2-braid groups and positivity phenomenons in Hecke and Temperley-Lieb algebras

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- Let $\mathcal{H}_n$ be the Iwahori-Hecke algebra of $\mathfrak{S}_n$, i.e., the associative, unital $\mathbb{Z}[v^{\pm 1}]$-algebra with generators T_{s_i} and relations the defining relations of B_n together with

$$T_{s_i}^2 = (v^{-2} - 1) T_{s_i} + v^{-2}, \forall i = 1, \ldots, n-1.$$

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- Starting from an arbitrary Coxeter group W, one can define B_W, H_W, canonical bases, ...

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- Let $\mathbf{NC}$ denote the simple dual elements. Their images under φ form a linearly independent subset of $\mathcal{H}_n$. In fact, they form a basis of a famous quotient of $\mathcal{H}_n$, the *Temperley-Lieb algebra* (Zinno 2002, Lee-Lee 2004).

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- What happens in other types ?

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- Define the *c-simple dual braids* (c-SDB) in B_c^* in the exact same way as for B_n^*.

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- Let ψ denote the composition $B_n \stackrel{\varphi}{\to} \mathcal{H}_n \stackrel{\theta}{\to} \mathrm{TL}_n$.

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Theorem (Zinno 2002, Lee-Lee 2004, G. 2016)

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Temperley-Lieb algebras and dual braid monoids

Theorem (Zinno 2002, Lee-Lee 2004, G. 2016)

Fix a Coxeter element c in $\mathfrak{S}_n$. The images of the c-SDB of B_c^* under ψ yield a basis of TL_n, and there is a triangular base change matrix to $\{b_w\}_w$.

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- Note that (1) can be asked for an arbitrary Coxeter group W with attached Artin-Tits group B_W...

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  To get the statement at least for all the finite *Coxeter* groups, we can also consider the following:

  Question 2: Is Dyer's conjecture true for arbitrary Coxeter systems ?

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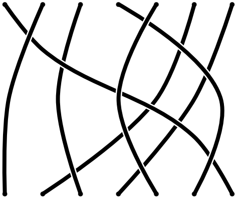

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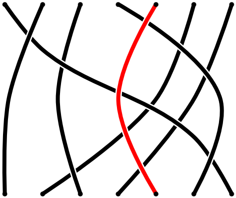

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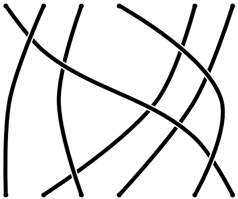

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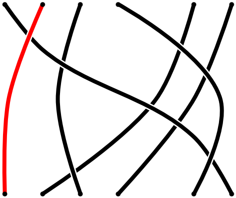

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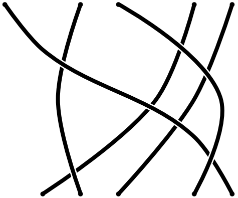

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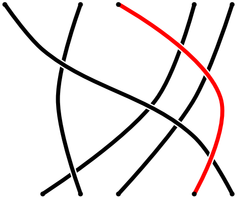

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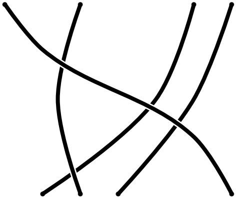

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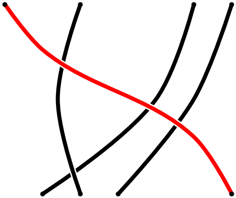

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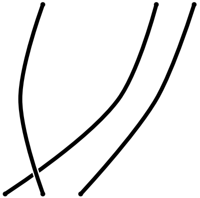

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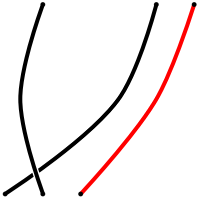

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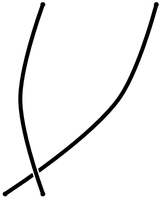

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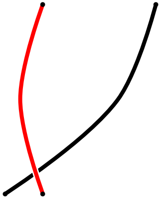

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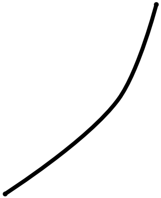

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A braid $\beta \in B_n$ is Mikado iff there are $x, y \in \mathfrak{S}_n$ such that $\beta = \mathbf{x}\mathbf{y}^{-1}$ (iff there are $u, v \in \mathfrak{S}_n$ such that $\beta = \mathbf{u}^{-1}\mathbf{v}$)

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- Dyer gave the following definition. Let (W, S) be a Coxeter group with set of reflections T and root system Φ. Let $A \subseteq \Phi^+$ be a biclosed set of positive roots and let $T_A \subseteq T$ be the corresponding set of reflections. Let $x \in W$, let $s_1 s_2 \cdots s_k$ be a reduced expression of x and let

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- Call an element x_A a *Mikado braid*. It is not hard to see that braids of the form $\mathbf{x}\mathbf{y}^{-1}$ or $\mathbf{u}^{-1}\mathbf{v}$ are Mikado, and that in spherical types both definitions are equivalent.

Mikado braids and SDB

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Theorem (Formula expressing the SDB in the standard Artin generators; G., 2018)

Let β be a SDB in a spherical type Artin group with choice of Coxeter element c. Let x be its image in W. Then $\beta = x_A$, where A can be explicitely defined using Reading's c-sortable elements.

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- In the first case, the coefficients are interpreted as graded multiplicities in filtrations of B_w (recall that $\langle B_w \rangle = C'_w$). In the second case, the coefficients count the number of occurrences of B_w's in a chain complex of Soergel bimodules categorifying T_x.

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- Rouquier (2004) showed that it defines a conjecturally faithful action of the Artin-Tits group B_W on $K^b(\mathcal{B})$.
- In this way one can attach to any braid $\beta \in B_W$ an object $F_\beta \in K^b(\mathcal{B})$, $\mathbf{s_1 s_2}^{-1} \cdots \mapsto F_{s_1} \otimes E_{s_2} \otimes \cdots$.

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- Given $\beta \in B_W$, one can compute F_β using any word for β. The obtained complex may have contractible summands. Removing them yields the (unique up to isomorphism of complexes) *minimal complex* $F_\beta^{\min}$ of β.
- A crucial step in the proof of inverse KL positivity is to show that $F_{\mathbf{x}}^{\min}$ is *linear*, i.e., that the indecomposable summands in homological degree i are all of the form $B_x(i)$ for various $x \in W$ (in other words, the object $F_{\mathbf{x}}$ lies in the heart of the canonical t-structure on $K^b(\mathcal{B})$).

Linearity

- **Example:** let $\beta = \sigma_2\sigma_1\sigma_3\sigma_2$, then $F_\beta^{\min}$ is of the form

$$\begin{array}{ccccccccc} & & B_{s_2s_1s_2}(1) & & B_{s_1s_2}(2) & & & & \\ & & B_{s_1s_3s_2}(1) & & B_{s_3s_2}(2) & & B_{s_1}(3) & & \\ \overset{\star}{B_{s_2s_1s_3s_2}} & \to & B_{s_2s_3s_2}(1) & \to & B_{s_2s_1}(2) & \to & B_{s_2}(3) & \to & R(4) \\ & & B_{s_2}(1) & & B_{s_1s_3}(2) & & B_{s_3}(3) & & \\ & & B_{s_2s_1s_3}(1) & & B_{s_2s_3}(2) & & & & \end{array}$$

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3. *Conversely, one has $C'_w \in \sum_{x \in W} \mathbb{Z}_{\geq 0}[v^{\pm 1}] T_x T_y^{-1}$, where coefficients are interpreted as graded multiplicities in a twisted filtration of B_w.*

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Thank you for your attention!