Noncrossing partitions and Bruhat order

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- Multiplicative homomorphism

$$B_{n+1} \to \mathrm{TL}_n$$

where $B_{n+1} =$ braid group on $n+1$ strands.

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- Partial order $<_\mathcal{T}$ on $\mathcal{W}$: $u <_\mathcal{T} v$ if and only if

$$\ell_\mathcal{T}(u) + \ell_\mathcal{T}(u^{-1}v) = \ell_\mathcal{T}(v),$$

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  where $\ell_{\mathcal{T}}$ is the reflection or absolute length. The poset $\mathcal{P}_c=\{x <_{\mathcal{T}} c\}$ is isomorphic to the lattice of noncrossing partitions.
- Dual braid monoid associated to $(\mathcal{W},\mathcal{T},c)$: it has one generator $i_c(t)$ per element t of $\mathcal{T}$ and relations

  $$i_c(t)i_c(t')=i_c(tt't)i_c(t) \text{ whenever } tt' <_{\mathcal{T}} c$$

  called dual braid relations.

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  $\rightarrow$ IDEA (Zinno): map the simple elements to the Temperley-Lieb algebra via the composition

$$B_c^* \hookrightarrow B_{n+1} \rightarrow \mathrm{TL}_n,$$

$$i_c(x) \mapsto Z_x.$$

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- Zinno shows that there exists a total order on the set $\mathcal{P}_c := \{x <_{\mathcal{T}} c\}$ and a bijection $a : \mathcal{P}_c \to \mathcal{W}_f$ such that if you endow $\mathcal{W}_f$ with the order induced by a, then for $x \in \mathcal{P}_c$,

$$Z_x = c_x b_{a(x)} + \sum_{y \in \mathcal{P}_c,\ y < x} c_{y,x} b_{a(y)},$$

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$$Z_x = c_x b_{a(x)} + \sum_{y \in \mathcal{P}_c,\ y < x} c_{y,x} b_{a(y)},$$

where c_x is invertible. From Zinno's work it is not difficult to see that

$$c_{y,x} \neq 0 \Rightarrow y <_{\mathcal{S}} x,$$

where $<_{\mathcal{S}}$ is the restriction of the Bruhat order to $\mathcal{P}_c$!

- There is another proof that Z_x is a basis by Lee and Lee; however they don't prove triangularity. As shown by Vincenti, one can then derive a proof that we get a basis by mapping the simple elements of any dual braid monoid (that is, for any Coxeter element c) to the TL algebra.

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- Question: It there a "general" explanation of these phenomenons (triangularity, positivity + preserved when changing the Coxeter element) by a nice categorification of the TL algebra? → open problem. Or in case you have one, please inform me ☎.

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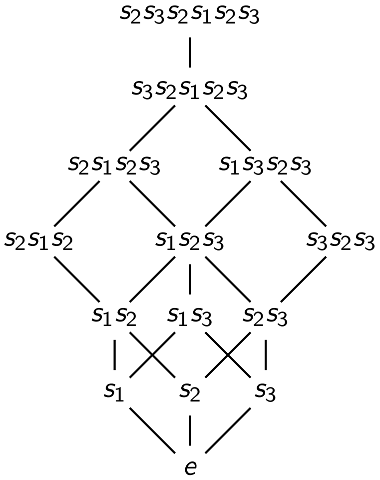
Bruhat order on noncrossing partitions, two examples



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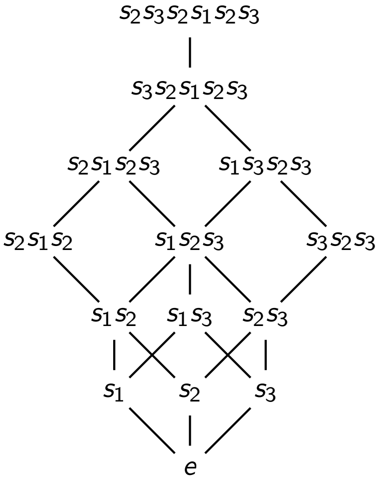

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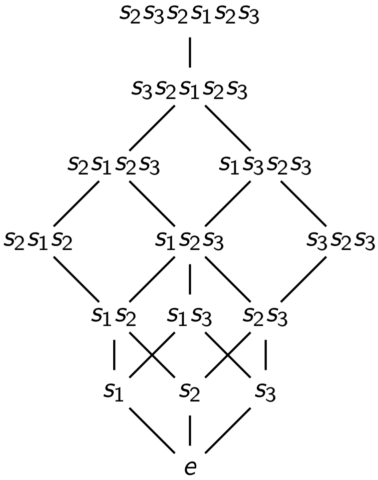

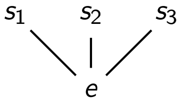

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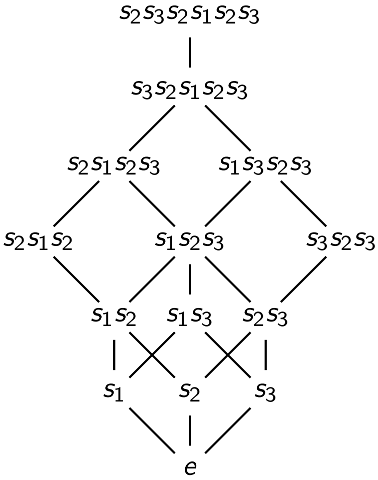

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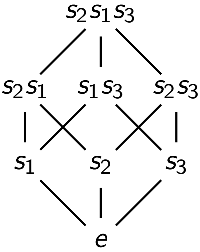

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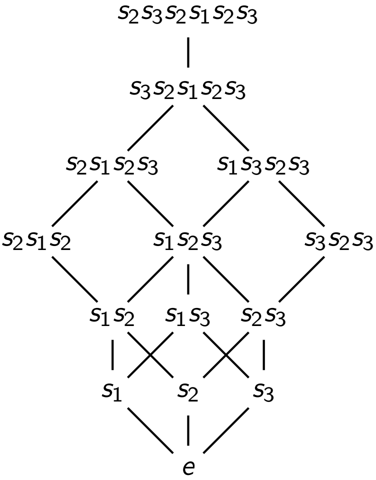

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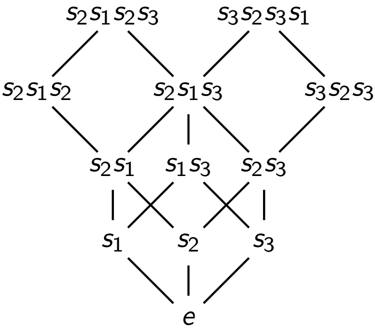

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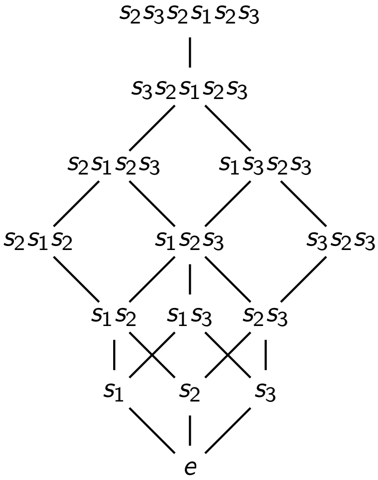

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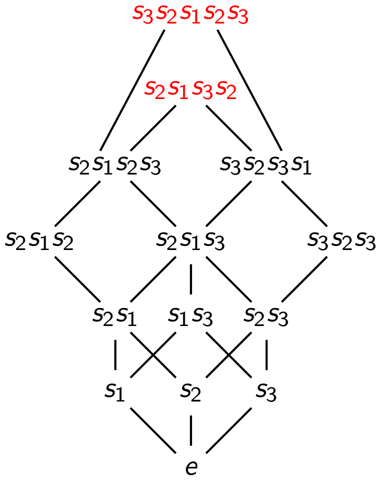

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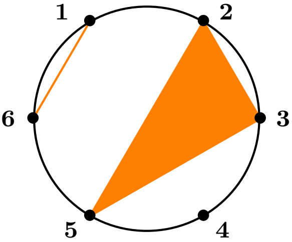
Let us assume that $c = s_1 s_2 \cdots s_n$. Noncrossing partitions are represented by disjoing unions of polygons having as vertices marked points on a circle:



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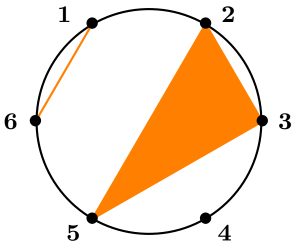

Geometric representation of the noncrossing partition $x = (1,6)(2,3,5)$. Here $n = 5$. We represent a polygon by the ordered sequence of numbers indexing its vertices. In the example above, there are two polygons [16] and [235].

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Let $x \in \mathcal{P}_c$. Consider any polygon $P = [i_1 i_2 \cdots i_k]$ occuring in the geometric representation of x. It corresponds to the cycle $y = (i_1, i_2, \ldots, i_k) \in \mathcal{P}_c$. One has

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We then replace each transposition (j, j'), $j < j'$ in the product above by the Coxeter word

$$[j, j'] := s_{j'-1} s_{j'-2} \cdots s_{j+1} s_j s_{j+1} \cdots s_{j'-2} s_{j'-1}$$

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Example

Let $x = (1,6)(2,3,5)$, $n = 5$. Let $P_1 = [235]$, $P_2 = [16]$. Let $y_1 = (2,3,5)$, $y_2 = (1,6)$. The standard form of y_1 is $s_2 s_4 s_3 s_4$. The standard form of y_2 is $s_5 s_4 s_3 s_2 s_1 s_2 s_3 s_4 s_5$.

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$$v_x = (1,3,3,4,2).$$

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- We have the following conditions on the differences $x_{k+1} - x_k$, for each $1 \leq k < n$:

$x_{m+1} - x_m$	x_{m+1} even	x_{m+1} odd
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Example

If $n = 2$, one has $\mathcal{V} = \{(0,0), (1,0), (0,1), (1,1), (1,2)\}$.

Example

If $n = 3$, one has
$\mathcal{V} = \{(0,0,0),(1,0,0),(0,1,0),$
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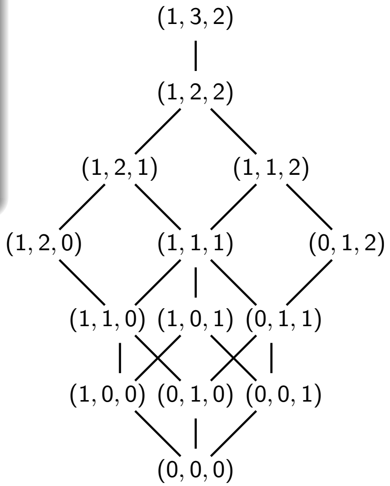

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Corollary

The poset $(\mathcal{P}_c, <_{\mathcal{S}})$ *is a lattice.*

- Is it a general fact that for $c = s_1 \cdots s_n$, the set of noncrossing partitions together with the restriction of the Bruhat order gives rise to the lattice structure coming from the root poset?
- If the answer is yes, why does it fail for other Coxeter elements? Is Bruhat order still the order to consider to prove triangularity of the change of basis matrix in the Temperley-Lieb algebras in case we change the Coxeter element?

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Such a word $m_{x'}^{c'}$ is called a *standard form* of x'. It is not unique in general (products of the words $[d_i, d_{i+1}]$ in different orders may yield words representing the same element of the Coxeter group). If $t \in \mathcal{T} \subset \mathcal{P}_c \cap \mathcal{P}_{c'}$ we have $m_t = m_t^{c'}$.

To summarize: if $x' \in \mathcal{P}_{c'}$ is a cycle, we can associate to x' a tuple $v_{x'}^{c'} = ({x'}_1^{c'}, \ldots, {x'}_n^{c'}) \in \mathcal{V}$ where ${x'}_i^{c'}$ is the number of occurrences of s_i in $m_{x'}^{c'}$.

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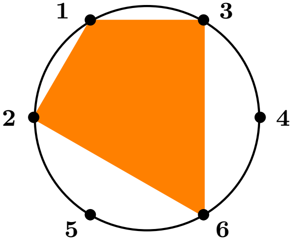

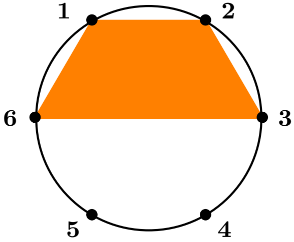

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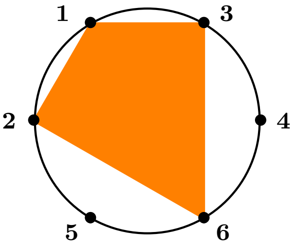

$c' = s_2 s_1 s_3 s_5 s_4 = (1, 3, 4, 6, 5, 2)$	$c = s_1 s_2 s_3 s_4 s_5$
$x' = (1, 3, 6, 2) = (2, 3)(3, 6)(2, 1)$	$x = (1, 2, 3, 6) = (1, 2)(2, 3)(3, 6)$
$m_{x'}^{c'} = s_2(s_5 s_4 s_3 s_4 s_5)s_1$	$m_x = s_1 s_2 (s_5 s_4 s_3 s_4 s_5)$
$v_{x'}^{c'} = (1, 1, 1, 2, 2)$	$v_x = (1, 1, 1, 2, 2)$

The aim now is to associate a tuple $v_{x'}^{c'} \in \mathcal{V}$ to any $x' \in \mathcal{P}_{c'}$. In that case we decompose x' into a product $y_1 y_2 \dots y_m$ of disjoint cycles and define a standard form of x' as the product

$$m_{y_1}^{c'} \cdots m_{y_2}^{c'}.$$

Such a word will be called a *standard form* of x'. Counting the number of simple reflections in it gives rise to a tuple $v_{x'}^{c'}$ but it is not clear that it lies in $\mathcal{V}$.

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Such a word will be called a *standard form* of x'. Counting the number of simple reflections in it gives rise to a tuple $v_{x'}^{c'}$ but it is not clear that it lies in $\mathcal{V}$. To prove that $v_{x'}^{c'} \in \mathcal{V}$ for any $x' \in \mathcal{P}_{c'}$, we first define a map $\mathcal{P}_{c'} \to \mathcal{P}_c$.

Step 1: let $y_i = (i_1, \ldots, i_k)$ be any cycle in the decomposition of x'. Write $\{i_1, \ldots, i_k\} = \{d_1, \ldots, d_k\}$ where $d_j < d_{j+1}$. We represent each y_i on a line with marked points from 1 to $n+1$ by arcs joining the point d_j to the point d_{j+1}, $j = 1, \ldots, k-1$. The resulting diagram may have crossings.

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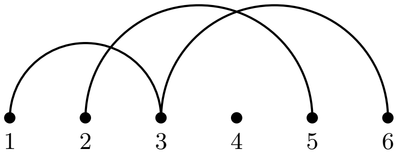
Example

Let $c' = s_4 s_3 s_1 s_2 s_5 = (1,2,5,6,4,3)$ and $x' = \underbrace{(2,5)}_{y_1}\underbrace{(1,6,3)}_{y_2}$. Then we represent x' in the following way

$x' \rightsquigarrow$



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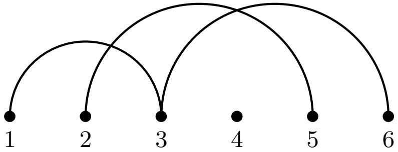

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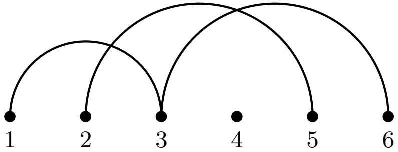

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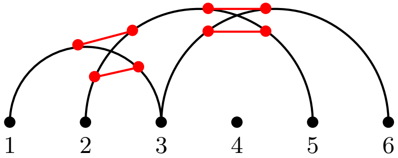

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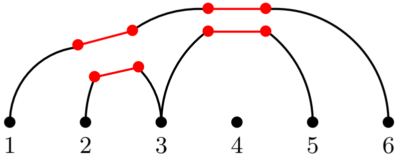

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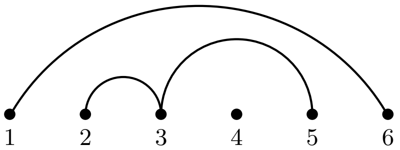

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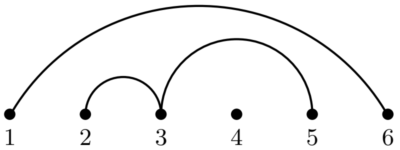


$\rightsquigarrow x$

Lemma

The geometrical process described above defines a bijective map $\phi_{c',c} : \mathcal{P}_{c'} \to \mathcal{P}_c$ *which fixes the set of reflections. Moreover, one has that* $x_i = {x'}_i^{c'}$ *for any* $i \in \{1, \ldots, n\}$.

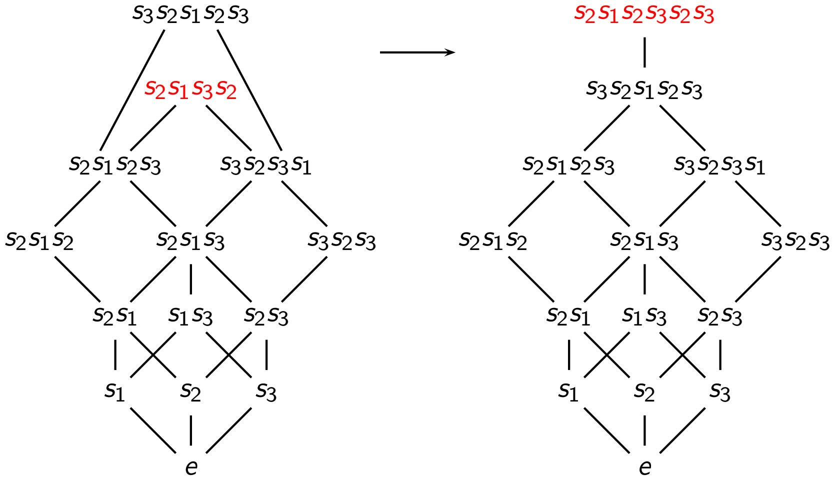
As a consequence,

Corollary

The map $\mathcal{P}_{c'} \to \mathcal{V}, x' \mapsto ({x'}_1^{c'}, \ldots, {x'}_n^{c'})$ *is a well-defined bijection.*

We therefore can consider the order $<$ induced on $\mathcal{P}_{c'}$ by the natural order on $\mathcal{V}$. In case $c' = c$, this is the Bruhat order.

A new order on $\mathcal{P}_{c'}$



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Theorem

For any Coxeter element c', there exist inverse bijections

$$\psi_{c'} : \mathcal{W}_f \to \mathcal{P}_{c'}, \varphi_{c'} : \mathcal{P}_{c'} \to \mathcal{W}_f$$

such that

$$Z_x = c_x b_{\varphi_{c'}(x)} + \sum_{y \in \mathcal{P}_{c'}} c_{y,x} b_{\varphi_{c'}(y)},$$

where c_x is invertible. Moreover,

$$c_{y,x} \neq 0 \Rightarrow y < x,$$

where $<$ is the order induced by $\mathcal{V}$ on $\mathcal{P}_{c'}$.