SCALING PROPERTIES OF POISSON GERM-GRAIN MODELS WITH POWER-LAW GRAIN SIZE

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Abstract. We study properties of certain limiting random systems that arise by aggregation of spherical grains which are uniformly scattered according to a Poisson point process in d-dimensional space. The grains have random radius, independent and identically distributed, with a distribution which is assumed to have a power law behavior either in zero or at infinity. The resulting configurations of mass suitably centered and normalized are known to have a limit distribution under scaling, which is conveniently described in a random fields setting. The model with a singular radius distribution in zero and hence predominantly small grains yields a negatively dependent limiting field, whereas heavy-tailed grain size distribution generate positive dependence in the limit. The limit fields admit various similarity properties.

1 Introduction and Model Setting

We start with a family of grains $X_j + B(0, R_j)$ in $\mathbb{R}^d$ generated by a Poisson point process $(X_j, R_j)_j$ in $\mathbb{R}^d \times \mathbb{R}^+$. Equivalently one can start with a Poisson random measure N on $\mathbb{R}^d \times \mathbb{R}^+$ and associate with each random point $(x, r) \in \mathbb{R}^d \times \mathbb{R}^+$ the random ball of center x and radius r. We assume that the intensity measure of N is given by $dxF(dr)$ where F is a σ-finite non-negative measure on $\mathbb{R}^+$ such that all grains have finite expected volume, that is $\int_{\mathbb{R}^+} r^d F(dr) < +\infty$. In addition, we impose on F the following assumption of asymptotic power law behavior, near 0 or at infinity. For $\beta > 0$ with $\beta \neq d$,

$$\mathbf{H}(\beta) \ : F(dr) = f(r)dr \text{ with } f(r) \sim C_\beta\, r^{-\beta-1} \ , \text{ as } \ r \to 0^{d-\beta},$$

where by convention $0^\alpha = 0$ if $\alpha > 0$ and $0^\alpha = +\infty$ if $\alpha < 0$.

Our investigations are concerned with the asymptotic behavior of F around 0 for $\beta < d$ and at infinity for $\beta > d$. The case $\beta < d$ is studied in [1], the case $\beta > d$ in [4]. In [2] we have proposed first steps towards a unified frame including and extending both the situations of [1] and [4]. In a sense the asymptotics for the two cases are complementary to each other and yield limit models which we consider to be generic examples of random fields with strong negative and positive dependence.

We consider random fields defined on a space of measures. Let $\mathcal{M}_1$ denote the space of signed measures μ on $\mathbb{R}^d$ with finite total variation $||\mu||_1$. Since for all $\mu \in \mathcal{M}_1$,

$$\int_{\mathbb{R}^d\times\mathbb{R}^+} |\mu(B(x,r))| \ dxF(dr) \leq |B(0,1)| \, ||\mu||_1 \int_{\mathbb{R}^+} r^d F(dr) < +\infty,$$

one can introduce the generalized random field X defined on $\mathcal{M}_1$ as

$$\mu \mapsto \langle X, \mu \rangle = \int_{\mathbb{R}^d\times\mathbb{R}^+} \mu(B(x,r)) \ N(dx, dr).$$

Let us multiply the intensity measure by $\lambda > 0$ and the radii by $\rho > 0$. We denote by $F_\rho(dr)$ the image measure of $F(dr)$ by the change of scale $r \mapsto \rho r$, and consider the associated Poisson random measure $N_{\lambda,\rho}(dx, dr)$ with intensity measure $\lambda dx F_\rho(dr)$. Considering $\lambda = \lambda(\rho)$ as a function of ρ, we define on $\mathcal{M}_1$ the random field

$$X_\rho(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}^+} \mu(B(x, r))\ N_{\lambda(\rho),\rho}(dx, dr).$$

2 Scaling Limit Results

To study the limiting behavior of the centered and normalized field $(X_\rho(.) - \mathbf{E}(X_\rho(.)))/n(\rho)$ of this scaled random balls model when $\rho \to 0$ or $\rho \to +\infty$. we need to impose some assumptions on the measure $\mu \in \mathcal{M}_1$. For $\alpha > 0$ with $\alpha \neq d$ let us define the space of measures

$$\mathcal{M}_\alpha = \begin{cases} \left\{\mu \in \mathcal{M}_1 : \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |z - z'|^{-(\alpha-d)}|\mu|(dz)|\mu|(dz') < +\infty\right\} & \text{if } \alpha > d \\ \\ \left\{\mu \in \mathcal{M}_1 : \int_{\mathbb{R}^d} |z|^{-(\alpha-d)}|\mu|(dz) < +\infty \text{ and } \int_{\mathbb{R}^d} \mu(dz) = 0\right\} & \text{if } \alpha < d \end{cases},$$

where $|\mu|$ is the total variation measure of μ. Let us consider the kernel on $\mathbb{R}^d \times \mathbb{R}^d$,

$$K_\alpha(z, z') = \begin{cases} |z - z'|^{-(\alpha-d)}, \text{ for } z \neq z' & \text{if } \alpha > d \\ \\ \frac{1}{2}\left(|z|^{-(\alpha-d)} + |z'|^{-(\alpha-d)} - |z - z'|^{-(\alpha-d)}\right) & \text{if } \alpha < d \end{cases}.$$

Then, for $\alpha \in (d - 1, 2d)$ with $\alpha \neq d$, for any $\mu \in \mathcal{M}_\alpha$, the kernel K_α is defined and non negative $|\mu| \times |\mu|$ everywhere, with

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K_\alpha(z, z')|\mu|(dz)|\mu|(dz') < +\infty.$$

For $\beta \in (d - 1, 2d)$ with $\beta \neq d$, we finally define the enlarged spaces

$$\overline{\mathcal{M}_\beta} = \bigcup_{\alpha \in (\beta, 2d)} \mathcal{M}_\alpha \text{ if } \quad \beta > d, \qquad \overline{\mathcal{M}_\beta} = \bigcup_{\alpha \in (d-1, \beta)} \mathcal{M}_\alpha \text{ if } \quad \beta < d$$

Theorem 2.1. [1, 2, 4] *Let F be a non-negative measure on $\mathbb{R}^+$ satisfying $\mathbf{H}(\beta)$ for $\beta \in (d - 1, 2d)$ with $\beta \neq d$. For all $\mu \in \overline{\mathcal{M}_\beta}$ the following limit results holds in the sense of convergence of finite dimensional distributions of the random functionals:*

1) For all positive functions λ such that $n(\rho) := \sqrt{\lambda(\rho)\rho^\beta} \underset{\rho \to 0^{\beta-d}}{\longrightarrow} +\infty$,

$$\frac{X_\rho(\mu) - \mathbf{E}(X_\rho(\mu))}{n(\rho)} \underset{\rho \to 0^{\beta-d}}{\overset{fdd}{\longrightarrow}} c_\beta\, W_\beta(\mu)$$

where c_β is a positive constant and W_β is the centered Gaussian random linear functional on $\overline{\mathcal{M}_\beta}$ with

$$\mathbb{E}\left(W_\beta(\mu)W_\beta(\nu)\right) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K_\beta(z, z')\mu(dz)\nu(dz').$$

2) When $\lambda(\rho)\rho^\beta \to \sigma_0^{d-\beta}$ as $\rho \to 0^{\beta-d}$,

$$X_\rho(\mu) - \mathbf{E}(X_\rho(\mu)) \underset{\rho \to 0^{\beta-d}}{\overset{fdd}{\longrightarrow}} J_\beta(\mu_{\sigma_0}),$$

where μ_{σ_0} is defined by $\mu_{\sigma_0}(A) = \mu(\sigma_0^{-1}A)$ and J_β is the centered random linear functional

$$J_\beta(\mu) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^+} \mu\left(B(x,r)\right)\widetilde{N_\beta}(dx,dr), \quad \mu \in \overline{\mathcal{M}_\beta},$$

where $\widetilde{N_\beta}$ is a compensated Poisson random measure with intensity $C_\beta\, dx\, r^{-\beta-1}dr$.

3 Similarity properties

The Gaussian limit field W_β is self-similar in the sense that $\langle W_\beta, \mu_s\rangle \overset{fdd}{=} s^{(d-\beta)/2}\langle W_\beta, \mu\rangle$ for all $s > 0$. In the terminology of "ponctual random fields" used in [2] such a field has self-similarity index $\frac{d-\beta}{2} < 0$ for $\beta \in (d, 2d)$ and $\frac{d-\beta}{2} \in (0, 1/2)$ for $\beta \in (d-1, d)$. An alternative convention is applied in [4] which renders W_β H-self-similar with index $H = (3d-\beta)/2d \in (1/2, 1)$ for $d < \beta < 2d$. The limit field $J_\beta(\mu)$ is not self-similar. A similarity property which applies more generally to long-range dependent processes is discussed in [3]. The following is a version for spatial models.

Definition 3.1. *A centered random field is aggregate-similar with rigidity-index ρ, if for each $m \geq 1$, letting $(X^{(i)})_{i\geq 1}$ be i.i.d. copies of X, we have the distributional identity*

$$\sum_{i=1}^m X^{(i)}(\mu) \overset{\text{f.d.d.}}{=} X(\mu_{m^{-\rho}})$$

It is straightforward to check using the characteristic functionals that W_β and J_β are both aggregate-similar with rigidity index $\rho = d/(\beta - d)$.

This property provides an interpretation of the scaling parameter σ_0 in the second half of the theorem. Choose σ_0 so that the number $\sigma_0^{\beta/d-1}$ is an integer m. Then the limit field has the representation

$$J_\beta(\mu_{\sigma_0}) \overset{f.d.d.}{=} \sum_{i=1}^m J_\gamma^i(\mu).$$

The guiding asymptotic quantity $\lambda\rho^\beta$ may be interpreted as the expected number of very large ($\beta > d$) or very small ($\beta < d$) balls which cover a point asymptotically. Thus, the more such extreme grains are allowed asymptotically, the larger number of i.i.d. copies of the basic field J_β appears in the limit.

References

[1] Biermé, H. and Estrade, A.: *Poisson overlapping microballs: self-similarity and X-ray images.* Preprint, 2005.

[2] Biermé, H., Estrade, A. and Kaj, I: *About scaling behavior of random balls models. Proceed. 6th Int. Conf. on Stereology, Spatial Statistics and Stochastic Geometry*, Prague, June 2006.

[3] Kaj, I.: *Limiting fractal random processes in heavy-tailed systems*, 2005. In: *Fractals in Engineering, New Trends in Theory and Applications*, p 199-218, Springer-Verlag London.

[4] Kaj, I., Leskelä, L., Norros, I. and Schmidt, V. *Scaling limits for random fields with long-range dependence.* To appear: *Ann. Probab.*, 2006.