On positivity properties in Hecke algebras of arbitrary Coxeter groups

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Coxeter groups and their Artin-Tits groups

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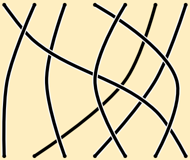

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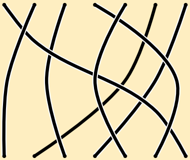

- Given $w = s_1 s_2 \cdots s_k$ with $\ell(w) = k$, the lift $\mathbf{s_1 s_2} \cdots \mathbf{s_k}$ in $B(W)$ is well-defined and denoted by $\mathbf{w}$.

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- Each T_w is invertible and $\{T_{w^{-1}}^{-1}\}_{w \in W}$ is also a basis of $\mathcal{H}(W)$, called *costandard*.

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Conjecture (Soergel 2007; proven by Elias and Williamson 2014)

$\mathcal{E}(C'_w) = \langle B_w \rangle$ *for all* $w \in W$.

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- It follows from Soergel's conjecture that these multiplicities categorify the KL polynomials when $B = B_w$.

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$$u \leq_y v \Leftrightarrow uy \leq vy.$$

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- Rouquier showed that the complexes $F_s := 0 \to B_s \to R(1) \to 0$, $s \in S$ (with B_s in cohom. degree zero) admit an inverse E_s for $\otimes_R^{\mathrm{tot}}$ in $K^b(\mathcal{B})$ and that they satisfy the braid relations of W.

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2. *The coefficient $q^y_{x,w}$ gives the multiplicity of B_w in all cohom. degrees of $C^{\bullet,\min}_{\beta(x,y)}$ together.* $\Rightarrow q^A_{x,w} \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$.

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- Open problem: Understand the perverse cohomology groups of the Rouquier complexes $C^{\bullet,\min}_{\beta(x,y)}$.

Thank you !