Theta functions in enumerative combinatorics
Week 1: Background on theta functions

Andrew Elvey Price

Université de Bordeaux

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ENUMERATIVE COMBINATORICS

General question: Let Ω be a set of combinatorial objects where each object $\sigma \in \Omega$ has some integer size $s(\sigma)$.

- **Aim:** Determine the number $|\Omega_n|$ of objects of size n.
- **Equivalent Aim:** Determine the generating function

$$F_\Omega(t) = \sum_{n=0}^{\infty} |\Omega_n| t^n = \sum_{\sigma \in \Omega} t^{s(\sigma)}.$$

Enumerative combinatorics

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$$F_\Omega(t) = \sum_{n=0}^{\infty} |\Omega_n| t^n = \sum_{\sigma \in \Omega} t^{s(\sigma)}.$$

- **Algebraicity:** Is $F_\Omega(t)$ the solution to a
  - Algebraic equation? (Algebraic)
  - Linear differential equation? (D-finite)
  - Algebraic differential equation? (D-Algebraic)
- **Asymptotics:** How does $|\Omega_n|$ behave as $n \to \infty$?

Problems solvable with theta functions

- Quadrant walk models with small steps (all D-algebraic cases)
- Square lattice walks weighted by winding number
- Enumeration of Eulerian orientations / height functions on quadrangulations
- Properly coloured triangulations

STRATEGY: RECURSIVE METHOD

- **Step 1:** Find a recursive decomposition of each object $\sigma \in \Omega$
- **Step 2:** Write functional equations defining $F_\Omega(t)$
- **Step 3:** Solve the functional equations

Strategy: Theta function method

- **Step 1:** Find a recursive decomposition of each object $\sigma \in \Omega$
- **Step 2:** Write functional equations defining $F_{\Omega}(t)$
- **Step 3:** Solve the functional equations using theta functions!

THE FUNCTION(S)

Define the series $T(u,q) \in \mathbb{F}(z)[[q]]$ by

$$T(u,q) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}(u^{n+1} - u^{-n})$$
$$= (u-1)\prod_{n=1}^{\infty}(1-q^n u)(1-q^n/u)(1-q^n).$$

For $\tau, z \in \mathbb{C}$, $\mathrm{im}(\tau) > 0$, the Jacobi theta function $\vartheta(z,\tau)$ is:

$$\vartheta(z,\tau) = e^{\frac{(\pi\tau-2z)i}{2}} T(e^{2iz}, e^{2i\pi\tau}) = \sum_{n=-\infty}^{\infty} (-1)^n e^{\left(\frac{2n+1}{2}\right)^2 i\pi\tau + (2n+1)iz}.$$

- $\vartheta(z,\tau)$ has nice analytic properties
- $T(u,q)$ looks nice for combinatorics

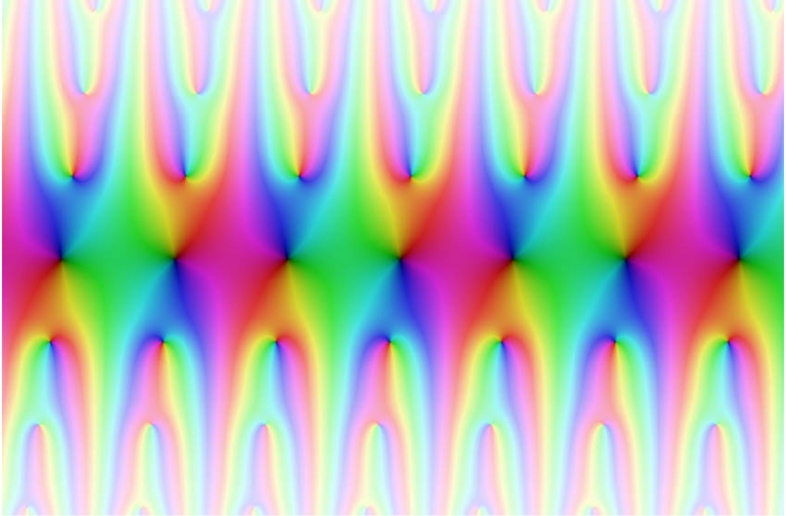
Part 1: Nice properties of $\vartheta(z, \tau)$

NICE PROPERTIES OF $\vartheta(z,\tau)$

- $\vartheta(-z,\tau) = -\vartheta(z,\tau)$
- $\vartheta(z+\pi,\tau) = -\vartheta(z,\tau)$
- $\vartheta(z+\pi\tau,\tau) = -e^{-2iz-i\pi\tau}\vartheta(z,\tau)$
- $\vartheta(z,\tau+1) = e^{i\pi/4}\vartheta(z,\tau)$
- $\vartheta\left(\frac{z}{\tau},-\frac{1}{\tau}\right) = -i(-i\tau)^{\frac{1}{2}}\exp\left(\frac{i}{\pi\tau}z^2\right)\vartheta(z,\tau)$
- $\frac{4i}{\pi}\frac{\partial}{\partial\tau}\vartheta(z,\tau) = \vartheta''(z,\tau)$
- $\{z|\vartheta(z,\tau)=0\} = \pi\mathbb{Z}+\pi\tau\mathbb{Z}$
- $\vartheta(z,\tau)$ is holomorphic in $(z,\tau)\in\mathbb{C}\times\mathbb{H}$.
- $\wp(z;\pi,\pi\tau) = \frac{\vartheta'(z,\tau)^2}{\vartheta(z,\tau)^2} - \frac{\vartheta''(z,\tau)}{\vartheta(z,\tau)} + \frac{\vartheta'''(0,\tau)}{3\vartheta'(0,\tau)}$
- Meromorphic functions $f(z)$ with π and $\pi\tau$ as periods are

$$f(z) = c\prod_{j=1}^{k}\frac{\vartheta(z+\alpha_j,\tau)}{\vartheta(z+\beta_j,\tau)}.$$

Part 1a: $\vartheta(z, \tau)$ for fixed τ



$\vartheta(z,\tau)$ FOR FIXED τ

Fix $\tau \in \mathbb{H}$. Then we write

$$\vartheta(z) \equiv \vartheta(z,\tau) = \sum_{n=-\infty}^{\infty} (-1)^n e^{\left(\frac{2n+1}{2}\right)^2 i\pi\tau + (2n+1)iz}.$$

Useful equations:

- $\vartheta(z+\pi) = \vartheta(-z) = -\vartheta(z)$
- $\vartheta(z+\pi\tau) = -e^{-2iz-i\pi\tau}\vartheta(z)$

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Proof:

$$\vartheta(-z) = \sum_{n=-\infty}^{\infty} (-1)^n e^{\left(\frac{2n+1}{2}\right)^2 i\pi\tau - (2n+1)iz} \qquad (n \to -n-1)$$

$$= \sum_{n=-\infty}^{\infty} (-1)^{-n-1} e^{\left(\frac{-2n-1}{2}\right)^2 i\pi\tau + (2n+1)iz}$$

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Proof:

$$\vartheta(z+\pi) = \sum_{n=-\infty}^{\infty} (-1)^n e^{\left(\frac{2n+1}{2}\right)^2 i\pi\tau + (2n+1)iz + (2n+1)i\pi}$$

$$= - \sum_{n=-\infty}^{\infty} (-1)^n e^{\left(\frac{2n+1}{2}\right)^2 i\pi\tau + (2n+1)iz}$$

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Proof:

$$\vartheta(z+\pi\tau) = \sum_{n=-\infty}^{\infty} (-1)^n e^{\left(\frac{2n+1}{2}\right)^2 i\pi\tau + (2n+1)iz + (2n+1)i\pi\tau}$$

$$= \sum_{n=-\infty}^{\infty} (-1)^n e^{\left(\frac{2n+3}{2}\right)^2 i\pi\tau + (2n+3)iz - 2iz - i\pi\tau}$$

$\vartheta(z,\tau)$ FOR FIXED τ

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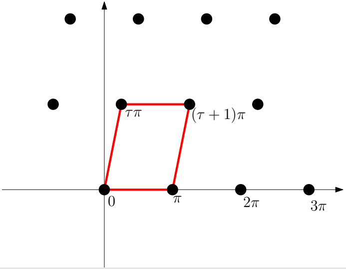
From these equations, we see that $\vartheta(z)$ is determined by its value on some "fundamental domain" $\{a\pi\tau + b\pi | a,b \in [0,1]\}$.



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Useful equations:

- $\vartheta(z+\pi) = \vartheta(-z) = -\vartheta(z)$
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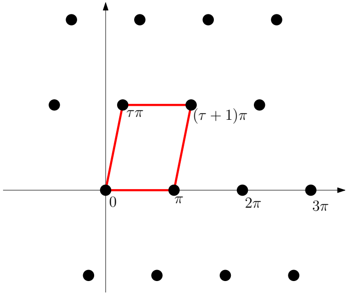
Also, the roots of ϑ are exactly the points of the lattice $(\mathbb{Z}+\tau\mathbb{Z})\,\pi$.



BUILDING ELLIPTIC FUNCTIONS

Recall: $\vartheta(z+\pi) = -\vartheta(z)$ and $\vartheta(z+\pi\tau) = -e^{-2iz-i\pi\tau}\vartheta(z)$.

Definition: An *elliptic* function is a meromorphic function $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ with two independent periods (for us: π and $\pi\tau$).

Theorem (Liouville): The only *holomorphic* elliptic functions are constant functions.

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Let $\alpha, \beta \in \mathbb{C}$ and let

$$f(z) = \frac{\vartheta(z-\alpha)}{\vartheta(z)},$$

$$g(z) = \frac{f(z-\beta)}{f(z)} = \frac{\vartheta(z-\alpha-\beta)\vartheta(z)}{\vartheta(z-\alpha)\vartheta(z-\beta)}$$

Then

- $f(z+\pi) = f(z)$
- $f(z+\pi\tau) = e^{2i\alpha}f(z)$
- $g(z+\pi) = g(z)$
- $g(z+\pi\tau) = g(z)$

So $g(z)$ is elliptic.

BUILDING ELLIPTIC FUNCTIONS

Recall: $\vartheta(z+\pi) = -\vartheta(z)$ and $\vartheta(z+\pi\tau) = -e^{-2iz-i\pi\tau}\vartheta(z)$.

Let k be a positive integer, let $c, \alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k$ be complex numbers and define

$$g(z) = c\prod_{j=1}^{k}\frac{\vartheta(z-\alpha_j)}{\vartheta(z-\beta_j)}.$$

Then

- $g(z+\pi) = g(z)$
- $g(z+\pi\tau) = \exp\left(\sum_{j=1}^{k} 2(\alpha_j - \beta_j)\right) g(z)$

So, if the sum is 0 then $g(z)$ is elliptic.

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All elliptic functions with periods π and $\pi\tau$ can be written like this: $\alpha_1, \ldots, \alpha_k$ are the roots of $g(z)$ and $\beta_1, \ldots, \beta_k$ are the poles.

Building more elliptic functions

Recall: $\vartheta(z+\pi) = -\vartheta(z)$ and $\vartheta(z+\pi\tau) = -e^{-2iz-i\pi\tau}\vartheta(z)$.

BUILDING MORE ELLIPTIC FUNCTIONS

Recall: $\vartheta(z+\pi) = -\vartheta(z)$ and $\vartheta(z+\pi\tau) = -e^{-2iz-i\pi\tau}\vartheta(z)$.

Let $f(z) = \frac{\vartheta'(z)}{\vartheta(z)}$. Then

- $f(z+\pi) = f(z)$
- $f(z+\pi\tau) = f(z) - 2i$

Building more elliptic functions

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Let $f(z) = \frac{\vartheta'(z)}{\vartheta(z)}$. Then

- $f(z+\pi) = f(z)$
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So $f'(z)$ is elliptic since

- $f'(z+\pi) = f'(z)$
- $f'(z+\pi\tau) = f'(z)$

Related to the Weierstrass function $\wp(z)$.

Side note: Weierstrass function

The Weierstrass function $\wp(z) \equiv \wp(z; \pi, \pi\tau)$ is the unique meromorphic function satisfying

- $\wp(z)$ is elliptic with periods π and $\pi\tau$.
- $\wp(z)$ has a double pole at $z = 0$ and no other poles.
- $\wp(z) \sim z^{-2} + O(z^2)$ around $z = 0$.

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So, there is some $\alpha, c \in \mathbb{C}$ satisfying

$$\wp(z) = c\frac{\vartheta(z-\alpha)\vartheta(z+\alpha)}{\vartheta(z)^2}.$$

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Alternatively we can write

$$\wp(z; \pi, \pi\tau) = \frac{\vartheta'''(0)}{3\vartheta'(0)} - f'(z) = \frac{\vartheta'(z)^2}{\vartheta(z)^2} - \frac{\vartheta''(z)}{\vartheta(z)} + \frac{\vartheta'''(0)}{3\vartheta'(0)}.$$

SIDE NOTE: WEIERSTRASS FUNCTION

Theorem: Every elliptic function $g(z)$ with periods π and $\pi\tau$ can be written as

$$g(z) = R_1(\wp(z)) + R_2(\wp(z))\wp'(z)$$

for some rational functions R_1 and R_2.

In particular

$$\wp'(z)^2 = \wp(z)^3 + g_1\wp(z) + g_2,$$

for some $g_1, g_2 \in \mathbb{C}$.

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Proof idea:

- Every *even* elliptic function with poles only at 0 is a polynomial of $\wp(z)$
- Every *even* elliptic function is a rational function of $\wp(z)$
- The theorem follows by writing $\frac{g(z)+g(-z)}{2} = R_1(\wp(z))$ and $\frac{g(z)-g(-z)}{\wp'(z)} = R_2(\wp(z))$.

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In particular

$$\wp'(z)^2 = \wp(z)^3 + g_1\wp(z) + g_2,$$

for some $g_1, g_2 \in \mathbb{C}$.

Corollary:

Any two elliptic functions (with periods π and $\pi\tau$) are related by an algebraic equation

Part 1b: Differential algebraicity

HEAT EQUATION

Recall the definition of ϑ:

$$\vartheta(z,\tau) = \sum_{n=-\infty}^{\infty} (-1)^n e^{\left(\frac{2n+1}{2}\right)^2 i\pi\tau + (2n+1)iz}.$$

The *heat equation* for $\vartheta(z,\tau)$ is

$$4\frac{\partial}{\partial\tau}\vartheta(z,\tau) = \vartheta''(z,\tau).$$

DIFFERENTIAL ALGEBRAICITY

We saw that $\vartheta(z,\tau)$ is differentially algebraic in z since

$$\wp(z) = \frac{\vartheta'(z)^2}{\vartheta(z)^2} - \frac{\vartheta''(z)}{\vartheta(z)} + \frac{\vartheta'''(0)}{3\vartheta'(0)}$$

and

$$\wp'(z)^2 = \wp(z)^3 + g_1\wp(z) + g_2.$$

For differential algebraicity in τ: Use heat equation

$$4\frac{\partial}{\partial\tau}\vartheta(z,\tau) = \vartheta''(z,\tau)$$

to convert τ derivatives to z derivatives.

Meaning in combinatorics: Generating functions with theta function solutions are always D-algebraic.

Part 1c: Conformal maps with theta functions

Conformal maps with theta functions

Useful theorem: Any doubly connected domain except for the punctured disk and the punctured plane is conformally equivalent to some cylinder (rectangle with right and left sides glued)

To make conformal map unique: fix image of a boundary point.

Reference: See for example Ahlfors' Complex Analysis, section 5, chapter 6 for equivalent statement mapping to annulus

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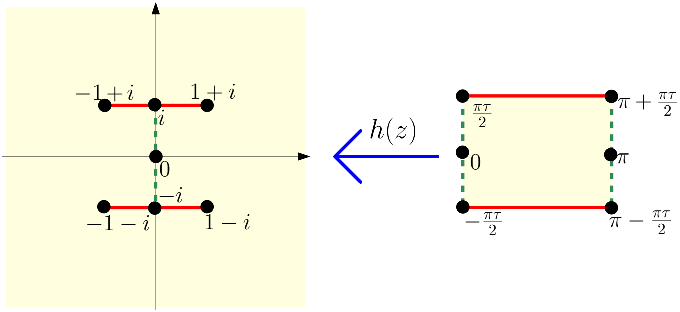

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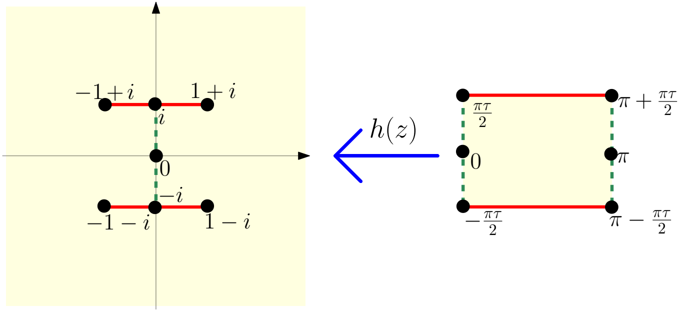
Conformal map example

Aim: Find $h(z)$ and τ.



Assumptions: $h\left(\frac{\pi\tau}{2}\right) = i$ and $h\left(z+\pi\right) = h(z)$

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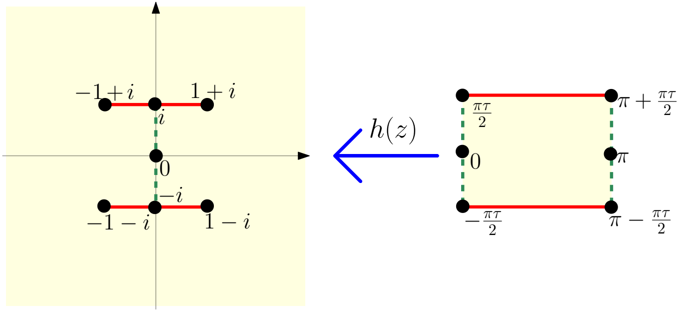

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- $h\left(n\pi\right)=h\left(0\right)=0$, and these are the only roots in the cylinder
- h has a simple pole at $\pi/2$ and no other poles in the rectangle.
- for $r\in\mathbb{R}$, we have $h\left(r+\frac{\pi\tau}{2}\right)=h\left(r-\frac{\pi\tau}{2}\right)+2i$

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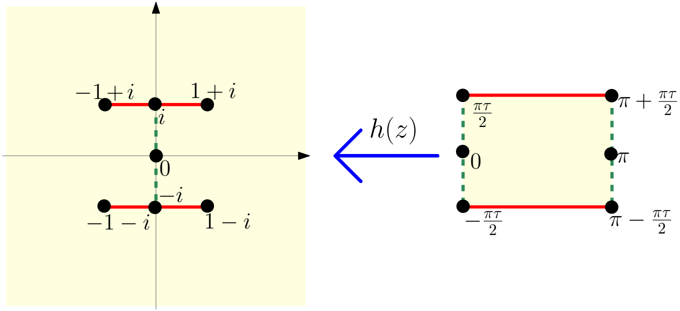

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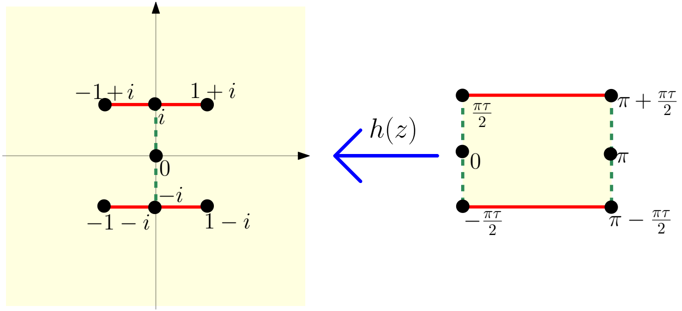

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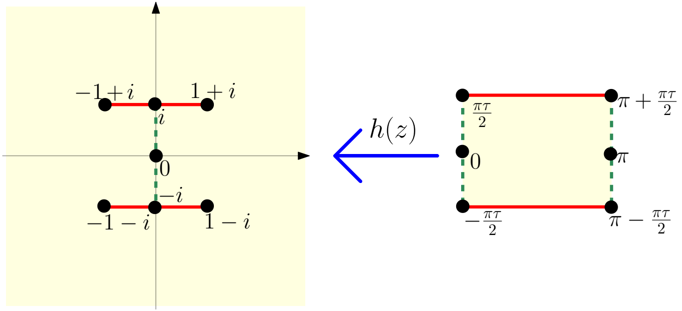

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Solution:

- $h_1(z) = -\frac{\vartheta'(z,\tau)}{\vartheta(z,\tau)}$ satisfies the red equations.

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Solution:

- $h_1(z) = -\frac{\vartheta'(z,\tau)}{\vartheta(z,\tau)}$ satisfies the red equations.
- The function $f(z) = h(z) - h_1(z-\pi/2)$ is elliptic and has at most one simple pole
  $\Rightarrow f$ must be constant
  $\Rightarrow f(z) = f(0) = 0.$

$$h(z) = -\frac{\vartheta'(z-\pi/2,\tau)}{\vartheta(z-\pi/2,\tau)}.$$

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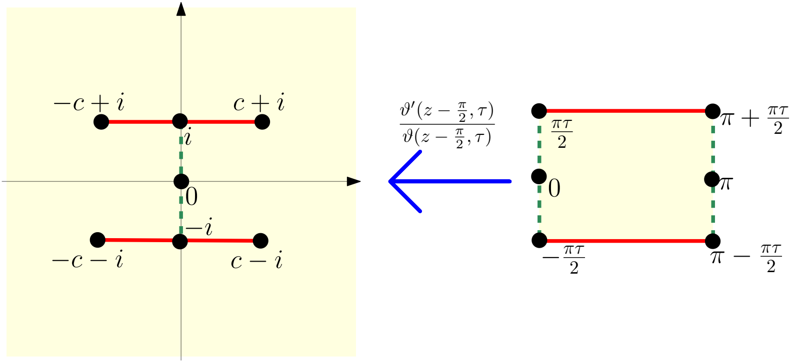
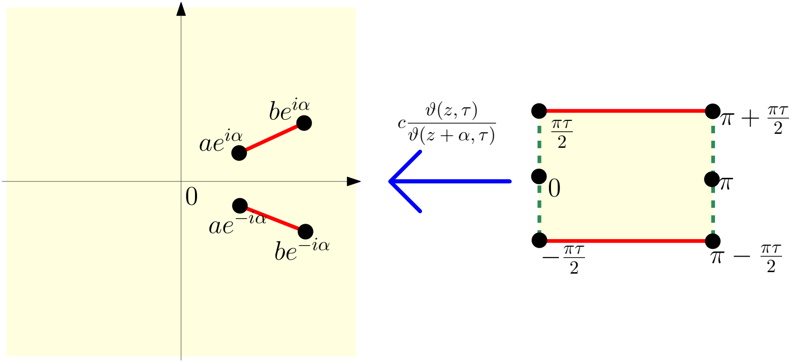
Solving for τ is annoying.

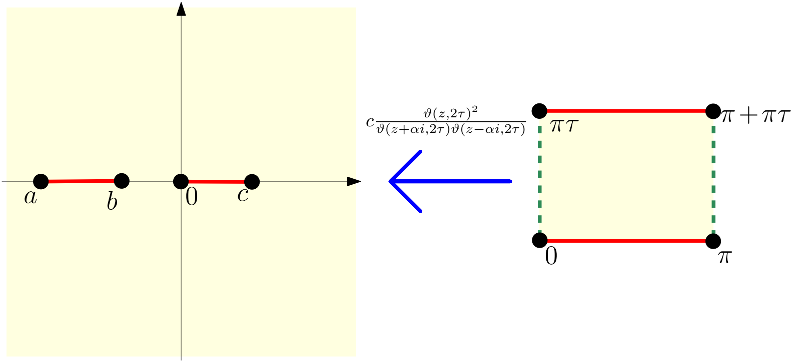
Conformal map example



Conformal map more examples



CONFORMAL MAP MORE EXAMPLES



Part 1d: More identities

Making new theta function identities

Recall: $\vartheta(z+\pi) = -\vartheta(z)$ and $\vartheta(z+\pi\tau) = -e^{-2iz-i\pi\tau}\vartheta(z)$.
Let's try to write $\vartheta(2z)$ as a product of functions like $\vartheta(z-\alpha)$.

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So consider

$$f(z) = \frac{\vartheta(2z)}{\vartheta(z)\vartheta(z-\pi/2)\vartheta(z-\pi\tau/2)\vartheta(z-\pi(\tau+1)/2)},$$

then $f(z)$ has no roots (or poles).

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So $h(z) = e^{2iz}f(z)$ has π and $\pi\tau$ as periods.

MAKING NEW THETA FUNCTION IDENTITIES

From the last slide: The function

$$h(z) = \frac{e^{2iz}\vartheta(2z)}{\vartheta(z)\vartheta\left(z-\frac{\pi}{2}\right)\vartheta\left(z-\frac{\pi\tau}{2}\right)\vartheta\left(z-\frac{\pi(\tau+1)}{2}\right)}$$

is holomorphic and has π and $\pi\tau$ as periods...

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is holomorphic and has π and $\pi\tau$ as periods... hence, $h(z)$ is constant by Liouville's theorem:

Theorem (Liouville): The only holomorphic elliptic functions are the constant functions.

It follows that

$$\vartheta(2z) = -\frac{2e^{-2iz}\vartheta(z)\vartheta(z-\frac{\pi}{2})\vartheta(z-\frac{\pi\tau}{2})\vartheta\left(z-\frac{\pi(\tau+1)}{2}\right)}{\vartheta(\frac{\pi}{2})\vartheta(\frac{\pi\tau}{2})\vartheta\left(\frac{\pi(\tau+1)}{2}\right)}$$