Anisotropic texture modeling and applications to mammograms

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Fractional Brownian fields

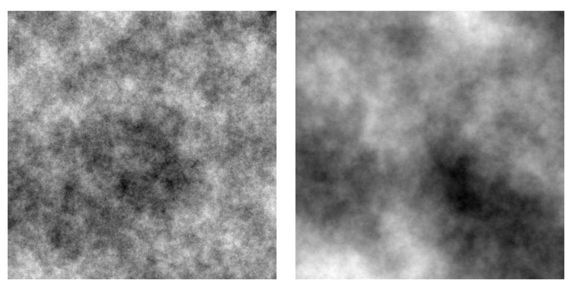
Kolmogorov (1940), Mandelbrot et Van Ness (1968)

For $H \in (0,1)$, the fractional Brownian motion $B_H = \left\{B_H(x); x \in \mathbb{R}^d\right\}$ of Hurst parameter H is the only Gaussian centered field that vanishes a.s. at 0

- with stationary increments: $\forall x_0 \in \mathbb{R}^d, \left\{B_H(x+x_0) - B_H(x_0); x \in \mathbb{R}^d\right\} \overset{fdd}{=} \left\{B_H(x) - B_H(0); x \in \mathbb{R}^d\right\}$;
- self-similar of order H: for all $\lambda > 0$, $\left\{B_H(\lambda x); x \in \mathbb{R}^d\right\} \overset{fdd}{=} \lambda^H \left\{B_H(x); x \in \mathbb{R}^d\right\}$;
- isotropic: for all rotation R, $\left\{B_H(Rx); x \in \mathbb{R}^d\right\} \overset{fdd}{=} \left\{B_H(x); x \in \mathbb{R}^d\right\}$.

Restriction along straight lines: for all direction $\theta \in S^{d-1}$

$$\{B_H(x_0 + t\theta) - B_H(x_0); t \in \mathbb{R}\} = \text{ fractional Brownian motion of order } H.$$



$H = 0.3$ $H = 0.7$

matlab code on `http://ciel.ccsd.cnrs.fr`
Fast and exact simulation of fractional Brownian surface, M. L. Stein, 2002

Hurst parameter

- H= Hurst parameter linked with
  - self-similarity order
  - Hölder regularity of sample paths ➥ Numerous estimators of H.
  - fractal dimension of graph
- The law is characterized by the variogram $v(x) = \text{Var}(B_H(x)) = c_{H,d}\|x\|^{2H}$.
- **Spectral representation** $v(x) = \int_{\mathbb{R}^d} \left|e^{-ix\cdot\omega} - 1\right|^2 \|\omega\|^{-2H-d} d\xi$ ➥ **spectral density** = $\|\omega\|^{-2H-d}$.

1D Estimation

Generalized quadratic variations (Istas, Lang, 97)

$$V_u := \frac{1}{r-2u+1}\sum_{t=0}^{r-2u}\left(B_H\left(\frac{t+2u}{r}\right) - 2B_H\left(\frac{t+u}{r}\right) + B_H\left(\frac{t}{r}\right)\right)^2$$

➥ $\mathbb{E}(V_u) = c_H r^{-2H} u^{2H}$.

$$\boxed{\widehat{H} = \frac{1}{2\log(2)}\log\left(\frac{V_2}{V_1}\right) \underset{r\to+\infty}{\longrightarrow} H \text{ a.s.} + \text{ asymptotic normality.}}$$

Anisotropic fractional Brownian fields

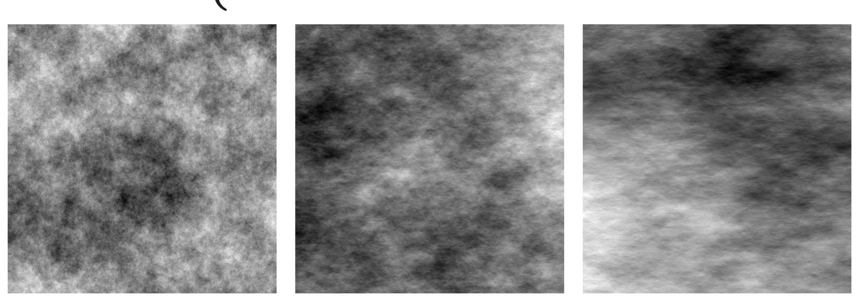
Bonami and Estrade, 2003

An anisotropic fractional Brownian field $X = \left\{X(x); x \in \mathbb{R}^d\right\}$ is a Gaussian centered field that vanishes a.s. at 0 with stationary increments and variogram

$$v(x) = \text{Var}(X(x)) = \int_{\mathbb{R}^d}\left|e^{-ix\cdot\omega} - 1\right|^2 f(\omega) d\omega, \quad \text{with} \quad f(\omega) = \|\omega\|^{-2h(\theta)-d},$$

where h is a Hurst parameter which depends on the direction $\theta = \arg(\omega)$.

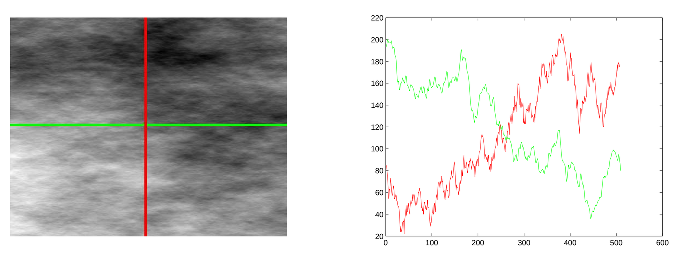
Example: AFBF (h_1, h_2) with $f(\omega) = \begin{cases} \|\omega\|^{-2h_1-2}, & \text{if } |\omega_2| < |\omega_1| \\ \|\omega\|^{-2h_2-2}, & \text{else.} \end{cases}$



$h_1 = h_2 = 0.3$ $h_1 = 0.5, h_2 = 0.3$ $h_1 = 0.7, h_2 = 0.3$

Hölder regularity

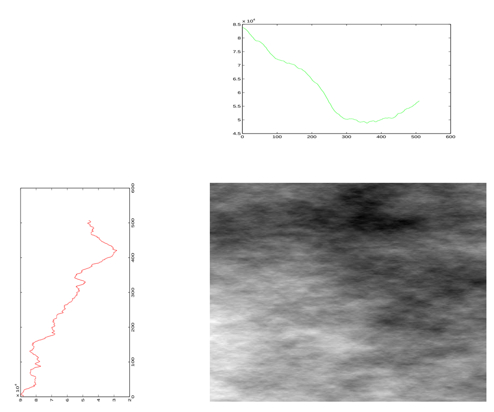
Restriction along straight lines



for all direction $\theta \in S^{d-1}$ $\{X(x_0 + t\theta); t \in \mathbb{R}\}$ has the same Hölder regularity given by

$$\boxed{H = \min_{\theta} h(\theta).}$$

Projections along hyperplanes



for all direction $\theta \in S^{d-1}$ the process $R_\theta X$ obtained by projection of X on $\langle\theta\rangle$ along $\langle\theta\rangle^\perp$ has Hölder regularity given by

$$\boxed{h(\theta) + \frac{d-1}{2}.}$$

Methodology

- We consider an image $I(n,m)$ as a realization of an AFBF (h_1, h_2) on a grid: $\left\{X\left(\frac{n}{r}, \frac{m}{r}\right); 0 \le n, m \le r-1\right\}$
- Oriented fractal analysis: Quadratic variations of $I(.,m)$ $\widehat{h_{01}} \to H$ and of $I(n,.)$ $\widehat{h_{02}} \to H$
- Projections of the image
  - on the horizontal direction $R_1(n) = \frac{1}{r}\sum_m I(n,m) \approx \int_0^1 X\left(\frac{n}{r}, y\right) dy$
  - on the vertical direction $R_2(m) = \frac{1}{r}\sum_n I(n,m) \approx \int_0^1 X\left(y, \frac{m}{r}\right) dy$
- Quadratic variations of subsamples $(R_1(2^\nu n))$ and $(R_2(2^\nu m))$

$$\boxed{\widehat{h_1}^{\nu} \longrightarrow h_1 \text{ and } \widehat{h_2}^{\nu} \longrightarrow h_2 \text{ a.s. + asymptotic normality}}$$

Anisotropy Tests

First test: $\mathcal{H}_0$: $h_1 = h_2$ (isotropy) against $\mathcal{H}_1$: $h_1 \neq h_2$ (anisotropy). Statistic $\widehat{d}^\nu = \left|\widehat{h}_1^\nu - \widehat{h}_2^\nu\right|$. Rejection interval at the level $\alpha = 5\%$: $\mathcal{R}^\nu = \{\widehat{d}^\nu > 1.96\sigma^\nu\}$ with σ^ν empirical standard deviation of d^ν under $\mathcal{H}_0$

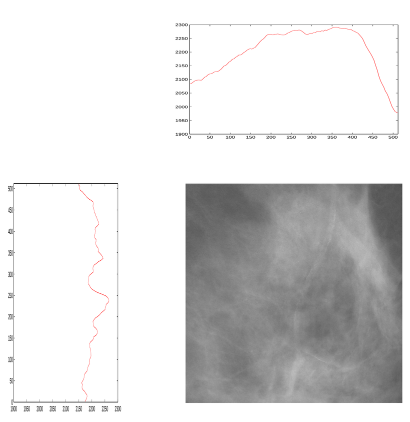
$$\boxed{\mathcal{R}^0 = \{\widehat{d}^0 > 0.16\}}$$

Second test: $\mathcal{H}_0$: $h_1 = h_2 = H$ (isotropy) against $\mathcal{H}_1$: $h_1 \neq H$ or $h_2 \neq H$ (anisotropy). Statistic $\widehat{\delta}^\nu = \left|\max(\widehat{h}_1^\nu, \widehat{h}_2^\nu) - \min(\widehat{h_{01}}, \widehat{h_{02}})\right|$. Rejection interval at the level $\alpha = 5\%$: $\mathcal{R}^\nu = \{\widehat{\delta}^\nu > c^\nu\}$ with c^ν empirically determined under $\mathcal{H}_0$

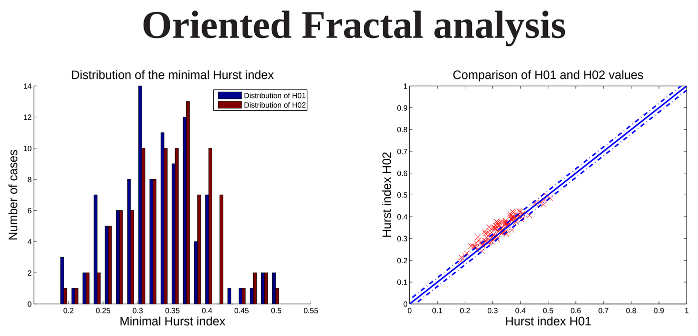
$$\boxed{\mathcal{R}^2 = \{\widehat{\delta}^2 > 0.2\}}$$

Application to mammograms

Analysis of ROI

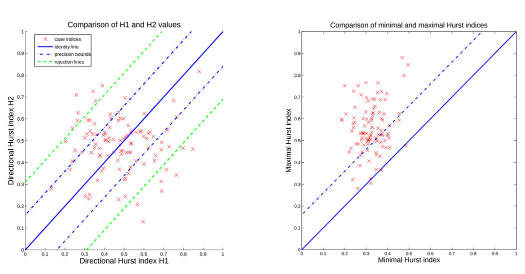


Oriented Fractal analysis



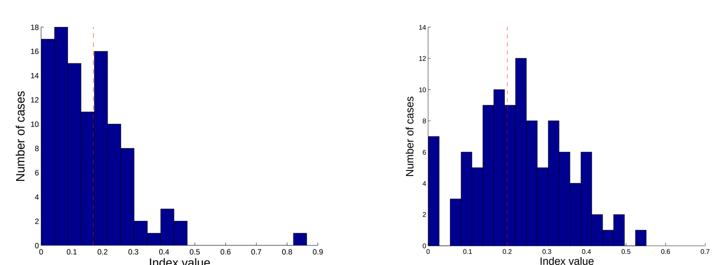
$\widehat{H} = \min(\widehat{h_{01}}, \widehat{h_{02}}) = 0.31 \pm 0.05$
(58 cases, 116 images)
Heine et al, 2002 $H \in [0.33, 0.42]$

Anisotropy of mammograms



$(\widehat{h}_1^2, \widehat{h}_2^2)$ $(\widehat{H}, \max(\widehat{h}_1^2, \widehat{h}_2^2))$

Anisotropy test



$\widehat{d}^0 = |\widehat{h}_1^0 - \widehat{h}_2^0|$ $\widehat{\delta}^2 = \left|\max(\widehat{h}_1^2, \widehat{h}_2^2) - \widehat{H}\right|$

43% Anisotropic 60% Anisotropic