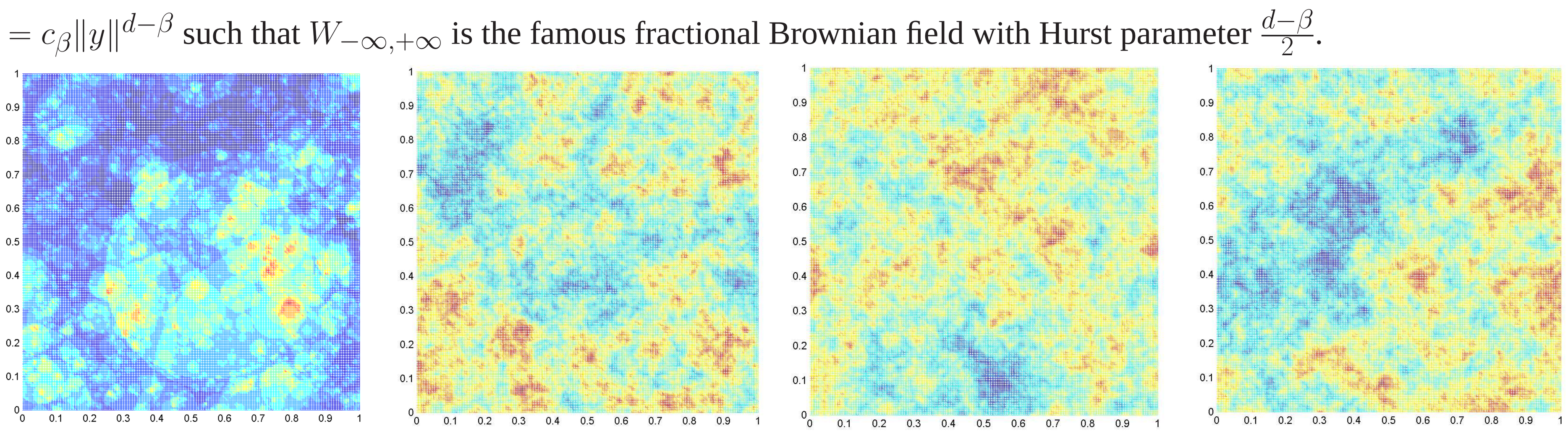
Let $\left(F^k_{j_{min},j_{max}}\right)_k$ iid copies of $F_{j_{min},j_{max}}$. According to the central limit theorem

$$\left\{\frac{1}{\sqrt{K}} \sum_{k=1}^{K} F^k_{jmin,jmax}(y) \,;\, y \in \mathbb{R}^d\right\} \xrightarrow[K \to +\infty]{fdd} \left\{W_{j_{min},j_{max}}(y) \,;\, y \in \mathbb{R}^d\right\},$$

where $W_{j_{min},j_{max}}$ is a centered Gaussian process with stationary increments and covariance function

$$\mathrm{Cov}\left(W_{j_{min},j_{max}}(y), W_{j_{min},j_{max}}(y')\right) = \frac{1}{2}\left(v_{j_{min},j_{max}}(y) + v_{j_{min},j_{max}}(y') - v_{j_{min},j_{max}}(y - y')\right), \text{ with } \; v_{j_{min},j_{max}}(y) = \int_{\mathbb{R}^d \times (2^{-j_{max}-1}, 2^{-j_{min}})} \mathbf{1}_{B(y,r)\Delta B(0,r)}(x) r^{-\beta-1} dx dr.$$

In particular for $\beta \in (d-1, d)$, we get $v_{-\infty,+\infty}(y) = c_\beta ||y||^{d-\beta}$ such that $W_{-\infty,+\infty}$ is the famous fractional Brownian field with Hurst parameter $\frac{d-\beta}{2}$.



$K = 1$ $K = 5$ $K = 10$ $K = 20$