Theta functions in enumerative combinatorics
Week 4: New solutions

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RECAP (WEEK 1+): PROPERTIES OF $\vartheta(z, \tau)$

Definition: For $\tau, z \in \mathbb{C}$, $\mathrm{im}(\tau) > 0$,

$$\vartheta(z,\tau) = \sum_{n=-\infty}^{\infty} (-1)^n e^{\left(\frac{2n+1}{2}\right)^2 i\pi\tau + (2n+1)iz}$$

Useful equations involving $\vartheta(z, \tau)$:

- $\vartheta(-z,\tau) = -\vartheta(z,\tau)$
- $\vartheta(z+\pi,\tau) = -\vartheta(z,\tau)$
- $\vartheta(z+\pi\tau,\tau) = -e^{-2iz-i\pi\tau}\vartheta(z,\tau)$
- $\frac{4i}{\pi}\frac{\partial}{\partial\tau}\vartheta(z,\tau) = \vartheta''(z,\tau)$
- $\vartheta(z,\tau+1) = e^{i\pi/4}\vartheta(z,\tau)$
- $\vartheta\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = -i(-i\tau)^{\frac{1}{2}}\exp\left(\frac{i}{\pi\tau}z^2\right)\vartheta(z,\tau)$

RECAP (WEEK 1+): PROPERTIES OF $\vartheta(z,\tau)$

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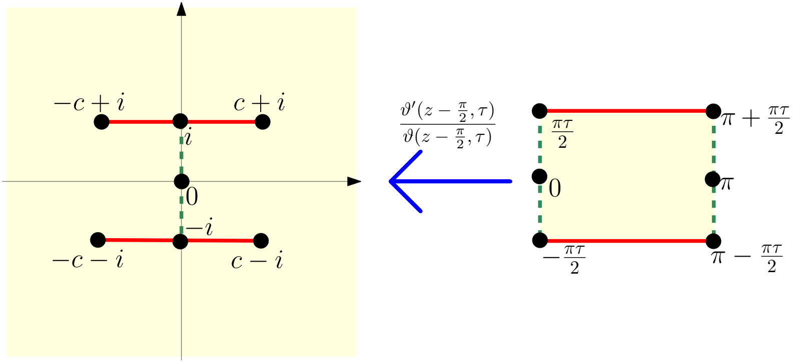
Other facts about $\vartheta(z,\tau)$:

- $\vartheta(z,\tau)$ is differentially algebraic (in both variables)
- **Liouville's elliptic theorem:** Any elliptic function with no poles is constant
- All elliptic functions $f(z)$ with periods π and $\pi\tau$ can be written as

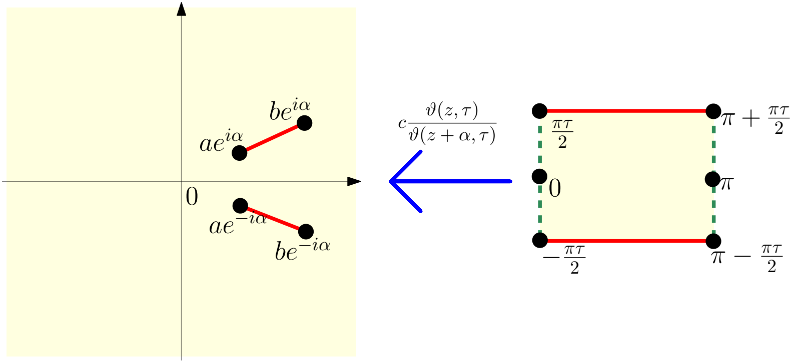
$$f(z) = c\prod_{j=1}^{k} \frac{\vartheta(z+\alpha_j,\tau)}{\vartheta(z+\beta_j,\tau)}.$$

- Any two elliptic functions with the same periods are algebraically related

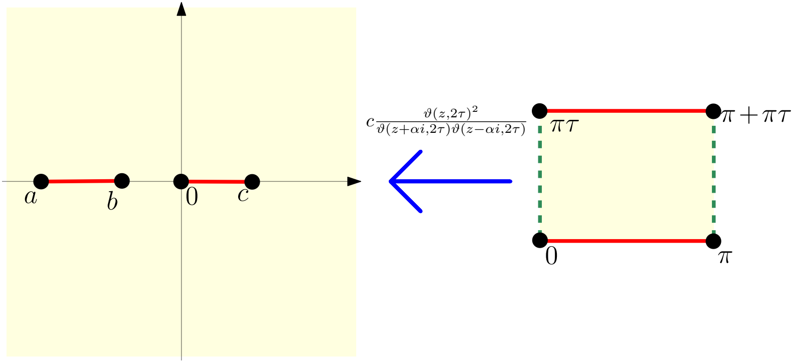
Recap (Week 1+): Conformal parametrizations



RECAP (WEEK 1+): CONFORMAL PARAMETRIZATIONS



RECAP (WEEK 1+): CONFORMAL PARAMETRIZATIONS



RECAP (WEEK 2): KREWERAS PATHS

Kreweras paths:

$$Q(x,y) = 1 + xytQ(x,y) + \frac{t}{x}\left(Q(x,y) - Q(0,y)\right) + \frac{t}{y}\left(Q(x,y) - Q(x,0)\right).$$

Kernel equation $K(x,y) = 1 - xyt - \frac{t}{x} - \frac{t}{y} = 0$ is parametrised by

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}}\vartheta(z,3\tau)\vartheta\left(z-\pi\tau,3\tau\right)}{\vartheta\left(z+\pi\tau,3\tau\right)\vartheta\left(z-2\pi\tau,3\tau\right)}, \qquad Y(z) = X(z+\pi\tau).$$

Using we solved for $Q(x,0)$:

$$t = e^{-\frac{\pi\tau i}{3}}\frac{\vartheta'(0,3\tau)}{4i\vartheta(\pi\tau,3\tau) + 6\vartheta'(\pi\tau,3\tau)}$$

$$I(t,X(z)) = e^{\frac{\pi\tau i}{3}}\frac{\vartheta(\pi\tau,3\tau)}{\vartheta'(0,3\tau)}\left(\frac{\vartheta'(z,2\tau)}{\vartheta(z,2\tau)} - \frac{\vartheta'(z+\pi\tau,2\tau)}{\vartheta(z+\pi\tau,2\tau)} - i\right) + \frac{1}{2t}.$$

$$Q(x,0) = \frac{I(x)}{tx} - \frac{1}{tx^2}.$$

Recap (Week 2): Quadrant walks

Any small (non-trivial) step set: Equation to solve:

$$Q(x,y)K(x,y) = xy - P_1(y)\mathsf{Q}(0,y) - P_2(x)\mathsf{Q}(x,0) + c.$$

The kernel equation $K(x,y) = 0$ is parameterised by

$$X(z) = c_1\frac{\vartheta(z-\alpha_1)\vartheta(z-\beta_1)}{\vartheta(z-\gamma_1)\vartheta(z-\delta_1)} \quad \text{and} \quad Y(z) = c_2\frac{\vartheta(z-\alpha_2)\vartheta(z-\beta_2)}{\vartheta(z-\gamma_2)\vartheta(z-\delta_2)}.$$

- **D-finite case:** $\alpha_2 + \beta_2 - \alpha_1 - \beta_1 \in \pi\mathbb{Q}$.
- **D-algebraic case:** $X(z)Y(z) = R_1(X(z)) + R_2(Y(z))$.

RECAP (WEEK 3): ANALYSING SOLUTIONS

- Deriving asymptotics using the equation

$$\vartheta\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = -i(-i\tau)^{\frac{1}{2}} \exp\left(\frac{i}{\pi\tau} z^2\right) \vartheta(z, \tau)$$

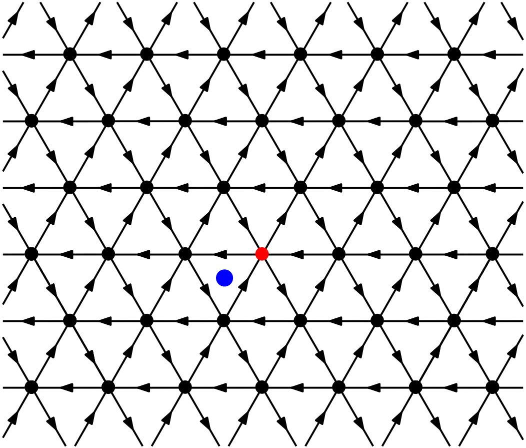
- Relation of $\vartheta(\frac{m}{n}\pi, \tau)$ to modular functions
- Some situations where theta function parametrisations are Algebraic or D-finite

TODAY (FINAL WEEK):

New work: More problems solvable with theta functions:

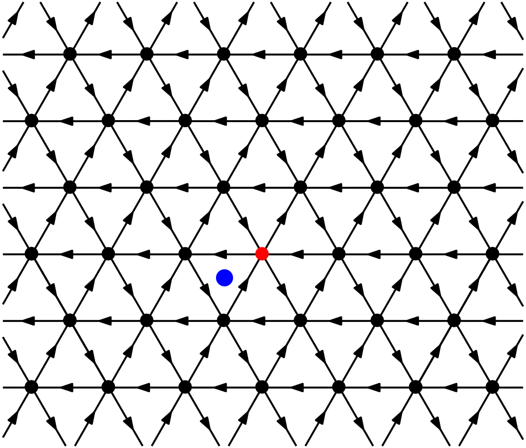
- Directed triangular lattice walks (tandem walks) by winding number
- Properly coloured triangulations (Previously shown to be D-algebraic by Tutte)
- Six vertex model on 4-valent maps (with Zinn-Justin, following Kostov)

Part 4a: Tandem walks by winding number



Tandem walks by winding number

The model: count walks starting at the red point by end point and number of times winding around the blue point.



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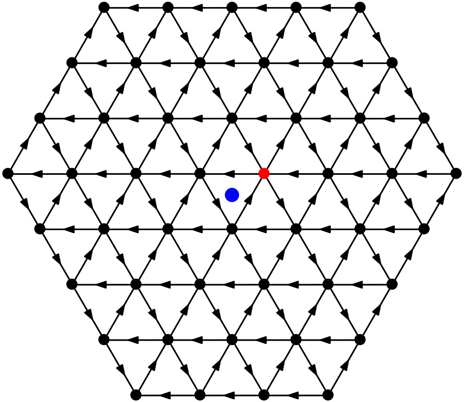

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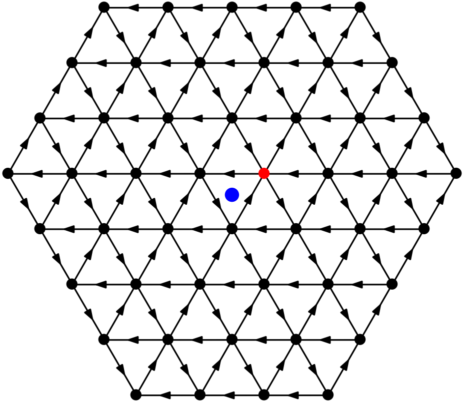

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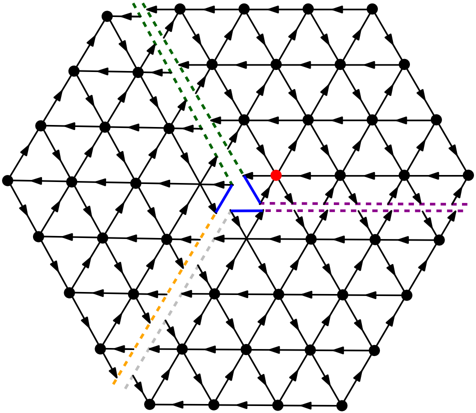

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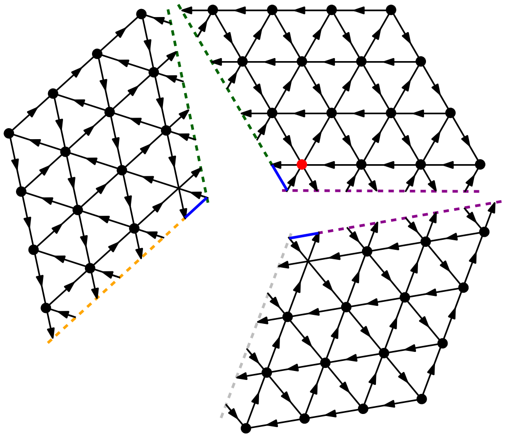

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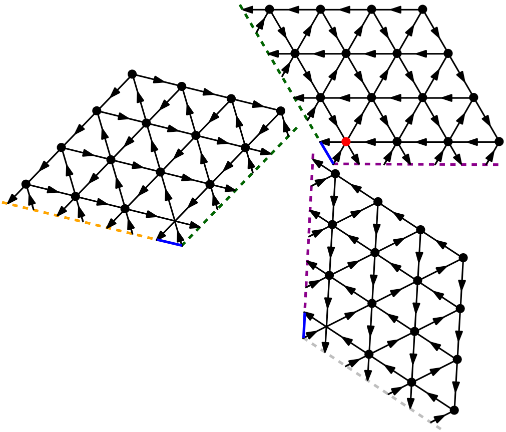

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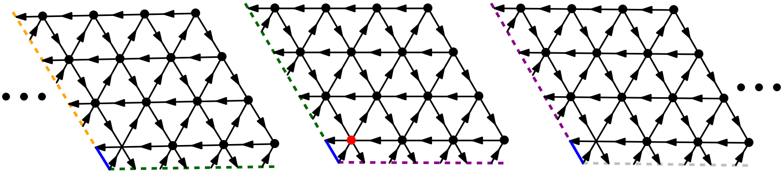

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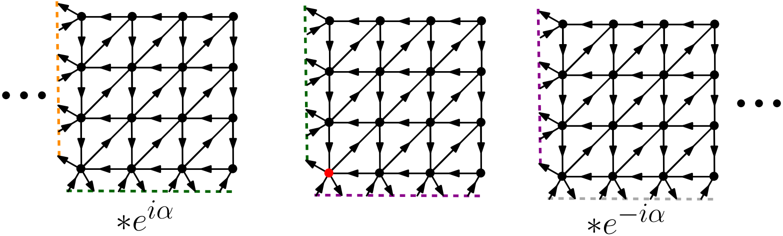

definition: $Q(t,\alpha,x,y) \equiv Q(x,y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)}$

TANDEM WALKS BY WINDING NUMBER

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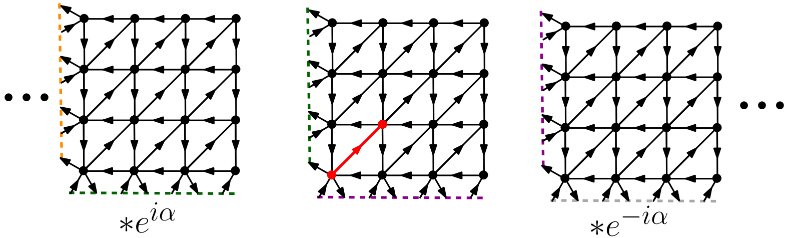

definition: $Q(t, \alpha, x, y) \equiv Q(x, y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)}$

This example contributes txy.

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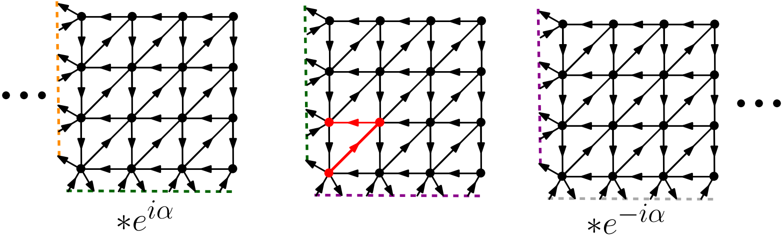

definition: $Q(t,\alpha,x,y) \equiv Q(x,y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)}$

This example contributes $t^2 y$.

Tandem walks by winding number

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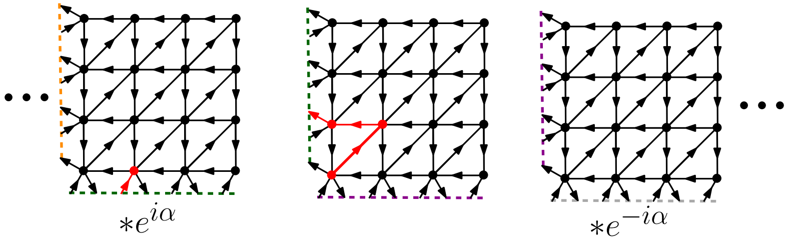

definition: $Q(t, \alpha, x, y) \equiv Q(x, y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)}$

This example contributes $t^3 x e^{i\alpha}$.

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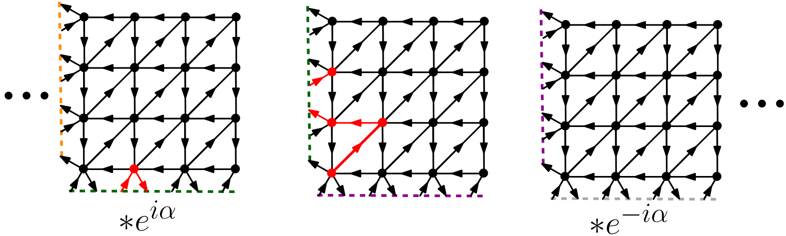

definition: $Q(t,\alpha,x,y) \equiv Q(x,y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)}$

This example contributes t^4y^2.

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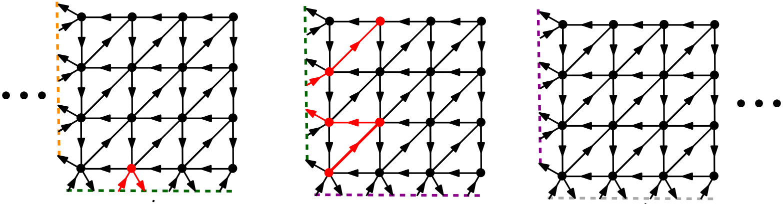

$$*e^{i\alpha} \qquad\qquad *e^{-i\alpha}$$

definition: $Q(t,\alpha,x,y) \equiv Q(x,y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)}$

This example contributes t^5xy^3.

Tandem walks by winding number

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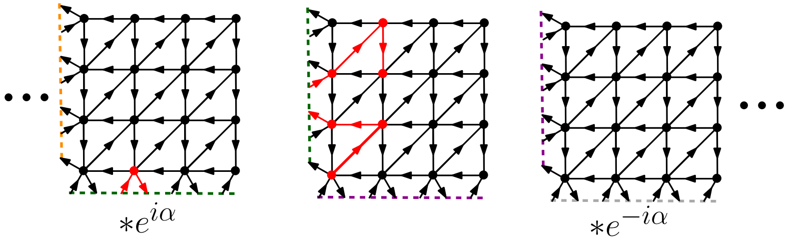

definition: $Q(t, \alpha, x, y) \equiv Q(x, y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)}$

This example contributes $t^6 xy^2$.

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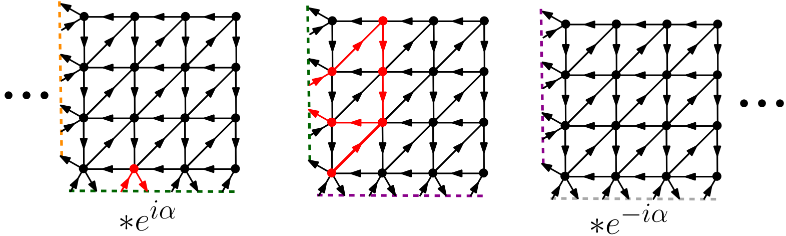

definition: $Q(t,\alpha,x,y) \equiv Q(x,y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)}$

This example contributes t^7xy.

Tandem walks by winding number

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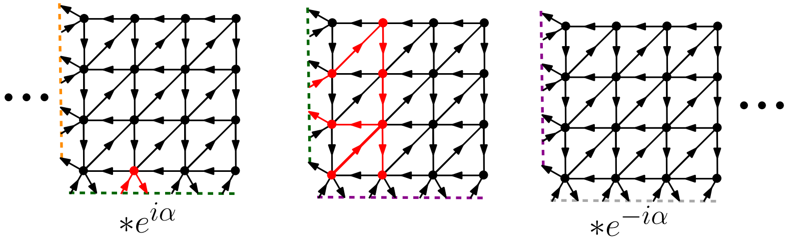

definition: $Q(t,\alpha,x,y) \equiv Q(x,y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)}$

This example contributes $t^8 x$.

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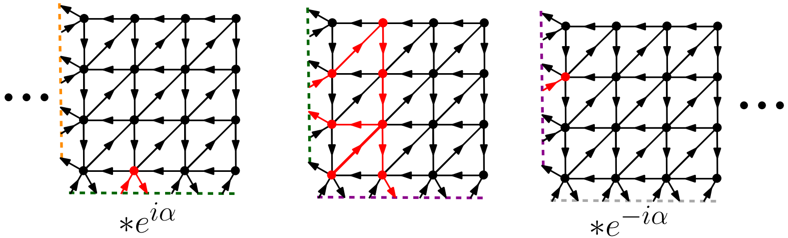

definition: $Q(t,\alpha,x,y) \equiv Q(x,y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)}$

This example contributes $t^9 y^2 e^{-i\alpha}$.

Tandem walks by winding number

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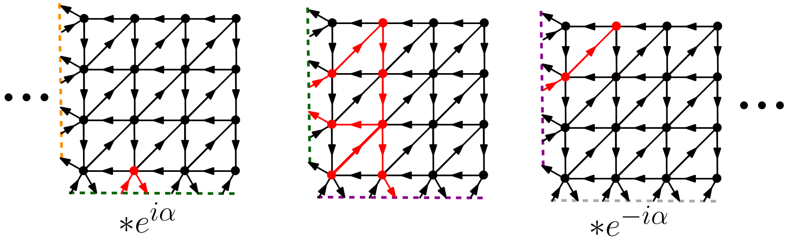

definition: $Q(t,\alpha,x,y) \equiv Q(x,y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)}$

This example contributes $t^{10}xy^3e^{-i\alpha}$.

Tandem walks by winding number

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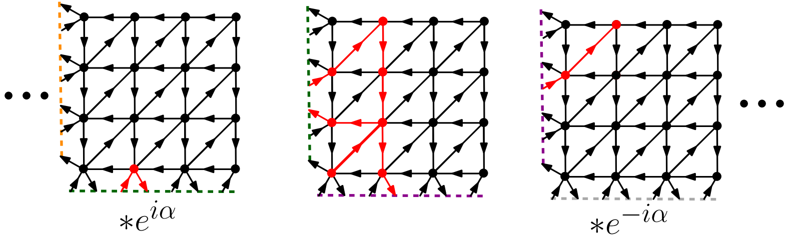

definition: $Q(t,\alpha,x,y) \equiv Q(x,y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)}$

Characterised by:

$$Q(x,y) = 1 + txyQ(x,y) + t\frac{Q(x,y)-Q(0,y)}{x} + t\frac{Q(x,y)-Q(x,0)}{y}$$

$$+e^{i\alpha}tQ(0,x) + e^{-i\alpha}tyQ(y,0).$$

TANDEM WALKS BY WINDING NUMBER

Equation to solve:

$$Q(x,y) = 1 + txyQ(x,y) + t\frac{Q(x,y) - Q(0,y)}{x} + t\frac{Q(x,y) - Q(x,0)}{y}$$
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$$+e^{i\alpha}tQ(0,x) + e^{-i\alpha}tyQ(y,0).$$

As in Kreweras solution:

Fix $t \in [0,1/3]$, $\alpha \in \mathbb{R}$. The series converge for $|x|, |y| < 1$.
Parametrise $K(x,y) = 1 - txy - t/y - t/x = 0$ by

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}}\vartheta(z,3\tau)\vartheta(z-\pi\tau,3\tau)}{\vartheta(z+\pi\tau,3\tau)\vartheta(z-2\pi\tau,3\tau)} \quad \text{and} \quad Y(z) = X(z+\pi\tau),$$

where τ is determined by $t = e^{-\frac{\pi\tau i}{3}}\frac{\vartheta'(0,3\tau)}{4i\vartheta(\pi\tau,3\tau)+6\vartheta'(\pi\tau,3\tau)}$.
For z near 0 the series converge, so when $x = X(z)$ and $y = Y(z)$:

$$1 = \frac{t}{x}Q(0,y) + \frac{t}{y}Q(x,0) - e^{i\alpha}tQ(0,x) - e^{-i\alpha}tyQ(y,0).$$

Tandem walks by winding number

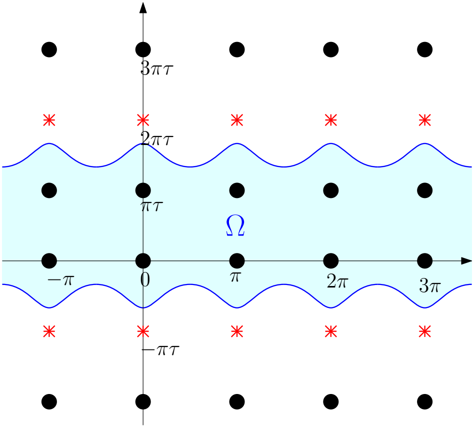
Equation to solve:

$$\begin{aligned}1 =& \frac{t}{X(z)}Q(0, X(z+\pi\tau)) + \frac{t}{X(z+\pi\tau)}Q(X(z), 0)\\ &- e^{i\alpha}tQ(0, X(z)) - e^{-i\alpha}tX(z+\pi\tau)Q(X(z+\pi\tau), 0),\end{aligned}$$

where

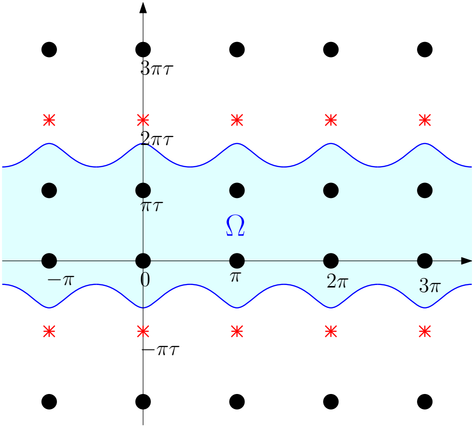
$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}}\vartheta(z, 3\tau)\vartheta(z-\pi\tau, 3\tau)}{\vartheta(z+\pi\tau, 3\tau)\vartheta(z-2\pi\tau, 3\tau)}.$$

Tandem walks by winding number



Recall: For $z \in \Omega$, $|X(z)| < 1$, so $Q(X(z), 0)$ and $Q(0, X(z))$ are well defined and holomorphic.

Tandem walks by winding number



Recall: For $z \in \Omega$, $|X(z)| < 1$, so $Q(X(z), 0)$ and $Q(0, X(z))$ are well defined and holomorphic. Near $z = 0$, $Q(X(z + \pi\tau), 0)$ and $Q(0, X(z + \pi\tau))$ are also well defined.

Tandem walks by winding number

Equation to solve:

$$1 = \frac{t}{X(z)}Q(0, X(z+\pi\tau)) + \frac{t}{X(z+\pi\tau)}Q(X(z), 0)$$
$$- e^{i\alpha}tQ(0, X(z)) - e^{-i\alpha}tX(z+\pi\tau)Q(X(z+\pi\tau), 0),$$

where

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}}\vartheta(z, 3\tau)\vartheta(z-\pi\tau, 3\tau)}{\vartheta(z+\pi\tau, 3\tau)\vartheta(z-2\pi\tau, 3\tau)}.$$

Recall: For $z \in \Omega$, $|X(z)| < 1$, so $Q(X(z), 0)$ and $Q(0, X(z))$ are well defined and holomorphic. Near $z = 0$, $Q(X(z+\pi\tau), 0)$ and $Q(0, X(z+\pi\tau))$ are also well defined.

TANDEM WALKS BY WINDING NUMBER

Equation to solve:

$$1 = \frac{t}{X(z)}Q(0, X(z+\pi\tau)) + \frac{t}{X(z+\pi\tau)}Q(X(z), 0) - e^{i\alpha}tQ(0, X(z)) - e^{-i\alpha}tX(z+\pi\tau)Q(X(z+\pi\tau), 0),$$

where

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}}\vartheta(z, 3\tau)\vartheta(z-\pi\tau, 3\tau)}{\vartheta(z+\pi\tau, 3\tau)\vartheta(z-2\pi\tau, 3\tau)}.$$

For z near 0, define

$$L(z) = \frac{t}{X(z+\pi\tau)}Q(X(z), 0) - e^{i\alpha}tQ(0, X(z)).$$

Both $L(z)$ and $L(z+\pi\tau)$ converge.

Recall: For $z \in \Omega$, $|X(z)| < 1$, so $Q(X(z), 0)$ and $Q(0, X(z))$ are well defined and holomorphic. Near $z = 0$, $Q(X(z+\pi\tau), 0)$ and $Q(0, X(z+\pi\tau))$ are also well defined.

Tandem walks by winding number

Equation to solve:

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where

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}}\vartheta(z, 3\tau)\vartheta(z - \pi\tau, 3\tau)}{\vartheta(z + \pi\tau, 3\tau)\vartheta(z - 2\pi\tau, 3\tau)}.$$

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Both $L(z)$ and $L(z + \pi\tau)$ converge.

Tandem walks by winding number

Equation to solve:

$$1 = \frac{t}{X(z)}Q(0, X(z+\pi\tau)) + L(z) - e^{-i\alpha}tX(z+\pi\tau)Q(X(z+\pi\tau), 0),$$

where

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}}\vartheta(z, 3\tau)\vartheta(z-\pi\tau, 3\tau)}{\vartheta(z+\pi\tau, 3\tau)\vartheta(z-2\pi\tau, 3\tau)}.$$

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Both $L(z)$ and $L(z+\pi\tau)$ converge.

Tandem walks by winding number

Equation to solve:

$$1 = \frac{t}{X(z)} Q(0, X(z+\pi\tau)) + L(z) - e^{-i\alpha} t X(z+\pi\tau) Q(X(z+\pi\tau), 0),$$

where

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}} \vartheta(z, 3\tau) \vartheta(z-\pi\tau, 3\tau)}{\vartheta(z+\pi\tau, 3\tau)\vartheta(z-2\pi\tau, 3\tau)}.$$

For z near 0, define

$$L(z) = \frac{t}{X(z+\pi\tau)} Q(X(z), 0) - e^{i\alpha} t Q(0, X(z)).$$

Both $L(z)$ and $L(z+\pi\tau)$ converge.

TANDEM WALKS BY WINDING NUMBER

Equation to solve:

$$1 = \frac{t}{X(z)} Q(0, X(z + \pi\tau)) + L(z) - e^{-i\alpha} t X(z + \pi\tau) Q(X(z + \pi\tau), 0),$$

where

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}} \vartheta(z, 3\tau) \vartheta(z - \pi\tau, 3\tau)}{\vartheta(z + \pi\tau, 3\tau) \vartheta(z - 2\pi\tau, 3\tau)}.$$

For z near 0, define

$$L(z) = \frac{t}{X(z + \pi\tau)} Q(X(z), 0) - e^{i\alpha} t Q(0, X(z)).$$

Both $L(z)$ and $L(z + \pi\tau)$ converge.

Tandem walks by winding number

Equation to solve:

$$1 = \frac{t}{X(z)} Q(0, X(z+\pi\tau)) + L(z) - \frac{e^{-i\alpha} t}{X(z)X(z+2\pi\tau)} Q(X(z+\pi\tau), 0),$$

where

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}} \vartheta(z, 3\tau) \vartheta(z-\pi\tau, 3\tau)}{\vartheta(z+\pi\tau, 3\tau)\vartheta(z-2\pi\tau, 3\tau)}.$$

For z near 0, define

$$L(z) = \frac{t}{X(z+\pi\tau)} Q(X(z), 0) - e^{i\alpha} t Q(0, X(z)).$$

Both $L(z)$ and $L(z+\pi\tau)$ converge.

TANDEM WALKS BY WINDING NUMBER

Equation to solve:

$$1 = \frac{t}{X(z)} Q(0, X(z+\pi\tau)) + L(z) - \frac{e^{-i\alpha} t}{X(z) X(z+2\pi\tau)} Q(X(z+\pi\tau), 0),$$

where

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}} \vartheta(z, 3\tau) \vartheta(z-\pi\tau, 3\tau)}{\vartheta(z+\pi\tau, 3\tau) \vartheta(z-2\pi\tau, 3\tau)}.$$

For z near 0, define

$$L(z) = \frac{t}{X(z+\pi\tau)} Q(X(z), 0) - e^{i\alpha} t Q(0, X(z)).$$

Both $L(z)$ and $L(z+\pi\tau)$ converge.

Tandem walks by winding number

Equation to solve:

$$1 = \frac{e^{i\alpha}}{X(z)} L(z + \pi\tau) + L(z).$$

where

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}} \vartheta(z, 3\tau) \vartheta(z - \pi\tau, 3\tau)}{\vartheta(z + \pi\tau, 3\tau) \vartheta(z - 2\pi\tau, 3\tau)}.$$

For z near 0, define

$$L(z) = \frac{t}{X(z + \pi\tau)} Q(X(z), 0) - e^{i\alpha} t Q(0, X(z)).$$

Both $L(z)$ and $L(z + \pi\tau)$ converge.

TANDEM WALKS BY WINDING NUMBER

Equation to solve:

$$1 = \frac{e^{i\alpha}}{X(z)} L(z + \pi\tau) + L(z).$$

where

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}} \vartheta(z, 3\tau)\vartheta\left(z - \pi\tau, 3\tau\right)}{\vartheta\left(z + \pi\tau, 3\tau\right)\vartheta\left(z - 2\pi\tau, 3\tau\right)}.$$

For z near 0, define

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Equation to solve:

$$1 = \frac{e^{i\alpha}}{X(z)} L(z + \pi\tau) + L(z).$$

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TANDEM WALKS BY WINDING NUMBER

Equation to solve:

$$1 = \frac{e^{i\alpha}}{X(z)} L(z+\pi\tau) + L(z).$$

where

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}} \vartheta(z, 3\tau) \vartheta\left(z - \pi\tau, 3\tau\right)}{\vartheta\left(z+\pi\tau, 3\tau\right) \vartheta\left(z - 2\pi\tau, 3\tau\right)}.$$

We can solve this exactly:

$$L(z) = \frac{e^{3i\alpha}}{1-e^{3i\alpha}}\left(1 - \frac{e^{i\alpha}}{X(z)} - e^{2i\alpha} X(z-\pi\tau)\right)$$
$$- \frac{e^{i\alpha + \frac{2i\pi\tau}{3}} \vartheta(\pi\tau, 3\tau) \vartheta'(0,\tau)}{(1-e^{3i\alpha}) \vartheta(\frac{\alpha}{2} + \frac{\pi\tau}{3}, \tau) \vartheta'(0, 3\tau)} \frac{\vartheta(z+\pi\tau, 3\tau)\vartheta(z - \frac{\alpha}{2} - \frac{\pi\tau}{3}, \tau)}{\vartheta(z,\tau)\vartheta(z, 3\tau)}$$

Tandem walks by winding number

Equation to solve:

$$1 = \frac{e^{i\alpha}}{X(z)} L(z + \pi\tau) + L(z).$$

where

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}} \vartheta(z, 3\tau)\vartheta(z - \pi\tau, 3\tau)}{\vartheta(z + \pi\tau, 3\tau)\,\vartheta(z - 2\pi\tau, 3\tau)}.$$

We can solve this exactly:

$$L(z) = \frac{e^{3i\alpha}}{1 - e^{3i\alpha}} \left(1 - \frac{e^{i\alpha}}{X(z)} - e^{2i\alpha} X(z - \pi\tau)\right)$$
$$- \frac{e^{i\alpha + \frac{2i\pi\tau}{3}} \vartheta(\pi\tau, 3\tau)\vartheta'(0, \tau)}{(1 - e^{3i\alpha})\vartheta(\frac{\alpha}{2} + \frac{\pi\tau}{3}, \tau)\vartheta'(0, 3\tau)} \frac{\vartheta(z + \pi\tau, 3\tau)\vartheta(z - \frac{\alpha}{2} - \frac{\pi\tau}{3}, \tau)}{\vartheta(z, \tau)\vartheta(z, 3\tau)}$$

We can extract the solution for $Q(0, 0)$...

Tandem walks by winding number: Solution

τ is determined by

$$t = e^{-\frac{\pi\tau i}{3}} \frac{\vartheta'(0,3\tau)}{4i\vartheta(\pi\tau,3\tau) + 6\vartheta'(\pi\tau,3\tau)}.$$

Then $Q(0,0) \equiv Q(t,\alpha,0,0)$ is given by:

$$Q(0,0) = \frac{e^{\alpha i + \frac{2\pi\tau i}{3}}}{1 - a^{3\alpha i}} \left(e^{\alpha i - \frac{\pi\tau i}{3}} + \frac{\vartheta'(\pi\tau,3\tau)}{\vartheta'(0,3\tau)} - \frac{\vartheta(\pi\tau,3\tau)\vartheta'(\frac{\alpha}{2}+\frac{\pi\tau}{3},\tau)}{\vartheta'(0,3\tau)\vartheta(\frac{\alpha}{2}+\frac{\pi\tau}{3},\tau)} \right).$$

TANDEM WALKS BY WINDING NUMBER: SOLUTION

τ is determined by

$$t = e^{-\frac{\pi\tau i}{3}} \frac{\vartheta'(0, 3\tau)}{4i\vartheta(\pi\tau, 3\tau) + 6\vartheta'(\pi\tau, 3\tau)}.$$

Then $Q(0,0) \equiv Q(t, \alpha, 0, 0)$ is given by:

$$Q(0,0) = \frac{e^{\alpha i + \frac{2\pi\tau i}{3}}}{1 - a^{3\alpha i}} \left(e^{\alpha i - \frac{\pi\tau i}{3}} + \frac{\vartheta'(\pi\tau, 3\tau)}{\vartheta'(0, 3\tau)} - \frac{\vartheta(\pi\tau, 3\tau)\vartheta'(\frac{\alpha}{2} + \frac{\pi\tau}{3}, \tau)}{\vartheta'(0, 3\tau)\vartheta(\frac{\alpha}{2} + \frac{\pi\tau}{3}, \tau)} \right).$$

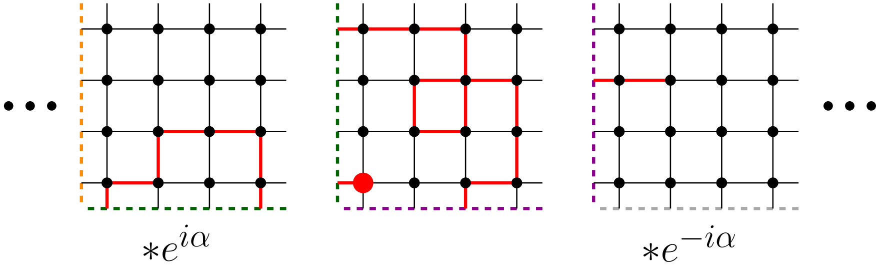
The generating function $Q(t, \alpha, 0, 0)$ is D-algebraic but *not* D-finite.

Asymptotics:

- For $\alpha \in i\mathbb{R} \setminus \{0\}$, the dominant singularity is a simple pole.
- Variance V_n for the winding angle of paths of size n behaves like

$$V_n \sim c \log(n)^2.$$

Part 4b: Square lattice walks by winding number

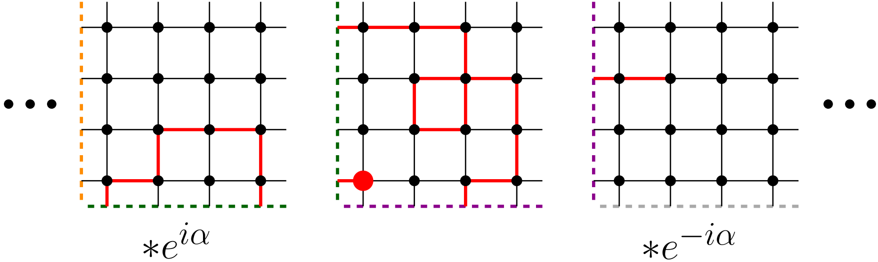


SQUARE LATTICE WALKS BY WINDING NUMBER

- Counted exactly in terms of theta functions (2017 Budd)
- Solution involved an eigenvalue decomposition of commuting Hilbert space operators.
- Corollaries include the exact enumeration of Gessel walks.
- **Alternative solution:** very similar to the last solution.

Functional equation:

$$Q(x,y) = 1 + txQ(x,y) + tyQ(x,y) + \frac{t}{x}\left(Q(x,y) - Q(0,y)\right) + \frac{t}{y}\left(Q(x,y) - Q(x,0)\right) + e^{i\alpha}tQ(0,x) + e^{-i\alpha}tQ(y,0)$$



Square lattice walks by winding number

Functional equation:

$$Q(x,y) = 1 + txQ(x,y) + tyQ(x,y) + \frac{t}{x}\left(Q(x,y) - Q(0,y)\right) + \frac{t}{y}\left(Q(x,y) - Q(x,0)\right) + e^{i\alpha}tQ(0,x) + e^{-i\alpha}tQ(y,0)$$

The kernel equation

$$K(x,y) = 1 - tx - ty - \frac{t}{x} - \frac{t}{y} = 0$$

is parametrised by

$$X(z) = \frac{e^{-\pi\tau i}\vartheta(z,4\tau)\vartheta(z-\pi\tau,4\tau)}{e^{-\pi\tau i}\vartheta(z+\pi\tau,4\tau)\vartheta(z-2\pi\tau,4\tau)} \quad \text{and} \quad Y(z) = X(z+\pi\tau),$$

where τ is determined by

$$t = \frac{e^{-\pi\tau i}\vartheta'(0,4\tau)\vartheta(\pi\tau,4\tau)}{2\vartheta(2\pi\tau,4\tau)\vartheta'(\pi\tau,4\tau) - \vartheta(\pi\tau,4\tau)\vartheta'(2\pi\tau,4\tau)}$$

SQUARE LATTICE WALKS BY WINDING NUMBER

Recall: τ is determined by

$$t = \frac{e^{-\pi\tau i}\vartheta'(0,4\tau)\vartheta(\pi\tau,4\tau)}{2\vartheta(2\pi\tau,4\tau)\vartheta'(\pi\tau,4\tau)-\vartheta(\pi\tau,4\tau)\vartheta'(2\pi\tau,4\tau)}$$

Solution: Expression for $Q(0,0)$ similar to tandem case, but bigger.

SQUARE LATTICE WALKS BY WINDING NUMBER

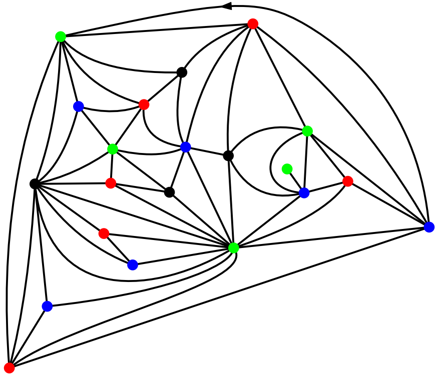
Recall: τ is determined by

$$t = \frac{e^{-\pi\tau i}\vartheta'(0,4\tau)\vartheta(\pi\tau,4\tau)}{2\vartheta(2\pi\tau,4\tau)\vartheta'(\pi\tau,4\tau) - \vartheta(\pi\tau,4\tau)\vartheta'(2\pi\tau,4\tau)}$$

Solution: Expression for $Q(0,0)$ similar to tandem case, but bigger.

As usual, asymptotic and algebraic information can be extracted.

Part 4c: Properly coloured triangulations



Coloured triangulations: Background

Tutte showed that the generating function for properly coloured triangulations counted by faces is $sT(t,s,1,0)-s(s-1)$, where $T(t,s,x,y)\equiv T(x,y)$ is determined by

$$T(x,y)=x(s-1)+xytT(1,y)T(x,y)$$
$$+\frac{xt}{y}(T(x,y)-T(x,0))-x^2yt\frac{T(x,y)-T(1,y)}{x-1}$$

Background:

- Tutte 1982: The series $H(t)=t^2T(\sqrt{t},s,1,0)$ satisfies

  $$2(1-s)t+(t+10H+6tH)H''+(4-s)(20H-18tH'+9t^2H'')=0.$$

- Guttmann and Bousquet-Mélou: predicted (proved in some cases) asymptotic behaviour for a range of fixed values of s
- *More precise* asymptotic behaviour can be *proven* with theta functions!

Coloured triangulations: Theta solution

Functional equation:

$$\begin{aligned} T(x,y) &= x(s-1) + xytT(1,y)T(x,y) \\ &\quad + \frac{xt}{y}(T(x,y) - T(x,0)) - x^2yt\frac{T(x,y)-T(1,y)}{x-1} \end{aligned}$$

Want to parametrize the Kernel equation

$$K(x,y) := -1 + xytT(1,y) + \frac{xt}{y} - \frac{x^2yt}{x-1} = 0,$$

as then the remainder

$$R(x,y) := x(s-1) - \frac{xt}{y}T(x,0) + x^2yt\frac{T(1,y)}{x-1} = 0.$$

Guess: There is some parametrization $(X(z), Y(z))$ satisfying

- $K(X(z), Y(z)) = 0$ and therefore $R(X(z), Y(z)) = 0$ (near $z = 0$).
- $X(z+\pi) = X(z)$ and $Y(z+\pi) = Y(z)$.
- $X(-z) = X(z)$ and $Y(\pi\tau - z) = Y(z)$.

COLOURED TRIANGULATIONS: THETA SOLUTION

Functional equation:

$$0 = T(x,y)K(x,y) + R(x,y),$$

where $$K(x,y) = -1 + xytT(1,y) + \frac{xt}{y} - \frac{x^2yt}{x-1}$$

and $$R(x,y) = x(s-1) - \frac{xt}{y}T(x,0) + x^2yt\frac{T(1,y)}{x-1}$$

Want to parametrize $K(x,y) = 0$ as then $R(x,y) = 0$.
Guess: There is some parametrization $(X(z), Y(z))$ satisfying

- $K(X(z), Y(z)) = 0$ and therefore $R(X(z), Y(z)) = 0$ (near $z = 0$).
- $X(z+\pi) = X(z)$ and $Y(z+\pi) = Y(z)$.
- $X(-z) = X(z)$ and $Y(\pi\tau - z) = Y(z)$.

Plan: Solve under this assumption then check the solution.

Coloured triangulations: Theta solution

Functional equation:

$$0 = T(x,y)K(x,y) + R(x,y),$$

$$\text{where} \qquad K(x,y) = -1 + xytT(1,y) + \frac{xt}{y} - \frac{x^2yt}{x-1}$$

$$\text{and} \qquad R(x,y) = x(s-1) - \frac{xt}{y}T(x,0) + x^2yt\frac{T(1,y)}{x-1}$$

Want to parametrize $K(x,y) = 0$ as then $R(x,y) = 0$.
Guess: There is some parametrization $(X(z), Y(z))$ satisfying

- $K(X(z), Y(z)) = 0$ and therefore $R(X(z), Y(z)) = 0$ (near $z = 0$).
- $X(z+\pi) = X(z)$ and $Y(z+\pi) = Y(z)$.
- $X(-z) = X(z)$ and $Y(\pi\tau - z) = Y(z)$.

Plan: Solve under this assumption then check the solution.
Kernel not explicit, but theta method still works

COLOURED TRIANGULATIONS: THETA SOLUTION

Equations to solve:

- $K(X(z), Y(z)) = 0.$
- $R(X(z), Y(z)) = 0.$
- $X(z+\pi) = X(z)$ and $Y(z+\pi) = Y(z).$
- $X(-z) = X(z)$ and $Y(\pi\tau - z) = Y(z)$

Solving the equations: Use an "invariant function" found by Tutte

$$I(z) = t^2T(X(z),0) + t^2\frac{X(z)}{X(z)-1} - \frac{X(z)-1}{X(z)^2}$$
$$= \frac{t}{Y(z)^2} - \frac{t}{Y(z)} + stY(z) + (2t - Y(z))tT(1,Y(z))$$
$$+t^2Y(z)^2T(1,Y(z))^2.$$

Then $I(z) = I(-z)$ and $I(z) = I(\pi\tau - z)$, so $I(z)$ is elliptic.

Coloured triangulations: Theta solution

We end up with a solution involving β, related to s by $s = e^{2i\beta} + 2 + e^{-2i\beta}$.

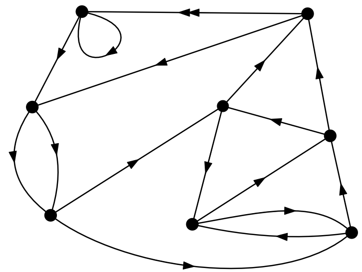
$$t^2 = -\frac{(e^{\beta i} + e^{-\beta i})^2 \vartheta(\beta)(\vartheta'(\beta)\vartheta^{(3)}(0) - \vartheta'(0)\vartheta^{(3)}(\beta))}{24(e^{i\beta} - e^{-i\beta})^4 \vartheta'(0)\vartheta'(\beta)^3}$$

$$\frac{1}{X(z)} = \frac{e^{2\beta i} - 1 + e^{-2\beta i}}{e^{\beta i} - e^{-\beta i}} + \frac{\vartheta'(0)\left(\vartheta(z+\beta) + \vartheta(z-\beta)\right)}{(e^{\beta i} - e^{-\beta i})^2 \vartheta'(\beta)\vartheta(z)}$$

$$Y(z) = -\frac{1}{t}\frac{(X(z)-1)(X(\pi\tau - z) - 1)}{X(z)X(\pi\tau - z)}$$

$T(1,0) =$ EXPRESSION TO BIG FOR SLIDE

Part 4d: Six vertex model on 4-valent planar maps



Functional equation:

- Kostov, 1999: Exactly, but non-rigorously solved in terms of elliptic functions
  - Solution involved matrix integrals and analysis of the limiting spectral density
- E.P. and Zinn-Justin, 2019+: Kostov's solution made rigorous
  - Mistake corrected
  - Matrix integrals converted to functional equations
  - Answer checked (using functional equations)
- E.P. and Bousquet-Mélou, 2019+: Another proof using completely different functional equations

Six vertex model, solution 1 (E.P., Zinn-Justin)

Equations to solve:

$$\mathsf{W}(x) = x^2 t\mathsf{W}(x)^2 + e^{i\alpha} xt\mathsf{H}(0,x) - e^{-i\alpha} xt\mathsf{H}(x,0) + 1$$

$$\mathsf{H}(x) = \mathsf{W}(x)\mathsf{W}(y) + \frac{e^{i\alpha}}{y}(\mathsf{H}(x,y) - \mathsf{H}(x,0)) - \frac{e^{-i\alpha}}{x}(\mathsf{H}(x,y) - \mathsf{H}(0,y))$$

Vague idea of solution: Parametrise Kernel equation

$$1 - \frac{e^{i\alpha}}{y} + \frac{e^{-i\alpha}}{x} = 0$$

by

$$\frac{1}{X(z)} = \frac{1}{e^{i\alpha} - e^{-i\alpha}} + c\frac{\vartheta(z+\alpha,\tau)}{\vartheta(z,\tau)} \quad \text{and} \quad Y(z) = X(z+\pi\tau).$$

In the end, $\omega X(z)W(X(z)) + \omega^{-1}Y(z)W(Y(z)) + R(X(z))$ is elliptic for some explicit rational function R.

SIX VERTEX MODEL, SOLUTION 2 (MBM, E.P.)

Lots of equations... after some manipulation we end up with
Equations to solve:

$$\mathsf{H}(x) = \frac{1 - \omega x - \sqrt{1 - \omega x + \omega^2 x^2 - 4\mathsf{F}(x)/x}}{2}$$
$$\mathsf{H}(x) = \mathsf{H}^{-1}(x)$$

Vague idea of solution: Find a parametrization $X(z)$ satisfying $X(z+\pi) = X(z) = X(-z)$ and $H(X(z)) = X(\pi\tau - z)$.
Using both equations, deduce

$$X(z+\pi\tau) + \omega X(z) + X(z-\pi\tau) = 1,$$

which is solvable in terms of theta functions, leading to a complete solution.

Six vertex model solution

With either solution we (of course) get the same answer:
τ is determined by

$$t = \frac{\cos\alpha}{64\sin^3\alpha}\left(\frac{\vartheta(\alpha,\tau)\vartheta'''(\alpha,\tau)}{\vartheta'(\alpha,\tau)^2} + \frac{\vartheta''(\alpha,\tau)}{\vartheta'(\alpha,\tau)}\right).$$

An auxiliary series $\mathsf{R}(t,\gamma)$ is determined by

$$\mathsf{R}(t,-2\cos(2\alpha)) = \frac{\cos^2\alpha}{96\sin^4\alpha}\frac{\vartheta(\alpha,q)^2}{\vartheta'(\alpha,q)^2}\left(-\frac{\vartheta'''(\alpha,q)}{\vartheta'(\alpha,q)} + \frac{\vartheta'''(0,q)}{\vartheta'(0,q)}\right).$$

Finally the series in question is

$$\mathsf{Q}(t,\omega) = \frac{1}{(\omega+2)t^2}\left(t - (\omega+2)t^2 - \mathsf{R}(t,\omega)\right).$$

Part 4e: Final comments

Reasons to try theta functions

- Functional equation has two catalytic variables.
- Kernel is quadratic, so equation $K(x, y) = 0$ is parametrized by elliptic functions.
- Both $Q(0, y)$ and $Q(0, x)$ appear in the equation (or something like that).
- a doubly connected domain is involved.
- You expect one (or both) of the following:
  - The solution simplifies when a weight is either $e^{i\pi\tau}$ or $\cos(\pi r)$ for $r \in \mathbb{Q}$.
  - The solution is D-algebraic.

FURTHER AIMS

- Solve more problems
- streamline these methods
- strengthen these methods
- Convert techniques to world of formal power series and rely less on complex analysis
- Find combinatorial interpretations of theta function expressions and prove formulas bijectively.

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- Sur quelques formules relatives à la transformation des fonctions elliptiques (1858, Hermite)
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- The six vertex model on planar maps (Bousquet-Mélou and EP).
- Walks by winding number on the Kreweras lattice (EP).
- Properly coloured triangulations (EP).
- Distribution of height functions on quadrangulations (EP).

Thank you!