A Soergel-like category for complex reflection groups of rank one

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$$B_s \cong \mathcal{O}(\mathrm{Gr}(e) \cup \mathrm{Gr}(s)) =: \mathcal{O}(e, s).$$

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- The bounded homotopy category of $\mathcal{B}$ can be used to produce a categorical action of the (generalized) braid group of (W, S) (Rouquier).

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- First basic observation: if W is a CRG and $s \in \mathrm{Ref}(W)$, then if $\mathrm{o}(s) \neq 2$, we have

$$B_s \not\cong \mathcal{O}(e, s).$$

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$$\begin{aligned} s^d &= 1, \\ C_iC_j = C_jC_i \ \forall i,j \text{ and } sC_i &= C_is \ \forall i, \\ C_1C_i = C_{i+1} + sC_{i-1} \ \forall i &= 1, \ldots, d-2, \\ C_1C_{d-1} &= \left(v + v^{-1}\right) C_{d-1}, \\ sC_{d-1} &= C_{d-1}, \end{aligned}$$

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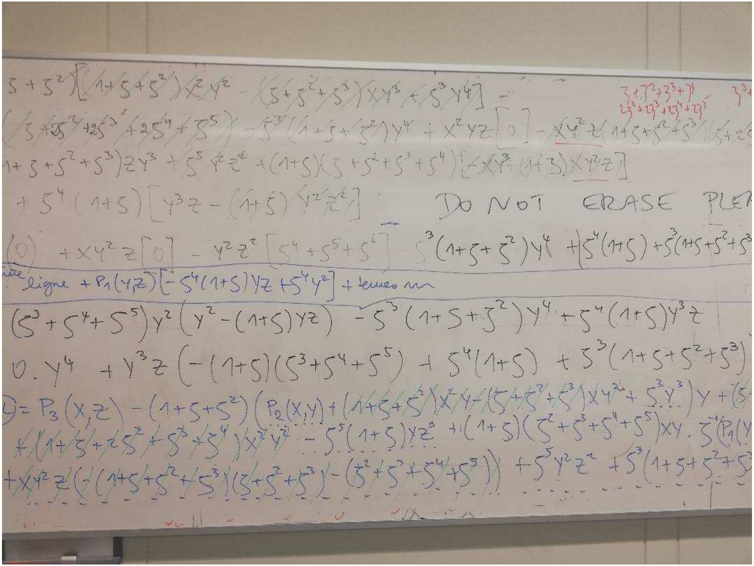

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- This allows one to decompose tensor powers of $\mathcal{O}(e,s)$ in direct sums of $\mathcal{O}(A)$, where $A \subseteq W$ is cyclically connected.

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Theorem (Semisimplicity)

If $v + v^{-1} \neq 2\cos(\frac{k\pi}{d})$ for all $k = 1, \ldots, d-1$, then $A_W^{\mathbb{C}}$ is a semisimple algebra.

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- In type A_2, this category gives rise to a Grothendieck ring of rank 25.