Theta functions in enumerative combinatorics

Week 4: New solutions

Andrew Elvey Price

Université de Bordeaux

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**Recap (Week 1+): Properties of $\vartheta(z, \tau)$**

**Definition:** For $\tau, z \in \mathbb{C}$, $\text{im}(\tau) > 0$,

$$\vartheta(z, \tau) = \sum_{n=-\infty}^{\infty} (-1)^n e^{\left(\frac{2n+1}{2}\right)^2 i\pi \tau + (2n+1)iz}$$

**Useful equations involving $\vartheta(z, \tau)$:**

- $\vartheta(-z, \tau) = -\vartheta(z, \tau)$
- $\vartheta(z + \pi, \tau) = -\vartheta(z, \tau)$
- $\vartheta(z + \pi \tau, \tau) = -e^{-2iz - i\pi \tau} \vartheta(z, \tau)$
- $\frac{4i}{\pi} \frac{\partial}{\partial \tau} \vartheta(z, \tau) = \vartheta''(z, \tau)$
- $\vartheta(z, \tau + 1) = e^{i\pi/4} \vartheta(z, \tau)$
- $\vartheta\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = -i(-i\tau)^{1/2} \exp\left(\frac{i}{\pi \tau} z^2\right) \vartheta(z, \tau)$
**Recap (Week 1+): Properties of \( \vartheta(z, \tau) \)**

**Definition:** For \( \tau, z \in \mathbb{C} \), \( \text{im}(\tau) > 0 \),

\[
\vartheta(z, \tau) = \sum_{n=-\infty}^{\infty} (-1)^n e^{(\frac{2n+1}{2})^2 i \tau + (2n+1)iz}
\]

**Other facts about \( \vartheta(z, \tau) \):**

- \( \vartheta(z, \tau) \) is differentially algebraic (in both variables)
- **Liouville’s elliptic theorem:** Any elliptic function with no poles is constant
- All elliptic functions \( f(z) \) with periods \( \pi \) and \( \pi \tau \) can be written as

\[
f(z) = c \prod_{j=1}^{k} \frac{\vartheta(z + \alpha_j, \tau)}{\vartheta(z + \beta_j, \tau)}.
\]

- Any two elliptic functions with the same periods are algebraically related
Recap (Week 1+): Conformal parametrizations

\[ \frac{\vartheta'(z - \frac{\pi}{2}, \tau)}{\vartheta(z - \frac{\pi}{2}, \tau)} \]
**Recap (Week 1+): Conformal parametrizations**

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Recap (Week 1+): Conformal parametrizations

\[ C \theta(z, 2\tau) \]
**Recap (Week 2): Kreheras paths**

**Kreheras paths:**

\[ Q(x, y) = 1 + xytQ(x, y) + \frac{t}{x} (Q(x, y) - Q(0, y)) + \frac{t}{y} (Q(x, y) - Q(x, 0)) . \]

Kernel equation \( K(x, y) = 1 - xyt - \frac{t}{x} - \frac{t}{y} = 0 \) is parametrised by

\[ X(z) = \frac{e^{-\frac{4\pi ti}{3}} \vartheta(z, 3\tau) \vartheta(z - \pi\tau, 3\tau)}{\vartheta(z + \pi\tau, 3\tau) \vartheta(z - 2\pi\tau, 3\tau)}, \quad Y(z) = X(z + \pi\tau) . \]

Using we solved for \( Q(x, 0) \):

\[ t = e^{-\frac{\pi ti}{3}} \frac{\vartheta'(0, 3\tau)}{4i\vartheta(\pi\tau, 3\tau) + 6\vartheta'(\pi\tau, 3\tau)} \]

\[ I(t, X(z)) = e^{\frac{\pi ti}{3}} \frac{\vartheta(\pi\tau, 3\tau)}{\vartheta'(0, 3\tau)} \left( \frac{\vartheta'(z, 2\tau)}{\vartheta(z, 2\tau)} - \frac{\vartheta'(z + \pi\tau, 2\tau)}{\vartheta(z + \pi\tau, 2\tau)} - i \right) + \frac{1}{2t} . \]

\[ Q(x, 0) = \frac{I(x)}{tx} - \frac{1}{tx^2} . \]
Recap (Week 2): Quadrant Walks

Any small (non-trivial) step set: Equation to solve:

\[ Q(x, y)K(x, y) = xy - P_1(y)Q(0, y) - P_2(x)Q(x, 0) + c. \]

The kernel equation \( K(x, y) = 0 \) is parameterised by

\[
X(z) = c_1 \frac{\vartheta(z - \alpha_1)\vartheta(z - \beta_1)}{\vartheta(z - \gamma_1)\vartheta(z - \delta_1)} \quad \text{and} \quad Y(z) = c_2 \frac{\vartheta(z - \alpha_2)\vartheta(z - \beta_2)}{\vartheta(z - \gamma_2)\vartheta(z - \delta_2)}. 
\]

- **D-finite case:** \( \alpha_2 + \beta_2 - \alpha_1 - \beta_1 \in \pi \mathbb{Q} \).
- **D-algebraic case:** \( X(z)Y(z) = R_1(X(z)) + R_2(Y(z)). \)
Recap (Week 3): Analysing Solutions

- Deriving asymptotics using the equation

\[ \vartheta \left( \frac{z}{\tau}, -\frac{1}{\tau} \right) = -i(-i\tau)^{\frac{1}{2}} \exp \left( \frac{i}{\pi \tau} z^2 \right) \vartheta(z, \tau) \]

- Relation of \( \vartheta \left( \frac{m}{n} \pi, \tau \right) \) to modular functions

- Some situations where theta function parametrisations are Algebraic or D-finite
TODAY (FINAL WEEK):

New work: More problems solvable with theta functions:

- Directed triangular lattice walks (tandem walks) by winding number
- Properly coloured triangulations (Previously shown to be D-algebraic by Tutte)
- Six vertex model on 4-valent maps (with Zinn-Justin, following Kostov)
Part 4a: Tandem walks by winding number
**TANDEM WALKS BY WINDING NUMBER**

**The model:** count walks starting at the red point by end point and number of times winding around the blue point.
**TANDEM WALKS BY WINDING NUMBER**

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**Tandem Walks by Winding Number**

**The model:** count walks starting at the red point by end point.
TANDEM WALKS BY WINDING NUMBER

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**Tandem walks by winding number**

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**Tandem walks by winding number**

**The model:** count walks starting at the red point by end point.

\[ Q(t, \alpha, x, y) \equiv Q(x, y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)} \]

**Definition:** The model involves counting walks starting at the red point by end point. The enumeration is given by the formula above, where \( Q(t, \alpha, x, y) \) is the generating function for counting walks starting at \( x \) and ending at \( y \), with the exponent \( n(p) \) representing the length of the walk. The factor \( e^{i\alpha} \) accounts for the winding number of the walk.
**Tandem walks by winding number**

**The model:** count walks starting at the red point by end point.

![Diagram of tandem walks](image)

**Definition:**

\[ Q(t, \alpha, x, y) \equiv Q(x, y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)} \]

This example contributes \( txy \).
**TANDEM WALKS BY WINDING NUMBER**

**The model:** count walks starting at the red point by end point.

![Diagram of tandem walks]

**definition:** \( Q(t, \alpha, x, y) \equiv Q(x, y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)} \)

This example contributes \( t^2 y \).
**TANDEM WALKS BY WINDING NUMBER**

**The model:** count walks starting at the red point by end point.

![Diagram showing tandem walks with arrows and labels](image)

**definition:** \( Q(t, \alpha, x, y) \equiv Q(x, y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)} \)

This example contributes \( t^3 xe^{i\alpha} \).
**TANDEM WALKS BY WINDING NUMBER**

**The model:** count walks starting at the red point by end point.

**definition:** \( Q(t, \alpha, x, y) \equiv Q(x, y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)} \)

This example contributes \( t^4 y^2 \).
**Tandem walks by winding number**

**The model:** count walks starting at the red point by end point.

**Definition:** \( Q(t, \alpha, x, y) \equiv Q(x, y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)} \)

This example contributes \( t^5 xy^3 \).
**Tandem Walks by Winding Number**

**The model:** count walks starting at the red point by end point.

![Diagram showing tandem walks](image)

**Definition:** \( Q(t, \alpha, x, y) \equiv Q(x, y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)} \)

This example contributes \( t^6xy^2 \).
**TANDEM WALKS BY WINDING NUMBER**

**The model:** count walks starting at the red point by end point.

\[ Q(t, \alpha, x, y) \equiv Q(x, y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)} \]

This example contributes \( t^7 xy \).
**TANDEM WALKS BY WINDING NUMBER**

**The model:** count walks starting at the red point by end point.

![Diagram of tandem walks](image)

**definition:** \( Q(t, \alpha, x, y) \equiv Q(x, y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)} \)

This example contributes \( t^8 x \).
**Tandem walks by winding number**

**The model:** count walks starting at the red point by end point.

![Diagram of tandem walks with labels $e^{i\alpha}$ and $e^{-i\alpha}$.

**Definition:** $Q(t, \alpha, x, y) \equiv Q(x, y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)}$

This example contributes $t^9 y^2 e^{-i\alpha}$.
**TANDEM WALKS BY WINDING NUMBER**

**The model:** count walks starting at the red point by end point.

![Diagram of tandem walks](image)

**definition:** \( Q(t, \alpha, x, y) \equiv Q(x, y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)} \)

This example contributes \( t^{10} x y^3 e^{-i\alpha} \).
**Tandem walks by winding number**

**The model:** count walks starting at the red point by end point.

![Diagram showing tandem walks with exponential factors $e^{i\alpha}$ and $e^{-i\alpha}$]

**Definition:**

$$Q(t, \alpha, x, y) \equiv Q(x, y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)}$$

**Characterised by:**

$$Q(x, y) = 1 + txyQ(x, y) + t \frac{Q(x, y) - Q(0, y)}{x} + t \frac{Q(x, y) - Q(x, 0)}{y}$$

$$+ e^{i\alpha} tQ(0, x) + e^{-i\alpha} tyQ(y, 0).$$
TANDEM WALKS BY WINDING NUMBER

Equation to solve:

\[ Q(x, y) = 1 + txyQ(x, y) + t \frac{Q(x, y) - Q(0, y)}{x} + t \frac{Q(x, y) - Q(x, 0)}{y} \]

\[ + e^{i\alpha} tQ(0, x) + e^{-i\alpha} tyQ(y, 0). \]
TANDEM WALKS BY WINDING NUMBER

Equation to solve:

\[ Q(x, y) = 1 + t x y Q(x, y) + t \frac{Q(x, y) - Q(0, y)}{x} + t \frac{Q(x, y) - Q(x, 0)}{y} + e^{i\alpha} t Q(0, x) + e^{-i\alpha} t y Q(y, 0). \]

As in Kreweras solution:

Fix \( t \in [0, 1/3], \alpha \in \mathbb{R} \). The series converge for \( |x|, |y| < 1 \).

Parametrise \( K(x, y) = 1 - t x y - t/y - t/x = 0 \) by

\[
X(z) = \frac{e^{-\frac{4\pi\tau i}{3}} \vartheta(z, 3\tau) \vartheta(z - \pi\tau, 3\tau)}{\vartheta(z + \pi\tau, 3\tau) \vartheta(z - 2\pi\tau, 3\tau)} \quad \text{and} \quad Y(z) = X(z + \pi\tau),
\]

where \( \tau \) is determined by \( t = e^{-\frac{\pi\tau i}{3}} \frac{\vartheta'(0, 3\tau)}{4i\vartheta(\pi\tau, 3\tau) + 6\vartheta'(\pi\tau, 3\tau)} \).

For \( z \) near 0 the series converge, so when \( x = X(z) \) and \( y = Y(z) \):

\[ 1 = \frac{t}{x} Q(0, y) + \frac{t}{y} Q(x, 0) - e^{i\alpha} t Q(0, x) - e^{-i\alpha} t y Q(y, 0). \]
**TANDEM WALKS BY WINDING NUMBER**

**Equation to solve:**

\[
1 = \frac{t}{X(z)} Q(0, X(z + \pi \tau)) + \frac{t}{X(z + \pi \tau)} Q(X(z), 0) \\
- e^{i\alpha t} Q(0, X(z)) - e^{-i\alpha t} X(z + \pi \tau) Q(X(z + \pi \tau), 0),
\]

where

\[
X(z) = \frac{e^{-\frac{4\pi \tau i}{3}} \vartheta(z, 3\tau) \vartheta(z - \pi \tau, 3\tau)}{\vartheta(z + \pi \tau, 3\tau) \vartheta(z - 2\pi \tau, 3\tau)}.
\]
Recall: For $z \in \Omega$, $|X(z)| < 1$, so $Q(X(z), 0)$ and $Q(0, X(z))$ are well defined and holomorphic.
Recall: For \( z \in \Omega, \quad |X(z)| < 1, \) so \( Q(X(z), 0) \) and \( Q(0, X(z)) \) are well defined and holomorphic. Near \( z = 0, \) \( Q(X(z + \pi \tau), 0) \) and \( Q(0, X(z + \pi \tau)) \) are also well defined.
**Equation to solve:**

\[ 1 = \frac{t}{X(z)} Q(0, X(z + \pi \tau)) + \frac{t}{X(z + \pi \tau)} Q(X(z), 0) \]

\[ - e^{i \alpha t} Q(0, X(z)) - e^{-i \alpha t} X(z + \pi \tau) Q(X(z + \pi \tau), 0), \]

where

\[ X(z) = \frac{e^{-\frac{4 \pi \tau i}{3}} \vartheta(z, 3 \tau) \vartheta(z - \pi \tau, 3 \tau)}{\vartheta(z + \pi \tau, 3 \tau) \vartheta(z - 2 \pi \tau, 3 \tau)}. \]

**Recall:** For \( z \in \Omega, \ |X(z)| < 1, \) so \( Q(X(z), 0) \) and \( Q(0, X(z)) \) are well defined and holomorphic. Near \( z = 0, \ Q(X(z + \pi \tau), 0) \) and \( Q(0, X(z + \pi \tau)) \) are also well defined.
Tandem walks by winding number

Equation to solve:

\[
1 = \frac{t}{X(z)} Q(0, X(z + \pi \tau)) + \frac{t}{X(z + \pi \tau)} Q(X(z), 0) \\
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\[
X(z) = \frac{e^{-\frac{4\pi i}{3} \vartheta(z, 3\tau) \vartheta(z - \pi \tau, 3\tau)}}{\vartheta(z + \pi \tau, 3\tau) \vartheta(z - 2\pi \tau, 3\tau)}.
\]

For \( z \) near 0, define

\[
L(z) = \frac{t}{X(z + \pi \tau)} Q(X(z), 0) - e^{i\alpha t} Q(0, X(z)).
\]

Both \( L(z) \) and \( L(z + \pi \tau) \) converge.

Recall: For \( z \in \Omega, \ |X(z)| < 1 \), so \( Q(X(z), 0) \) and \( Q(0, X(z)) \) are well defined and holomorphic. Near \( z = 0 \), \( Q(X(z + \pi \tau), 0) \) and \( Q(0, X(z + \pi \tau)) \) are also well defined.
TANDEM WALKS BY WINDING NUMBER

Equation to solve:

$$1 = \frac{t}{X(z)} Q(0, X(z + \pi \tau)) + \frac{t}{X(z + \pi \tau)} Q(X(z), 0)$$

$$- e^{i\alpha t} Q(0, X(z)) - e^{-i\alpha t} X(z + \pi \tau) Q(X(z + \pi \tau), 0),$$

where

$$X(z) = e^{-\frac{4\pi i}{3} \vartheta(z, 3\tau) \vartheta(z - \pi \tau, 3\tau)} \vartheta(z + \pi \tau, 3\tau) \vartheta(z - 2\pi \tau, 3\tau).$$

For $z$ near 0, define

$$L(z) = \frac{t}{X(z + \pi \tau)} Q(X(z), 0) - e^{i\alpha t} Q(0, X(z)).$$

Both $L(z)$ and $L(z + \pi \tau)$ converge.
Equation to solve:

\[
1 = \frac{t}{X(z)} Q(0, X(z + \pi \tau)) + L(z) \\
- e^{-i\alpha t} X(z + \pi \tau) Q(X(z + \pi \tau), 0),
\]

where

\[
X(z) = e^{-\frac{4\pi i \tau}{3}} \frac{\vartheta(z, 3\tau) \vartheta(z - \pi \tau, 3\tau)}{\vartheta(z + \pi \tau, 3\tau) \vartheta(z - 2\pi \tau, 3\tau)}.
\]

For \( z \) near 0, define

\[
L(z) = \frac{t}{X(z + \pi \tau)} Q(X(z), 0) - e^{i\alpha t} Q(0, X(z)).
\]

Both \( L(z) \) and \( L(z + \pi \tau) \) converge.
TANDEM WALKS BY WINDING NUMBER

Equation to solve:

\[ 1 = \frac{t}{X(z)} Q(0, X(z + \pi \tau)) + L(z) \]

\[- e^{-i\alpha t} X(z + \pi \tau) Q(X(z + \pi \tau), 0), \]

where

\[ X(z) = e^{-\frac{4\pi \tau i}{3}} \vartheta(z, 3\tau) \vartheta(z - \pi \tau, 3\tau) \]

\[ \vartheta(z + \pi \tau, 3\tau) \vartheta(z - 2\pi \tau, 3\tau). \]

For \( z \) near 0, define

\[ L(z) = \frac{t}{X(z + \pi \tau)} Q(X(z), 0) - e^{i\alpha t} Q(0, X(z)). \]

Both \( L(z) \) and \( L(z + \pi \tau) \) converge.
Equation to solve:

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1 = \frac{t}{X(z)} Q(0, X(z + \pi\tau)) + L(z)
\]

\[
- e^{-i\alpha t} X(z + \pi\tau) Q(X(z + \pi\tau), 0),
\]

where

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X(z) = e^{-\frac{4\pi i}{3} \vartheta(z, 3\tau) \vartheta(z - \pi\tau, 3\tau)} \vartheta(z + \pi\tau, 3\tau) \vartheta(z - 2\pi\tau, 3\tau).
\]

For \( z \) near 0, define

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L(z) = \frac{t}{X(z + \pi\tau)} Q(X(z), 0) - e^{i\alpha t} Q(0, X(z)).
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Both \( L(z) \) and \( L(z + \pi\tau) \) converge.
Equation to solve:

\[ 1 = \frac{t}{X(z)} Q(0, X(z + \pi \tau)) + L(z) \]

\[ - \frac{e^{-i\alpha t}}{X(z)X(z + 2\pi \tau)} Q(X(z + \pi \tau), 0), \]

where

\[ X(z) = \frac{e^{-\frac{4\pi i}{3} \vartheta(z, 3\tau) \vartheta(z - \pi \tau, 3\tau)}}{\vartheta(z + \pi \tau, 3\tau) \vartheta(z - 2\pi \tau, 3\tau)}. \]

For \( z \) near 0, define

\[ L(z) = \frac{t}{X(z + \pi \tau)} Q(X(z), 0) - e^{i\alpha t} Q(0, X(z)). \]

Both \( L(z) \) and \( L(z + \pi \tau) \) converge.
Equation to solve:

\[ 1 = \frac{t}{X(z)} Q(0, X(z + \pi \tau)) + L(z) \]

\[ - \frac{e^{-i\alpha t}}{X(z)X(z + 2\pi \tau)} Q(X(z + \pi \tau), 0), \]

where

\[ X(z) = e^{-\frac{4\pi \tau i}{3}} \frac{\vartheta(z, 3\tau) \vartheta(z - \pi \tau, 3\tau)}{\vartheta(z + \pi \tau, 3\tau) \vartheta(z - 2\pi \tau, 3\tau)}. \]

For \( z \) near 0, define

\[ L(z) = \frac{t}{X(z + \pi \tau)} Q(X(z), 0) - e^{i\alpha t} Q(0, X(z)). \]

Both \( L(z) \) and \( L(z + \pi \tau) \) converge.
TANDEM WALKS BY WINDING NUMBER

Equation to solve:

\[ 1 = \frac{e^{i\alpha}}{X(z)} L(z + \pi \tau) + L(z). \]

where

\[ X(z) = e^{-\frac{4\pi \tau i}{3}} \vartheta(z, 3\tau) \vartheta(z - \pi \tau, 3\tau) \vartheta(z - 2\pi \tau, 3\tau). \]

For \( z \) near 0, define

\[ L(z) = \frac{t}{X(z + \pi \tau)} Q(X(z), 0) - e^{i\alpha} t Q(0, X(z)). \]

Both \( L(z) \) and \( L(z + \pi \tau) \) converge.
TANDEM WALKS BY WINDING NUMBER

Equation to solve:

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For \( z \) near 0, define

\[ L(z) = \frac{t}{X(z + \pi \tau)} Q(X(z), 0) - e^{i\alpha} t Q(0, X(z)). \]

Both \( L(z) \) and \( L(z + \pi \tau) \) converge.
Equation to solve:

\[ 1 = \frac{e^{i\alpha}}{X(z)} L(z + \pi \tau) + L(z). \]

where

\[ X(z) = \frac{e^{-\frac{4\pi \tau i}{3}} \vartheta(z, 3\tau) \vartheta(z - \pi \tau, 3\tau)}{\vartheta(z + \pi \tau, 3\tau) \vartheta(z - 2\pi \tau, 3\tau)}. \]
Tandem walks by winding number

Equation to solve:

\[ 1 = \frac{e^{i\alpha}}{X(z)} L(z + \pi \tau) + L(z). \]

where

\[ X(z) = e^{-\frac{4\pi i}{3} \psi(z, 3\tau) \psi(z - \pi \tau, 3\tau)} \psi(z + \pi \tau, 3\tau) \psi(z - 2\pi \tau, 3\tau). \]

We can solve this exactly:

\[ L(z) = \frac{e^{3i\alpha}}{1 - e^{3i\alpha}} \left( 1 - \frac{e^{i\alpha}}{X(z)} - e^{2i\alpha} X(z - \pi \tau) \right) \]

\[ - \frac{e^{i\alpha + \frac{2i\pi \tau}{3}} \psi(\pi \tau, 3\tau) \psi'(0, \tau)}{(1 - e^{3i\alpha}) \psi(\frac{\alpha}{2} + \frac{\pi \tau}{3}, \tau) \psi'(0, 3\tau)} \frac{\psi(z + \pi \tau, 3\tau) \psi(z - \frac{\alpha}{2} - \frac{\pi \tau}{3}, \tau)}{\psi(z, \tau) \psi(z, 3\tau)}. \]
**Tandem Walks by Winding Number**

**Equation to solve:**

\[ 1 = \frac{e^{i\alpha}}{X(z)} L(z + \pi \tau) + L(z). \]

where

\[ X(z) = \frac{e^{-\frac{4\pi i}{3} \vartheta(z, 3\tau) \vartheta(z - \pi \tau, 3\tau)}}{\vartheta(z + \pi \tau, 3\tau) \vartheta(z - 2\pi \tau, 3\tau)}. \]

We can solve this exactly:

\[ L(z) = \frac{e^{3i\alpha}}{1 - e^{3i\alpha}} \left( 1 - \frac{e^{i\alpha}}{X(z)} - e^{2i\alpha} X(z - \pi \tau) \right) \]

\[- \frac{e^{i\alpha + \frac{2i\pi \tau}{3}} \vartheta(\pi \tau, 3\tau) \vartheta'(0, \tau) \vartheta(z + \pi \tau, 3\tau) \vartheta(z - \frac{\alpha}{2} - \frac{\pi \tau}{3}, \tau)}{(1 - e^{3i\alpha}) \vartheta(\frac{\alpha}{2} + \frac{\pi \tau}{3}, \tau) \vartheta'(0, 3\tau) \vartheta(z, \tau) \vartheta(z, 3\tau)} \]

We can extract the solution for \( Q(0, 0) \)...
Tandem walks by winding number: Solution

\( \tau \) is determined by

\[ t = e^{-\frac{\pi \tau i}{3}} \frac{\vartheta'(0, 3\tau)}{4i\vartheta(\pi \tau, 3\tau) + 6\vartheta'(\pi \tau, 3\tau)} . \]

Then \( Q(0, 0) \equiv Q(t, \alpha, 0, 0) \) is given by:

\[ Q(0, 0) = \frac{e^{\alpha i + \frac{2\pi \tau i}{3}}}{1 - a^3e^{\alpha i}} \left( e^{\alpha i - \frac{\pi \tau i}{3}} + \frac{\vartheta'(\pi \tau, 3\tau)}{\vartheta'(0, 3\tau)} - \frac{\vartheta(\pi \tau, 3\tau)\vartheta'(\frac{\alpha}{2} + \frac{\pi \tau}{3}, \tau)}{\vartheta'(0, 3\tau)\vartheta(\frac{\alpha}{2} + \frac{\pi \tau}{3}, \tau)} \right) . \]
Tandem walks by winding number: Solution

\( \tau \) is determined by

\[
t = e^{-\frac{\pi \tau i}{3}} \frac{\vartheta'(0, 3\tau)}{4i \vartheta(\pi \tau, 3\tau) + 6 \vartheta'(\pi \tau, 3\tau)}.
\]

Then \( Q(0, 0) \equiv Q(t, \alpha, 0, 0) \) is given by:

\[
Q(0, 0) = \frac{e^{\alpha i + \frac{2\pi \tau i}{3}}}{1 - a^3 \alpha i} \left( e^{\alpha i - \frac{\pi \tau i}{3}} + \frac{\vartheta'(\pi \tau, 3\tau)}{\vartheta'(0, 3\tau)} - \frac{\vartheta(\pi \tau, 3\tau) \vartheta'(\frac{\alpha}{2} + \frac{\pi \tau}{3}, \tau)}{\vartheta'(0, 3\tau) \vartheta(\frac{\alpha}{2} + \frac{\pi \tau}{3}, \tau)} \right).
\]

The generating function \( Q(t, \alpha, 0, 0) \) is D-algebraic but not D-finite.

Asymptotics:

- For \( \alpha \in i \mathbb{R} \setminus \{0\} \), the dominant singularity is a simple pole.
- Variance \( V_n \) for the winding angle of paths of size \( n \) behaves like

\[
V_n \sim c \log(n)^2.
\]
Part 4b: Square lattice walks by winding number

\[ e^{i\alpha} \]

\[ e^{-i\alpha} \]
Counted exactly in terms of theta functions (2017 Budd)
Solution involved an eigenvalue decomposition of commuting Hilbert space operators.
Corollaries include the exact enumeration of Gessel walks.
Alternative solution: very similar to the last solution.
Functional equation:

\[ Q(x, y) = 1 + txQ(x, y) + tyQ(x, y) + \frac{t}{x} (Q(x, y) - Q(0, y)) \]
\[ + \frac{t}{y} (Q(x, y) - Q(x, 0)) + e^{i\alpha}tQ(0, x) + e^{-i\alpha}tQ(y, 0) \]
**Functional equation:**

\[
Q(x, y) = 1 + txQ(x, y) + tyQ(x, y) + \frac{t}{x} (Q(x, y) - Q(0, y)) \\
+ \frac{t}{y} (Q(x, y) - Q(x, 0)) + e^{i\alpha} tQ(0, x) + e^{-i\alpha} tQ(y, 0)
\]

The kernel equation

\[
K(x, y) = 1 - tx - ty - \frac{t}{x} - \frac{t}{y} = 0
\]

is parametrised by

\[
X(z) = \frac{e^{-\pi \tau i} \vartheta(z, 4\tau) \vartheta(z - \pi \tau, 4\tau)}{e^{-\pi \tau i} \vartheta(z + \pi \tau, 4\tau) \vartheta(z - 2\pi \tau, 4\tau)}
\]

and \( Y(z) = X(z + \pi \tau) \),

where \( \tau \) is determined by

\[
t = \frac{e^{-\pi \tau i} \vartheta'(0, 4\tau) \vartheta(\pi \tau, 4\tau)}{2 \vartheta(2\pi \tau, 4\tau) \vartheta'(\pi \tau, 4\tau) - \vartheta(\pi \tau, 4\tau) \vartheta'(2\pi \tau, 4\tau)}
\]
**Recall:** $\tau$ is determined by

$$t = \frac{e^{-\pi \tau i} \theta'(0, 4\tau) \theta(\pi \tau, 4\tau)}{2 \theta(2\pi \tau, 4\tau) \theta'(\pi \tau, 4\tau) - \theta(\pi \tau, 4\tau) \theta'(2\pi \tau, 4\tau)}$$

**Solution:** Expression for $Q(0, 0)$ similar to tandem case, but bigger.
Recall: $\tau$ is determined by

$$t = \frac{e^{-\pi \tau i} \vartheta'(0, 4\tau) \vartheta(\pi \tau, 4\tau)}{2 \vartheta(2\pi \tau, 4\tau) \vartheta'(\pi \tau, 4\tau) - \vartheta(\pi \tau, 4\tau) \vartheta'(2\pi \tau, 4\tau)}$$

Solution: Expression for $Q(0, 0)$ similar to tandem case, but bigger.

As usual, asymptotic and algebraic information can be extracted.
Part 4c: Properly coloured triangulations
Coloured triangulations: Background

Tutte showed that the generating function for properly coloured triangulations counted by faces is $sT(t, s, 1, 0) - s(s - 1)$, where $T(t, s, x, y) \equiv T(x, y)$ is determined by

$$T(x, y) = x(s - 1) + xytT(1, y)T(x, y)$$

$$+ \frac{xt}{y} (T(x, y) - T(x, 0)) - x^2yt \frac{T(x, y) - T(1, y)}{x - 1}$$

Background:

- Tutte 1982: The series $H(t) = t^2T(\sqrt{t}, s, 1, 0)$ satisfies
  $$2(1-s)t + (t+10H+6tH)H'' + (4-s)(20H - 18tH' + 9t^2H'') = 0.$$  

- Guttmann and Bousquet-Mélou: predicted (proved in some cases) asymptotic behaviour for a range of fixed values of $s$

- More precise asymptotic behaviour can be proven with theta functions!
COLOURED TRIANGULATIONS: THETA SOLUTION

Functional equation:

\[ T(x, y) = x(s - 1) + xytT(1, y)T(x, y) \]
\[ + \frac{xt}{y}(T(x, y) - T(x, 0)) - x^2yt\frac{T(x, y) - T(1, y)}{x - 1} \]

Want to parametrize the Kernel equation

\[ K(x, y) := -1 + xytT(1, y) + \frac{xt}{y} - \frac{x^2yt}{x - 1} = 0, \]

as then the remainder

\[ R(x, y) := x(s - 1) - \frac{xt}{y}T(x, 0) + x^2yt\frac{T(1, y)}{x - 1} = 0. \]

Guess: There is some parametrization \((X(z), Y(z))\) satisfying

- \(K(X(z), Y(z)) = 0\) and therefore \(R(X(z), Y(z)) = 0\) (near \(z = 0\)).
- \(X(z + \pi) = X(z)\) and \(Y(z + \pi) = Y(z)\).
- \(X(\pi) = X(z)\) and \(Y(\pi z - \pi) = Y(z)\).
COLOURED TRIANGULATIONS: THETA SOLUTION

Functional equation:

\[ 0 = T(x, y)K(x, y) + R(x, y), \]

where \[ K(x, y) = -1 + xytT(1, y) + \frac{xt}{y} - \frac{x^2yt}{x - 1} \]

and \[ R(x, y) = x(s - 1) - \frac{xt}{y}T(x, 0) + x^2yt\frac{T(1, y)}{x - 1} \]

Want to parametrize \( K(x, y) = 0 \) as then \( R(x, y) = 0 \).

**Guess:** There is some parametrization \((X(z), Y(z))\) satisfying

- \( K(X(z), Y(z)) = 0 \) and therefore \( R(X(z), Y(z)) = 0 \) (near \( z = 0 \)).
- \( X(z + \pi) = X(z) \) and \( Y(z + \pi) = Y(z) \).
- \( X(-z) = X(z) \) and \( Y(\pi \tau - z) = Y(z) \).

**Plan:** Solve under this assumption then check the solution.
**Coloured triangulations: Theta solution**

**Functional equation:**

\[ 0 = T(x, y)K(x, y) + R(x, y), \]

where \( K(x, y) = -1 + xytT(1, y) + \frac{xt}{y} - \frac{x^2yt}{x - 1} \)

and \( R(x, y) = x(s - 1) - \frac{xt}{y}T(x, 0) + x^2yt \frac{T(1, y)}{x - 1} \)

Want to parametrize \( K(x, y) = 0 \) as then \( R(x, y) = 0 \).

**Guess:** There is some parametrization \((X(z), Y(z))\) satisfying

- \( K(X(z), Y(z)) = 0 \) and therefore \( R(X(z), Y(z)) = 0 \) (near \( z = 0 \)).
- \( X(z + \pi) = X(z) \) and \( Y(z + \pi) = Y(z) \).
- \( X(-z) = X(z) \) and \( Y(\pi \tau - z) = Y(z) \).

**Plan:** Solve under this assumption then check the solution.

Kernel not explicit, but theta method still works
Coloured triangulations: Theta solution

Equations to solve:

- \( K(X(z), Y(z)) = 0. \)
- \( R(X(z), Y(z)) = 0. \)
- \( X(z + \pi) = X(z) \) and \( Y(z + \pi) = Y(z). \)
- \( X(-z) = X(z) \) and \( Y(\pi \tau - z) = Y(z) \)

Solving the equations: Use an “invariant function” found by Tutte

\[
I(z) = t^2 T(X(z), 0) + t^2 \frac{X(z)}{X(z) - 1} - \frac{X(z) - 1}{X(z)^2}
\]

\[
= \frac{t}{Y(z)^2} - \frac{t}{Y(z)} + stY(z) + (2t - Y(z))tT(1, Y(z))
\]

\[+ t^2 Y(z)^2 T(1, Y(z))^2. \]

Then \( I(z) = I(-z) \) and \( I(z) = I(\pi \tau - z) \), so \( I(z) \) is elliptic.
We end up with a solution involving $\beta$, related to $s$ by

\[ s = e^{2i\beta} + 2 + e^{-2i\beta}. \]

\[
t^2 = -\frac{(e^{\beta i} + e^{-\beta i})^2 \vartheta(\beta)(\vartheta'(\beta)\vartheta^{(3)}(0) - \vartheta'(0)\vartheta^{(3)}(\beta))}{24(e^{i\beta} - e^{-i\beta})^4 \vartheta'(0)\vartheta'(\beta)^3}
\]

\[
\frac{1}{X(z)} = \frac{e^{2\beta i} - 1 + e^{-2\beta i}}{e^{\beta i} - e^{-\beta i}} + \frac{\vartheta'(0)(\vartheta(z + \beta) + \vartheta(z - \beta))}{(e^{\beta i} - e^{-\beta i})^2 \vartheta'(\beta)\vartheta(z)}
\]

\[
Y(z) = -\frac{1}{t} \frac{(X(z) - 1)(X(\pi\tau - z) - 1)}{X(z)X(\pi\tau - z)}
\]

\[ T(1, 0) = \text{EXPRESSION TO BIG FOR SLIDE} \]
Part 4d: Six vertex model on 4-valent planar maps
Six vertex model

Functional equation:

- Kostov, 1999: Exactly, but non-rigorously solved in terms of elliptic functions
  - Solution involved matrix integrals and analysis of the limiting spectral density
- E.P. and Zinn-Justin, 2019+: Kostov’s solution made rigorous
  - Mistake corrected
  - Matrix integrals converted to functional equations
  - Answer checked (using functional equations)
- E.P. and Bousquet-Mélou, 2019+: Another proof using completely different functional equations
SIX VERTEX MODEL, SOLUTION 1 (E.P., ZINN-JUSTIN)

Equations to solve:

\[ W(x) = x^2 t W(x)^2 + e^{i\alpha} x t H(0, x) - e^{-i\alpha} x t H(x, 0) + 1 \]

\[ H(x) = W(x) W(y) + \frac{e^{i\alpha}}{y} (H(x, y) - H(x, 0)) - \frac{e^{-i\alpha}}{x} (H(x, y) - H(0, y)) \]

Vague idea of solution: Parametrise Kernel equation

\[ 1 - \frac{e^{i\alpha}}{y} + \frac{e^{-i\alpha}}{x} = 0 \]

by

\[ \frac{1}{X(z)} = \frac{1}{e^{i\alpha} - e^{-i\alpha}} + c \frac{\vartheta(z + \alpha, \tau)}{\vartheta(z, \tau)} \quad \text{and} \quad Y(z) = X(z + \pi \tau). \]

In the end, \( \omega X(z) W(X(z)) + \omega^{-1} Y(z) W(Y(z)) + R(X(z)) \) is elliptic for some explicit rational function \( R \).
Lots of equations... after some manipulation we end up with

**Equations to solve:**

\[
H(x) = \frac{1 - \omega x - \sqrt{1 - \omega x + \omega^2 x^2 - 4F(x)/x}}{2}
\]

\[
H(x) = H^{-1}(x)
\]

**Vague idea of solution:** Find a parametrization \(X(z)\) satisfying \(X(z + \pi) = X(z) = X(-z)\) and \(H(X(z)) = X(\pi \tau - z)\).

Using both equations, deduce

\[
X(z + \pi \tau) + \omega X(z) + X(z - \pi \tau) = 1,
\]

which is solvable in terms of theta functions, leading to a complete solution.
With either solution we (of course) get the same answer: \( \tau \) is determined by

\[
t = \frac{\cos \alpha}{64 \sin^3 \alpha} \left( \frac{\vartheta(\alpha, \tau)\vartheta'''(\alpha, \tau)}{\vartheta'(\alpha, \tau)^2} + \frac{\vartheta''(\alpha, \tau)}{\vartheta'(\alpha, \tau)} \right).
\]

An auxiliary series \( R(t, \gamma) \) is determined by

\[
R(t, -2 \cos(2\alpha)) = \frac{\cos^2 \alpha \vartheta(\alpha, q)^2}{96 \sin^4 \alpha \vartheta'(\alpha, q)^2} \left( -\frac{\vartheta'''(\alpha, q)}{\vartheta'(\alpha, q)} + \frac{\vartheta'''(0, q)}{\vartheta'(0, q)} \right).
\]

Finally the series in question is

\[
Q(t, \omega) = \frac{1}{(\omega + 2)t^2} \left( t - (\omega + 2)t^2 - R(t, \omega) \right).
\]
Part 4e: Final comments
REASONS TO TRY THETA FUNCTIONS

- Functional equation has two catalytic variables.
- Kernel is quadratic, so equation $K(x, y) = 0$ is parametrized by elliptic functions.
- Both $Q(0, y)$ and $Q(0, x)$ appear in the equation (or something like that).
- A doubly connected domain is involved.
- You expect one (or both) of the following:
  - The solution simplifies when a weight is either $e^{i\pi \tau}$ or $\cos(\pi r)$ for $r \in \mathbb{Q}$.
  - The solution is D-algebraic.
FURTHER AIMS

- Solve more problems
- streamline these methods
- strengthen these methods
- Convert techniques to world of formal power series and rely less on complex analysis
- Find combinatorial interpretations of theta function expressions and prove formulas bijectively.
BIBLIOGRAPHY

General references:

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- A brief introduction to theta functions (1961, Bellmann)
- Sur quelques formules relatives à la transformation des fonctions elliptiques (1858, Hermite)
- Elliptic modular forms and their applications (2008, Zagier)
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**Quadrant model references:**

- Counting quadrant walks via Tutte’s invariant method (2017, Bernardi, Bousquet-Mélou and Raschel)
- Walks with small steps in the quarter plane (2010, Bousquet-Mélou and Mishna)
- Random walks in the quarter plane (1999, Fayolle, Iasnogorodski and Malyshev)
- On the functions counting walks in the quarter plane (2012, Kurkova and Raschel)
Other problem-specific references:

- Counting colored planar maps: algebraicity results (2011, Bernardi and Bousquet-Mélou).
- Winding of simple walks on the square lattice (2017, Budd).
- Exact solution to the six-vertex model on a random lattice (1999, Kostov).
- Chromatic sums for rooted planar triangulations: the cases $\lambda = 1$ and $\lambda = 2$ (1973, Tutte).
**Solutions to appear:**

- The six vertex model on planar maps (EP and Zinn-Justin).
- The six vertex model on planar maps (Bousquet-Mélou and EP).
- Walks by winding number on the Kreweras lattice (EP).
- Properly coloured triangulations (EP).
- Distribution of height functions on quadrangulations (EP).
Thank you!