A nontrivial order on the symmetric group: the Bruhat order, and generalizations

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Semaine des Jeunes de l'IDP,
Ferme de Courcimont,
20-23th July 2021.

Plan of the talk

Flags of vector spaces

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- Let $n \geq 1$. A *(complete) flag* in $V = \mathbb{C}^n$ is a sequence

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- Since the group $G = \mathrm{GL}_n(\mathbb{C})$ of complex invertible matrices of size $n \times n$ acts transitively on bases of $V = \mathbb{C}^n$, it also acts transitively on the set of flags in V (for if $\dim_{\mathbb{C}}(V_i) = i$ and $g \in G$, we have $\dim_{\mathbb{C}}(g(V_i)) = i$ as g is invertible).

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- Since $G = \mathrm{GL}_n(\mathbb{C})$ acts transitively on the set of flags, by basic group theory we can identify the set of flags with $G/\mathrm{Stab}_G(\mathbf{s})$.
- The stabilizer of the standard flag is nothing but the subgroup $B \subseteq G$ of upper-triangular matrices.
- Therefore, we have a one-to-one correspondence

$$\{\text{Complete flags in } V\} \overset{1:1}{\longleftrightarrow} G/B = \{gB \mid g \in G\}.$$

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- So, the action of B on $\mathbb{P}(V)$ has two orbits : the singleton $\{\mathbf{s} = (0 \subseteq \langle e_1 \rangle \subseteq V)\}$, and a dense orbit $\mathbb{C} = \mathbb{P}(V) \backslash \{\mathbf{s}\}$.

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- Similarly, one can define algebraic subsets of $\mathbb{P}(V)$: one just replaces polynomials by homogeneous polynomials.

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- Algebraic subsets of V are called *affine algebraic varieties*. Algebraic subsets of $\mathbb{P}(V)$ are called *projective algebraic varieties*.

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- Can we describe the partial order induced by inclusions of B-orbit closures ?

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Setting $w = s_1 = (1, 2)$, with corresponding matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we have $w \cdot \mathbf{s} = \mathbf{s}'$. In other words, the B-orbits on G/B are parametrized by the elements of $\mathfrak{S}_2$.

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Theorem (Bruhat decomposition of GL_n)

For $x \in G = \mathrm{GL}_n(\mathbb{C})$, let $BxB := \{bxb' \mid b, b' \in B\}$. We have

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Corollary

We have

$$G/B = \coprod_{w \in \mathfrak{S}_n} BwB/B$$

and hence, the B-orbits on G/B are parametrized by $\mathfrak{S}_n$.

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$$A = \underbrace{\begin{pmatrix} \frac{bc-ad}{c} & \frac{a}{c} \\ 0 & 1 \end{pmatrix}}_{\in B} \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{=s_1} \underbrace{\begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix}}_{\in B}.$$

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- If $\mathcal{O}_2$ was closed, then there would be a family $(P_i)_{i \in I}$ of two-variable homogeneous polynomials having as common vanishing set the complement $V \backslash L$ of the line $L := \langle e_1 \rangle$. Let $Q = P_i$ $(i \in I)$ be nonzero.

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- The above-defined partial order was first introduced by Ehresmann in 1934. It is called the *(strong) Bruhat order* in reference to the Bruhat decomposition of G.

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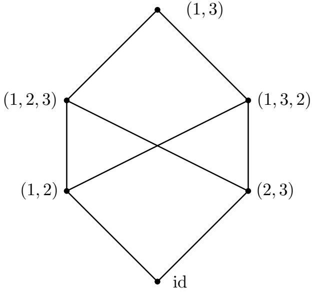

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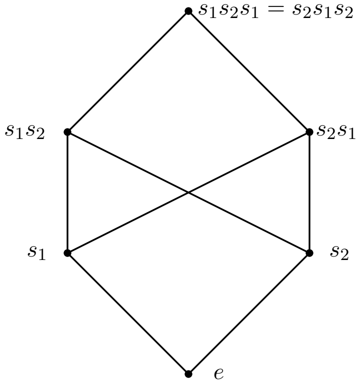

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- There are still finitely many B-orbits on G/B, and they are parametrized by a group W ("Weyl group") generalizing the symmetric group. The Weyl group is generated by a set S of involutions (in fact, it is a *Coxeter group*), and Deodhar's criterion is still valid to describe inclusions of orbit closures in G/B.

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For $w \in W_r$ we set $[w] := wP_r'$

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We have the following:

1. *The Z-orbits on G/B are parametrized by the set W_r. For $w \in W_r$ we denote by $\mathcal{O}_w$ the corresponding orbit.*

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Theorem (Boos-Reineke 2012, Bender-Perrin 2019, Chaput-Fresse-G. 2020)

We have the following:

1. *The Z-orbits on G/B are parametrized by the set W_r. For $w \in W_r$ we denote by $\mathcal{O}_w$ the corresponding orbit.*
2. *For $w, w' \in W_r$, we have $\mathcal{O}_w \subseteq \overline{\mathcal{O}_{w'}}$ if and only if there is $u \in [w] = wP_r'$ such that $u \leq w'$ (strong Bruhat order).*

Example : $n = 4$, $r = 2$.

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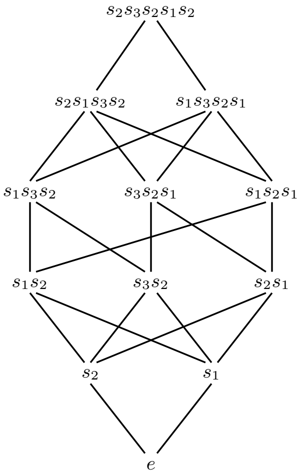

Figure: Partial order describing inclusions of orbit closures for $n = 4$, $r = 2$.