Zinno bases of Temperley-Lieb algebras

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The Temperley-Lieb algebra

Definition

Let δ a parameter, $n \in \mathbb{Z}_{>0}$. The *Temperley-Lieb algebra* TL_n is the associative unital $\mathbb{Z}[\delta]$-algebra with generators $b_1, \ldots, b_n$ and relations

$$b_j b_i b_j = b_j \text{ if } |i-j| = 1,$$

$$b_i b_j = b_j b_i \text{ if } |i-j| > 1,$$

$$b_i^2 = \delta b_i.$$

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Fact

The Temperley-Lieb algebra is often viewed as a $\mathbb{Z}[v, v^{-1}]$*-algebra with* $\delta = v + v^{-1}$*. This allows one to realize it as a quotient of the Iwahori-Hecke algebra* $\mathrm{H}(\mathfrak{S}_{n+1})$ *of type* A_n*.*

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Let $\mathcal{W} = \mathfrak{S}_{n+1}$, $S = \{s_i\}_{i=1}^{n}$ where $s_i = (i, i+1)$. An element $w \in \mathcal{W}$ is *fully commutative* if given any reduced expression $s_{i_1} s_{i_2} \cdots s_{i_k}$ of w (in the sense of Coxeter) and any $s \in S$,

$$n(s) = \#\{k \mid s_{i_k} = s\}$$

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1. *The element $b_{i_1} b_{i_2} \cdots b_{i_k}$ is independent of the choice of the reduced expression for w. We will therefore denote it by b_w.*
2. *The set $\{b_w\}_{w \in \mathcal{W}_f}$ is a $\mathbb{Z}[\delta]$-basis of $\mathrm{TL}_n(\delta)$.*

Noncrossing partitions

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A noncrossing partition is a partition of the set $\{1, 2, \ldots, n+1\}$ such that any two blocks B and B' never cross, that is, there exist no pairs of indices $i, j \in B$, $k, \ell \in B'$ such that $i < k < j < \ell$.

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A noncrossing partition can be seen as a permutation: one associates to each block $B = \{i_1, i_2, \ldots, i_k\}$ $(i_m < i_{m+1})$ the cycle $c_B = (i_1, i_2, \ldots i_k)$ and takes the product of the various c_B.

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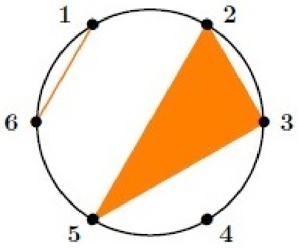

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Consider the braid group B_n on $n+1$ strands

$$B_n := \left\langle \mathbf{s_1}, \ldots, \mathbf{s_n} \;\middle|\; \begin{array}{c} \mathbf{s_i s_{i+1} s_i} = \mathbf{s_{i+1} s_i s_{i+1}}, \quad \forall i \in \{1, \ldots, n-1\} \\ \mathbf{s_i s_j} = \mathbf{s_j s_i}, \quad \text{if } |i-j| > 1 \end{array} \right\rangle.$$

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$$[i, j+1] := \mathbf{s_j}^{-1} \mathbf{s_{j-1}}^{-1} \cdots \mathbf{s_{i+1}}^{-1} \mathbf{s_i s_{i+1}} \cdots \mathbf{s_{j-1} s_j}.$$

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The submonoid of B_n generated by the equivalence classes of these braid words is the *Birman-Ko-Lee monoid*. To any cycle $c = (i_1, i_2, \ldots, i_k) \in \mathfrak{S}_{n+1}$ with $i_1 < i_2 < \cdots < i_k$, associate the braid word

$$\underline{c} = [i_1, i_2, i_3, \ldots, i_k] := [i_1, i_2][i_2, i_3] \cdots [i_{k-1}, i_k].$$

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The set D of equivalence classes of braid words $\underline{x}$ where x is a noncrossing partition is the set of *simple* elements of the Birman-Ko-Lee monoid. Since these elements are lifts of noncrossing partitions in the braid group we will also denote by D the set of noncrossing partitions.

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There is a homomorphism

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It turns out that $|\mathcal{W}_f| = |D| = C_{n+1} = \dim(\mathrm{TL}_n(v + v^{-1}))$. What happens if one maps the elements of D in $\mathrm{TL}_n(v + v^{-1})$?

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$$\alpha(x) = t^x_{a(x)} b_{a(x)} + \sum_{y \in \mathcal{W}_f,\ y < a(x)} t^x_y b_y,$$

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Corollary

The set $\{Z_x := \alpha(x) \mid x \in D\}$ is a $\mathbb{Z}[v, v^{-1}]$-basis of $\mathrm{TL}_n(v + v^{-1})$, which we will call Zinno basis.

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- What can be said on the coefficients of the change base matrix?
- Write $Z_x = \sum_{w \in \mathcal{W}_f} t'^{w}_{x} b_w$, where $t'^{w}_{x} \in \mathbb{Z}[v, v^{-1}]$. Computations in small cases show that $t'^{w}_{x} = (-1)^{\ell_S(w)} t^{w}_{x}$ where t^{w}_{x} is a polynomial with positive coefficients. Can we find an interesting interpretation of these coefficients or prove positivity using either combinatorial methods or categorification?

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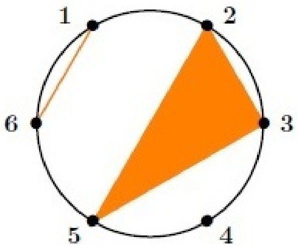

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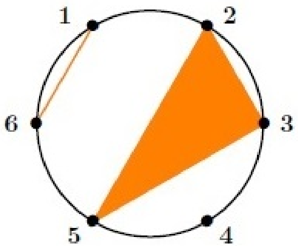

$$m_x = \mathbf{s_2}(\mathbf{s_4}^{-1}\mathbf{s_3}\mathbf{s_4})(\mathbf{s_5}^{-1}\mathbf{s_4}^{-1}\mathbf{s_3}^{-1}\mathbf{s_2}^{-1}\mathbf{s_1}\mathbf{s_2}\mathbf{s_3}\mathbf{s_4}\mathbf{s_5}).$$

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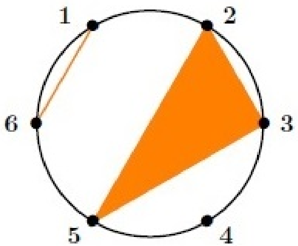

$$m_x = \mathbf{s}_2(\mathbf{s}_4{}^{-1}\mathbf{s}_3\mathbf{s}_4)(\mathbf{s}_5{}^{-1}\mathbf{s}_4{}^{-1}\mathbf{s}_3{}^{-1}\mathbf{s}_2{}^{-1}\mathbf{s}_1\mathbf{s}_2\mathbf{s}_3\mathbf{s}_4\mathbf{s}_5).$$

Zinno gives rules for extracting the fully commutative element $a(\underline{x})$ as a subword of m_x.

Zinno's bijection, II

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$\rightsquigarrow a(x) = (s_2 s_1)(s_4 s_3 s_2)(s_5 s_4 s_3) \in \mathcal{W}_f.$

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The algorithm: read the word m_x from the left to the right. If the first letter $\mathbf{s_i}^{\pm 1}$ occuring in m_x has positive (resp. negative) exponent, then all the occurrences of $\mathbf{s_i}^{\pm 1}$ in m_x with positive (resp. negative) exponent and only those must contribute to the subword $a(x)$. Apply the same process to the next generator $s_j^{\pm 1}$, $j \neq i$ occuring right to the first $s_i^{\pm 1}$ in m_x, until you have considered all the indices k such that $s_k^{\pm 1}$ occurs in m_x.

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- Zinno shows that such a process gives a well-defined map $a : D \to \mathcal{W}_f$ and shows that it is surjective; hence it is bijective since both sets have cardinality (equal to the $(n+1)$th Catalan number $C_{n+1} = \frac{1}{n+2}\binom{2(n+1)}{n+1}$). However surjectivity is proved indirectly, not allowing one to give the inverse bijection.

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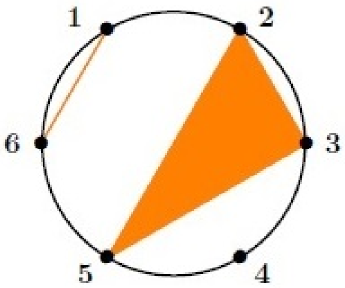
- It is not clear on how to generalize such a process to an arbitrary dual braid monoid since it needs the representation of $\underline{x}$ by a specific braid word m_x.

A new version of Zinno's bijection



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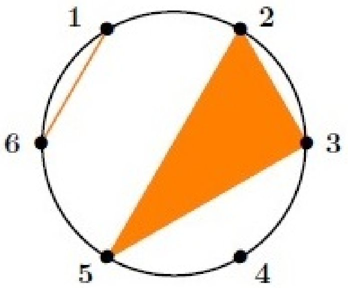

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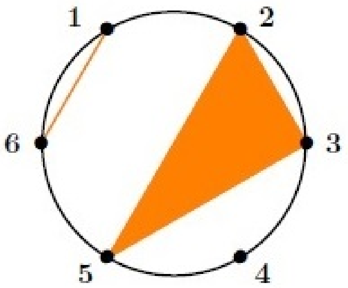

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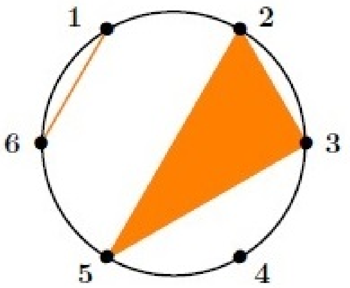

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3. *Such a process generalizes to dual braid monoids.*

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2. *One has $s\psi(w) = \psi(w)t \Leftrightarrow s = t$ or $s = s_i$, $t = s_{i-1}$.*

A new basis of the Temperley-Lieb algebra, II

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- $w = s_2 s_1$; $L(w) = \{s_2\}$, $R(w) = \{s_1\}$; set $x := \psi(w)$; $U_x = \{3\}$, $D_x = \{1\}$ hence $x = (1,3) = s_2 s_1 s_2$. One has $s_2 x = s_1 s_2 = x s_1$.

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Proposition

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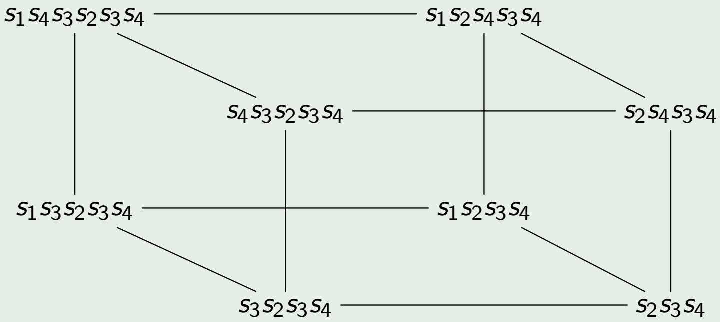

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Proposition

The set $\{X_w\}_{w \in \mathcal{W}_f}$ is a basis of $\mathrm{TL}_n(\delta)$; the change base matrix $\{Z_x\} \leftrightarrow \{X_w\}$ (and $\{X_w\} \leftrightarrow \{b_w\}$) is upper triangular with invertible coefficient on the diagonal (D is ordered by Bruhat order and $\mathcal{W}_f$ is ordered by the order induced by $\varphi = a$).

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Positivity results

Proposition

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Theorem

For any $x \in D$, $w \in \mathcal{W}_f$, t_w^x has only positive coefficients.