# Good permutation braids, dual braid monoids and positivity in Hecke algebras

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## Example

*The symmetric group $W = \mathfrak{S}_n$, is a Coxeter group with $S = \{s_i = (i, i+1) \mid i = 1, \ldots, n-1\}$, $m_{ij} = 3$ if $|i-j| = 1$, $m_{ij} = 2$ if $|i-j| > 1$.*

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- Let $(W, S)$ be finite. There is an embedding $B(W)^+ \hookrightarrow B(W)$ (Deligne, Brieskorn-Saito) and $B(W)^+$ is a *Garside monoid*. In particular, left divisibility is a partial order on $B(W)^+$.

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- The monoid $B(W)^+$ contains a set $\mathbf{W}$ of *simple elements* with are in one-to-one correspondence with $W$. They can be defined as follows: let $w \in W$, with $S$-reduced expression $s_1 s_2 \cdots s_k$. The corresponding element $\mathbf{w} \in \mathbf{W}$ is the product $\mathbf{s}_1 \mathbf{s}_2 \cdots \mathbf{s}_k$.

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- Idea: replace the set $S$ of generators of $W$ by the set of all reflections of $W$, that is, by the set $T = \bigcup_{w \in W} wSw^{-1}$, and study $W$ and $B(W)$ using this point of view.

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Problem: Unlike $<_S$, the partial order $<_T$ does not endow $W$ with a lattice structure.

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- One can show that $B_c^* \hookrightarrow \mathrm{Frac}(B_c^*) \cong B(W)$ and that $B_c^*$ is a Garside monoid. The image of $i_c(s_i)$ in $B(W)$ is $\mathbf{s}_i$.

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- Question: what can be said of the images of $\mathbf{P_c}$ in $\mathcal{H}(W)$? More generally, can we use $B_c^*$ and the dual theory to study $\mathcal{H}(W)$?

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- The conjecture is a straightforward computation for dihedral groups.

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**Conjecture**

*Any simple element of $B_c^*$, after embedding in $B(W)$, can be written in the form $\mathbf{x}^{-1}\mathbf{y}$, with $\mathbf{x}, \mathbf{y} \in \mathbf{W}$.*

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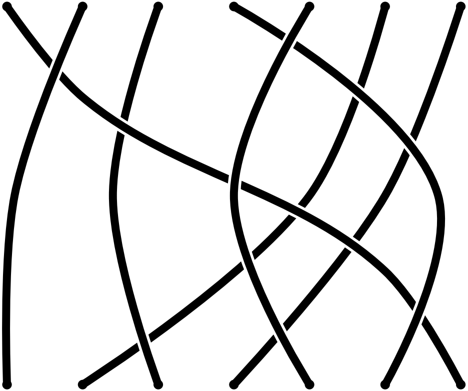
- A braid diagram is reduced if it has a minimal number of crossings. A braid $\beta$ is a permutation braid if it has a reduced diagram where any two strands cross at most once (equivalently if any reduced diagram is like that).
- A braid is a good permutation braid if one can inductively remove a strand which is over all the others (as the vocabulary suggests, such a braid must be a permutation braid).

# Example



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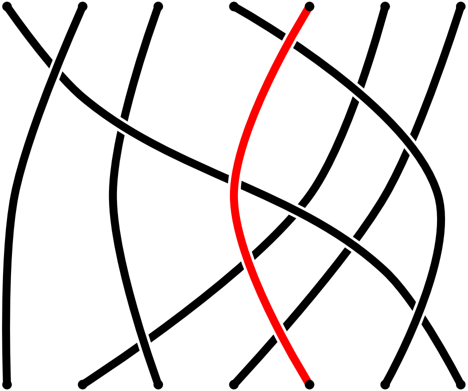

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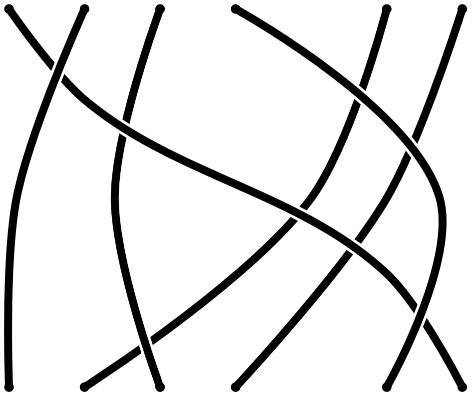

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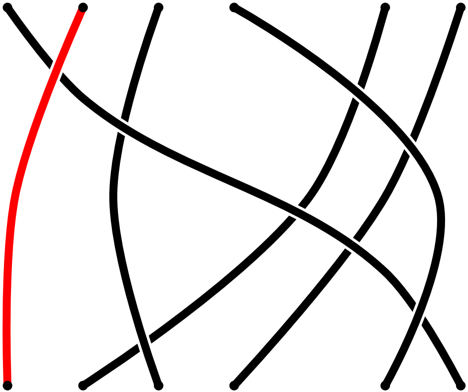

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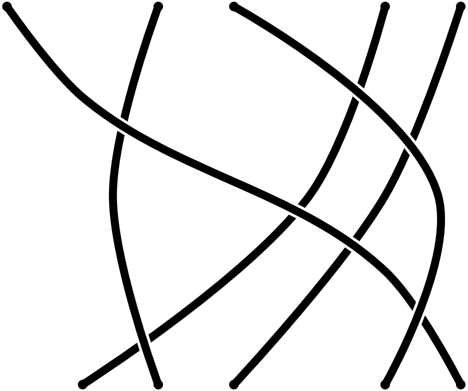

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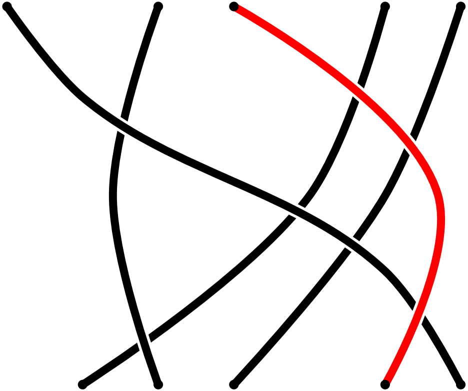

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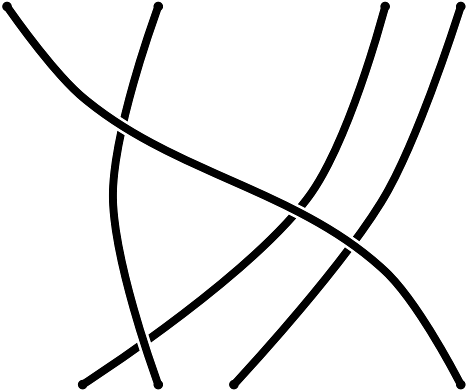

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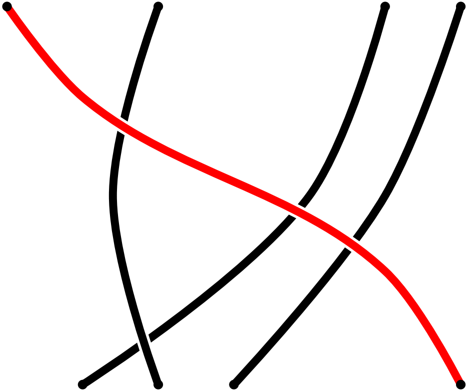

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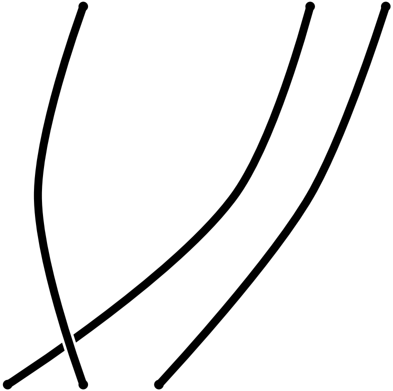

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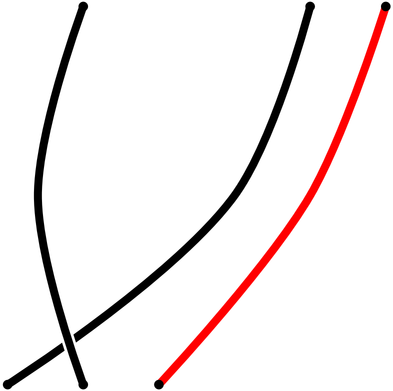

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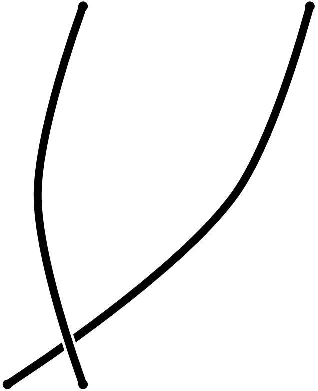

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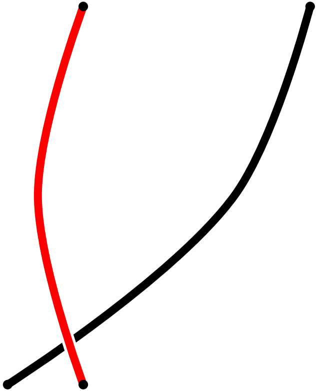

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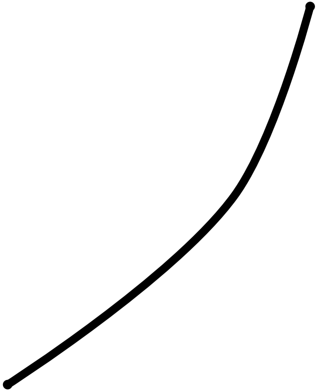

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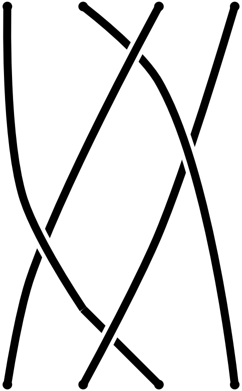
# Counterexample



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2. *One has $\Delta^{-1} \leq \beta \leq \Delta$, where $\Delta$ is the Garside element for the classical Garside structure and $u \leq v$ iff $u^{-1}v \in B^+(W)$.*

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## Corollary

*The positivity conjecture on the linear expansion of an element $x \in \mathbf{P_c} \subset B_c^*$ in the Kazhdan-Lusztig $C_w$-basis of $\mathcal{H}(\mathcal{W})$ holds in type $A_n$ for any choice of Coxeter element.*

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- What about type $D_n$? Computations in small ranks using the computer show that the conjecture should still hold, but to the best of our knowledge, there is no geometrical model for the Artin-Tits group of type $D_n$ similar to those used for types $A_n$ and $B_n$.

Thank you !