A categorification of the Temperley-Lieb algebra by analogues of Soergel bimodules

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Let δ be a parameter, $n \in \mathbb{Z}_{\geq 2}$. The *Temperley-Lieb algebra* $\mathrm{TL}_n(\delta)$ is the associative unital $\mathbb{Z}[\delta]$-algebra with generators $b_1, \ldots, b_n$ and relations

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$$\begin{aligned} b_j b_i b_j &= b_j \text{ if } |i-j| = 1, \\ b_i b_j &= b_j b_i \text{ if } |i-j| > 1, \\ b_i^2 &= \delta b_i. \end{aligned}$$

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Fact

The Temperley-Lieb algebra is often defined as a $\mathbb{Z}[v, v^{-1}]$-algebra with $\delta = v + v^{-1}$. This allows one to realize it as a quotient of the Iwahori-Hecke algebra $H(\mathfrak{S}_{n+1})$ of type A_n.

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Let $w \in \mathcal{W}_f$. One associates to any reduced decomposition $s_{i_1} s_{i_2} \cdots s_{i_k}$ of w the element $b_{i_1} b_{i_2} \cdots b_{i_k}$ of $\mathrm{TL}_n(\delta)$.

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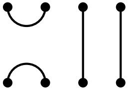
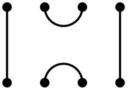
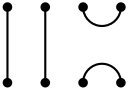
2. *The set $\{b_w\}_{w \in \mathcal{W}_f}$ is a $\mathbb{Z}[\delta]$-basis of $\mathrm{TL}_n(\delta)$.*
3. *Given any sequence $j_1 j_2 \cdots j_m$ of integers in $\{1, \ldots, n\}$, there exists a unique pair $(x, k) \in \mathcal{W}_f \times \mathbb{Z}_{\geq 0}$ such that*

$$b_{j_1} b_{j_2} \cdots b_{j_m} = \delta^k b_x.$$

Diagrammatic version of the Temperley-Lieb algebra

$b_1 =$  , $b_2 =$  , $b_3 =$ 

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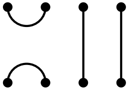
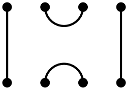
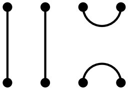
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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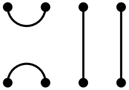
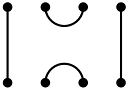
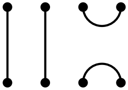
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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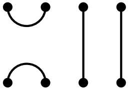
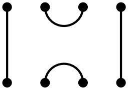
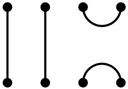
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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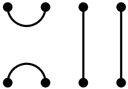
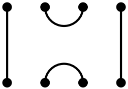
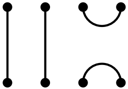
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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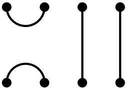
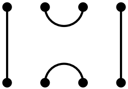
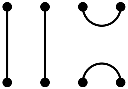
$b_1 =$ , $b_2 =$ , $b_3 =$ 

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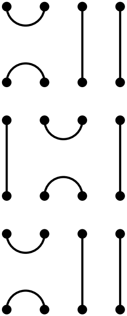
b_1



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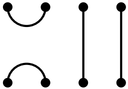
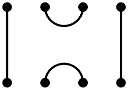
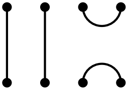
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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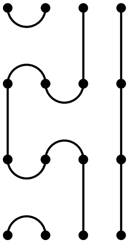

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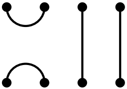
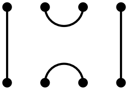
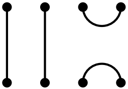
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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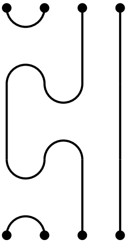

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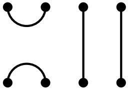
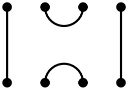
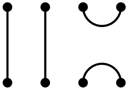
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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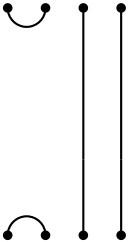

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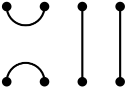
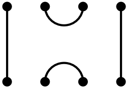
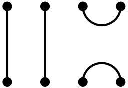
$b_1 =$ , $b_2 =$ , $b_3 =$ 

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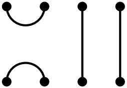

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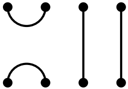
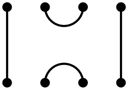
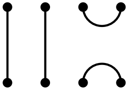
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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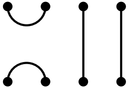
b_1 

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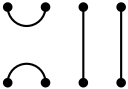
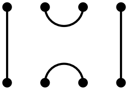
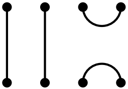
$b_1 =$ , $b_2 =$ , $b_3 =$ 

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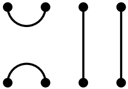
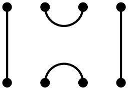
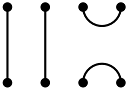
$b_1 =$ , $b_2 =$ , $b_3 =$ 

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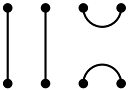
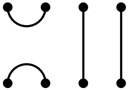
b_3 

b_1 

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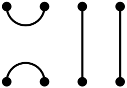
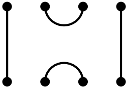
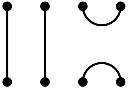
$b_1 =$  , $b_2 =$  , $b_3 =$ 

- Multiplication = concatenation of diagrams
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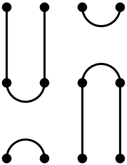

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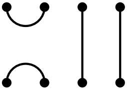
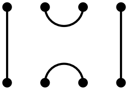
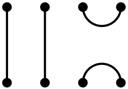
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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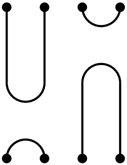

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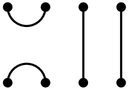
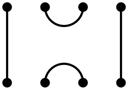
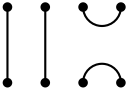
$b_1 =$ , $b_2 =$ , $b_3 =$ 

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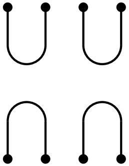

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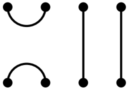
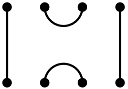
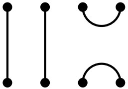
Diagrammatic version of the Temperley-Lieb algebra

$b_1 =$  , $b_2 =$  , $b_3 =$ 

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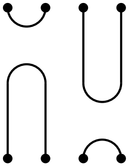

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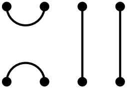
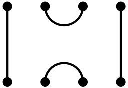
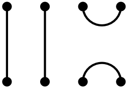
$b_1 =$ , $b_2 =$ , $b_3 =$ 

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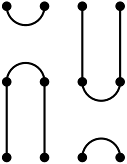

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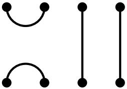
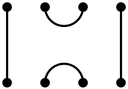
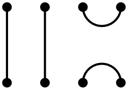
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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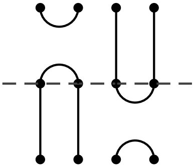

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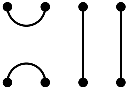
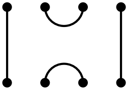
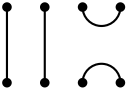
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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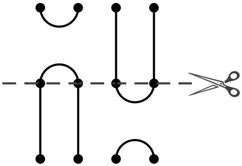

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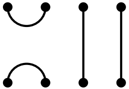
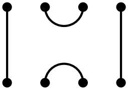
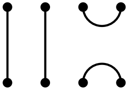
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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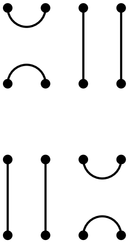

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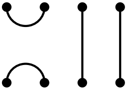
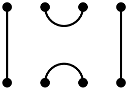
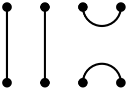
$b_1 =$ , $b_2 =$ , $b_3 =$ 

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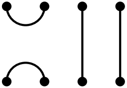
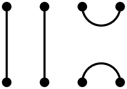
b_1 

b_3 

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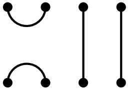
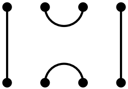
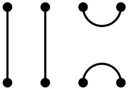
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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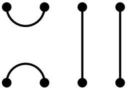
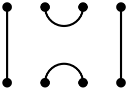
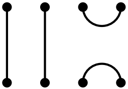
$$b_1^2 = \delta b_1$$

Diagrammatic version of the Temperley-Lieb algebra

$b_1 =$  , $b_2 =$  , $b_3 =$ 

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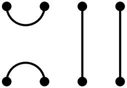
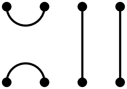
b_1 

b_1 

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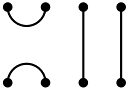
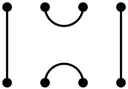
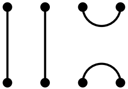
$b_1 =$ , $b_2 =$ , $b_3 =$ 

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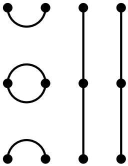

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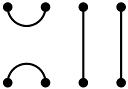
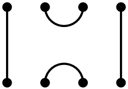
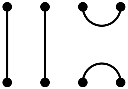
$b_1 =$ , $b_2 =$ , $b_3 =$ 

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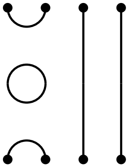

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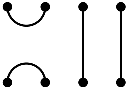
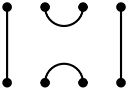
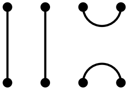
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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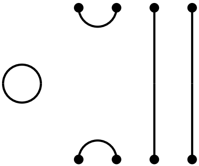

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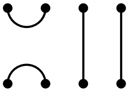
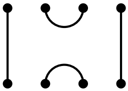
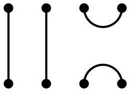
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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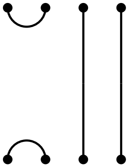
δ 

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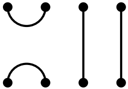
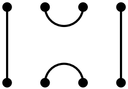
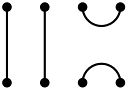
$b_1 =$  , $b_2 =$  , $b_3 =$ 

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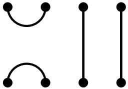
δ 

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Diagrammatic version of the Temperley-Lieb algebra

Multiplying generators b_i yields (linear combinations of) various diagrams. In case $n = 3$, there are 14 possible diagrams:

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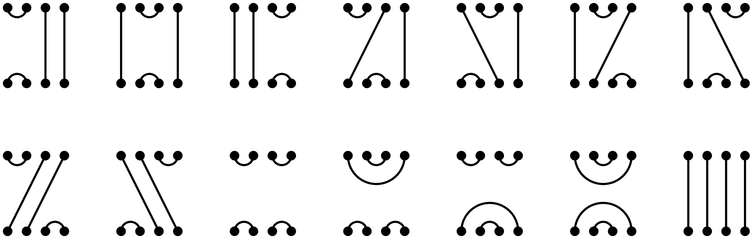
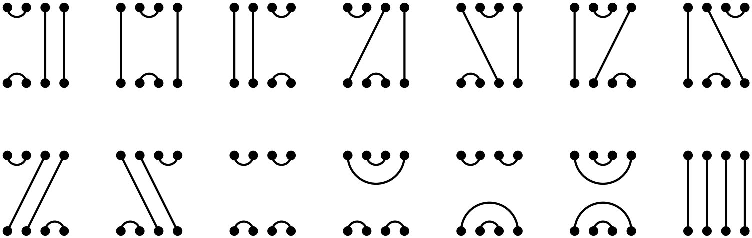


Diagrammatic version of the Temperley-Lieb algebra

Multiplying generators b_i yields (linear combinations of) various diagrams. In case $n = 3$, there are 14 possible diagrams:



These diagrams, which form a basis of the obtained diagram algebra, correspond exactly to the elements of the basis indexed by fully commutative elements.

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The aim now is to *categorify* the diagram basis, that is, to find a (graded) additive category $\mathcal{C}_{\mathrm{TL}_n}$ with a product $\star$ of objects endowing its split Grothendieck group $\langle \mathcal{C}_{\mathrm{TL}_n} \rangle$ with a ring structure such that there exists a homomorphism of rings

$$\psi : \mathrm{TL}_n(\delta) \to \langle \mathcal{C}_{\mathrm{TL}_n} \rangle ,$$

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with the following properties:

- The map ψ is an isomorphism of algebras,
- The image of the diagram basis matches with the set of classes of indecomposable objects, that is

$$\{\psi(b_w) \mid w \in \mathcal{W}_f\} = \{\langle B \rangle \mid B \text{ is indecomposable}\}.$$

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There is a $\mathcal{W}$-equivariant bijection

$$\left\{ \text{ Weyl lines of } (\mathcal{W}, V) \text{ } \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{Maximal parabolic} \\ \text{subgroups of } \mathcal{W} \end{array} \right\}, \quad L \mapsto \mathrm{Fix}_{\mathcal{W}}(L).$$

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From now on $(\mathcal{W}, \mathcal{S})$ will be of type A_n. The vector space V will be the geometric representation of $(\mathcal{W}, \mathcal{S})$.

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In type A_n, there are $2^n - 1$ Weyl lines.

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We write $\mathcal{V}_n$ for the set of closed subsets of Z obtained in this way. Notice that any element of $\mathcal{V}_n$ is a union of Weyl lines. One can show that $\{0\} \notin \mathcal{V}_n$.

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Regular functions on Weyl lines

Denote by $\mathcal{L}$ the set of Weyl lines and set $Z := \bigcup_{L \in \mathcal{L}} L$. The group $\mathcal{W}$ acts on Z, which is a Zariski closed subset of V.

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$$\mathcal{O}(V) \cong S(V^*) \cong k[f_1, f_2, \dots f_n],$$

where f_i is the equation of the hyperplane H_{s_i}. The algebra $R := \mathcal{O}(V)$ is, therefore, naturally graded (with $\deg f_i = 2$). The group $\mathcal{W}$ acts on R.

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$\Rightarrow \bar{R} := \mathcal{O}(Z) = R/I(Z)$ inherits a grading from R.

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The bimodule B_i is an analogue of the Bott-Samelson or Soergel bimodule $B_{s_i} := R \otimes_{R^{s_i}} R \in R - \mathrm{mod}_{\mathbb{Z}} - R$.

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Recall that the left (or right) annihilator of $B_w \cong B(\underline{s})$ is the ideal of $W(\underline{s})$ (or of $W(r(\underline{s}))$) and that there are unique sets $T(\underline{s})$, $T(r(\underline{s}))$ of pairwise commuting reflections such that

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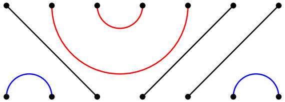
$$W(\underline{s}) = \bigcap_{t \in T(\underline{s})} V_t, \quad W(r(\underline{s})) = \bigcap_{t \in T(\underline{s})} V_t.$$

We have another way to associate to $w \in \mathcal{W}_f$ a pair of sets of pairwise commuting reflections:

$w \in \mathcal{W}_f \longrightarrow b_w \in \mathrm{TL}_n(\delta) \longrightarrow$ 

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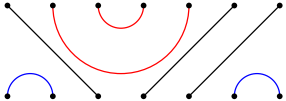

Proposition

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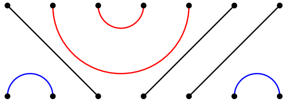

Proposition

One has $(T(\underline{s}), T(r(\underline{s}))) = (Q_w^{\mathrm{top}}, Q_w^{\mathrm{bottom}})$. As a consequence, if $w, w' \in \mathcal{W}_f$, $w \neq w'$, then $B_w \ncong B_{w'}$ (as $\bar{R}$-bimodules).

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Theorem

Each B_w is generated by $1 \otimes 1 \otimes \cdots \otimes 1 \otimes 1$. In particular, since $\dim_k (B_w)_0 = 1$, each B_w is indecomposable in $\bar{R} - \mathrm{mod}_{\mathbb{Z}} - \bar{R}$.

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As a consequence, we can extend our product $\star$ to the category $\mathcal{C}_{\mathrm{TL}_n}$ containing all the $B(\underline{s})$ and stable by $\oplus$.

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Let $\delta = 1 + v^{-2}$. There is an isomorphism of associative, unital $\mathbb{Z}[v, v^{-1}]$-algebras

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defined by the assignments $\mathcal{E}(b_w) = \langle B_w \rangle$, $\mathcal{E}(v) = \langle \bar{R}[1] \rangle$.

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- Proven for $w = s_i s_{i-1} \cdots s_j$, $i > j$ and in small cases.

Thank you for your attention!