Torus knot groups, Garside groups, complex reflection groups

Thomas Gobet

Institut Denis Poisson, Université de Tours

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Plan of the talk

Garside structures for Artin's braid group

Questions and motivations

A new Garside structure on torus knot groups

Application to DDGKM's monoid

Torus knot groups and braid groups of complex reflection groups

J-groups

"Braid groups" of J-groups

Perspectives

Artin's braid group and monoid

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- Let $n \geq 0$. Recall that *Artin's braid group* $\mathcal{B}_{n+1}$ (respectively *Artin's braid monoid* $\mathcal{B}_{n+1}^{+}$) on $n+1$ strands is defined by the following group (resp. monoid) presentation

$$\left\langle \sigma_1, \ldots, \sigma_n \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j\sigma_i & = \sigma_j\sigma_i\sigma_j \text{ if } |i-j| = 1, \\ \sigma_i\sigma_j & = \sigma_j\sigma_i \text{ if } |i-j| > 1. \end{array} \right\rangle$$

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- The monoid $\mathcal{B}_{n+1}^+$ is a *Garside monoid*. Among other properties, it implies that it is a lattice for left and right divisibility, that it embeds into the group with the same presentation, i.e. $\mathcal{B}_{n+1}$, and that every element in $\mathcal{B}_{n+1}$ can be uniquely written as an irreducible fraction xy^{-1} with $x, y \in \mathcal{B}_{n+1}^+$.

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- A Garside monoid satisfies Ore's conditions (cancellativity + existence of common multiples), hence can be embedded into a group of left-fractions $G(M)$ (called a *Garside group*) having the same presentation as M. Garside groups have solvable word and conjugacy problems, are torsion free, have a non-trivial center, a finite $K(\pi, 1)$ space, are biautomatic, ...

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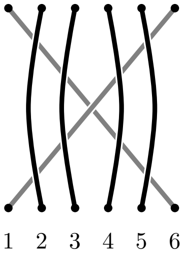

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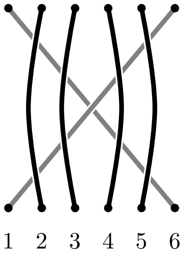

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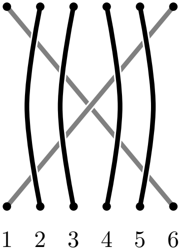

- For $n = 3$, there are (at least) 4 known Garside structures on $\mathcal{B}_n$. For $n > 3$, up to slight variations there are no other known Garside structures than the above two.

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- What about the "exotic" structure ?

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- Is there a generalization of the exotic monoid to "higher ranks" ?

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- If $n = 2$, we obtain the presentation

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2. *The Garside group $\mathcal{G}(n)$ is isomorphic to $G(n, n+1)$, the knot group of the torus knot $T_{n,n+1}$.*

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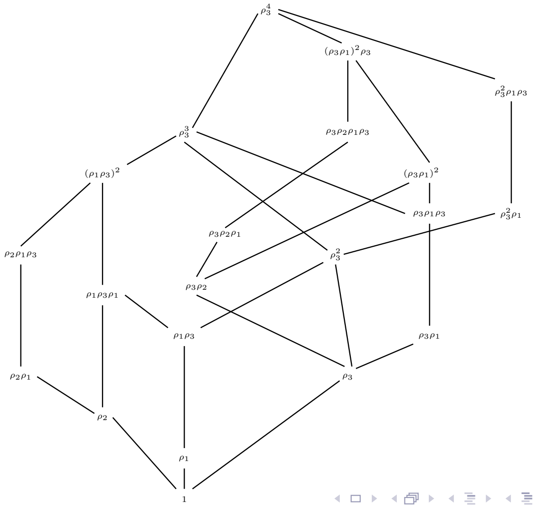

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Torus knot groups and braid groups of complex reflection groups

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Theorem

Let $n, m \geq 2$ be two relatively prime integers. The Garside structure introduced above for $\mathcal{G}(n) \cong G(n, n+1)$ can be generalized to all torus knot groups $G(n, m)$. That is, there is a Garside monoid $\mathcal{M}(n, m)$ generalizing the construction given above (hence with $\mathcal{M}(n) = \mathcal{M}(n, n+1)$), such that the corresponding Garside group $\mathcal{G}(n, m)$ is isomorphic to the torus knot group $G(n, m)$.

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- Moreover, a Garside structure analogous to the one introduced above for torus knot groups can be constructed for a few additional braid groups of complex reflection groups which are not isomorphic to torus knot groups (for instance G_{13}, and the dihedral Artin-Tits groups of even type).

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- **Question 1** : Can torus knot groups be realized as "braid groups" of (generalizations of) complex reflection groups ?
- **Question 2** : Can the Garside structure introduced above be defined for a more general class of groups than torus knot groups ?
- **Question 3** : In the above mentioned examples where a torus knot group is isomorphic to the braid group of a finite complex reflection group, the reflection group is obtained as quotient of $\mathcal{G}(n,m)$ by a single relation of the form $\rho_1^k = 1$ $(k \geq 2)$. For arbitrary coprime $n, m \geq 2$, does this quotient (which is infinite in general) admit a natural structure of complex reflection group ?

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- Achar and Aubert introduced in 2008 a family of (in general infinite) groups, called *J-groups*, which generalize the finite complex reflection groups of rank two.

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$$J\begin{pmatrix} a & b & c \\ 1 & 1 & 1 \end{pmatrix} = J\begin{pmatrix} a & b & c \\ & & \end{pmatrix}.$$

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- This result is reminiscent of the following theorem : let W be a Coxeter group. Then W is a real reflection group if and only if W is finite.

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k	n	m	$\mathcal{W}(n,m,k)$
2	3	4	G_{12}
2	3	5	G_{22}
3	2	3	G_4
4	2	3	G_8
5	2	3	G_{16}
3	2	5	G_{20}
2	2	≥ 3, odd	$G(m,m,2) = I_2(m)$

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In this way, we can associate a Garside group to every J-group in the above defined family, in such a way that whenever the J-group is finite, one recovers the braid group of the finite complex reflection group isomorphic to the J-group.

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In this way, we can associate a Garside group to every J-group in the above defined family, in such a way that whenever the J-group is finite, one recovers the braid group of the finite complex reflection group isomorphic to the J-group. As a byproduct, we get presentations by generators and relations for the J-groups $\mathcal{W}(n, m, k)$.

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- One can define a Hecke algebra of $\mathcal{W}(n, m, k)$ in a natural way. What are its properties ?

Thank you for your attention !