Theta functions in enumerative combinatorics
Week 3: Analysis of solutions

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STRATEGY: THETA FUNCTION METHOD

- **Step 1:** Find a recursive decomposition of each object in your class
- **Step 2:** Write functional equations which characterise the generating function $F(t)$
- **Step 3:** Solve the functional equations using theta functions!

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- **Step 4 (today):** Analyse the solutions' algebraic and asymptotic properties

Problems solvable with theta functions

Quadrant walk models with small steps:

- All "genus 1" cases written as integrals involving Weierstrass $\wp$. (2012 Kurkova, Raschel)
- Walks categorised into 79 non-trivial cases and All D-finite cases (except Gessel walks) solved (2009 Bousquet-Mélou, Mishna)
- D-algebraic, but non-D-finite cases solved in terms of $\wp$ (2017 Bernardi, Bousquet-Mélou, Raschel)
- Last week: I described (vaguely) how to solve D-algebraic "genus 1" cases with the theta function $\vartheta(z, \tau)$.

Whole plane walks by winding number:

- Counted exactly in terms of theta functions (2019 Budd)
- Solution involved an eigenvalue decomposition of commuting Hilbert space operators.
- Corollaries include the exact enumeration of Gessel walks.
- Next week: Same problem solved with the recursive method.

Problems solvable with theta functions

More solutions next week:

- Six vertex model on 4-valent maps (Kostov/Bousquet-Mélou, E.P., Zinn-Justin)
- height functions on quadrangulations NEW!
- Properly coloured triangulations NEW! (Previously shown to be D-algebraic by Tutte)
- Triangular lattice walks by winding number NEW!

Two weeks ago: the theta function

Definition: For $\tau, z \in \mathbb{C}$, $\text{im}(\tau) > 0$,

$$\vartheta(z,\tau) = \sum_{n=-\infty}^{\infty} (-1)^n e^{\left(\frac{2n+1}{2}\right)^2 i\pi\tau + (2n+1)iz}$$

Useful facts about $\vartheta(z,\tau)$:

- $\vartheta(z+\pi,\tau) = \vartheta(-z,\tau) = -\vartheta(z,\tau)$
- $\vartheta(z+\pi\tau,\tau) = -e^{-2iz-i\pi\tau}\vartheta(z,\tau)$
- Roots of $\vartheta(z,\tau)$ form the lattice $\pi\mathbb{Z}+\pi\tau\mathbb{Z}$
- All elliptic functions $f(z)$ with periods π and $\pi\tau$ can be written as

$$f(z) = c\prod_{j=1}^{k} \frac{\vartheta(z+\alpha_j,\tau)}{\vartheta(z+\beta_j,\tau)}.$$

- $\frac{4i}{\pi}\frac{\partial}{\partial\tau}\vartheta(z,\tau) = \vartheta''(z,\tau)$
- $\vartheta(z,\tau)$ is differentially algebraic (in both variables)

Last week: Quarter plane walks

The generating function $Q(x,y)$ for Kreweras paths is characterised by:

$$Q(x,y) = 1 + xytQ(x,y) + \frac{t}{x}\left(Q(x,y) - Q(0,y)\right) + \frac{t}{y}\left(Q(x,y) - Q(x,0)\right).$$

Last week, we solved this using theta functions:

$$t = e^{-\frac{\pi\tau i}{3}} \frac{\vartheta'(0,3\tau)}{4i\vartheta(\pi\tau,3\tau) + 6\vartheta'(\pi\tau,3\tau)}$$

$$x = \frac{e^{-\frac{4\pi\tau i}{3}}\vartheta(z,3\tau)\vartheta\left(z-\pi\tau,3\tau\right)}{\vartheta\left(z+\pi\tau,3\tau\right)\vartheta\left(z-2\pi\tau,3\tau\right)}$$

$$I(t,x) = e^{\frac{\pi\tau i}{3}} \frac{\vartheta(\pi\tau,3\tau)}{\vartheta'(0,3\tau)}\left(\frac{\vartheta'(z,2\tau)}{\vartheta(z,2\tau)} - \frac{\vartheta'(z+\pi\tau,2\tau)}{\vartheta(z+\pi\tau,2\tau)} - i\right) + \frac{1}{2t}.$$

$$Q(t,x,0) = \frac{I(x)}{tx} - \frac{1}{tx^2}.$$

Today: τ transformations and analysing solutions

TODAY: τ TRANSFORMATIONS AND ANALYSING SOLUTIONS

- **Part 3a:** A new equation:

$$\vartheta\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = -i(-i\tau)^{\frac{1}{2}} \exp\left(\frac{i}{\pi\tau}z^2\right)\vartheta(z,\tau)$$

- **Part 3b:** Asymptotic analysis
- **Part 3c:** Modular functions and modular properties of ϑ
- **Part 3d:** Algebraicity for Kreweras paths

Part 3a: a new equation

$$\vartheta\left(\frac{z}{\tau},-\frac{1}{\tau}\right)=-i(-i\tau)^{\frac{1}{2}}\exp\left(\frac{i}{\pi\tau}z^{2}\right)\vartheta(z,\tau)$$

A NEW EQUATION

Recall: For fixed τ:

- $\vartheta(z+\pi,\tau) = -\vartheta(z,\tau)$
- $\vartheta(z+\pi\tau,\tau) = -e^{-2iz-i\pi\tau}\vartheta(z,\tau)$

Determining c_τ: Use heat equation to get $c_\tau = c(-i\tau)^{1/2}$.
sub $\tau = i$ and take derivative at $z = 0$ to get $c = -i$.

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$$-2iz - i\pi\tau = \frac{i}{\pi\tau}(z^2 - (z+\pi\tau)^2).$$

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We see that:

- $f(z) = e^{\frac{iz^2}{\pi\tau}}\vartheta(z,\tau)$ satisfies

$$f(z+\pi\tau) = -f(z) \quad \text{and} \quad f(z+\pi) = -e^{-\frac{2iz+\pi}{\tau}}f(z).$$

- $g(z) = f(z\tau)$ satisfies

$$g(z+\pi) = -g(z) \quad \text{and} \quad g(z+\pi/\tau) = -e^{-2iz-\frac{\pi}{\tau}}g(z).$$

- This implies $g(z) = c\vartheta(z, -\frac{1}{\tau})$

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Part 3b: Asymptotic analysis

Kreweras paths

Exact solution for Kreweras paths:

$$t = e^{-\frac{\pi\tau i}{3}} \frac{\vartheta'(0,3\tau)}{4i\vartheta(\pi\tau,3\tau) + 6\vartheta'(\pi\tau,3\tau)}$$

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$$I(x) = e^{\frac{\pi\tau i}{3}} \frac{\vartheta(\pi\tau,3\tau)}{\vartheta'(0,3\tau)} \left(\frac{\vartheta'(z,2\tau)}{\vartheta(z,2\tau)} - \frac{\vartheta'(z+\pi\tau,2\tau)}{\vartheta(z+\pi\tau,2\tau)} - i\right) + \frac{1}{2t}.$$

$$Q(x,0) = \frac{I(x)}{tx} - \frac{1}{tx^2}.$$

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Transform each ϑ using

$$\vartheta(z,\tau) = i(-i\tau)^{-\frac{1}{2}} \exp\left(-\frac{i}{\pi\tau}z^2\right)\vartheta\left(\frac{z}{\tau},-\frac{1}{\tau}\right)$$

Exact solution for Kreweras paths:

$$t = \frac{\vartheta'(0, -\frac{1}{3\tau})}{6\vartheta'(\frac{\pi}{3}, -\frac{1}{3\tau})}$$

$$x = \frac{\vartheta(\frac{z}{3\tau}, -\frac{1}{3\tau})\vartheta(\frac{z}{3\tau} - \frac{\pi}{3}, -\frac{1}{3\tau})}{\vartheta(\frac{z}{3\tau} + \frac{\pi}{3}, -\frac{1}{3\tau})\vartheta(\frac{z}{3\tau} - \frac{2\pi}{3}, -\frac{1}{3\tau})}$$

$$I(x) = \frac{3}{2}\frac{\vartheta(\frac{\pi}{3}, -\frac{1}{3\tau})}{\vartheta(\frac{\pi}{3}, -\frac{1}{3\tau})}\left(\frac{\vartheta'(\frac{z}{2\tau}, -\frac{1}{2\tau})}{\vartheta(\frac{z}{2\tau}, -\frac{1}{2\tau})} - \frac{\vartheta'(\frac{z}{2\tau} + \frac{\pi}{2}, -\frac{1}{2\tau})}{\vartheta(\frac{z}{2\tau} + \frac{\pi}{2}, -\frac{1}{2\tau})}\right) + \frac{1}{2t}.$$

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Write in terms of $\hat{\tau} = -\frac{1}{6\tau}$ and $\hat{z} = \frac{z}{3\tau}$

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Write as series in $\hat{q} = e^{2\pi i\hat{\tau}}$ and $\hat{u} = e^{i\hat{z}}$ using

$$T_k(u, q) = \sum_{n=0}^{\infty}(-1)^n(2n+1)^k q^{n(n+1)/2}(u^{n+1} - (-1)^k u^{-n}).$$

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$$t = \frac{1}{3} - 3\hat{q}^2 + 18\hat{q}^4 + O(\hat{q}^5)$$

$$E(t) = Q(0, 0) = \frac{9}{8} - \frac{81}{8}\hat{q}^2 + 81\hat{q}^3 + O(\hat{q}^4)$$

KREWERAS PATHS

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So

$$[t^{3n}]E(t) \sim \frac{\sqrt{3}}{4\sqrt{\pi}} 3^{3n} n^{-5/2}$$

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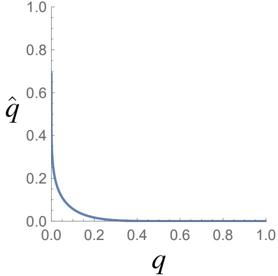
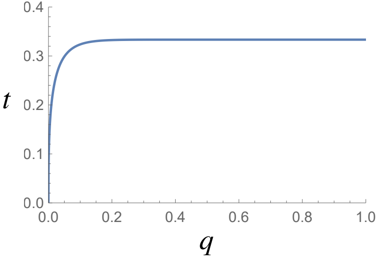
But why is this on the radius of convergence??





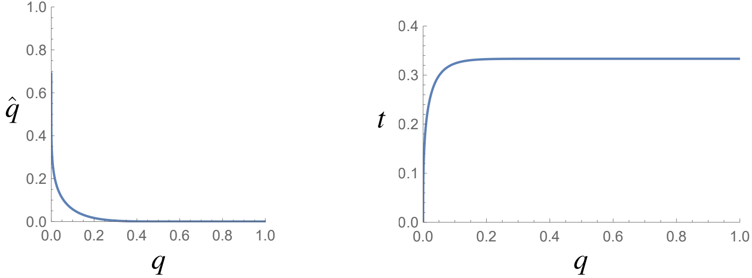
Recall (last week): $E(t) = Q(0,0,t)$ and t are series in q:

$$t = q^{1/3} - 5q^{4/3} + 32q^{7/3} + O(q^{10/3})$$

$$E(t) = 1 + 2q - 14q^2 + O(q^3).$$

$\hat{q} = e^{2\pi i \hat{\tau}} = e^{-\frac{\pi i}{3\tau}} = e^{\frac{2\pi^2}{3\log(q)}}$ is 0 at $q = 1$.

By the **implicit function theorem** we just need $t(q)$ to be increasing for $0 \le q \le 1$.



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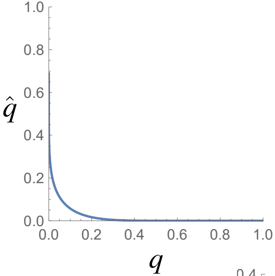
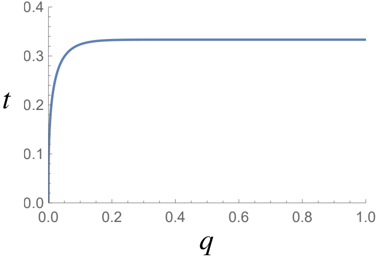
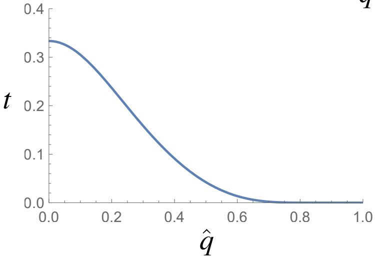
Asymptotics of Kreweras paths







Exact solution for Kreweras paths:

$$\begin{aligned} t &= \frac{1}{3} - 3\hat{q}^2 + 18\hat{q}^4 + O(\hat{q}^5) \\ Q(0,0) &= \frac{9}{8} - \frac{81}{8}\hat{q}^2 + 81\hat{q}^3 + O(\hat{q}^4) \\ &= \frac{9}{8} - \frac{9}{8}(1-3t) + 3\,(1-3t)^{3/2} + O\left((1-3t)^2\right). \end{aligned}$$

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Asymptotics using theta functions

Exact solution for Kreweras paths:

$$t = \frac{1}{3} - 3\hat{q}^2 + 18\hat{q}^4 + O(\hat{q}^5)$$

$$\begin{aligned} Q(0,0) &= \frac{9}{8} - \frac{81}{8}\hat{q}^2 + 81\hat{q}^3 + O(\hat{q}^4) \\ &= \frac{9}{8} - \frac{9}{8}(1-3t) + 3\,(1-3t)^{3/2} + O\left((1-3t)^2\right). \end{aligned}$$

Other possible asymptotics:

$$t = \frac{1}{4\sqrt{3}\pi} + \frac{q^{1/3}(8\log(q) - 24)}{16\sqrt{3}\pi} + O(q^{1/2})$$

$$\begin{aligned} R(t) &= \frac{1}{27} - q^{1/3} + 15q^{2/3} + O(q) \\ &= \frac{1}{27} - \frac{1}{96\sqrt{3}\pi}\frac{(1-4\sqrt{3}\pi t)}{\log(1-4\sqrt{3}\pi t)} + \dots \end{aligned}$$

Asymptotics using theta functions

Another possibility:

- Sometimes $t(q)$ is not increasing for $q \in [0, 1]$.
- $\rightarrow$ critical point occurs when $t'(q_0) = 0$ i.e., $t'(\tau_0) = 0$.
- We get asymptotics by analysing series near $\tau = \tau_0$.

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Annoying mystery:

For any fixed z, the functions

$$F_0(\tau) = \frac{\vartheta'(0,\tau)\vartheta(2z,\tau)}{\vartheta'(z,\tau)\vartheta(z,\tau)}$$

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satisfy $F_j'(\tau_0) = 0$ for *the same* τ_0.

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satisfy $F_j'(\tau_0) = 0$ for *the same* τ_0.

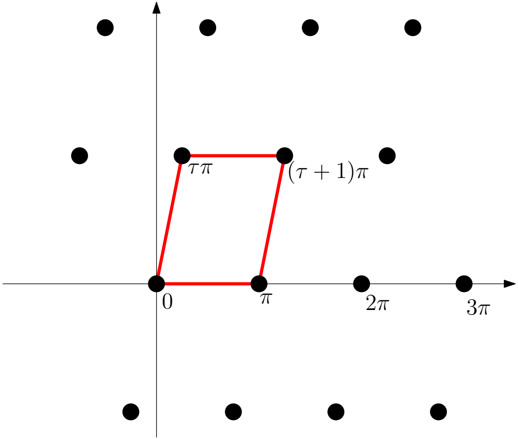
This effects asymptotics (eg. when $R(t) = F_2(\tau)$ and $t = F_1(\tau)$).

Part 3c: Modular properties and algebraicity

Main reference: Elliptic Modular Forms and Their Applications (Zagier 2008)

VARYING τ

Aim: relate $\vartheta(z,\tau)$ to other τ values
Natural transformations: $\tau \to \tau + 1$ and $\tau \to -\frac{1}{\tau}$ (these preserve the shape of the root lattice $\pi\mathbb{Z} + \pi\tau\mathbb{Z}$)



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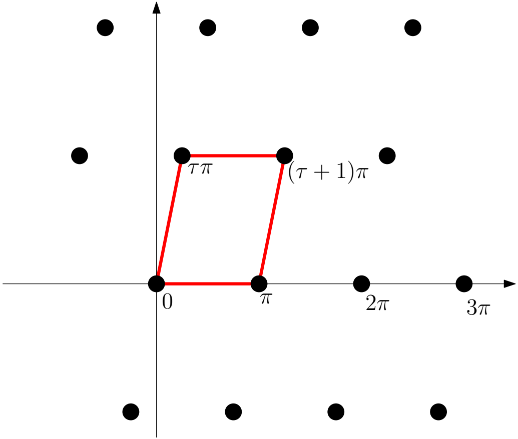

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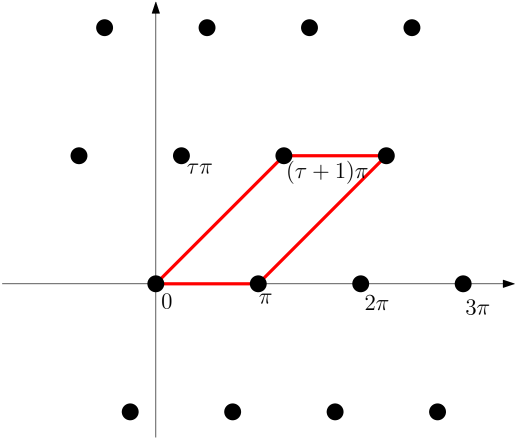

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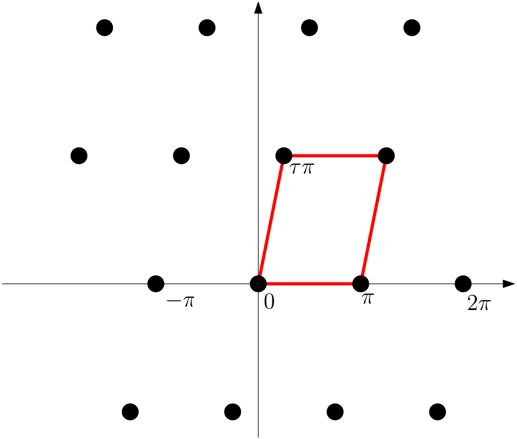

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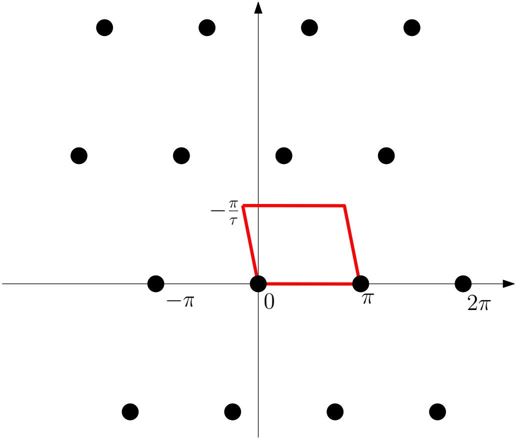

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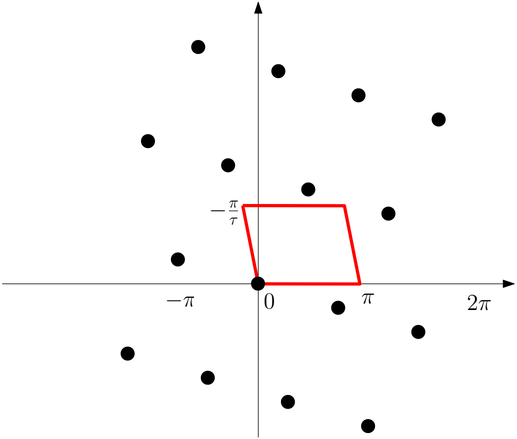

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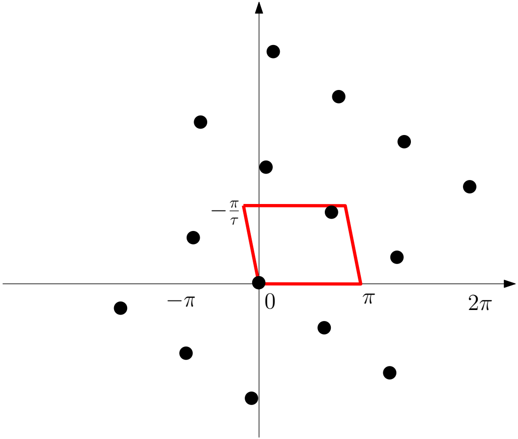

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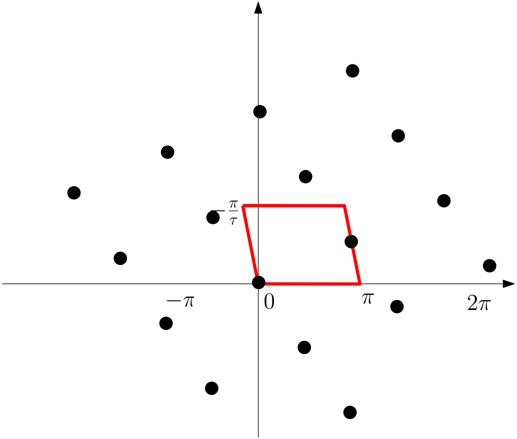

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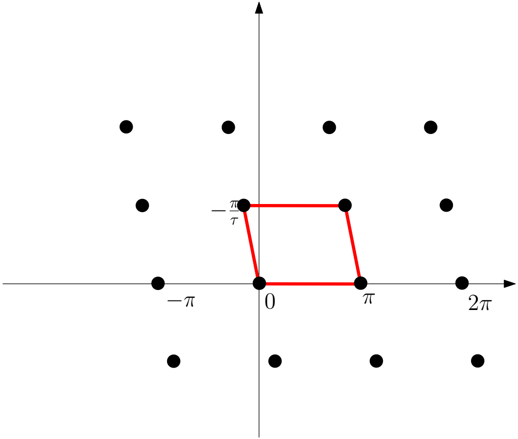

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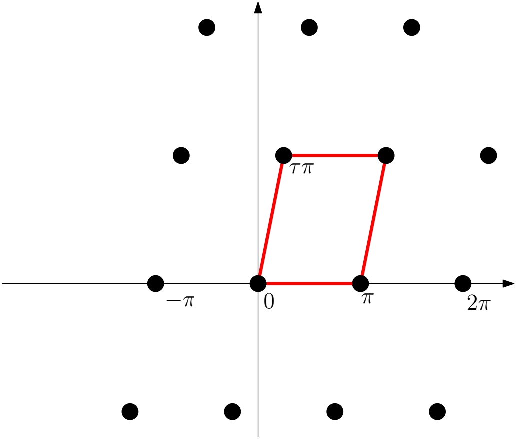

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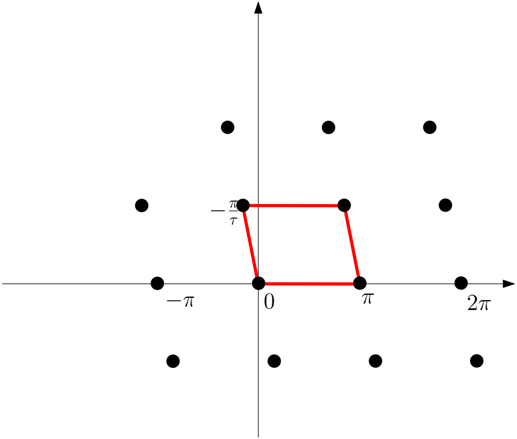

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Natural transformations: $\tau \to \tau+1$ and $\tau \to -\frac{1}{\tau}$ (these preserve the shape of the root lattice $\pi\mathbb{Z}+\pi\tau\mathbb{Z}$)
Equations:

- $\vartheta(z,\tau+1) = e^{i\pi/4}\vartheta(z,\tau)$
- $\vartheta\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = -i(-i\tau)^{\frac{1}{2}}\exp\left(\frac{i}{\pi\tau}z^2\right)\vartheta(z,\tau)$

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These transformations generate the group of transformations

$$\tau \to \frac{a\tau + b}{c\tau + d},$$

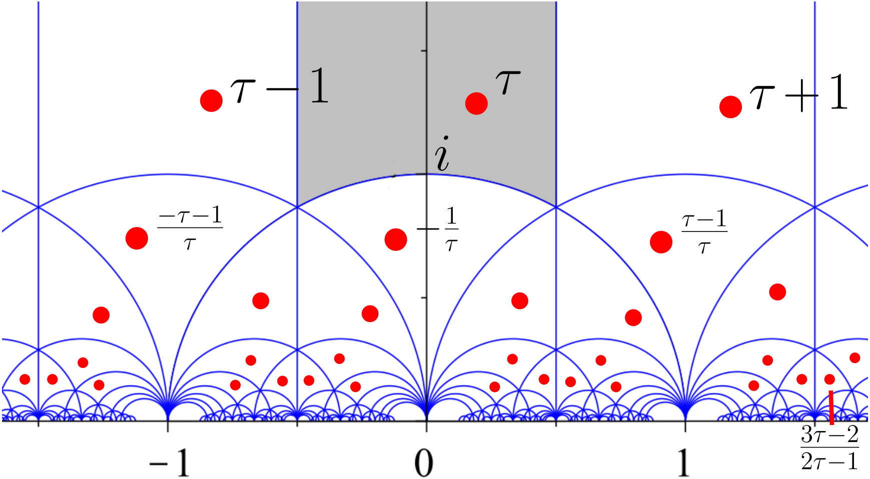
satisfying $ad - bc = 1$.

This is isomorphic to the group $SL_2(\mathbb{Z})$ of matrices

$$\begin{bmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{bmatrix},$$

with determinant 1.

Orbit of $SL_2(\mathbb{Z})$



MODULAR FUNCTIONS

Definition: $SL_2(\mathbb{Z})$ is the group of matrices $\begin{bmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{bmatrix}$ with determinant 1.

Action on upper half plane $\mathbb{H} = \{z \in \mathbb{C} | \mathbf{im}(z) > 0\}$**:**

$$\begin{bmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{bmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$

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definition: Let Γ be a finite index subgroup of $SL_2(\mathbb{Z})$. A *modular function* of weight n is a meromorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfying the following for all $\rho \in \Gamma$:

$$f(\rho \cdot \tau) := f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^n f(\tau).$$

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Theorem: All modular functions of weight 0 are algebraically related

Theorem: If $f(\tau)$ and $g(\tau)$ are modular functions of weights $n \in \mathbb{N}$ and 0, and $f(\tau) = \phi(g(\tau))$ then ϕ is D-finite.

Modular-like theta equation

In general, if

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}),$$

then

$$(c\tau+d)^{-\frac{1}{2}}\vartheta\left(z, \frac{a\tau+b}{c\tau+d}\right) = e^{K\pi i/4}\exp\left(\frac{ic(c\tau+d)}{\pi}z^2\right)\vartheta(z(c\tau+d),\tau).$$

for some $K \in \mathbb{Z}$ depending on a, b, c, d.

MODULAR-LIKE THETA EQUATION FOR $z \in \pi\mathbb{Q}$

In general, if

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for some $K \in \mathbb{Z}$ depending on a, b, c, d.

In the case $zc, z(d-1) \in \pi\mathbb{Z}$**:**

$$(c\tau+d)^{-\frac{1}{2}}\vartheta\left(z, \frac{a\tau+b}{c\tau+d}\right) = e^{\hat{K}\pi i/4}\exp\left(\frac{ic(d-2)}{\pi}z^2\right)\vartheta(z,\tau).$$

So: on some subgroup $\Gamma \subset SL_2(\mathbb{Z})$,

$$(c\tau+d)^{-\frac{1}{2}}\vartheta\left(z, \frac{a\tau+b}{c\tau+d}\right) = \vartheta(z,\tau).$$

Modular properties for $z \in \pi\mathbb{Q}$

On some subgroup $\Gamma \subset SL_2(\mathbb{Z})$,

$$(c\tau + d)^{-\frac{1}{2}}\vartheta\left(z, \frac{a\tau + b}{c\tau + d}\right) = \vartheta(z, \tau),$$

so $\vartheta(z, \tau)$ is a modular function of weight $1/2$.
Corollary: For $z_1, z_2 \in \pi\mathbb{Q}$ the function

$$f(\tau) = \frac{\vartheta(z_1, \tau)}{\vartheta(z_2, \tau)}$$

is modular of weight 0.
Corollary: Any two such functions $f(\tau)$ are algebraically related.
Meaning in combinatorics: Generating functions with theta function solutions are sometimes algebraic.

MODULAR PROPERTIES FOR $z \in \pi\mathbb{Q} + \pi\tau\mathbb{Q}$

Assume $z_1, z_2 \in \pi\mathbb{Q}$. Then the function

$$f(\tau) = e^{\frac{i}{\pi} z_2^2 \tau} \vartheta(z_1 + z_2\tau, \tau)$$

is modular of weight $1/2$ on some finite index subgroup Γ of $SL_2(\mathbb{Z})$:

$$(c\tau + d)^{-1/2} f\left(\frac{a\tau + b}{c\tau + d}\right) = f(\tau).$$

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Even more generally: the function $g(\tau) = f(n\tau)$ has this property (for a different finite index subgroup Γ of $SL_2(\mathbb{Z})$.)

Derivatives: The function $h(\tau) = \dfrac{\vartheta'(z_1 + z_2\tau, \tau)}{\vartheta(z_1 + z_2\tau, \tau)} + \dfrac{2iz_2}{\pi}$ is modular of weight 1.

Higher derivatives: Use heat equation to find similar properties.

ALGEBRAICITY FOR KREWERAS PATHS

Recall:

$$t(\tau) = e^{-\frac{\pi\tau i}{3}} \frac{\vartheta'(0,3\tau)}{4i\vartheta(\pi\tau,3\tau)+6\vartheta'(\pi\tau,3\tau)}$$

$$X(\tau,z) = \frac{e^{-\frac{4\pi\tau i}{3}}\vartheta(z,3\tau)\vartheta\left(z-\pi\tau,3\tau\right)}{\vartheta\left(z+\pi\tau,3\tau\right)\vartheta\left(z-2\pi\tau,3\tau\right)}$$

$$I(\tau,z) = e^{\frac{\pi\tau i}{3}} \frac{\vartheta(\pi\tau,3\tau)}{\vartheta'(0,3\tau)} \left(\frac{\vartheta'(z,2\tau)}{\vartheta(z,2\tau)} - \frac{\vartheta'(z+\pi\tau,2\tau)}{\vartheta(z+\pi\tau,2\tau)} - i\right) + \frac{1}{2t}.$$

$$Q(X(z),0) = \frac{I(z)}{tX(z)} - \frac{1}{tX(z)^2}.$$

Since $X(z)$ and $I(z)$ are both elliptic: There exists $P \in \mathbb{R}((t))[x,y]$ such that $P(t,X(z),I(z)) = 0$.
Modular properties: We can show that $t(\tau)$ is a modular function of weight 0, as are $X(z)$ and $I(z)$ for $z \in \pi\mathbb{Q}$.

Algebraicity for Kreweras paths

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So we have: Dense set of algebraic $x(t), I(t)$ satisfying $P(t,x(t),I(t)) = 0$.

Corollary: Coefficients of $P(x(t),I(t))$ are algebraic in t.

NEXT WEEK

Next week: New results - examples showing more types of equations that can be solved

- Walks on a directed triangular lattice by winding number
- Properly coloured triangulations
- Six vertex model on 4-valent maps (Joint work with Mireille Bousquet-Mélou and Paul Zinn-Justin)

Part ???: More identities

Recall: $\vartheta(z+\pi)=-\vartheta(z)$ and $\vartheta(z+\pi\tau)=-e^{-2iz-i\pi\tau}\vartheta(z)$.

Let's try to write $\vartheta(2z)$ as a product of functions like $\vartheta(z-\alpha)$.

Making new theta function identities

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Roots of $\vartheta(2z)$ (inside fundamental domain):

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So consider

$$f(z) = \frac{\vartheta(2z)}{\vartheta(z)\vartheta(z-\pi/2)\vartheta(z-\pi\tau/2)\vartheta(z-\pi(\tau+1)/2)},$$

then $f(z)$ has no roots (or poles).

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then $f(z)$ has no roots (or poles). Then

- $f(z+\pi) = f(z)$
- $f(z+\pi\tau) = e^{2\pi(\tau+1)i}f(z) = e^{2\pi\tau i}f(z)$

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So $h(z) = e^{2iz}f(z)$ has π and $\pi\tau$ as periods.

Making new theta function identities

From the last slide: The function

$$h(z) = \frac{e^{2iz}\vartheta(2z)}{\vartheta(z)\vartheta\left(z-\frac{\pi}{2}\right)\vartheta\left(z-\frac{\pi\tau}{2}\right)\vartheta\left(z-\frac{\pi(\tau+1)}{2}\right)}$$

is holomorphic and has π and $\pi\tau$ as periods...

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is holomorphic and has π and $\pi\tau$ as periods... hence, $h(z)$ is constant by Liouville's theorem:

Theorem (Liouville): The only holomorphic elliptic functions are the constant functions.

It follows that

$$\vartheta(2z) = -\frac{2e^{-2iz}\vartheta(z)\vartheta(z-\frac{\pi}{2})\vartheta(z-\frac{\pi\tau}{2})\vartheta\left(z-\frac{\pi(\tau+1)}{2}\right)}{\vartheta(\frac{\pi}{2})\vartheta(\frac{\pi\tau}{2})\vartheta\left(\frac{\pi(\tau+1)}{2}\right)}$$