Theta functions in enumerative combinatorics
Week 2: Quadrant walks

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STRATEGY: RECURSIVE METHOD

- **Step 1:** Find a recursive decomposition of each object in your class
- **Step 2:** Write functional equations which characterise the generating function $F(t)$
- **Step 3:** Solve the functional equations

Strategy: Theta function method

- **Step 1:** Find a recursive decomposition of each object in your class
- **Step 2:** Write functional equations which characterise the generating function $F(t)$
- **Step 3:** Solve the functional equations using theta functions!

Problems solvable with theta functions

- Quadrant walk models with small steps (all D-algebraic cases) (Bernardi, Bousquet-Mélou, Raschel)
- Square lattice walks weighted by winding number (Budd)
- Six vertex model on 4-valent maps (Kostov/Bousquet-Mélou, E.P., Zinn-Justin)
- height functions on quadrangulations NEW!
- Properly coloured triangulations NEW! (Previously shown to be D-algebraic by Tutte)
- Triangular lattice walks by winding number NEW!

Last time: the theta function

Definition: For $\tau, z \in \mathbb{C}$, $\mathrm{im}(\tau) > 0$,

$$\begin{aligned}\vartheta(z,\tau) &= \sum_{n=-\infty}^{\infty} (-1)^n e^{\left(\frac{2n+1}{2}\right)^2 i\pi\tau + (2n+1)iz} \\ &= 2\sum_{n=0}^{\infty} (-i)^{2n+1} e^{\left(\frac{2n+1}{2}\right)^2 i\pi\tau} \sin\left((2n+1)z\right) \\ &= e^{\frac{\pi\tau i}{4}} \left(e^{iz} - e^{-iz}\right) \prod_{n=1}^{\infty} \left(1 - e^{2\pi\tau ni + 2iz}\right)\left(1 - e^{2\pi\tau ni - 2iz}\right)\left(1 - e^{2\pi\tau ni}\right)\end{aligned}$$

Useful facts for fixed τ:

- $\vartheta(-z,\tau) = -\vartheta(z,\tau)$
- $\vartheta(z+\pi,\tau) = -\vartheta(z,\tau)$
- $\vartheta(z+\pi\tau,\tau) = -e^{-2iz-i\pi\tau}\vartheta(z,\tau)$
- Roots of $\vartheta(z,\tau)$ form the lattice $\pi\mathbb{Z} + \pi\tau\mathbb{Z}$
- All elliptic functions $f(z)$ with periods π and $\pi\tau$ can be written as

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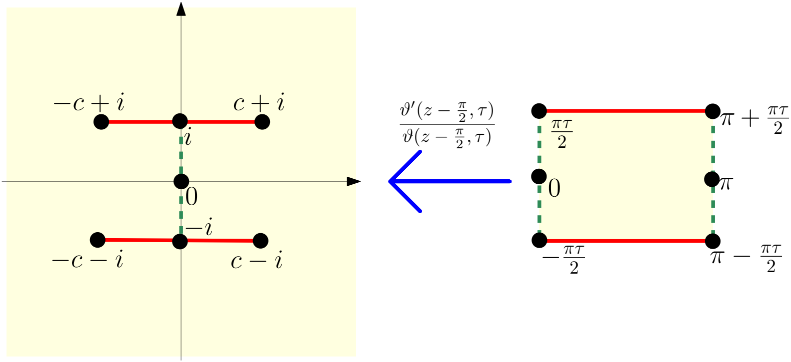
- All elliptic functions $f(z)$ with periods π and $\pi\tau$ can be written as

$$f(z) = c\prod_{j=1}^{k}\frac{\vartheta(z+\alpha_j,\tau)}{\vartheta(z+\beta_j,\tau)}.$$

Other useful facts:

- $\frac{4i}{\pi}\frac{\partial}{\partial\tau}\vartheta(z,\tau) = \vartheta''(z,\tau)$
- $\vartheta(z,\tau)$ is differentially algebraic (in both variables)

Last time: Conformal parametrizations



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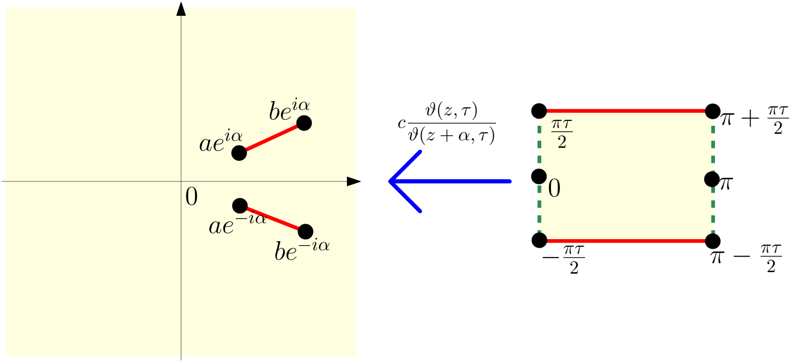

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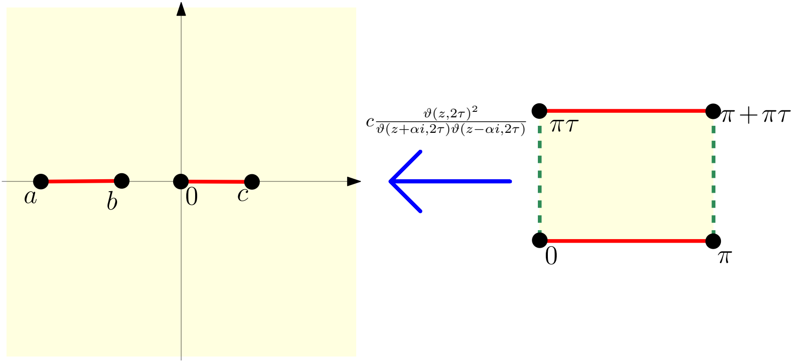
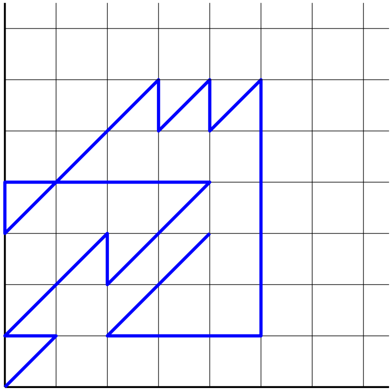
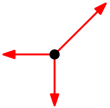


Today (Part 2): Quadrant walks with small steps

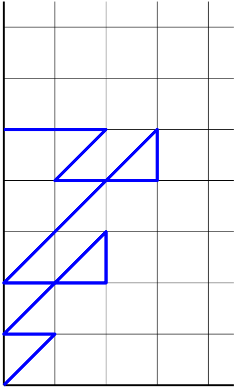
KREWERAS PATHS





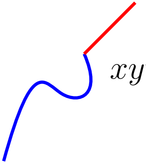
$$Q(x,y) \equiv Q(t,x,y) := \sum_{a,b=0}^{\infty} \sum_{\substack{\text{paths from} \\ (0,0) \text{ to } (a,b)}} t^{\#\text{steps}} x^a y^b.$$

KREWERAS PATHS



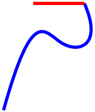
$$Q(x,y) = \quad 1$$

$$+$$



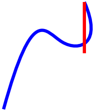
$$xytQ(x,y)$$

$$+$$



$$\frac{t}{x}(Q(x,y) - Q(0,y))$$

$$+$$



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Kreweras paths

The generating function $Q(t,x,y) \equiv Q(x,y)$ is characterised by

$$Q(x,y) = 1 + xytQ(x,y) + \frac{t}{x}\left(Q(x,y) - Q(0,y)\right) + \frac{t}{y}\left(Q(x,y) - Q(x,0)\right).$$

Aim: Solve this equation

KREWERAS PATHS

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Aim: Solve this equation using theta functions!

Quadrant walks

More generally: Take a step set $S \subset \{-1, 0, 1\}^2$ and write

$$P_S(x, y) = \sum_{(i,j)\in S} x^{i+1} y^{j+1}.$$

The generating function $Q_S(t, x, y) \equiv Q(x, y)$ is characterised by

$$\begin{aligned} xyQ(x, y) = xy + tP_S(x, y)Q(x, y) - tP_S(0, y)Q(0, y) \\ - tP_S(x, 0)Q(x, 0) + tP_S(0, 0)Q(0, 0). \end{aligned}$$

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Solving:

- Fix $t < 1/9$ and think of $Q(x,y)$ as an analytic function of x, y (the series converges for $|x|, |y| < 1$).
- **Kernel method:** write $K(x,y)Q(x,y) + R(x,y) = 0$. Then whenever $K(x,y) = 0$, we have $R(x,y) = 0$.

$$K(x,y) = xy - tP_S(x,y),$$
$$R(x,y) = xy - tP_S(0,y)Q(0,y) - tP_S(x,0)Q(x,0) + tP_S(0,0)Q(0,0).$$

Part 2a: Elliptic parameterisation of $K(x, y) = 0$

QUADRANT WALKS: PARAMETERIZING $K(x, y) = 0$

Write $K(x, y) = A(x)y^2 + B(x)y + C(x)$, then

$$Y(x) = \frac{-B(x) \pm \sqrt{B(x)^2 - 4A(x)C(x)}}{2A(x)}$$

parameterizes $K(x, Y(x)) = 0$. Typically, $Y_+(x)$ is meromorphic on:

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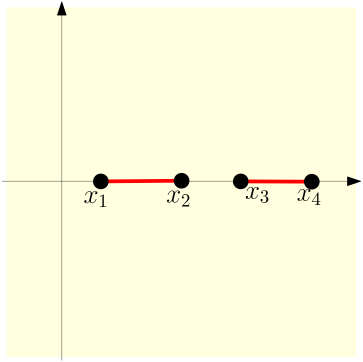

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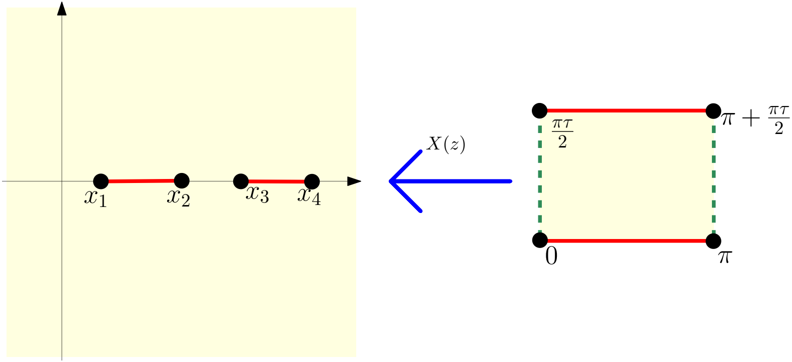

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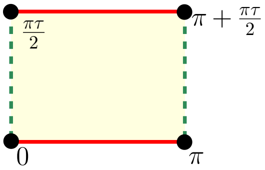

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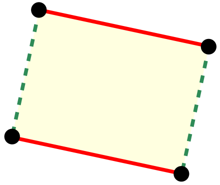

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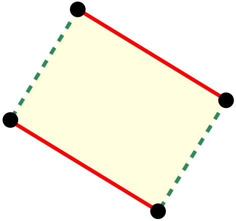

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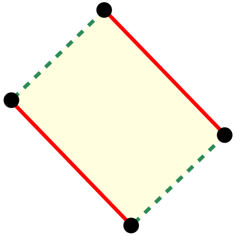

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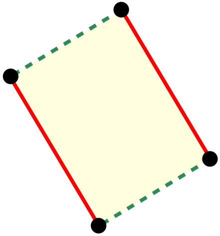

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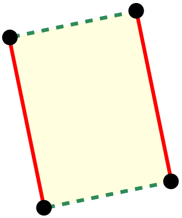

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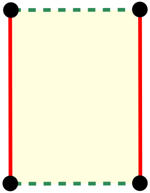

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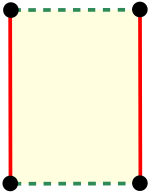

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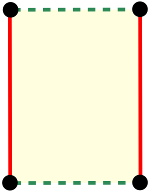

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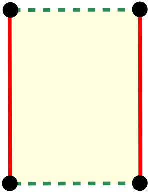

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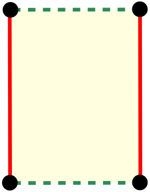

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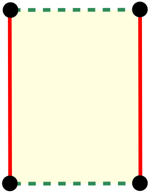

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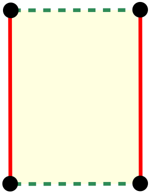

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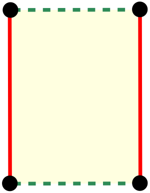

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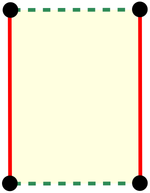

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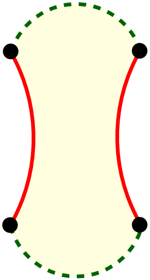

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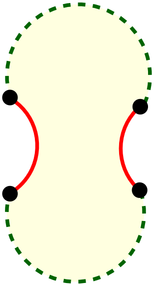

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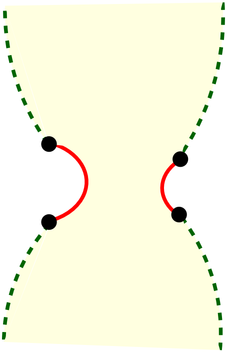

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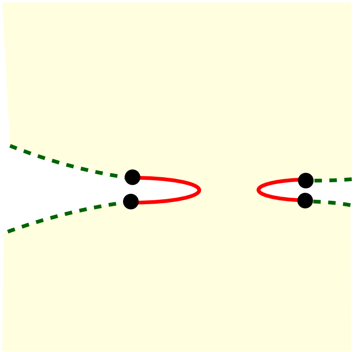

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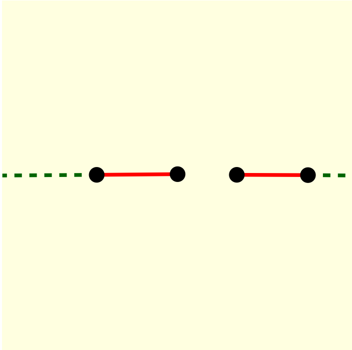

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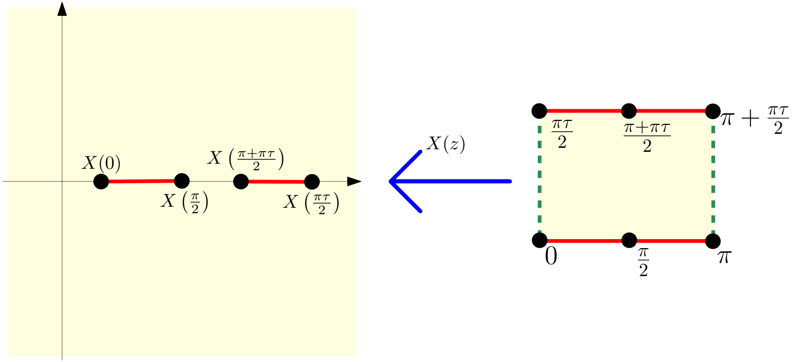

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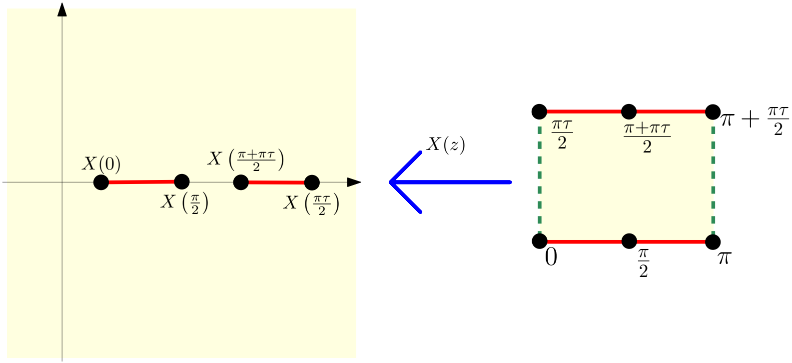
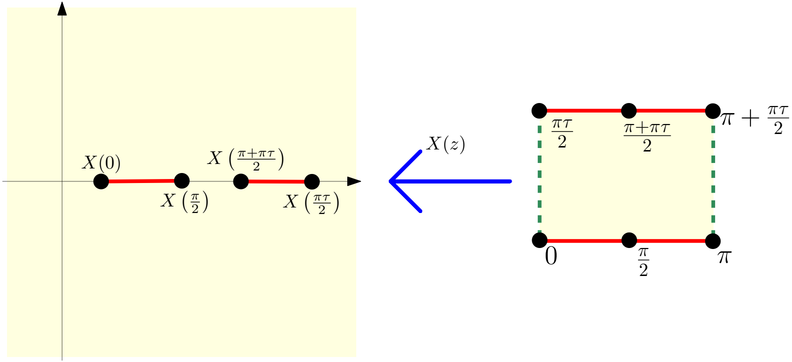


By symmetry, for $r \in \mathbb{R}$:

- $X(r) = X(\pi - r) = X(-r)$
- $X(\frac{\pi\tau}{2} + r) = X(\frac{\pi\tau}{2} - r)$

Quadrant walks: parameterizing $K(x, y) = 0$



For $z \in \mathbb{C}$:

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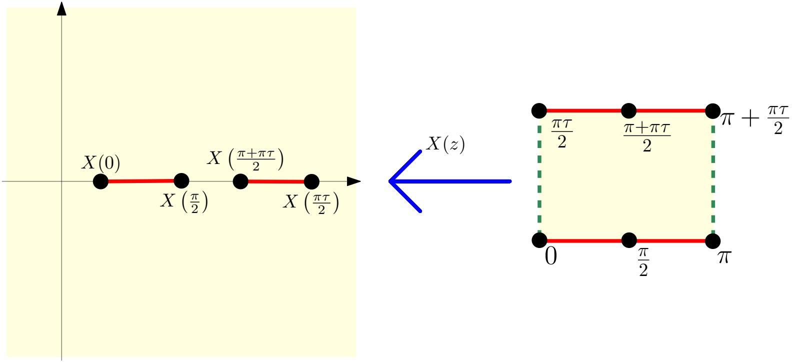
QUADRANT WALKS: PARAMETERIZING $K(x, y) = 0$



For $z \in \mathbb{C}$:

- $X(z) = X(\pi - z) = X(-z) = X(\pi\tau + z)$
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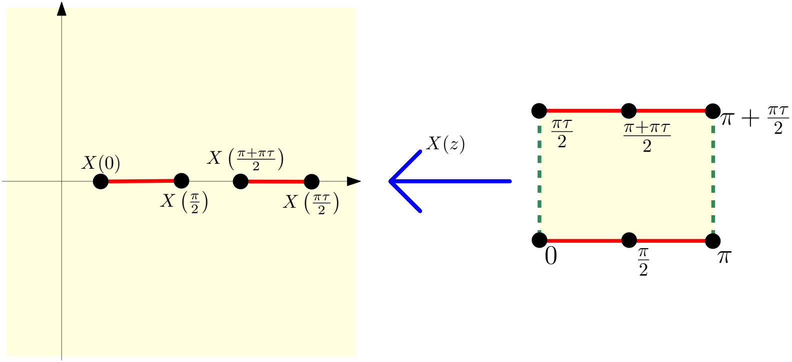
QUADRANT WALKS: PARAMETERIZING $K(x, y) = 0$



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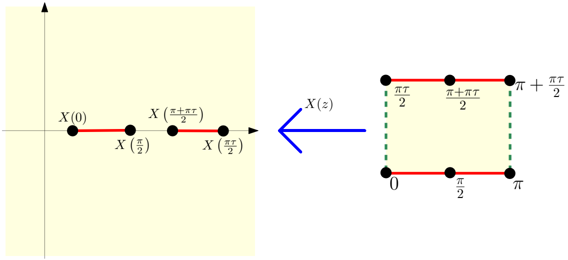
- $X(z) = X(\pi - z) = X(-z) = X(\pi\tau + z)$

$$X(z) = c\frac{\vartheta(z-\alpha)\vartheta(z+\alpha)}{\vartheta(z-\beta)\vartheta(z+\beta)}$$

QUADRANT WALKS: PARAMETERIZING $K(x, y) = 0$



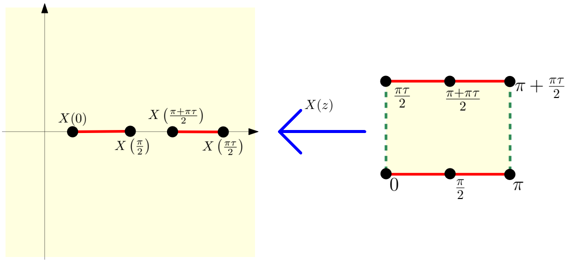
Recall:

$$y(x) = \frac{-B(x) \pm \sqrt{B(x)^2 - 4A(x)C(x)}}{2A(x)}.$$

Consider $Y(z) = y(X(z))$. By symmetry, for $r \in \mathbb{R}$:

- $X(r) = X(-r)$, so $Y(r) + Y(-r) = -\frac{B(X(r))}{A(X(r))}$.
- Similarly, $Y\left(\frac{\pi\tau}{2} + r\right) + Y\left(\frac{\pi\tau}{2} - r\right) = -\frac{B\left(X\left(\frac{\pi\tau}{2} + r\right)\right)}{A\left(X\left(\frac{\pi\tau}{2} + r\right)\right)}$.

QUADRANT WALKS: PARAMETERIZING $K(x, y) = 0$



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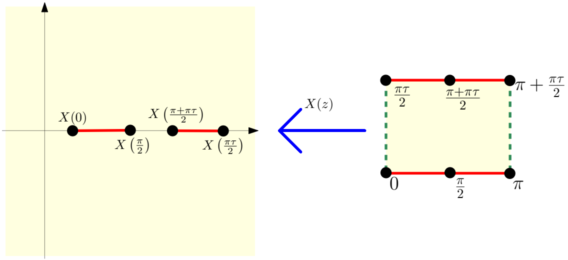
Consider $Y(z) = y(X(z))$. For $z \in \mathbb{C}$:

- $Y(z) + Y(-z) = -\frac{B(X(z))}{A(X(z))}$.
- $Y(z) + Y(\pi\tau - z) = -\frac{B(X(z))}{A(X(z))}$.

QUADRANT WALKS: PARAMETERIZING $K(x, y) = 0$



For $z \in \mathbb{C}$:

- $Y(z) + Y(-z) = -\frac{B(X(z))}{A(X(z))}$.
- $Y(z) + Y(\pi\tau - z) = -\frac{B(X(z))}{A(X(z))}$.

So $Y(z) = Y(z + \pi\tau) = Y(z + \pi)$

$$\Rightarrow Y(z) = c\frac{\vartheta(z-\gamma)\vartheta(z-\delta)}{\vartheta(z-\epsilon)\vartheta(z-\gamma-\delta+\epsilon)}.$$

PARAMETERIZATION OF $K(x,y)$

Equation characterising $Q(x,y) \equiv Q(t,x,y)$ for quadrant walks:

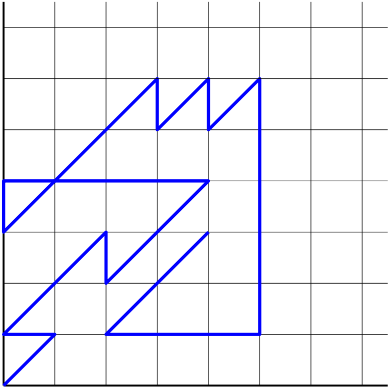
$$K(x,y)Q(x,y) + R(x,y) = 0.$$

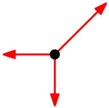
$K(x,y) = 0$ is parameterised by

$$X(z) = c_1 \frac{\vartheta(z-\alpha_1)\vartheta(z-\beta_1)}{\vartheta(z-\gamma_1)\vartheta(z-\delta_1)} \quad \text{and} \quad Y(z) = c_2 \frac{\vartheta(z-\alpha_2)\vartheta(z-\beta_2)}{\vartheta(z-\gamma_2)\vartheta(z-\delta_2)},$$

where the constants satisfy $\alpha_j + \beta_j = \gamma_j + \delta_j$ for $j = 1, 2$.
So, $R(X(z), Y(z)) = 0$.

Part 2b: Solution for Kreweras paths





KREWERAS PATHS

In general: $K(x, y) = 0$ is parameterised by

$$X(z) = c_1 \frac{\vartheta(z-\alpha_1)\vartheta(z-\beta_1)}{\vartheta(z-\gamma_1)\vartheta(z-\delta_1)} \quad \text{and} \quad Y(z) = c_2 \frac{\vartheta(z-\alpha_2)\vartheta(z-\beta_2)}{\vartheta(z-\gamma_2)\vartheta(z-\delta_2)},$$

with $\alpha_j + \beta_j = \gamma_j + \delta_j$ for $j = 1, 2$.

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For Kreweras paths:

$$Q(x,y) = 1 + xytQ(x,y) + \frac{t}{x}\left(Q(x,y) - Q(0,y)\right) + \frac{t}{y}\left(Q(x,y) - Q(x,0)\right).$$

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Change of variables

Then $K(x,y) = xy - tx^2y^2 - tx - ty = 0$ is parameterised by

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}}\vartheta(z,3\tau)\vartheta\left(z-\pi\tau,3\tau\right)}{\vartheta\left(z+\pi\tau,3\tau\right)\vartheta\left(z-2\pi\tau,3\tau\right)} \quad \text{and} \quad Y(z) = X(z+\pi\tau),$$

where

$$t = \frac{1}{X(z)Y(z) + X(z)^{-1} + Y(z)^{-1}}.$$

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Now: As $t \to 0$, we have $\tau \to i\infty$.

KREWERAS PATHS: DETERMINING $Q(x,0)$ AND $Q(0,y)$

Recall: $Q(x,y)$ is characterised by

$$K(x,y)Q(x,y) = R(x,y),$$

where

$$K(x,y) = xy - tx^2y^2 - tx - ty$$
$$R(x,y) = xy - tyQ(0,y) - txQ(x,0).$$

Since $K(X(z),Y(z)) = 0$,

$$R(X(z),Y(z)) = X(z)Y(z) - tY(z)Q(0,Y(z)) - tX(z)Q(X(z),0) = 0.$$

KREWERAS PATHS: DETERMINING $Q(x,0)$ AND $Q(0,y)$

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$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}}\vartheta(z,3\tau)\vartheta(z-\pi\tau,3\tau)}{\vartheta(z+\pi\tau,3\tau)\vartheta(z-2\pi\tau,3\tau)} \quad \text{and} \quad Y(z) = X(z+\pi\tau)$$

Aim: Solve for $Q(x,0)$ and $Q(0,y)$ using

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But what does $Q(X(z),0)$ mean?

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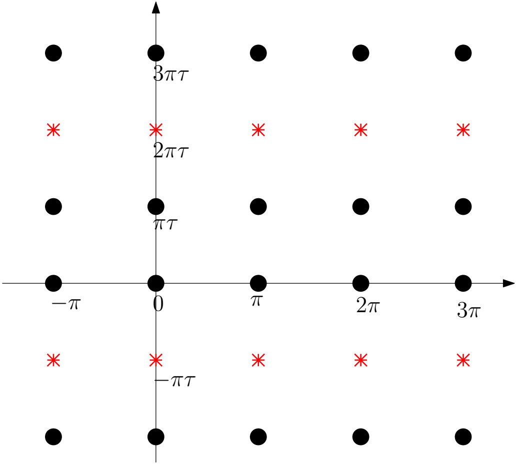
But what does $Q(X(z),0)$ mean?

Recall: $Q(x,0)$ converges for $|x| < 1$.

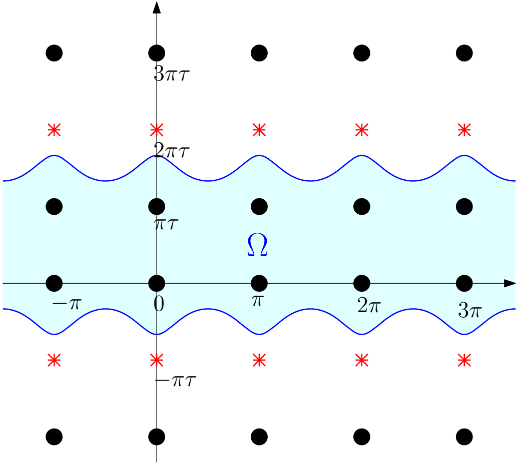
UNDERSTANDING $Q(X(z), 0)$ AND $Q(0, Y(z))$

Poles and roots of $X(z)$:



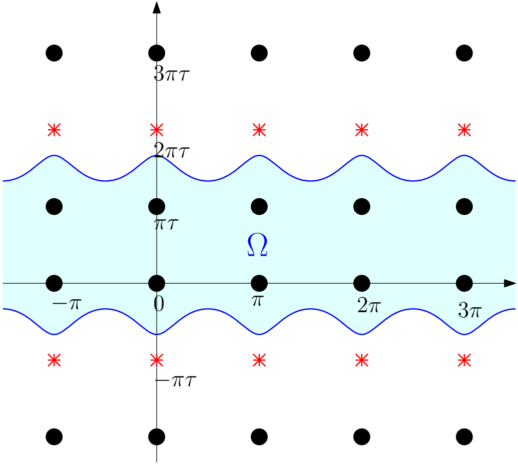
UNDERSTANDING $Q(X(z), 0)$ AND $Q(0, Y(z))$

Let Ω be the largest neighbourhood of 0 in which $|X(z)| < 1$.



UNDERSTANDING $Q(X(z), 0)$ AND $Q(0, Y(z))$

Let Ω be the largest neighbourhood of 0 in which $|X(z)| < 1$. For $z \in \Omega$, define $A(z) = Q(X(z), 0)$. Then $A(\pi\tau - z) = A(z)$.



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Similarly, defining $B(z) = Q(0, Y(z))$ around $z = 0$ implies $B(-\pi\tau - z) = B(z)$.
The equation becomes:

$$X(z)Y(z) - tX(z)A(z) - tY(z)B(z) = 0.$$

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From $K(X(z),Y(z))=0$, we have

$$X(z)Y(z)=\frac{1}{t}-\frac{1}{X(z)}-\frac{1}{Y(z)}.$$

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The equation becomes:

$$\frac{1}{t} - \frac{1}{Y(z)} - tY(z)B(z) = \frac{1}{X(z)} + tX(z)A(z).$$

UNDERSTANDING $Q(X(z),0)$ AND $Q(0,Y(z))$

Let Ω be the largest neighbourhood of 0 in which $|X(z)|<1$. For $z\in\Omega$, define $A(z)=Q(X(z),0)$. Then $A(\pi\tau-z)=A(z)$.
Similarly, defining $B(z)=Q(0,Y(z))$ around $z=0$ implies $B(-\pi\tau-z)=B(z)$.
The equation becomes:

$$I(z)=\frac{1}{t}-\frac{1}{Y(z)}-tY(z)B(z)=\frac{1}{X(z)}+tX(z)A(z).$$

From LHS: $I(z)=I(-\pi\tau-z)$. From RHS: $I(z)=I(\pi\tau-z)$.

UNDERSTANDING $Q(X(z), 0)$ AND $Q(0, Y(z))$

Let Ω be the largest neighbourhood of 0 in which $|X(z)| < 1$. For $z \in \Omega$, define $A(z) = Q(X(z), 0)$. Then $A(\pi\tau - z) = A(z)$.
Similarly, defining $B(z) = Q(0, Y(z))$ around $z = 0$ implies $B(-\pi\tau - z) = B(z)$.
The equation becomes:

$$I(z) = \frac{1}{t} - \frac{1}{Y(z)} - tY(z)B(z) = \frac{1}{X(z)} + tX(z)A(z).$$

From LHS: $I(z) = I(-\pi\tau - z)$. From RHS: $I(z) = I(\pi\tau - z)$.
Now $I(z)$ has π and $2\pi\tau$ as periods, so we can solve it:

$$I(z) = e^{\frac{\pi\tau i}{3}} \frac{\vartheta(\pi\tau, 3\tau)}{\vartheta'(0, 3\tau)} \left(\frac{\vartheta'(z, 2\tau)}{\vartheta(z, 2\tau)} - \frac{\vartheta'(z + \pi\tau, 2\tau)}{\vartheta(z + \pi\tau, 2\tau)} - i \right) + \frac{1}{2t}.$$

Then $Q(x, 0)$ is given by

$$Q(X(z), 0) = \frac{I(z)}{tX(z)} - \frac{1}{tX(z)^2}.$$

Solution for Kreweras paths

Recall: $Q(x,0)$ is determined by

$$t = e^{-\frac{\pi\tau i}{3}} \frac{\vartheta'(0,3\tau)}{4i\vartheta(\pi\tau,3\tau)+6\vartheta'(\pi\tau,3\tau)}$$

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}}\vartheta(z,3\tau)\vartheta\left(z-\pi\tau,3\tau\right)}{\vartheta\left(z+\pi\tau,3\tau\right)\vartheta\left(z-2\pi\tau,3\tau\right)}$$

$$I(z) = e^{\frac{\pi\tau i}{3}} \frac{\vartheta(\pi\tau,3\tau)}{\vartheta'(0,3\tau)}\left(\frac{\vartheta'(z,2\tau)}{\vartheta(z,2\tau)} - \frac{\vartheta'(z+\pi\tau,2\tau)}{\vartheta(z+\pi\tau,2\tau)} - i\right) + \frac{1}{2t}.$$

$$Q(X(z),0) = \frac{I(z)}{tX(z)} - \frac{1}{tX(z)^2}.$$

All three have periods $6\pi\tau$ and π, so $X(z)$ and $Q(X(z),0)$ are algebraically related.
Therefore: $Q(x,0)$ is algebraic in x.

Solution as formal power series

Recall:

$$\vartheta(z,\tau) = e^{\frac{(\pi\tau-2z)i}{2}} T(e^{2iz}, e^{2i\pi\tau}),$$

where

$$T(u,q) = \sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}(u^{n+1} - u^{-n}).$$

More generally:

$$\vartheta^{(k)}(z,\tau) = e^{\frac{(\pi\tau-2z)i}{2}} i^k T_k(e^{2iz}, e^{2i\pi\tau}),$$

where

$$T_k(u,q) = \sum_{n=0}^{\infty}(-1)^n (2n+1)^k q^{n(n+1)/2}(u^{n+1} - (-1)^k u^{-n}).$$

We can rewrite the solution using $q = e^{2\pi\tau i}$ and $u = e^{2iz}$

Solution as formal power series

$$T_k(u,q) = \sum_{n=0}^{\infty}(-1)^n(2n+1)^k q^{n(n+1)/2}(u^{n+1} - (-1)^k u^{-n}).$$

We can rewrite the solution using $q = e^{2\pi\tau i}$ and $u = e^{2iz}$

$$t = q^{1/3}\frac{T_1(1,q^3)}{4T(q,q^3) + 6T_1(q,q^3)}$$

$$x = q^{-2/3}\frac{T(u,q^3)T\left(u/q,q^3\right)}{T\left(uq,q^3\right)\vartheta\left(uq^{-2},q^3\right)}$$

$$I(x) = q^{-1/3}\frac{T(q,q^3)}{T_1(1,q^3)}\left(\frac{T_1(u,q^2)}{T(u,q^2)} - \frac{T_1(uq,q^2)}{T(uq,q^2)} - 1\right) + \frac{1}{2t}.$$

$$Q(x,0) = \frac{I(x)}{tx} - \frac{1}{tx^2}.$$

Part 2c: Solutions for other quadrant models

SOLUTION IN GENERAL

Equation to solve:

$$Q(x,y)K(x,y) = R(x,y),$$

where

$$R(x,y) = xy - P_1(y)Q(0,y) - P_2(x)Q(x,0) + c$$

- $K(x,y) = 0$ is parameterised by

$$X(z) = c_1 \frac{\vartheta(z-\alpha_1)\vartheta(z-\beta_1)}{\vartheta(z-\gamma_1)\vartheta(z-\delta_1)} \quad \text{and} \quad Y(z) = c_2 \frac{\vartheta(z-\alpha_2)\vartheta(z-\beta_2)}{\vartheta(z-\gamma_2)\vartheta(z-\delta_2)}.$$

SOLUTION IN GENERAL

Equation to solve:

$$X(z)Y(z) - P_1(Y(z))Q(0,Y(z)) - P_2(X(z))Q(X(z),0) + c = 0$$

- $K(x,y) = 0$ is parameterised by

  $$X(z) = c_1\frac{\vartheta(z-\alpha_1)\vartheta(z-\beta_1)}{\vartheta(z-\gamma_1)\vartheta(z-\delta_1)} \text{ and } Y(z) = c_2\frac{\vartheta(z-\alpha_2)\vartheta(z-\beta_2)}{\vartheta(z-\gamma_2)\vartheta(z-\delta_2)}.$$

SOLUTION IN GENERAL

Equation to solve:

$$X(z)Y(z) - P_1(Y(z))Q(0,Y(z)) - P_2(X(z))Q(X(z),0) + c = 0$$

- $K(x,y) = 0$ is parameterised by

  $$X(z) = c_1\frac{\vartheta(z-\alpha_1)\vartheta(z-\beta_1)}{\vartheta(z-\gamma_1)\vartheta(z-\delta_1)} \quad \text{and} \quad Y(z) = c_2\frac{\vartheta(z-\alpha_2)\vartheta(z-\beta_2)}{\vartheta(z-\gamma_2)\vartheta(z-\delta_2)}.$$
- $X(z)$ and $Q(X(z),0)$ are fixed under $z \to \alpha_1 + \beta_1 - z$.
- $Y(z)$ and $Q(0,Y(z))$ are fixed under $z \to \alpha_2 + \beta_2 - z$.
- Write $A(z) = P_2(X(z))Q(X(z),0)$ and
  $B(z) = P_1(Y(z))Q(0,Y(z)) - c$

SOLUTION IN GENERAL

Equation to solve:

$$X(z)Y(z) = A(z) + B(z)$$

- $K(x, y) = 0$ is parameterised by

  $$X(z) = c_1 \frac{\vartheta(z-\alpha_1)\vartheta(z-\beta_1)}{\vartheta(z-\gamma_1)\vartheta(z-\delta_1)} \quad \text{and} \quad Y(z) = c_2 \frac{\vartheta(z-\alpha_2)\vartheta(z-\beta_2)}{\vartheta(z-\gamma_2)\vartheta(z-\delta_2)}.$$
- $X(z)$ and $Q(X(z), 0)$ are fixed under $z \to \alpha_1 + \beta_1 - z$.
- $Y(z)$ and $Q(0, Y(z))$ are fixed under $z \to \alpha_2 + \beta_2 - z$.
- Write $A(z) = P_2(X(z))Q(X(z), 0)$ and $B(z) = P_1(Y(z))Q(0, Y(z)) - c$

Solution in general

Equation to solve for $A(z)$ and $B(z)$:

$$X(z)Y(z) = A(z) + B(z)$$

where

$$X(z) = c_1\frac{\vartheta(z-\alpha_1)\vartheta(z-\beta_1)}{\vartheta(z-\gamma_1)\vartheta(z-\delta_1)} \quad \text{and} \quad Y(z) = c_2\frac{\vartheta(z-\alpha_2)\vartheta(z-\beta_2)}{\vartheta(z-\gamma_2)\vartheta(z-\delta_2)}.$$

- $X(z)$ and $A(z)$ are fixed under $z \to \alpha_1 + \beta_1 - z$.
- $Y(z)$ and $B(z)$ are fixed under $z \to \alpha_2 + \beta_2 - z$.

SOLUTION IN D-FINITE CASES ($\alpha_1 + \beta_1 - \alpha_2 - \beta_2 \in \pi\tau\mathbb{Q}$)

Equation to solve for $A(z)$ and $B(z)$:

$$X(z)Y(z) = A(z) + B(z)$$

where

$$X(z) = c_1 \frac{\vartheta(z-\alpha_1)\vartheta(z-\beta_1)}{\vartheta(z-\gamma_1)\vartheta(z-\delta_1)} \quad \text{and} \quad Y(z) = c_2 \frac{\vartheta(z-\alpha_2)\vartheta(z-\beta_2)}{\vartheta(z-\gamma_2)\vartheta(z-\delta_2)}.$$

- $X(z)$ and $A(z)$ are fixed under $z \to \alpha_1 + \beta_1 - z$.
- $Y(z)$ and $B(z)$ are fixed under $z \to \alpha_2 + \beta_2 - z$.

Solution in D-finite cases ($\alpha_1+\beta_1-\alpha_2-\beta_2\in\pi\tau\mathbb{Q}$)

Equation to solve for $A(z)$ and $B(z)$:

$$X(z)Y(z)=A(z)+B(z)$$

$$(X(z)-X(\alpha_2+\beta_2-z))Y(z)=A(z)-A(\alpha_2+\beta_2-z)$$

where

$$X(z)=c_1\frac{\vartheta(z-\alpha_1)\vartheta(z-\beta_1)}{\vartheta(z-\gamma_1)\vartheta(z-\delta_1)}\quad\text{and}\quad Y(z)=c_2\frac{\vartheta(z-\alpha_2)\vartheta(z-\beta_2)}{\vartheta(z-\gamma_2)\vartheta(z-\delta_2)}.$$

- $X(z)$ and $A(z)$ are fixed under $z\to\alpha_1+\beta_1-z$.
- $Y(z)$ and $B(z)$ are fixed under $z\to\alpha_2+\beta_2-z$.

Solution in D-finite cases ($\alpha_1+\beta_1-\alpha_2-\beta_2\in\pi\tau\mathbb{Q}$)

Equation to solve for $A(z)$ and $B(z)$:

$$X(z)Y(z)=A(z)+B(z)$$

$$(X(z)-X(\alpha_2+\beta_2-z))Y(z)=A(z)-A(\alpha_1+\beta_1-\alpha_2-\beta_2+z)$$

where

$$X(z)=c_1\frac{\vartheta(z-\alpha_1)\vartheta(z-\beta_1)}{\vartheta(z-\gamma_1)\vartheta(z-\delta_1)}\quad\text{and}\quad Y(z)=c_2\frac{\vartheta(z-\alpha_2)\vartheta(z-\beta_2)}{\vartheta(z-\gamma_2)\vartheta(z-\delta_2)}.$$

- $X(z)$ and $A(z)$ are fixed under $z\to\alpha_1+\beta_1-z$.
- $Y(z)$ and $B(z)$ are fixed under $z\to\alpha_2+\beta_2-z$.

SOLUTION IN D-FINITE CASES ($\alpha_1 + \beta_1 - \alpha_2 - \beta_2 \in \pi\tau\mathbb{Q}$)

Equation to solve for $A(z)$:

$$(X(z) - X(\alpha_2 + \beta_2 - z))Y(z) = A(z) - A(\alpha_1 + \beta_1 - \alpha_2 - \beta_2 + z)$$

where

$$X(z) = c_1 \frac{\vartheta(z-\alpha_1)\vartheta(z-\beta_1)}{\vartheta(z-\gamma_1)\vartheta(z-\delta_1)} \quad \text{and} \quad Y(z) = c_2 \frac{\vartheta(z-\alpha_2)\vartheta(z-\beta_2)}{\vartheta(z-\gamma_2)\vartheta(z-\delta_2)}.$$

- $X(z)$ and $A(z)$ are fixed under $z \to \alpha_1 + \beta_1 - z$.
- $Y(z)$ and $B(z)$ are fixed under $z \to \alpha_2 + \beta_2 - z$.

Solution in D-finite cases ($\alpha_1+\beta_1-\alpha_2-\beta_2\in\pi\tau\mathbb{Q}$)

Equation to solve for $A(z)$:

$$(X(z)-X(\alpha_2+\beta_2-z))Y(z)=A(z)-A(\alpha_1+\beta_1-\alpha_2-\beta_2+z)$$

where $X(z)$ and $Y(z)$ have π and $\pi\tau$ as periods.

SOLUTION IN D-FINITE CASES ($\alpha_1+\beta_1-\alpha_2-\beta_2\in\pi\tau\mathbb{Q}$)

Equation to solve for $A(z)$:

$$(X(z)-X(\alpha_2+\beta_2-z))Y(z)=A(z)-A(\alpha_1+\beta_1-\alpha_2-\beta_2+z)$$

where $X(z)$ and $Y(z)$ have π and $\pi\tau$ as periods.
Since $\alpha_1+\beta_1-\alpha_2-\beta_2\in\pi\tau\mathbb{Q}$, we get:

$$F(z)=A(z)-A(n\pi\tau+z),$$

where $F(z)$ has periods π and $\pi\tau$.

SOLUTION IN D-FINITE CASES $(\alpha_1+\beta_1-\alpha_2-\beta_2\in\pi\tau\mathbb{Q})$

Equation to solve for $A(z)$:

$$(X(z)-X(\alpha_2+\beta_2-z))Y(z)=A(z)-A(\alpha_1+\beta_1-\alpha_2-\beta_2+z)$$

where $X(z)$ and $Y(z)$ have π and $\pi\tau$ as periods.
Since $\alpha_1+\beta_1-\alpha_2-\beta_2\in\pi\tau\mathbb{Q}$, we get:

$$F(z)=A(z)-A(n\pi\tau+z),$$

where $F(z)$ has periods π and $\pi\tau$.
Then $U(z)=\frac{A(z)}{F(z)}$ satisfies $U(z+n\pi\tau)=U(z)-1$.
So the following all have π and $\pi\tau$ as periods:

$$U'(z),\ F(z),\ X(z) \text{ and } Y(z)$$

It follows that $A(z)=U(z)F(z)$ is D-finite in $X(z)$

Solution in general

Equation to solve for $A(z)$ and $B(z)$:

$$X(z)Y(z) = A(z) + B(z)$$

where

$$X(z) = c_1 \frac{\vartheta(z-\alpha_1)\vartheta(z-\beta_1)}{\vartheta(z-\gamma_1)\vartheta(z-\delta_1)} \quad \text{and} \quad Y(z) = c_2 \frac{\vartheta(z-\alpha_2)\vartheta(z-\beta_2)}{\vartheta(z-\gamma_2)\vartheta(z-\delta_2)}.$$

- $X(z)$ and $A(z)$ are fixed under $z \to \alpha_1 + \beta_1 - z$.
- $Y(z)$ and $B(z)$ are fixed under $z \to \alpha_2 + \beta_2 - z$.

Solution in D-algebraic (non-D-finite) cases

Equation to solve for $A(z)$ and $B(z)$:

$$X(z)Y(z) = A(z) + B(z)$$

where

$$X(z) = c_1 \frac{\vartheta(z-\alpha_1)\vartheta(z-\beta_1)}{\vartheta(z-\gamma_1)\vartheta(z-\delta_1)} \quad \text{and} \quad Y(z) = c_2 \frac{\vartheta(z-\alpha_2)\vartheta(z-\beta_2)}{\vartheta(z-\gamma_2)\vartheta(z-\delta_2)}.$$

- $X(z)$ and $A(z)$ are fixed under $z \to \alpha_1 + \beta_1 - z$.
- $Y(z)$ and $B(z)$ are fixed under $z \to \alpha_2 + \beta_2 - z$.

In these cases:

- $X(z)Y(z)$ splits as $X(z)Y(z) = R_1(X(z)) + R_2(Y(z))$, for explicit rational functions R_1, R_2.

SOLUTION IN D-ALGEBRAIC (NON-D-FINITE) CASES

Equation to solve for $A(z)$ and $B(z)$:

$$R_1(X(z)) + R_2(Y(z)) = A(z) + B(z)$$

where

$$X(z) = c_1 \frac{\vartheta(z-\alpha_1)\vartheta(z-\beta_1)}{\vartheta(z-\gamma_1)\vartheta(z-\delta_1)} \quad \text{and} \quad Y(z) = c_2 \frac{\vartheta(z-\alpha_2)\vartheta(z-\beta_2)}{\vartheta(z-\gamma_2)\vartheta(z-\delta_2)}.$$

- $X(z)$ and $A(z)$ are fixed under $z \to \alpha_1 + \beta_1 - z$.
- $Y(z)$ and $B(z)$ are fixed under $z \to \alpha_2 + \beta_2 - z$.

In these cases:

- $X(z)Y(z)$ splits as $X(z)Y(z) = R_1(X(z)) + R_2(Y(z))$, for explicit rational functions R_1, R_2.

Solution in D-algebraic (non-D-finite) cases

Equation to solve for $A(z)$ and $B(z)$:

$$I(z) = R_1(X(z)) - A(z) = B(z) - R_2(Y(z))$$

where

$$X(z) = c_1 \frac{\vartheta(z-\alpha_1)\vartheta(z-\beta_1)}{\vartheta(z-\gamma_1)\vartheta(z-\delta_1)} \quad \text{and} \quad Y(z) = c_2 \frac{\vartheta(z-\alpha_2)\vartheta(z-\beta_2)}{\vartheta(z-\gamma_2)\vartheta(z-\delta_2)}.$$

- $X(z)$ and $A(z)$ are fixed under $z \to \alpha_1 + \beta_1 - z$.
- $Y(z)$ and $B(z)$ are fixed under $z \to \alpha_2 + \beta_2 - z$.

In these cases:

- $X(z)Y(z)$ splits as $X(z)Y(z) = R_1(X(z)) + R_2(Y(z))$, for explicit rational functions R_1, R_2.

Solution in D-algebraic (non-D-finite) cases

Equation to solve for $A(z)$ and $B(z)$:

$$I(z) = R_1(X(z)) - A(z) = B(z) - R_2(Y(z))$$

where

$$X(z) = c_1 \frac{\vartheta(z-\alpha_1)\vartheta(z-\beta_1)}{\vartheta(z-\gamma_1)\vartheta(z-\delta_1)} \quad \text{and} \quad Y(z) = c_2 \frac{\vartheta(z-\alpha_2)\vartheta(z-\beta_2)}{\vartheta(z-\gamma_2)\vartheta(z-\delta_2)}.$$

- $X(z)$ and $A(z)$ are fixed under $z \to \alpha_1 + \beta_1 - z$.
- $Y(z)$ and $B(z)$ are fixed under $z \to \alpha_2 + \beta_2 - z$.

In these cases:

- $X(z)Y(z)$ splits as $X(z)Y(z) = R_1(X(z)) + R_2(Y(z))$, for explicit rational functions R_1, R_2.
- $I(z)$ is fixed under $z \to \alpha_1 + \beta_1 - z$ and $z \to \alpha_2 + \beta_2 - z$, so it has $\alpha_1 + \beta_1 - \alpha_2 - \beta_2$ and π as periods.
- We can then solve for $I(z)$.

NEXT WEEKS

Next week:

- Asymptotic analysis
- Algebraicity properties in t
- Modular properties of $\vartheta(\tau, z)$

Following week: More problems that we can solve with theta functions