Reflection groups, braid groups, Hecke algebras and categories

Mémoire d'habilitation à diriger des recherches

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Alles das schien mir eine Art Algebra,
zu der ich keinen Schlüssel fand.

Stefan ZWEIG, *Schachnovelle.*

Contents

Chapter 1

Introduction

1.1 Field of research and organization of the manuscript

This manuscript gathers a series of works all realized after my PhD thesis (defended in September 2014 in Amiens) and recently achieved for some of them. All these papers address several problems through various motivations, nonetheless they all aim to study the structure and properties of some classes of algebraic or sometimes slightly geometric objects all built using (generalizations of) *reflection groups*.

Let $\mathbb{K}$ be a field which, for the sake of simplicity, we may assume of characteristic zero. Let V be a finite-dimensional $\mathbb{K}$-vector space. A *reflection* on V is a transformation $s \in \mathrm{GL}(V)$ such that

1. $\mathrm{codim}(V^s) = 1$, where $V^s = \{v \in V \mid s(v) = v\}$,

2. s is of finite order.

If $\mathbb{K} = \mathbb{R}$ or $\mathbb{Q}$, then a reflection is necessarily of order 2, while for $\mathbb{K} = \mathbb{C}$ the order may be any integer $k \geq 2$. A *finite reflection group* is a finite subgroup $W \subseteq \mathrm{GL}(V)$ generated by reflections.

A fundamental example is given by the symmetric group $\mathfrak{S}_n$. Indeed, letting $\mathfrak{S}_n$ act on $\mathbb{K}^n$ by permutation of the vectors of a chosen basis yields a realization of $\mathfrak{S}_n$ as a reflection group, since every transposition acts by a reflection, and the set of transpositions generates $\mathfrak{S}_n$. Note that, when working over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we may assume that W preserves an inner product.

A reflection group is often written W because a historically fundamental example of reflection groups is given by *Weyl groups* of connected reductive algebraic groups. In fact, finite Weyl groups are examples of finite *real* reflection groups, that is, finite reflection groups over a real vector space. The family of finite real reflection groups coincides with the family of *finite Coxeter groups*. Coxeter groups are defined by generators and relations, as groups admitting a presentation of a certain form where the generators are involutions. These generators turn out to form a distinguished subset of the set of reflections in the case where the group is finite. For the symmetric group $\mathfrak{S}_n$, the presentation as a Coxeter group is the presentation

$$\left\langle s_1, s_2, \ldots, s_{n-1} \;\middle|\; \begin{array}{c} s_i^2 = 1, \forall i = 1, \ldots, n-1, \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \text{ for } 1 \leq i < n-1, \\ s_i s_j = s_j s_i \text{ for all } i, j \text{ with } |i-j| > 1. \end{array} \right\rangle \tag{1.1.1}$$

In terms of permutations, the generator s_i corresponds to the simple transposition $(i, i+1)$.

The theory of Coxeter groups is extremely powerful. Indeed, many fundamental tools that originated in the study of finite Weyl groups, such as the (strong) Bruhat order, the technology of root systems, the exchange lemma, and so forth, admit generalizations to *all* Coxeter groups, that is, groups defined by the same kind of presentations as the one above, but without the finiteness assumption. These groups still admit a realization as subgroups of some $\mathrm{GL}(V)$ generated by reflections, which preserve a symmetric bilinear form (which is not an inner product in general).

A second generalization of finite real reflection groups is given by finite complex reflection groups, that is, finite reflection groups over a complex, finite-dimensional vector space.

Given a finite complex reflection group W acting on V, denoting by $\mathrm{Ref}(W)$ the set of reflections of W, one can consider the subset $V^{\mathrm{reg}} = V \setminus \bigcup_{s \in \mathrm{Ref}(W)} V^s$ of V (which is stable by W), and the quotient V^{reg}/W. By a theorem of Steinberg, the quotient map $p : V^{\mathrm{reg}} \twoheadrightarrow V^{\mathrm{reg}}/W$ is a Galois covering, yielding a short exact sequence

$$1 \longrightarrow P_W := \pi_1(V^{\mathrm{reg}}) \longrightarrow B_W := \pi_1(V^{\mathrm{reg}}/W) \longrightarrow W \longrightarrow 1.$$

The group P_W is called the *pure braid group* of W, while the group B_W is called the (generalized) *braid group* of W.

For $W = \mathfrak{S}_n$ acting on $V = \mathbb{C}^n$, one has

$$V^{\mathrm{reg}} = \{(x_1, \ldots, x_n) \in V \;\; | \; x_i \neq x_j \text{ if } i \neq j\},$$

and V^{reg} is thus identified with the set of subsets of $\mathbb{C}^n$ with n elements, that is,

$$V^{\mathrm{reg}}/W = \{\{x_1, x_2, \ldots, x_n\} \mid x_i \neq x_j \text{ if } i \neq j\}.$$

An element of $B_W = \pi_1(V^{\mathrm{reg}}/W)$ can thus be represented by a *geometric braid* β on n strands as in Figure 1.1, the intersection of the horizontal plane $z = t \in [0,1]$ with the geometric braid yielding the n elements of $\mathbb{C}$ given by $\beta(t)$.

Originally, geometric braids were introduced by Artin [8] (see [9] for a more rigorous treatment), who gave a definition of the n-strand braid group $\mathcal{B}_n$ as the group consisting of geometric braids on n strands up to a suitable notion of isotopy, and where the product is given by vertical concatenation of braids. As explained above, this group turns out to be isomorphic to $B_{\mathfrak{S}_n}$. Artin gave the following presentation by generators and relations for $\mathcal{B}_n$.

$$\left\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{c} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for } 1 \leq i < n-1, \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for all } i, j \text{ with } |i-j| > 1. \end{array} \right\rangle \tag{1.1.2}$$

Observe that this presentation is obtained from the presentation of $\mathfrak{S}_n$ from (1.1.1) above by removing the torsion relations $s_i^2 = 1$ for all $i = 1, \ldots, n-1$. It turns out that for all finite real

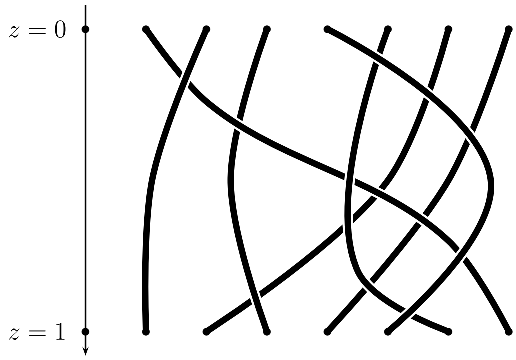


Figure 1.1: A geometric braid on 7 strands.

reflection groups, that is, those finite complex reflection groups which can be realized over $\mathbb{R}$, the group B_W is always obtained from the Coxeter presentation of W by removing the relations $s^2 = 1$ where s is any generator. The situation is unfortunately more complicated in general when the group is not real, where there is not a single canonical "Coxeter-like" presentation of W in general.

Nevertheless, the fact that the presentation of a finite Coxeter group is canonical is true without the finiteness assumption, and allows one to define a "braid group" attached to any Coxeter group by removing torsion relations from the Coxeter presentation. Such a group is usually called the *Artin-Tits group* or simply *Artin group* attached to W, still denoted B_W.

The works we chose to present in this manuscript were collected in five different chapters, corresponding to five (different, but often connected) families of mathematical structures attached to (generalizations of) reflection groups and their generalized braid groups, coming after a chapter of preliminaries.

1. **Dual Coxeter systems and dual braid monoids** (Chapter 3): we study (mostly finite) Coxeter groups and their Artin groups via a so-called *dual* approach, which consists of viewing a finite Coxeter group (respectively its Artin group) as generated by the whole set of reflections (respectively a copy of the set of reflections), instead of just a Coxeter generating set. Typically, in the Coxeter presentation of the symmetric group given above, the set of generators is given by the simple transpositions, while the reflections are all the transpositions. This approach has many applications in the study or Artin groups (word problem, $K(\pi, 1)$-conjecture, ...), and in combinatorics (generalizations of noncrossing partitions and other Catalan enumerated objects, ...).

2. **Soergel bimodules** (Chapter 4): Soergel bimodules form a monoidal category of graded bimodules over a polynomial ring, categorifying the Iwahori-Hecke algebra of an arbitrary Coxeter system (a deformation of the group algebra of W, which is a central object in representation theory and low-dimensional topology). It contains many information on representation theories of Lie theoretic objects, and can also be used to construct a categorical action of any Artin group on a suitable triangulated category.

3. **Reflection subgroups: structure, normalizer, braid group, Hecke algebra, ...** (Chapter 5): A reflection subgroup of a Coxeter group is a subgroup generated by a subset

of the set of conjugates of the Coxeter generators. In the case where the group is finite, this is nothing but a subgroup generated by reflections, and such subgroups can also be considered in the complex case. In the case of an arbitrary Coxeter group, a reflection subgroup still admits a structure of Coxeter group. It is natural, in all these situations, to study various algebraic structures attached to a reflection subgroup (braid group, Hecke algebra, normalizer, ...), and to study their relationship to the corresponding structures of the ambient group.

4. **On Garside structures for torus knot groups** (Chapter 6): A Garside group is the group of fractions of a monoid satisfying certain good "divisibility" conditions (called a *Garside monoid*). Such a group has a solvable word problem, and is torsion-free, among other highly nontrivial properties. Artin's braid group $\mathcal{B}_n$ and more generally all Artin groups of spherical type (that is, attached to a finite Coxeter group) are Garside groups, and so are most of the complex braid groups. A Garside structure is not unique in general, in the sense that one can have several nonisomorphic Garside monoids with the same group of fractions. Another example of Garside groups is given by torus knot groups. This provides one of the most accessible family of Garside groups, where natural questions can be asked: classification of Garside structures, study of quotients playing the role of "generalized reflection groups" as the symmetric group does for $\mathcal{B}_n$, ...

5. **Bruhat order on quotients** (Chapter 7): The Bruhat order is a subtle way of ordering a finite Weyl group of a connected reductive algebraic group G, given by the inclusion of orbit closures of a Borel subgroup B acting on the flag variety G/B (these orbits are indexed by the Weyl group W of G). It admits a generalization to arbitrary Coxeter groups, via a combinatorial definition which can be given even in the case where there is no algebraic group. There are several natural combinatorial generalizations or variations of the Bruhat order on an arbitrary Coxeter group that one can define and study, and in the case where the Coxeter group is the Weyl group of a reductive group, one can also consider the action of *spherical subgroups* of G on the flag variety G/B, that is, subgroups acting with finitely orbits. In such situations, it is natural to seek for a parametrization of the orbits, and for a combinatorial criterion to describe the inclusion or orbit closures, and see to what extent such orderings can be given for an arbitrary Coxeter group.

The five aforementioned topics where we present our results may seem of rather different nature, and we confess that trying to present them in a unified way may look a bit artificial. We therefore end this introductory chapter by briefly presenting the questions we were interested in in a more personal and chronological way, after listing the relevant publications.

Some of the objects under study are then briefly presented in Chapter 2 below, before the five chapters corresponding to the above five topics. In these chapters, the statements in which I am involved are presented in gray coloured boxes, to locate them more easily.

1.2 Selection of publications

Here is the list of publications or preprints presented in this thesis.

1. B. Baumeister and T. Gobet, *Simple dual braids, noncrossing partitions and Mikado braids of type D_n*, Bull. Lond. Math. Soc. **49** (2017), no.6, 1048-1065.
2. B. Baumeister, T. Gobet, K. Roberts, P. Wegener, *On the Hurwitz action in finite Coxeter groups*, J. Group Theory **20** (2017), no.1, 103-131.
3. N. Chapelier-Laget and T. Gobet, *Elements of minimal length and Bruhat order on fixed-point cosets of Coxeter groups*, preprint (2023). `https://arxiv.org/abs/2311.06827`.
4. P.-E. Chaput, L. Fresse, and T. Gobet, *Parametrization, structure and Bruhat order of certain spherical quotients*, Represent. Theory **25** (2021), 935-974.
5. F. Digne and T. Gobet, *Dual braid monoids, Mikado braids and positivity in Hecke algebras*, Math. Z. **285** (2017), no. 1-2, 215-238.
6. T. Gobet, *Twisted filtrations of Soergel bimodules and linear Rouquier complexes*, J. Algebra **484** (2017), 275–309.
7. T. Gobet, *On cycle decompositions in Coxeter groups*, Sém. Lothar. Combin. **78B** (2017), Art. 45, 12 pp.
8. T. Gobet, *Dual Garside structures and Coxeter sortable elements*, J. Comb. Algebra **4** (2020), no. 2, 167–213.
9. T. Gobet, *On some torus knot groups and submonoids of the braid groups*, J. Algebra **607** (2022), Part B, 260-289.
10. T. Gobet, *Addendum to "On some torus knot groups and submonoids of the braid groups" [J. Algebra 607 (Part B) (2022) 260–289]*, J. Algebra **633** (2023), 666–667.
11. T. Gobet, *Toric reflection groups*, *Journal of the Australian Mathematical Society*, online first (2023).
12. T. Gobet, *A new Garside structure on torus knot groups and some complex braid groups*, to appear in *Journal of Knot Theory and its Ramifications* (2023).
13. T. Gobet, *On maximal dihedral reflection subgroups and generalized noncrossing partitions*, preprint (2023), `https://arxiv.org/abs/2307.16791`.
14. T. Gobet, A. Henderson, and I. Marin, *Braid groups of normalizers of reflection subgroups*, Ann. Inst. Fourier **71** (2021), no. 6, 2273-2304.
15. T. Gobet and I. Marin, *Hecke Algebras of Normalizers of Parabolic Subgroups*, Algebr. Represent. Theor. **26** (2023), 1609-1639.
16. T. Gobet and B. Rognerud, *Odd and even Fibonacci lattices arising from a Garside monoid*, preprint (2023). `https://arxiv.org/abs/2301.00744`.
17. T. Gobet and A.-L. Thiel, *On generalized categories of Soergel bimodules in type $A2$*, C. R. Math. Acad. Sci. Paris **356** (2018), no. 3, 258–263.
18. T. Gobet and A.-L. Thiel, *A Soergel-like category for complex reflection groups of rank one*, Math. Z. **295**, (2020), 643-665.

1.3 Chronological overview of results and approaches

Most of the objects considered in this overview are defined in the following chapters.

1.3.1 Questions raised by certain bases of Temperley-Lieb algebras

In my PhD thesis [72], I studied certain bases of Temperley-Lieb algebras, together with their categorifications. Let $\mathcal{B}_n$ denote Artin's n-strand braid group. There is a well-known group homomorphism $\mathcal{B}_n \longrightarrow \mathrm{TL}_n^\times$, where TL_n denotes the Temperley-Lieb algebra. Birman, Ko and Lee considered a particular set of elements of $\mathcal{B}_n$, which we will call the simple dual braids, which are counted by the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. The simple dual braids generate $\mathcal{B}_n$, and a submonoid with particularly nice properties (called a *Garside monoid*), allowing one to improve the solution to the word problem in $\mathcal{B}_n$. The Catalan number C_n is also the dimension of TL_n, and it was observed by Zinno [143] that mapping the simple dual braids to TL_n through the above homomorphism yields a basis of TL_n. My PhD supervisor had observed on specific examples and conjectured that expanding elements of Zinno's basis in the canonical diagrammatic basis of TL_n gives rise to some positivity properties (i.e., that under suitable conventions, the polynomials occurring as coefficients of the base change matrix have nonnegative coefficients). One of the aims of my PhD project was to prove this conjecture.

At that time (2011-2014), the so-called "categorification" program was being developed, with the (already widely used, and not really new) idea that one way to show that a polynomial has nonnegative coefficient is to interpret it as the (graded) character of a module, or in some cases, as the Euler-Poincaré characteristic of a chain complex. This includes for instance, in particular cases, Kazhdan-Lusztig polynomials [99], and such phenomenons also occur for instance in the framework of cluster algebras [92].

Kazhdan-Lusztig polynomials appear in the framework of Hecke algebras attached to Coxeter groups [98]. Given a Coxeter group W, the Hecke algebra H_W is an algebra over $\mathcal{A} = \mathbb{Z}[v^{\pm 1}]$ that deforms the group algebra $\mathbb{Z}[X]$, and has many applications in representation theory and low-dimensional topology. In the particular case of the symmetric group (which is a Coxeter group), the above group homomorphism $\mathcal{B}_n \longrightarrow \mathrm{TL}_n^\times$ factors through the Hecke algebra $H_n = H_{\mathfrak{S}_n}$. At the categorified level, one has geometric realizations of H_n as a convolution algebra on the flag variety, and the theory of Soergel bimodules [136, 137, 65], which was also being developed at that time, was providing an interesting tool to consider. Roughly speaking, Soergel's category is a monoidal, graded, Karoubian category of bimodules over a polynomial ring R constructed using only some data attached to the (arbitrary) Coxeter system (W, S), stable by direct sums, and such that the indecomposable bimodules are indexed up to isomorphism and grading shifts by the elements of w. Denoting by B_w the indecomposable Soergel bimodule attached to $w \in W$, Soergel showed that the split Grothendieck ring of this category is isomorphic to the Hecke algebra H_W (with the parameter interpreted as a grading shift), and conjectured that under this isomorphism, the class $\langle B_w \rangle$ of the bimodule B_w in the Grothendieck ring coincides with the element C'_w of the canonical basis $\{C'_u\}_{u\in W}$ of Kazhdan and Lusztig.

I did not manage to prove the positivity of Zinno's basis in my PhD work, were I provided a categorification of the Temperley-Lieb algebra by analogues of Soergel bimodules [73], and also established various combinatorial properties of Zinno's bases [74, 76, 84] using properties of reduced expressions of (fully commutative) permutations. The positivity was treated shortly after the end of my PhD, using results which can be formulated in a more general framework than the one introduced above.

To be more precise, concerning positivity, the following result of Dyer and Lehrer from 1990 [63], valid in an arbitrary finite Weyl group, previously conjectured by Dyer [55] in 1987 for an arbitrary Coxeter group, turned out to be useful. It states that, in the Hecke algebra H_W of a finite Weyl group, one has

$$T_x^{-1}T_y \in \sum_{w\in W} \mathbb{Z}_{\geq 0}[v^{\pm 1}]C_w, \ \forall x, y \in W, \tag{1.3.1}$$

where $\{T_u\}_{u\in W}$ denotes the standard basis of H_W (deforming the basis of $\mathbb{Z}[W]$ given by the elements of W) and $\{C_u\}_{u\in W}$ is another canonical basis (a slightly different, but similar basis, to $\{C'_u\}_{u\in W}$–see Subsection 2.6 below for definitions). Three observations can be made in relation with the aforementioned positivity question:

- As pointed out above, this property, conjectured for an arbitrary Coxeter group, was a theorem for the symmetric group (and more generally for any finite Weyl group) [63],
- In the case where $W = \mathfrak{S}_n$, the Temperley-Lieb algebra TL_n is a quotient of $H_n := H_W$, and under suitable conventions on the quotient map, the diagrammatic basis of TL_n is the projection of the canonical basis $\{C_u\}_{u\in W}$ of H_n,
- The elements $T_x^{-1}T_y$ are in the image of the group homomorphism $\mathcal{B}_n \longrightarrow H_n^{\times}$; in other words, they "come from braids" (uniquely determined if one believes the conjecture that the above group homomorphism is injective).

One natural program, supported by several examples, to prove the positivity of Zinno's basis was thus to verify that the Birman, Ko and Lee simple dual braids in $\mathcal{B}_n$ can be written under the form $\mathbf{x}^{-1}\mathbf{y}$, where for $w \in \mathfrak{S}_n$, we denote by $\mathbf{w}$ its canonical positive lift in the Artin group $B_W \cong \mathcal{B}_n$ attached to W; the lift $\mathbf{w}$ has image T_w inside H_n, so that $\mathbf{x}^{-1}\mathbf{y}$ has image $T_x^{-1}T_y$, and Dyer and Lehrer's result then applies. Note that, since Birman, Ko and Lee's monoid was generalized by Bessis [18] to arbitrary finite Coxeter groups, this question makes sense for an arbitrary finite Coxeter group, not only for $W = \mathfrak{S}_n$.

During my PhD thesis, I was aware of 1.3.1, but had not been able to apply the above program. I was also focused on the specific type A situation, and had not realized how rich the content of 1.3.1 was; let me say a few more words about it, as it motivated an important portion of my works from my first years of postdoc:

1. Specializing at $y = 1$, the content of 1.3.1 becomes the so-called *positivity of inverse Kazhdan-Lusztig polynomials*, which had just been proven in full generality, i.e., for an arbitrary Coxeter system, by Elias and Williamson [65] (where the positivity of classical Kazhdan-Lusztig polynomials is also proven). Elias and Williamson's proof uses the bounded homotopy category of Soergel bimodules, and the positivity is obtained by taking the Euler-Poincaré caracteristic of a chain complex of Soergel bimodules categorifying the braid $\mathbf{x}^{-1}\mathbf{y}$, yielding an explicit description of the coefficient of C_w as the (graded) number of occurrences of the indecomposable Soergel bimodule B_w in the chain complex,

2. The positive lifts $\mathbf{w} \in B_W$ of the elements $w \in W$ play a central role, when W is finite, in the study of combinatorial problems on B_W (word problem, conjugacy problem, ...), as they are the simple elements of the classical Garside structure on B_W. This means that they provide a generating set for B_W which is of particular importance in the aforementioned questions.

In view of these observations, there were (at least) two natural problems to investigate: one problem was to try to show 1.3.1 (and other positivity properties) for arbitrary Coxeter systems, using the breakthrough of Elias and Williamson [65] (see also [130] for a survey) relying on Soergel's framework [136, 137]. This is not related to the original problem of positivity of Zinno's basis anymore. Another problem was to try to explicit the somewhat mysterious link between *simple dual braids* (generalizing Birman, Ko and Lee's generators) and *simple classical braids* (that is, elements of the form $\mathbf{w}$, $w \in W$), and especially to determine in spherical type if simple dual braids can always be written as a quotient of two positive simple braids, which implies the original motivating question on the positivity of Zinno's basis in the particular type A situation. We explain these two questions a bit more in details in the next two subsections.

1.3.2 Dyer's conjectures, Soergel bimodules, and 2-braid groups

The results from this subsection are stated precisely in Section 4.1 in Chapter 4 below.

In addition to 1.3.1, Dyer also conjectured the following:

$$C'_w T_y \in \sum_{x \in W} \mathbb{Z}_{\geq 0}[v^{\pm 1}] T_x, \ \forall w, y \in W. \tag{1.3.2}$$

Specializing at $y = 1$, one gets nothing but the positivity of ordinary Kazhdan-Lusztig polynomials. When W is finite, Dyer and Lehrer [63] showed in 1990 that (1.3.1) and (1.3.2) are equivalent, and showed (1.3.2) to hold for any finite Weyl group.

At $y = 1$, property (1.3.2) is obtained by interpreting the expansion of C'_w in the basis $\{T_u\}_{u \in W}$ as the "character" of the indecomposable Soergel bimodule B_w attached to w, that is, the coefficients of T_x is the graded multiplicity of some module R_x occurring in a canonical filtration of B_w. To be defined, this filtration requires one to fix a total order refining the (strong) Bruhat order, but the multiplicities are then shown to be independent of the chosen total order.

During a conference in Bad Honnef in Germany in 2015, Wolfgang Soergel suggested to me that I consider "twisted" filtrations of Soergel bimodules to establish (1.3.2) for arbitrary Coxeter systems, that is, that I mimick his contruction of the filtrations, but replace the Bruhat order by a "twisted" one. This led to a proof of (1.3.2) in general.

Unfortunately, for infinite W, there is no proof that (1.3.2) and (1.3.1) are equivalent. It was thus interesting to also seek for a proof of (1.3.1) for arbitrary Coxeter systems. We can obtain (1.3.1) in the same spirit as Elias and Williamson, using Rouquier's categorical action of the Artin group B_W attached to an arbitrary Coxeter system on the triangulated category given by the bounded homotopy category of Soergel bimodules [133, 132]. To each element of B_W, one can associate (an equivalence class of) chain complexes, and the coefficient of C_w in (1.3.1) counts the number of occurrences of the bimodule B_w appearing in a specific complex, called *minimal*, in the equivalence class of complexes attached to $\mathbf{x}^{-1}\mathbf{y}$. Two properties are crucial to deduce the positivity: a property of *perversity* or *linearity* of the minimal complex, and the fact that, for a fixed $w \in W$, the indecomposable bimodule B_w appears either only in odd or only in even cohomological degrees of the minimal complex.

1.3.3 Classical and dual simple generators of Artin groups of spherical type

The results from this subsection are stated precisely in Section 3.1 from Chapter 3 below.

I understood a few weeks after the end of my PhD thesis how to establish, in type A, that simple dual braids of $\mathcal{B}_n$ can always be written under the form $\mathbf{x}^{-1}\mathbf{y}$. This was using a geometric approach to the problem: in Artin's n-strand braid group $\mathcal{B}_n$, the elements of the form $\mathbf{x}^{-1}\mathbf{y}$ can be characterized as those braids admitting a braid diagram where one can inductively remove a strand lying above all the others until reaching the empty diagram–see Figure 1.2 for an illustration. Using the "noncrossing partition" type models for simple dual braids in type A, it was then easy to show that every simple dual braid has this geometric property, which we called the "Mikado" property because of the similarity with the so-called Mikado game (which seems to have another name in English...).

This gave rise to a joint article with François Digne [53], where we proved that simple dual braids can be written under the form $\mathbf{x}^{-1}\mathbf{y}$ in Artin groups attached to any finite irreducible Coxeter group except possibly type D, where we conjectured the result to still hold. Type D was then obtained independently by Licata and Queffelec [104] using categorical actions of Artin groups of types ADE on a suitable triangulated category, which also yields a new proof of the result in types A and E, and by Baumeister and myself [13] using the same kind of techniques as used with Digne in type A; this relies on the realization of the Artin group of type D as an index two subgroup of a suitable quotient of an Artin group of type B, due to Allcock [4].

The approaches to solve the problem in the various works mentioned in the previous paragraph all furnish an *algorithm* to express a simple dual braid under the form $\mathbf{x}^{-1}\mathbf{y}$, rather than a closed formula. It should also be noted that the pair $(\mathbf{x}, \mathbf{y})$ is not unique in general. The geometric characterization of braids of the form $\mathbf{x}^{-1}\mathbf{y}$ in type A allows one to establish the following, a priori not trivial, property: denoting by $\mathbf{S}$ the set of Artin generators of $\mathcal{B}_n$, every braid of the form $\beta = \mathbf{x}^{-1}\mathbf{y}$ in type A admits a word in the generators $\mathbf{S} \cup \mathbf{S}^{-1}$ representing β, which has the same length as the image of β in $\mathfrak{S}_n$ with respect to the Coxeter generating set of simple transpositions. Indeed, if we are given a braid diagram with the Mikado property, we can readily convince ourselves that it is isotopic to a diagram where any two pairs of strands cross at most once. And in fact, it is not difficult to show that *every* reduced expression of the image of β in the symmetric group can be lifted to a word for β, where a Coxeter generator is replaced by the corresponding Artin generator or its inverse (not every choice of exponents is possible, of course).

Matthew Dyer gave me an algebraic explanation of this phenomenon, as well as a definition of braids of the form $\mathbf{x}^{-1}\mathbf{y}$ in the spirit of the aforementioned principle of "lifting reduced expressions". When W is finite, Dyer's definition precisely yields the braids that can be written

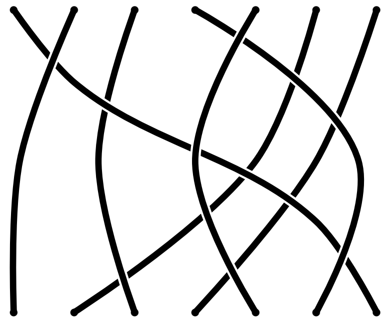


Figure 1.2: A Mikado braid in $\mathcal{B}_7$.

under the form $\mathbf{x}^{-1}\mathbf{y}$ (or $\mathbf{x}\mathbf{y}^{-1}$, which is equivalent when W is finite), while for infinite W it yields many more braids, which can still be expected to share the same kind of properties–for instance positivity properties. Dyer's definition involves choosing an element $v \in W$, and a set A of positive roots with suitable properties (called *biclosed*). One then lifts any reduced expression of v to B_W, replacing a Coxeter group generator by the corresponding Artin one or its inverse, using a rule involving the set A. The obtained element is independent of the chosen reduced expression. In other words, these braids are "those for which Matsumoto's Lemma applies", in the sense that any two reduced words for these braids in $\mathbf{S} \cup \mathbf{S}^{-1}$ where $\mathbf{S}$ denotes the set of Artin generators of B_W can be related using a sequence of (mixed) braid relations. These elements thus have their set of reduced words with respect to $\mathbf{S} \cup \mathbf{S}^{-1}$ in canonical bijection with the set of S-reduced expressions of their image in the Coxeter group.

Even though the question of whether simple dual braids are of the form $\mathbf{x}^{-1}\mathbf{y}$ or not was solved, I was not satisfied by the fact that one only had an algorithm to convert a simple dual braid into one of the form $\mathbf{x}^{-1}\mathbf{y}$. Using Dyer's framework, one wonders on how to obtain a simple dual braid β by "lifting" its image $v = p(\beta)$ in W to B_W using a biclosed set of roots. In spherical type, biclosed sets of roots coincide with inversion sets of elements of W, hence there is a bijection between W and such sets, and one thus looks for a canonical element $u \in W$ such that lifting $v = p(\beta)$ using the inversion set of u yields the simple dual braid β. Here again, such a u is not unique in general, but one can wonder if there exists a canonical one or not.

I positively answered this question as follows. Dual braid monoids are defined using a choice c of standard Coxeter element in W, and for a simple dual braid β in this monoid, the element $v = p(\beta)$ turns out to be a generalized c-noncrossing partition, that is, an element lying below c for the absolute order on W (a natural order on W required in the construction of dual braid monoids–see Subsection 2.1.5 below for precise definitions). These c-noncrossing partitions are in bijection with another subset of W introduced by Reading [126] and also depending on c, the so-called *c-sortable elements*. I showed that, given a simple dual braid β and $v = p(\beta)$, one can take as element u the c-sortable element attached to the Kreweras complement $v^{-1}c$ of v (which is also a c-noncrossing partition).

1.3.4 On the Hurwitz action in finite Coxeter groups

The results from this subsection are stated precisely in Section 3.2 from Chapter 3 below.

Given a group G, one can define an action of the n-strand braid group $\mathcal{B}_n$ on the set of n-tuples of elements of G by letting the standard Artin generator σ_i act as follows

$$\sigma_i : (g_1, g_2, \dots, g_n) \mapsto (g_1, \dots, g_{i-1}, g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, g_{i+2}, \dots, g_n).$$

First observe that such an action preserves the product from left to right of the elements in the tuple. In particular, if $G \neq \{1\}$, there are always several orbits. It is thus natural to restrict this action to the set of decompositions of a given element g as a product of n elements of G (or of a generating set of G stable by conjugation).

In Bessis' construction of the dual braid monoids [18] already mentioned in Subsection 1.3.3 above, such an action appears in the following specific situation. Consider a finite Coxeter system (W, S), and let $c = s_1 s_2 \cdots s_n$ be a standard Coxeter element; here $S = \{s_1, s_2, \dots, s_n\}$. Then the product $s_1 s_2 \cdots s_n$ is reduced with respect to the generating set S of W, but it can be shown to also be reduced with respect to the larger generating set $T = \bigcup_{w \in W} wSw^{-1}$ of all conjugates of the elements of S, the so-called *reflections* of W. The set

$$\mathsf{Red}_T(c) = \{(t_1, t_2, \dots, t_n) \in T^n \mid t_1 t_2 \cdots t_n = c\}$$

of T-reduced decompositions of c is stable by the Hurwitz action, since T is stable by conjugation. Bessis showed that when W is finite, the Hurwitz action on $\mathsf{Red}_T(c)$ is transitive [18].

There are at least two reasons to consider this as an interesting result:

- Bessis uses it to show that a natural map between B_W and the dual Artin group, defined as the group of fractions $G(B_c^*)$ of the dual braid monoid B_c^* (which is a Garside monoid), is surjective. Both groups appear to be isomorphic and this is an important step in the proof[1].

- An application of σ_i or its inverse replaces a subexpression (t_i, t_{i+1}) of a T-reduced expression by (t_{i+1}, t'), where $t' = t_{i+1} t_i t_{i+1}$, or by (t', t_i), where $t' = t_i t_{i+1} t_i$. These relations, together with the relations $t^2 = 1$ whenever t appears in a T-reduced expression of c (which happens for all t in T when W is finite), yield a presentation of W. The transitivity of the Hurwitz action can then be considered as a kind of "dual"[2] Matsumoto Lemma for Coxeter elements, as it states that one can relate any two T-reduced expressions of a Coxeter element using only these relations, sometimes called "dual braid relations".

The "dual Matsumoto property" mentioned in the second point above is only deduced for Coxeter elements, and parabolic versions of such elements. It is not difficult to show that the Hurwitz action is not transitive in general on $\mathsf{Red}_T(w)$, $w \in W$. For instance, in type B_2 with classical generators s and t, for w equal to the longest element $w_0 = stst = tsts$ of W one has two orbits $\{(s, tst), (tst, s)\}$ and $\{(t, sts), (sts, t)\}$.

[1] For arbitrary W, one can still define a group generalizing $G(B_c^*)$, which is not Garside in general, but can still be conjectured to be isomorphic to B_W. Bessis' result on the transitivity on the Hurwitz action on $\mathsf{Red}_T(c)$ can be shown to hold in arbitrary Coxeter systems [96, 12], and the surjectivity $B_W \twoheadrightarrow G(B_c^*)$ can be deduced for arbitrary Coxeter systems (whether it is an isomorphism or not is open in general).

[2] The word "dual" here is used because, since Bessis' paper [18], viewing a Coxeter group W as being generated by all the reflections occuring in T-reduced expressions of a Coxeter element is sometimes called the "dual approach" to Coxeter groups. For the origin of this terminology see [18, Section 5.1].

In a joint work with Barbara Baumeister, Kieran Roberts and Patrick Wegener [14], initiated after two visits in Bielefeld in 2014 and 2015, we characterized those elements w of finite Coxeter groups such that the Hurwitz action is transitive on $\mathsf{Red}_T(c)$. This happens if and only if w is a so-called *parabolic quasi-Coxeter element*, that is, an element admitting a T-reduced expression such that the reflections in this expression generate a parabolic subgroup of W.

In a short note [77], I then showed that parabolic quasi-Coxeter elements are precisely those elements of finite Coxeter groups admitting a decomposition generalizing the decomposition of permutations in the symmetric group into products of cycles with disjoint support. In the symmetric group, where every element has such a decomposition, every element is indeed a parabolic quasi-Coxeter element (and even a parabolic Coxeter element).

1.3.5 Extended and generalized Soergel categories

The results from this subsection are stated precisely from Section 4.2 in Chapter 4 below.

During a winterschool in Denmark in March 2013, Elias and Williamson presented their results from [65]; they gave several open questions and the following one was source of interest to me. Given a Coxeter system (W, S), the monoidal category $\mathcal{B}$ of Soergel bimodules already mentioned in Subsection 1.3.2 above is a category of graded R-bimodules over some polynomial ring R depending on W and equipped with an action of W respecting the grading; this category is generated in a suitable sense by R-bimodules of the form $B_s := R \otimes_{R^s} R$, where $s \in S$ and R^s denotes the graded subring of elements of R which are fixed by s. It is possible to define such a bimodule for every $t \in T$, i.e., for every reflection. Elias and Williamson indicated that it was unknown what the split Grothendieck ring of the category $\mathcal{B}^T$ generated (in the same sense as $\mathcal{B}$) by this larger set of generating bimodules was, even in type A_2.

The "dual" nature of this question, even if, unlike in the previous subsections, there is no choice of standard Coxeter element here, naturally triggered my curiosity. Shortly after the winterschool, I convinced myself that it was possible to give a full description of the indecomposable objets in the category $\mathcal{B}^T$ in type A_2, but could not obtain a description for A_3 and not even for any other dihedral group. Two years later, during the AMS and EMS joint meeting in Porto in June 2015, I discussed with Anne-Laure Thiel, who turned out to also have thought about this question, and had also obtained a description of the indecomposable objects of $\mathcal{B}^T$ in type A_2, using a different method. We decided to join our efforts.

After several attempts and many extensive computations, we realized that we were not able to give any precise description of the split Grothendieck ring of $\mathcal{B}^T$ beyond type A_2, and that it was not even clear that such a ring would be of finite rank as a $\mathbb{Z}[v^{\pm 1}]$-module in types A_3 or B_2. By the time I am writing these lines, I would be tempted to conjecture that this module is always of infinite rank if W is irreducible and of types different from A_1 or A_2. We decided to submit a short note with the full description of the indecomposables and a presentation by generators and relations of the split Grothendieck ring of $\mathcal{B}^T$ in type A_2 [86].

To some extent, the aforementioned question consists of adopting a point of view where we see a Coxeter group W as being generated by the whole set of reflections T, instead of just a simple system S. When considering more generally finite *complex* reflection groups, the notion of "simple system" that we have for real groups is not canonically defined in general, and it is thus more natural in several cases to consider the whole set of reflections, instead of a minimal one which may not be canonical. Our failure to describe the category $\mathcal{B}^T$ outside type A_2 led us to finally wonder if a first step would not be to consider non necessarily real groups, but of *rank one*, that is, in the complex case, *finite cyclic groups*.

We thus began to be interested in constructing and describing a Soergel category for cyclic groups in 2018. When viewed as a reflection group, a cyclic group is generated by a reflection s of order $d \geq 2$. Already in the very particular case of such groups, it is not clear at all what the correct definition of the equivalent of the Soergel bimodule $B_s = R \otimes_{R^s} R$ should be. One still has a natural polynomial ring $R = S(V^*)$, where V is the natural module for W, but the naive algebraic definition of B_s as $R \otimes_{R^s} R$ gives rise to a category whose K_0 is a free $\mathbb{Z}[v^{\pm 1}]$-module of rank 2 for every cyclic group, while one would expect something related to the Hecke algebra of W as defined in [34], which is a deformation of the group algebra of W. We therefore defined B_s as a ring of regular functions on a union of graphs, also generalizing a description of B_s holding in the real case, but which yields a bimodule nonisomorphic to $R \otimes_{R^s} R$ when $d > 2$.

We obtained a complete description of the split Grothendieck ring of the category generated by such a bimodule in this situation, yielding a commutative algebra A_W that is an extension of a version of the Hecke algebra H_W of the cyclic group W, of rank $|W|(|W| - 1) + 1$ when $|W| > 2$. The algebra A_W also contains a copy of the group algebra of W, and is generically semisimple if defined over the complex numbers. This gave rise to the article [87]. Note that this category has a single generating bimodule, that is, we took one generator per *distinguished* reflection (see Subsection 2.2 below for precise definitions).

The category that we defined for cyclic groups can be defined for an arbitrary complex reflection group W; whether it can be understood or not is a story for another day.

1.3.6 Reflection subgroups: structure, normalizer, Hecke algebras, ...

The results from this subsection are stated precisely in Chapter 5 below.

Finite real reflection groups are finite subgroups of $\mathrm{GL}_n(\mathbb{R})$ which can be generated by reflections. They coincide with finite Coxeter groups. The latter are defined by generators and relations, and many features of finite real reflection groups can be generalized to arbitrary, in particular not necessarily finite, Coxeter groups. This includes properties such as the exchange lemma, the weak order, the Bruhat order, etc. Arbitrary Coxeter groups still admit a faithful representation as subgroups of $\mathrm{GL}_n(\mathbb{R})$ generated by reflections (but they do not preserve an inner product in general). See Subsection 2.1 below for a short introduction to Coxeter groups.

When the Coxeter group W is finite, the conjugates of its generating set S is precisely the set of elements acting as reflections if the group is viewed as a real reflection group. In the infinite case, one still calls *reflections* the conjugates of the elements of the simple system S. A fundamental theorem independently proven by Deodhar [50] and Dyer [57] states that every subgroup of a Coxeter group W generated by a (possibly infinite) set of reflection still admits a canonical structure of Coxeter group–note that, for finite Coxeter groups, this is just a consequence of the characterization of finite Coxeter groups as finite real reflection groups.

Another possible generalization of finite real reflection groups, still mentioned in the previous section, is given by finite *complex* reflection groups. That is, finite subgroups of $\mathrm{GL}_n(\mathbb{C})$ generated by reflections. Recall that a reflection is an invertible linear transformation of finite order, whose set of fixed points is a hyperplane. In particular, when working over a complex vector space, reflections need not have order 2. In this setting as well, one can consider *reflection subgroups*, that is, subgroups generated by a set of reflections from the ambient group. Of course, every real reflection group can be realized over the complex numbers, but the converse is not true.

The rank of a Coxeter group W is the size of its set S of generators coming from its Coxeter presentation. For a complex reflection group, one defines the rank as the dimension of the complex vector space on which the group acts as a reflection group, after removing if necessary the subspace consisting of those vectors which are fixed by the whole group. In the case of finite real reflection groups which form the intersection of the above two families, one can show that these two notions coincide.

In the above two generalizations of finite real reflection groups, namely arbitrary Coxeter groups and finite complex reflection groups, several natural questions on properties of their reflection subgroups can be formulated, including for instance the following (we do not pretend this list to be exhaustive at all):

- What can be said about the rank of a reflection subgroup? Are there reflection subgroups of a given rank which are maximal with respect to certain properties?
- What is the link between the Hecke algebra or (generalized) braid group of a reflection subgroup and those of the ambiant group?
- What are the properties of left or right cosets modulo a reflection subgroup?
- What is the structure of the normalizer of a reflection subgroup?

Some of the above questions have their origins in questions coming from representation theory, where certain families of reflection subgroups (such as standard parabolic subgroups of Weyl groups) naturally appear.

My first interest for reflection subgroups came when trying to better understand the *quasi-Coxeter elements* mentioned in Subsection 1.3.4 above. In fact, every element w of a finite Coxeter group is a quasi-Coxeter element in at least one reflection subgroup of W, and the number of Hurwitz orbits on $\mathsf{Red}_T(w)$ is in bijection with the number of reflection subgroups in which w is a quasi-Coxeter element.

The structure of normalizers of parabolic subgroups of Coxeter groups has been widely studied (see Section 5.1 below for a bit of context). In connection with questions on Yokonuma-Hecke algebras, Ivan Marin defined a notion of braid group and Hecke algebra of a normalizer of a (full) reflection subgroup of a finite complex reflection group [110]. During a visit of Marin when I was a postdoc in Sydney in 2019, together with Anthony Henderson we tried to elucidate the structure of this braid group and Hecke algebra, in the case of a reflection subgroup of a finite *real* reflection group, i.e., a finite Coxeter group.

It is easily seen that the normalizer of a reflection subgroup W_0 of an arbitrary Coxeter group W can always be written as a semidirect product of W_0 and a complement U_0 stabilizing a set of roots. When W is finite, we showed that the braid group of the normalizer always decomposes as a semidirect product of the braid group of W_0 with U_0. This allowed us to construct a "standard basis" of the Hecke algebra of the normalizer.

In the case of finite complex reflection groups, a reflection subgroup does not necessarily admit a complement inside its normalizer, but it does admit a complement if it is parabolic, as shown by Muraleedaran and Taylor [118]. But unlike in the real case, even in the parabolic case, the braid group of the normalizer does not necessarily decompose as a semidirect product. Together with Ivan Marin, we showed that, when the reflection subgroup is parabolic, one always has a semidirect (or crossed) product decomposition at the level of the Hecke algebra of the normalizer, provided the field of definition of the Hecke algebra is large enough. Explicit conditions on the field to ensure such a decomposition were provided for the infinite series of finite complex reflection groups, and for some of the exceptional groups.

Another family of reflection subgroups which interested me, in connection with "dual" questions, is the family of dihedral reflection subgroups of a Coxeter group W. Dyer showed that, given any pair t, t' of distinct reflections of W, there is a unique maximal dihedral reflection subgroup of W containing both t and t' (dihedral means here that the Coxeter generating set has two generators). Such a subgroup can be seen as a generalization of the "parabolic closure" of a reflection subgroup of rank 2. Dyer recently generalized this result [62].

I gave a new proof of Dyer's theorem on the existence of maximal dihedral reflection subgroups, not using root systems, and used it to give a new proof of a recent theorem of Delucchi, Paolini and Salvetti [47], stating that the interval between the identity and a Coxeter element in the absolute order on a Coxeter group of rank three is a lattice. This was achieved by showing the more general result that any interval of length 3 in the absolute order on a Coxeter group is a lattice.

1.3.7 On torus knot groups, their Garside structures, their quotients

The results from this subsection are stated precisely in Chapter 6 below.

Dual braid monoids, which are fundamental examples of Garside monoids, were a central object in study in my PhD thesis (in type A). I have thus been rapidly exposed to Garside theory and related questions, without however getting closely interested at that time in the general aspects of Garside groups and monoids.

Roughly speaking, a Garside group is an infinite group which is the group of fractions of a finitely generated monoid, called Garside monoid, with particularly good divisibility properties. These properties ensure, among others, that the word and conjugacy problems in the Garside group are solvable, and that the group is torsion free. See section 2.5 below for precise definitions and some properties.

Shortly after I began my postdoc in Sydney in January 2018, a question from the reference book on Garside theory [43] triggered my attention:

Question 1.3.1 ([43, Chapter IX, Question 30])**.** Is the submonoid $\mathcal{M}_n$ of Artin's braid group $\mathcal{B}_n$ generated by the elements $\sigma_1, \sigma_1\sigma_2, \dots, \sigma_1\sigma_2\cdots\sigma_{n-1}$ a Garside monoid? If not, is it finitely presented?

For $n = 3$, the answer is positive. This is related to an open problem suggested by Birman and Brendle [22, Open Problem 10], which asks if there are other Garside structures for $\mathcal{B}_n$ that the classical and dual one (for sufficiently large n). In fact, a given Garside group G can have several nonisomorphic Garside monoids which have G as group of fractions, and it is natural to try to classify such monoids. Depending for instance on the chosen monoid, one can get a better solution to the word problem in the Garside group. For $n = 3$, at least four Garside monoids are known, but for $n \geq 4$, to the best of our knowledge, only the classical and dual one are known.

I had unsuccessfully tried to obtain answers to Question 1.3.1 at the beginning of 2018. Two years later, I was asked if I would like to contribute to a special issue of Journal of Algebra in honour of Patrick Dehornoy. I decided to have a look at my notes from two years before, and began to investigate this question again. I realized that I was able to say more.

I constructed a new Garside structure on a group which was already known to be a Garside group, namely the torus knot group $G(n-1,n)$. This group is an extension of Artin's braid group $\mathcal{B}_n$, and the Garside monoid $\mathcal{M}(n-1,n)$ that I constructed projects onto $\mathcal{M}_n$ via the quotient map $G(n-1,n) \twoheadrightarrow \mathcal{B}_n$ (which is an isomorphism when $n = 3$). It is not hard to see that $\mathcal{M}_n$ cannot be a Garside monoid when $n \geq 4$ because, even if every pair of elements has common multiples, in general we do not have a least one. Somehow what the above construction says is that in order to get a Garside structure, one needs to "enlarge" $\mathcal{B}_n$ into $G(n-1,n)$. This also suggests that the "correct" framework to generalize the exotic Garside structure given by $\mathcal{M}_3$ is not Artin's braid groups $\mathcal{B}_n$, but rather torus knot groups (the same happens with another known Garside structure on $\mathcal{B}_3$).

I later generalized my Garside structure to all torus knot groups $G(n,m)$, where $n, m \geq 2$ are coprime and $n < m$ [82]. It is worth mentioning that in the constructed Garside monoid $\mathcal{M}(n,m)$, the Garside element is central, and the left and right lcms of the atoms are never equal to the Garside element (in some cases also, the left and right lcms of the atoms are not equal). The lattice of divisors of the Garside element is mysterious in general. Together with Baptiste Rognerud, we described this lattice for the monoids $\mathcal{M}(n-1,n)$, and showed that in this case, the number of elements in the lattice is given by F_{2n}, where $F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, \dots$ is the Fibonacci sequence.

In the article [79] published in the volume in honour of Patrick Dehornoy, I gave a conjectural presentation of $\mathcal{M}_n$ as a quotient of $\mathcal{M}(n-1,n)$, generalizing a conjecture formulated by Dehornoy for $n = 4$. I realized in September 2022 that my conjecture (and thus Dehornoy's conjecture as well) was false. This was published as an addendum [80] to the article [79].

I have also been interested by certain quotients of torus knot groups, obtained by adding torsion on atoms in certain homogeneous Garside structures (which are *not* the ones I introduced in [79, 82]). In fact, when submitting the article for the volume in honour of Dehornoy, I had not realized that the extension of $\mathcal{B}_n$ I was considering was isomorphic to the torus knot group $G(n-1,n)$; this was pointed out to me later by a referee of the paper. In between, I had noticed that, for $n = 4$, the natural quotient of my group which was corresponding, when $n = 3$, to the symmetric group $\mathfrak{S}_3$, was the exceptional complex reflection group G_{12}, and that for $n = 5$ the analogous quotient was infinite. This motivated the following question; to be a bit more precise, recall that torus knot groups $G(n,m)$ are usually defined by the presentation $\langle\, x, y \mid x^n = y^m \,\rangle$ which has a minimal number of generators (and is Garside), but if one wants a presentation generalizing Artin's presentation of $\mathcal{B}_3$, one rather considers the following presentation, where in the knot groups the generators correspond to meridians:

$$G(n,m) \cong \langle\, x_1, x_2, \ldots, x_n \mid \underbrace{x_1x_2\cdots}_{m \text{ factors}} = \underbrace{x_2x_3\cdots}_{m \text{ factors}} = \cdots = \underbrace{x_nx_1\cdots}_{m \text{ factors}} \,\rangle, \tag{1.3.3}$$

where indices are taken modulo n. The aforementioned quotients are thus of the following form. Let $k \geq 2$ and define

$$W(k,n,m) := \left\langle\, x_1, x_2, \ldots, x_n \;\middle|\; \begin{array}{c} x_i^k = 1 \text{ for all } i = 1, \ldots, n, \\ \underbrace{x_1x_2\cdots}_{m \text{ factors}} = \underbrace{x_2x_3\cdots}_{m \text{ factors}} = \cdots = \underbrace{x_nx_1\cdots}_{m \text{ factors}} \end{array} \right\rangle. \tag{1.3.4}$$

I called such groups *toric reflection groups*. These groups are infinite in general. For instance, for any $k \geq 2$ they include the quotient of the 3-strand braid group $\mathcal{B}_3$ by the relations $\sigma_1^k = \sigma_2^k = 1$, which is nothing but $W(k,2,3)$. This quotient was shown by Coxeter to be infinite [39] if and only if $k \geq 6$.

I initiated a study of these groups [81]. To be more precise, I have shown that they belong to a family of groups generalizing complex reflection groups of rank two, called J*-groups*, introduced by Achar and Aubert [2] and defined as certain normal closures inside certain groups defined by generators and relations. In particular, this gives a presentation by generators and relations for certain J-groups. Achar and Aubert showed that a group is a finite J-group if and only if it is a finite complex reflection group of rank 2. This motivates the name *toric reflection groups*. One can deduce that a group G is a finite toric reflection group if and only if it is a finite complex reflection group of rank two with a single orbit of reflecting hyperplanes.

I also showed that every toric reflection group has a cyclic center, and that the quotient of a toric reflection group by its center is isomorphic to the alternating subgroup of a Coxeter group of rank three. This allows one to classify toric reflection groups. As a corollary of the classification result, one shows that no two nonisomorphic torus knot groups can have the same toric reflection group as quotient obtained by adding torsion on the meridians, and thus, that the torus knot group can be seen to play the role of the "braid group" of a toric reflection group. In the case where the toric reflection group is finite, one recovers its complex braid group.

It is tempting to try to generalize the study of such groups. For instance, toric reflection groups are certain J-groups, and it is natural to wonder if one can obtain presentations by generators and relations for other families of J-groups (ideally for *all* J-groups), or generalize these results to complex reflection groups of rank greater than 2. One can then try to construct an analogue of the "braid group" of such groups, and check whether it is a Garside group or not. This is the topic of the PhD thesis of my student Igor Haladjian (begun in September 2022).

1.3.8 Spherical quotients and generalized Bruhat orders

The last chapter of this thesis is devoted to a topic where reflection groups still appear. I began working on this when I was a postdoc in Nancy in 2017, and together with my collaborators there we are still currently working on the many open questions remaining. The results are stated precisely in Chapter 7 below.

The strong Bruhat order is a partial order which can be defined on an arbitrary Coxeter group, but its original definition was given in the particular case where the Coxeter group is the finite Weyl group of a connected, reductive algebraic group. Given such a reductive group, one can consider the quotient by a Borel subgroup; it can be equipped with a structure of smooth, projective algebraic variety, called the *generalized flag variety*. The Borel subgroup acts on the flag variety with finitely many orbits, parametrized by the elements of the Weyl group. The Bruhat order then describes the inclusion of orbits closures. Several critera involving only the combinatorics of the Weyl group can be taken as a generalization to arbitrary Coxeter systems[3].

Instead of considering combinatorial generalizations of the Bruhat order as above to cases where there is no more (known) flag variety, one can also consider other closed subgroups of a connected, reductive algebraic group acting with finitely orbits on the flag variety (such a subgroup is called *spherical*), and

- Seek for a parametrization of the orbits,
- Find a combinatorial criterion for the description of orbit closures, possibly involving (a subset, or a quotient of) the Weyl group.

Together with Pierre-Emmanuel Chaput and Lucas Fresse, we considered the above "generalized Bruhat order" obtained by letting the centralizer of a 2-nilpotent matrix act on the flag variety of the general linear group over an algebraically closed field of characteristic zero. This subgroup is known to be spherical, and we gave a parametrization of the orbits in a family of situations including the above one, as well as a combinatorial criterion for orbit closures involving a quotient of the Weyl group of type A by a subgroup also admitting a structure of Coxeter group, given by the subgroup of fixed points of an automorphism of Coxeter group of a disconnected standard parabolic subgroup. This has a natural generalization to arbitrary Coxeter groups together with a choice of standard parabolic subgroup with automorphism of Coxeter group, in the sense that one can define a "Bruhat order" on the set of cosets of such a subgroup of fixed points. It is natural to wonder in which cases such a "Bruhat order" comes from an inclusion of orbit closures...

Together with Nathan Chapelier, we investigated in 2021 some natural questions involving such fixed-point subgroups, such as the behaviour of elements of minimal length in cosets, and the restriction of the Bruhat order on the ambient group to such cosets [36].

It is natural to extend these results to "subgroups of a Coxeter group admitting a canonical structure of Coxeter group" (a family of subgroups which is yet yo define precisely and unambigously...), and to see under which conditions one can mimick the constructions of Deodhar [49] of parabolic Kazhdan-Lusztig polynomials.

[3] We do not know what a reasonable generalization of the Bruhat order to complex reflection groups could be.

Chapter 2

Preliminaries

The aim of this section is to recall some of the objects used in the next chapters.

2.1 Coxeter groups and their Coxeter subgroups

Most of the results cited here may be found in basic references on Coxeter groups, such as [27, 1, 24, 95].

2.1.1 Coxeter matrices and Coxeter groups

Definition 2.1.1 (Coxeter matrix)**.** Let S be a set. A *Coxeter matrix* is a matrix $(m_{s,t})_{s,t\in S}$ with coefficients in $\mathbb{Z}_{\geq 1} \cup \{\infty\}$, where

- $m_{s,s} = 1, \forall s \in S$,
- $m_{s,t} = m_{t,s} \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$ if $s, t \in S$ with $s \neq t$.

One can attach to a Coxeter matrix a *Coxeter graph* Γ, which is a labelled graph whose vertices are in one-to-one correspondence with the elements of S, and there is an edge between s and t if and only if $m_{s,t} \geq 3$; this edge has no label if $m_{s,t} = 3$, and label $m_{s,t}$ otherwise.

Given two letters a, b and an integer $m \geq 2$, denote by $P(a, b; m)$ the word defined by

$$P(a, b; m) = \begin{cases} (ab)^k & \text{if } m = 2k \\ (ab)^k a & \text{if } m = 2k+1. \end{cases}$$

Definition 2.1.2 (Coxeter group)**.** Let S be a set. Fix a Coxeter matrix $(m_{s,t})_{s,t\in S}$. The *Coxeter group* attached to this matrix is the group defined by the presentation

$$W = \langle\ s \in S \mid (st)^{m_{s,t}} = 1, \forall s, t \in S \text{ such that } m_{s,t} \neq \infty\ \rangle. \tag{2.1.1}$$

We may rewrite the presentation in the form

$$\left\langle\ s \in S\ \middle|\ \begin{array}{l} s^2 = 1,\ \forall s \in S, \\ P(s, t; m_{s,t}) = P(t, s; m_{t,s}),\ \forall s, t \in S \text{ s.t. } s \neq t \text{ and } m_{s,t} \neq \infty \end{array} \right\rangle. \tag{2.1.2}$$

The pair (W, S) is called a *Coxeter system*, and S is the *simple system* (or *Coxeter generating set*) of W. The *rank* of W is $|S|$. The group W is *irreducible* if the corresponding Coxeter graph is connected. It is *simply-laced* if $m_{s,t} \leq 3$ for all $s, t \in S$. It is *universal* if $m_{s,t} = \infty$ whenever $s \neq t$.

Example 2.1.3. The symmetric group $\mathfrak{S}_n$ with its presentation (1.1.1) is a Coxeter group.

Remark 2.1.4. It is clear from the definition that there is a group homomorphism $W \longrightarrow \mathbb{Z}/2\mathbb{Z}$, sending every generator s to 1. In particular, $s \neq 1$ in a Coxeter group. It is less obvious that distinct elements s, t from the set S we started with are not identified in the group W. This can be seen for instance using a faithful representation of W inside $\mathrm{GL}_{|S|}(\mathbb{R})$ (introduced in Theorem 2.1.13 below), as a corollary of the fact that the order of st inside W is precisely equal to $m_{s,t}$. Note that an algebraic proof of this fact avoiding the use of the faithful representation is also possible (see [116]).

Definition 2.1.5 (Isomorphism of Coxeter groups)**.** Let (W_1, S_1) and (W_2, S_2) be Coxeter systems. We say that W_1 and W_2 are *isomorphic as Coxeter groups*, if there is a bijection $\varphi : S_1 \longrightarrow S_2$ extending to a group isomorphism.

It follows from Remark 2.1.4 that two Coxeter groups are isomorphic as Coxeter groups if and only if, up to a permutation of the elements of their generating sets, they have the same Coxeter matrix.

Remark 2.1.6. To be isomorphic as Coxeter groups is much stronger than to be isomorphic as abstract groups. It can happen that Coxeter groups which are nonisomorphic as Coxeter groups become isomorphic as abstract groups. For finite Coxeter groups, there is a classification of such isomorphisms, but in general the question is still open, and known as "The Isomorphism Problem for Coxeter Groups".

Definition 2.1.7 (Reflection)**.** Let (W, S) be a Coxeter system. The set $T := \bigcup_{w \in W} wSw^{-1}$ is the set of *reflections* of W.

One can show that T is infinite if and only if W is infinite. The reason for such a terminology comes from the following result of Coxeter.

Theorem 2.1.8. *Let G be a group. Then G is a finite Coxeter group if and only if it can be realized as a finite real reflection group, that is, a finite subgroup of $\mathrm{GL}_n(\mathbb{R})$ for some n generated by elements of order 2 fixing a hyperplane in $\mathbb{R}^n$. In this case, the set of elements acting as reflections coincide with the set T.*

Denote by $\ell = \ell_S$ the length function $W \longrightarrow \mathbb{Z}_{\geq 0}$ with respect to the generating set S of W, and by ℓ_T the one with respect to the generating set T.

Definition 2.1.9 (Reduced expressions)**.** Let (W, S) be a Coxeter system. Let $s_1, s_2, \ldots, s_k$ (respectively $t_1, t_2, \ldots, t_k$) be elements of S (respectively T), and let $w \in W$ such that $w = s_1 s_2 \cdots s_k$ (respectively $t_1 t_2 \cdots t_k$). If $\ell(w) = k$ (respectively $\ell_T(w) = k$), we say that the word $s_1 s_2 \cdots s_k$ (respectively the word $t_1 t_2 \cdots t_k$) is an *S-reduced expression* of w (respectively a *T-reduced expression* of w).

Note that, since the defining relations of (W, S) preserve the parity of the length of a word, for every $t \in T$ we get that $\ell(t)$ is odd. Hence, for $w \in W$, we cannot have $\ell(tw)$ (or $\ell(wt)$) equal to $\ell(w)$. Also note that $\ell_T(w) \leq \ell_S(w)$, and that $(-1)^{\ell(w)} = (-1)^{\ell_T(w)}$, for all $w \in W$.

Definition 2.1.10 (Alternating subgroup)**.** The kernel of the group homomorphism $W \longrightarrow \mathbb{Z}/2\mathbb{Z}$ sending each generator $s \in S$ to 1 is the *alternating subgroup* of W, denoted W^+. It is also the subgroup of W consisting of those $w \in W$ such that $\ell(w)$ is even.

Proposition 2.1.11 (Inversions)**.** *Given a Coxeter system (W, S), there is a unique application $N : W \longrightarrow \mathcal{P}(T)$ such that*

1. *$N(s) = \{s\}$ for all $s \in S$,*
2. *$N(xy) = N(x) + (xN(y)x^{-1})$ for all $x, y \in W$, where $+$ denotes symmetric difference.*

The set $N(w)$ is the set of left inversions *of w. The* right inversions *of $w \in W$ may be defined as the left inversions of w^{-1}.*

Proposition 2.1.12. *Let $w \in W$. Then*

$$N(w) = \{t \in T \mid \ell(tw) < \ell(w)\},$$

and $|N(w)| = \ell(w)$.

2.1.2 Geometric representation and root systems

Let (W, S) be a Coxeter system. Let $V = \bigoplus_{s \in S} \mathbb{R}\alpha_s$ be a vector space with a basis indexed by the elements of S. Define a symmetric bilinear form B on V by defining it on the basis vectors by

$$B(\alpha_s, \alpha_t) = \begin{cases} -\cos(\pi/m_{s,t}) & \text{if} \quad m_{s,t} \neq \infty \\ -1 & \text{if} \quad m_{s,t} = \infty \end{cases}.$$

Assign to each $s \in S$ the linear transformation σ_s of V defined by

$$v \mapsto v - 2B(\alpha_s, v)\alpha_s.$$

Theorem 2.1.13 (Geometric representation)**.** *The assignment $s \mapsto \sigma_s$ extends to a faithful representation ρ of W in $\mathrm{GL}(V) \cong \mathrm{GL}_{|S|}(\mathbb{R})$.*

Note that σ_s (and hence any conjugate) acts on V by a reflection (in the sense that $\sigma_s^2 = 1$ and that σ_s fixes exactly a hyperplane). This explains the terminology from Definition 2.1.7.

We abusively still denote by w the endomorphism $\rho(w)$. Define

$$\Phi = \{w(\alpha_s) \mid s \in S, w \in W\},$$

and

$$\Phi^+ = \{\beta \in \Phi \mid \beta = \sum_{s \in S} \lambda_s \alpha_s \text{ with } \lambda_s \geq 0 \ \forall s \in S\}.$$

Theorem 2.1.14 (Generalized root systems). *One has*

1. $\Phi = \Phi^+ \dot{\cup} (-\Phi^+)$,
2. *The set T is in bijection with Φ^+. To $t \in T$ written under the form wsw^{-1}, $s \in S$, $w \in W$, one associates the unique element in $\{w(\alpha_s), -w(\alpha_s)\}$ lying in Φ^+.*

Denote by α_t the image of $t \in T$ under the bijection $T \xrightarrow{\simeq} \Phi^+$.
The following proposition justifies the use of the term "inversions" for elements of $N(w)$:

Proposition 2.1.15 (Geometric interpretation of inversions). *For $w \in W$, we have $N(w) = \{t \in T \mid w^{-1}(\alpha_t) \in \Phi^-\}$.*

Definition 2.1.16. Let $A \subseteq \Phi^+$. We say that A is *closed* if for all $\alpha, \beta \in A$, we have $(\mathbb{R}_{>0}\alpha + \mathbb{R}_{>0}\beta) \cap \Phi^+ \subseteq A$. It is *coclosed* if $\Phi^+ \backslash A$ is closed. It is *biclosed* if it is both closed and coclosed. We say that $A \subseteq T$ is *biclosed* if $\Phi_A := \{\alpha_t \mid t \in A\}$ is biclosed.

One has the following characterization of finite biclosed sets of roots (see for instance [61, Lemma 4.1 (iv)]).

Lemma 2.1.17. *Let $A \subseteq T$ be finite. Then A is biclosed if and only if there exists $w \in W$ such that $A = N(w)$.*

2.1.3 Finite Coxeter groups

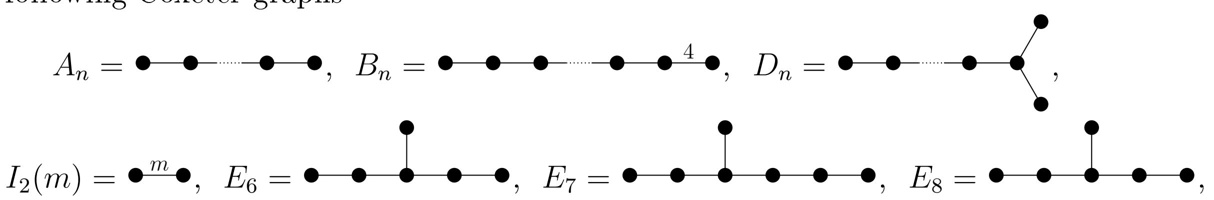
Finite Coxeter groups can be characterized as follows.

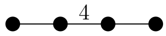
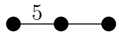
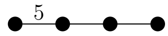
Proposition 2.1.18. *Let (W, S) be a Coxeter system. Let $w_0 \in W$. The following are equivalent*

1. *$\ell(w_0 s) < \ell(w_0)$, for all $s \in S$,*
2. *$\ell(w_0 w) = \ell(w_0) - \ell(w)$, for all $w \in W$,*
3. *w_0 has maximal length among the elements of W.*

If such an element w_0 exists, then it is unique and it is an involution, and W is finite. The element w_0 is then called the longest element *of W.*

An alternative terminology for a finite Coxeter group is a *spherical* Coxeter group, or a Coxeter group *of spherical type*. Finite irreducible Coxeter groups are classified in four infinite families A_n $(n \geq 2)$, B_n $(n \geq 2)$, D_n $(n \geq 4)$, $I_2(m)$ $(m \geq 3)$, and six exceptional groups $E_6, E_7, E_8, F_4, H_3, H_4$, given by the following diagrams (the subscript denotes the rank of the Coxeter system, hence the number of vertices of the graph). These types are defined by the following Coxeter graphs



$F_4 =$ , $H_3 =$ , $H_4 =$ .

Note that $A_2 = I_2(3)$ and $B_2 = I_2(4)$. Note that the symmetric group $\mathfrak{S}_n$ given in Example 2.1.3 corresponds to type A_{n-1}.

2.1.4 Reflection subgroups and other "Coxeter subgroups"

Let (W, S) be an arbitrary Coxeter system.

Definition 2.1.19. A *reflection subgroup* of W is a subgroup of W generated by a subset of T.

It is well-known that the reflection subgroup W_I generatedy by a subset $I \subseteq S$, called *standard parabolic subgroup*, is again a Coxeter group, with simple system I. Hence such reflection subgroups, as well as their conjugates, called *parabolic subgroups*, are themselves Coxeter groups.

In general, there are many more reflection subgroups than parabolic subgroups, and they turn out to still admit a canonical structure of Coxeter group.

Theorem 2.1.20 (Deodhar, 1989, [50], Dyer, 1990, [57])**.** *Let $A \subseteq T$, and let W_A be the subgroup of W generated by A. Then W_A is a Coxeter group, with canonical set of Coxeter generators given by*

$$\chi(W_A) := \{t \in T \cap W_A \mid N(t) \cap W_A = \{t\}\}.$$

Moreover, the set T_A of reflections of W_A is equal to $W_A \cap T$.

Ideally, one would like to have a good notion of *Coxeter subgroup* of a Coxeter group. Considering every abstract subgroup H of W admitting a generating set yielding a Coxeter presentation seems to be a too general question, and some compatibility between the generating system of H and that of W should be expected; it seems that such a theory is still missing, while there are several important families of subgroups of Coxeter groups admitting canonical structures of Coxeter groups (and often, sharing similar properties). Parabolic subgroups, and more generally reflection subgroups, form such a family. Another well-known family can be constructed as follows. Let (W, S) be a Coxeter system, and let θ be an automorphism of Coxeter group of (W, S). Consider the subgroup W^θ of θ-fixed points of W, that is, let

$$W^\theta = \{w \in W \mid \theta(w) = w\}.$$

It was observed by several authors that W^θ is again a Coxeter system, with a simple system built as follows. Let $\{\mathcal{O}_i\}_{i \in I}$ denote the set of orbits for the action of θ on S. Let

$$J := \{i \in I \mid \text{the standard parabolic subgroup } W_{\mathcal{O}_i} \text{ is finite}\},$$

where $W_{\mathcal{O}_i}$ is the standard parabolic subgroup generated by $\mathcal{O}_i \subseteq S$.

Let $S^\theta := \{w_{0,j} \mid j \in J\}$, where $w_{0,j}$ denotes the longest element of $W_{\mathcal{O}_j}$.

Theorem 2.1.21 (Steinberg, 1967, [140], Hée, 1991, [91], Mühlherr, 1993, [117], Lusztig, 2003, [106])**.** *The pair (W^θ, S^θ) is a Coxeter system.*

In general, the subgroup W^θ is *not* a reflection subgroup of W. Such Coxeter systems naturally appear for instance in certain representation theoretic questions. They are also frequently used to generalize constructions known for simply-laced Coxeter systems to non simply-laced ones, by realizing the latter as W^θ's inside simply laced W's.

2.1.5 Partial orders

Let (W, S) be a Coxeter system. There are several ways to partially order a Coxeter group.

Definition 2.1.22 (Weak order)**.** The (right) weak order on W is defined by $u \leq_S v$ if and only if $\ell_S(u) + \ell_S(u^{-1}v) = \ell_S(v)$. In other words, we have $u \leq_S v$ if and only if v admits an S-reduced expression with a prefix which is an S-reduced expression of u. One can similarly define a left weak order $\leq'_S$ by $u \leq'_S v$ if and only if $\ell_S(vu^{-1}) + \ell_S(u) = \ell_S(v)$.

Definition 2.1.23 (Bruhat order)**.** The *(strong) Bruhat order* on W is the transitive closure of the relation defined by $x < xt$ whenever $x \in W$, $t \in T$, and $\ell(x) < \ell(xt)$.

Originally, the Bruhat order appeared in the study of inclusions of Schubert varieties. These are Zariski closures of B-orbits on $\mathcal{B} = G/B$, where G is a connected reductive algebraic group and B is a Borel subgroup. As a consequence of the Bruhat decomposition, the B-orbits (and hence their closures) are parametrized by the Weyl group W of G, which is a Coxeter group. Denoting by C_w (resp. X_w) the corresponding B-orbit (resp. its closure), for $u, v \in W$ one has $u \leq v$ if and only if $X_u \subseteq X_v$.

There are several characterizations of the strong Bruhat order (see for instance [24, Corollary 2.2.3]).

Proposition 2.1.24. *Let $u, v \in W$. The following are equivalent:*

1. *$u \leq v$,*
2. *There is an S-reduced expression of v admitting a subexpression which is an S-reduced expression of u,*
3. *Every S-reduced expression of v admits a subexpression which is an S-reduced expression of u.*

The above characterization shows that, even if the whole set of reflections is used to define the strong Bruhat order, it has a natural description in terms of the classical generating system S of W.

On the other hand, every Coxeter group is generated by the bigger set T of reflections, and there are some contexts in which it is useful to see a Coxeter group as being generated by the whole set T of reflections. One can then mimick the definition of the (left) weak order:

Definition 2.1.25 (Absolute order)**.** Let $u, v \in W$. We define a partial order $\leq_T$ on W by $u \leq_T v$ if and only if $\ell_T(u) + \ell_T(u^{-1}v) = \ell_T(v)$.

Since the set T is invariant under conjugation, there is no need to distinguish between a "left" and a "right" weak order: both yield the same partial order. Concerning a "strong" order, the following proposition shows that the absolute order in fact satisfies the analogue of the second characterization of Proposition 2.1.24.

Lemma 2.1.26. *Let $u, v \in W$. The following are equivalent:*

1. *$u \leq_T v$,*
2. *There is a T-reduced expression of v admitting a subexpression which is a T-reduced expression of u.*

2.1.6 "Dual" results

Several results presented in this thesis are in the spirit of what is sometimes called the "dual" approach to Coxeter systems. It was initiated by Bessis [18] in link with Garside structures on spherical type Artin groups (see Sections 2.3 and 2.5 below). Roughly speaking, it consists of vieweing W, where (W, S) is a Coxeter system, as being generated by the whole set of reflections T, instead of just a simple system S. While the original motivation for this was lying with Garside structures on Artin groups, it revealed several rich combinatorial structures associated to a (finite) Coxeter group, such as generalizations of the noncrossing partition lattices, and other interesting combinatorial objects. See for instance [7].

In the dual setting, let us begin by giving criteria to determine $\ell_T(w)$, $w \in W$. In the case of a finite Coxeter group, we have the following, which is due to Carter. It is perhaps the first result which is "dual" in nature.

Theorem 2.1.27 (Carter's Lemma, 1972, [35, Lemmas 1 - 3])**.** *Let (W, S) be a finite Coxeter group, and let V be its geometric representation as defined in 2.1.2 above. For $w \in W$, let $V^w = \{v \in V \mid w(v) = v\}$. Then*

1. *$\ell_T(w) = \dim(V) - \dim(V^w)$, for all $w \in W$.*
2. *For all $t \in T$, $w \in W$, we have $t \leq_T w \Leftrightarrow V^w \subseteq V^t$.*
3. *Given $w \in W$ and $t_1, t_2, \ldots, t_k \in T$ such that $w = t_1 t_2 \cdots t_k$, we have $\ell_T(w) = k$ if and only if the roots $\{\alpha_{t_i}\}_{i=1}^k$ are linearly independent.*

This theorem does not generalize to arbitrary W, where typically the reflection length can be unbounded in general [54]. In affine types one can derive an analogous formula (see [103]).

For arbitrary Coxeter systems, we have the following.

Theorem 2.1.28 (Dyer, 2001, [60, Theorem 1.1])**.** *Let (W, S) be an arbitrary Coxeter system. Let $w \in W$. Let $s_1 s_2 \cdots s_k$ be any S-reduced expression of w. Then $\ell_T(w)$ is equal to the minimal number of letters to delete in the word $s_1 s_2 \cdots s_k$ to get a word representing the identity.*

Example 2.1.29. Let W be of type $\widetilde{A}_2$, that is, $S = \{s, t, u\}$ and $m_{qr} = 3$ for all $q \neq r$. Let $w = stustu$. One checks using Theorem 2.1.28 above that $\ell_T(w) = 4$. In particular, the reflection length of an element can exceed the rank $|S|$ of the Coxeter system, which by Theorem 2.1.27 cannot happen when W is finite, as $|S| = \dim(V)$.

One may consider the absolute order $\leq_T$ as the analogue of (both the left and) right weak order $\leq_S$ in the dual approach. It is a basic result in the classical approach that the poset $(W, \leq_S)$ is a lattice when W is a finite Coxeter group, with maximal element given by w_0.

In the dual setting, however, there are several maximal elements in the poset $(W, \leq_T)$ in general when W is finite.

Definition 2.1.30. Let (W, S) be a Coxeter system. A *standard Coxeter element* c in W is a product of all the elements of S in some order. A Coxeter element is a product of the form

$s_1s_2\cdots s_n$, where $S' := \{s_1, s_2, \cdots, s_n\}$ is a subset of $T = \bigcup_{w\in W} wSw^{-1}$ such that (W, S') is a Coxeter system[1].

If c is a standard Coxeter element, then $\ell_S(c) = \ell_T(c) = |S|$ (this can be seen for instance using Theorem 2.1.28 above). In particular, by Theorem 2.1.27, the reflection length of c in a finite Coxeter group is maximal, and any two distinct Coxeter elements are maximal in $(W, \leq_T)$. When W is finite, every reflection $t \in T$ satisfies $t \leq_T c$.

In the classical approach, we thus have that $(W, \leq_S)$ is the interval $[1, w_0]_S$ in the right weak order on a finite Coxeter group W. It is part of the elementary theory of finite Coxeter groups that $(W, \leq_S)$ forms a lattice. In the dual approach, the lattice $(W, \leq_T)$ cannot be a lattice, but fixing a choice of standard Coxeter element c and denoting by $[1, c]_T$ the interval between 1 and c in the absolute order, we have the following.

Theorem 2.1.31 (Bessis, 2003, [18, Fact 2.3.1], Brady-Watt, 2008, [28, Theorem 7.8], Reading, 2011, [127, Corollary 8.6])**.** *Let (W, S) be finite. The poset $[1, c]_T$ is a lattice.*

Unlike for the lattice property of $(W, \leq_S)$, the above result is very hard to show. Bessis' original proof is case-by-case, relying on the classification of finite Coxeter groups. Brady and Watt's and Reading's proofs are uniform.

Theorem 2.1.31 is especially interesting for at least two reasons

- It implies that the interval group $G([1, c]_T)$ built from the poset $[1, c]_T$ is a Garside group (see Section 2.5 below for definitions), and it turns out to be isomorphic to the Artin group B_W of W (see Section 2.3 below),
- In type A_n, as previously observed by Biane [21], the lattice $[1, c]_T$ is isomorphic to the lattice of noncrossing partitions of $\{1, 2, \ldots, n+1\}$. It therefore defines an analogue of the noncrossing partition lattice for every finite Coxeter group.

Remark 2.1.32. In a finite Coxeter group, any two standard Coxeter elements happen to be conjugate to each other [71, Theorem 3.1.4]. Since the generating set T is invariant under conjugation, the isomorphy type of the poset $[1, c]_T$ is thus independent of the choice of standard Coxeter element.

2.2 Complex reflection groups

The results cited here may be found in basic references on complex reflection groups, such as [32, 101].

Let V be a finite-dimensional $\mathbb{C}$-vector space. A *reflection* on V is an element $s \in \mathrm{GL}(V)$ such that s has finite order and $H_s := \ker(s - \mathrm{Id})$ has codimension 1.

A finite complex reflection group is a finite subgroup $W \subseteq \mathrm{GL}(V)$ generated by reflections. We say that W is *irreducible* if the W-module V is irreducible. Every finite complex reflection group is a direct product of irreducible ones.

[1]There are many possible definitions of Coxeter elements, all of which are not equivalent. We do not want to discuss this here as it is not really relevant for the results that will be presented here, but the interested reader can look for instance at [129].

Note that, unlike in the real case, reflections may not be of order 2. We denote by $\mathrm{Ref}(W)$ the set of reflections of W.

The finite irreducible complex reflection groups were classified by Shephard and Todd [135] in 1954. They fall into an infinite, three-parameter family, and 34 "exceptional" groups.

To be more precise, let d, e, r be three positive integers. Let $\mathrm{Diag}_r(de)$ be the group of $r \times r$ diagonal matrices with diagonal entries in the group μ_{de} de (de)-th roots of unity. Consider the group homomorphism $\det^d : \mathrm{Diag}_r(de) \longrightarrow \mu_e,\ X \mapsto (\det(X))^d$. The kernel $K(de, e, r)$ of this map is then the subgroup of diagonal $r \times r$ matrices with nonzero entries in μ_{de} and determinant in μ_d.

Now one defines $G(de, e, r) := \langle K(de, e, r), \mathfrak{S}_r \rangle = K(de, e, r) \rtimes \mathfrak{S}_r$, where we abuse notation and denote by $\mathfrak{S}_r$ the isomorphic finite subgroup of $\mathrm{GL}_r(\mathbb{C})$ of permutation matrices.

The group $G(de, e, r)$ can thus be identified with the group of monomial $r \times r$ matrices with entries in μ_{de} and product of the nonzero entries in μ_d.

Theorem 2.2.1 (Shephard-Todd classification, 1954, [135])**.** *The finite irreducible complex reflection groups are precisely*

1. *The $G(de, e, n)$,*
2. *34 exceptional groups denoted $G_4, G_5, \dots, G_{37}$.*

Note that every finite Coxeter group is a real reflection group, which can thus be realized as well over the complex numbers. All the finite Coxeter groups are thus part of the above classification: A_n is $G(1, 1, n+1)$, B_n is $G(2, 1, n)$, D_n is $G(2, 2, n)$, $I_2(m)$ is $G(m, m, 2)$, $E_6 = G_{35}$, $E_7 = G_{36}$, $E_8 = G_{37}$, $F_4 = G_{28}$, $H_3 = G_{23}$, $H_4 = G_{30}$.

Definition 2.2.2. Let W be a complex reflection group. A *reflection subgroup* of W is a subgroup of W generated by reflections.

Definition 2.2.3. Let W be a complex reflection group. Let $X \subseteq V$ be a subset. A subgroup of the form $\mathrm{Fix}_W(X) := \{w \in W \mid w(x) = x, \forall x \in X\}$ is called a *parabolic subgroup* of W.

Theorem 2.2.4 (Steinberg, 1964, [139, Theorem 1.5])**.** *Let W be a complex reflection group in $V = \mathbb{C}^n$. Let $v \in V$. Then $\mathrm{Fix}_W(\{v\})$ is a reflection subgroup of W, generated by the reflections in W whose reflecting hyperplane contains v.*

Corollary 2.2.5. *A parabolic subgroup $\mathrm{Fix}_W(X)$ of a complex reflection W is a reflection subgroup of W, generated by the reflections in W whose hyperplane contains X.*

In the real case, that is, in the case where W can be realized over $\mathbb{R}^n$, by Theorem 2.1.8 W is a Coxeter group, and in that case one can show that the above definition of parabolic subgroups coincides with the one given in section 2.1.

2.3 Artin groups

Definition 2.3.1 (Artin group)**.** Let S be a finite set and $(m_{s,t})_{s,t \in S}$ be a Coxeter matrix over S. The *Artin-Tits group* or simply *Artin group* $B_W = B(W, S)$ attached to the Coxeter system

(W, S) is the group generated by a copy $\mathbf{S} = \{\mathbf{s} \mid s \in S\}$ of S and defined by the presentation obtained from Presentation 2.1.2 by removing the relations ($s^2 = 1$, $\forall s \in S$), that is

$$B_W = \langle \ \mathbf{s}, s \in S \ \big| \, P(\mathbf{s}, \mathbf{t}; m_{s,t}) = P(\mathbf{t}, \mathbf{s}; m_{t,s}), \ \forall s, t \in S \text{ s.t. } s \neq t \text{ and } m_{s,t} \neq \infty \ \rangle. \quad (2.3.1)$$

The quotient map $B_W \twoheadrightarrow W$ that sends $\mathbf{s}$ to s for every $s \in S$ will usually be denoted p. If W is finite, the group B_W is said to be of *spherical type*.

Example 2.3.2. If W is the symmetric group $\mathfrak{S}_n$, then B_W is isomorphic to Artin's braid group $\mathcal{B}_n$ on n-strands.

Example 2.3.3. If the Coxeter system (W, S) is universal, that is, if $m_{s,t} = \infty$ whenever $s \neq t$, $s, t \in S$, then B_W is isomorphic to the free group on $|S|$ generators.

Artin groups are conjectured to have solvable word and conjugacy problems, to be torsion free, to satisfy the $K(\pi, 1)$-conjecture. They are also conjectured to have a trivial center when associated to an infinite, irreducible Coxeter group. It is also an open problem to determine whether they are linear or not. All the above questions are solved for Artin's braid group on n strands $\mathcal{B}_n$, and more generally for Artin groups of *spherical type*, i.e., attached to a finite Coxeter group, as well as for other important families of Artin groups. But all of them are open in general.

In fact, there are very few results which, at the time of writing, are solved for arbitrary Artin groups. One of them is the following. Let

$$B_W^+ = \langle \ \mathbf{s}, s \in S \ \big| \, P(\mathbf{s}, \mathbf{t}; m_{s,t}) = P(\mathbf{t}, \mathbf{s}; m_{t,s}), \ \forall s, t \in S \text{ s.t. } s \neq t \text{ and } m_{s,t} \neq \infty \ \rangle^+, \quad (2.3.2)$$

where the $+$ as exponent indicates that we consider the *monoid* defined by this presentation. It is the *positive Artin monoid*.

Theorem 2.3.4 (Paris, 2002, [121, Theorem 1.1])**.** *The natural map $B_W^+ \longrightarrow B_W$ is an embedding.*

Definition 2.3.5 (Canonical positive lifts of elements of W in B_W)**.** An important feature of Artin groups is the following. In a Coxeter group, the so-called *Matsumoto Lemma* states that any two S-reduced expressions of an element $w \in W$ can be related by a sequence of *braid moves*, that is, a sequence of applications of relations of the form $P(s, t; m_{s,t}) = P(t, s; m_{t,s})$ for $s, t \in S$, $s \neq t$. A solution to the word problem in an arbitrary Coxeter group can easily be deduced from this property.

Since these relations are the defining relations of B_W, it follows that the canonical quotient map p has a set-theoretic section $W \hookrightarrow B_W$ (or B_W^+), where any w with S-reduced expression $s_1 \cdots s_k$ is sent to $\mathbf{s_1} \cdots \mathbf{s_k}$. We denote by $\mathbf{W}$ the image of W under this injection, and by $\mathbf{w}$ the image of $w \in W$. We call such an element the *(canonical) positive lift* of w.

2.4 Complex braid groups

Let $W \subseteq \mathrm{GL}_n(\mathbb{C})$ be a finite complex reflection group with set $\mathcal{H}$ of reflecting hyperplanes. There is an action of W on the complement $V^{\mathrm{reg}} := \mathbb{C}^n \setminus \bigcup_{H \in \mathcal{H}} H$, and the (complex) *braid* *group* B_W of W is defined by $B_W := \pi_1(V^{\mathrm{reg}}/W)$. The covering $V^{\mathrm{reg}} \twoheadrightarrow V^{\mathrm{reg}}/W$ is Galois as a corollary of Steinberg's theorem 2.2.4, hence there is a short exact sequence

$$1 \longrightarrow P_W := \pi_1(V^{\mathrm{reg}}) \longrightarrow B_W = \pi_1(V^{\mathrm{reg}}/W) \longrightarrow W \longrightarrow 1.$$

We denote by $p : B_W \twoheadrightarrow W$ the quotient map. Its kernel P_W is the *pure braid group* of W. Finally, $\mathcal{H}$ is in bijection with the set of distinguished reflections, namely the reflections $s \in W$ whose non-trivial eigenvalue is equal to ζ_m for $\zeta_m = e^{2\pi i/m} \in \mathbb{C}^\times$ and m is equal to the order of the cyclic subgroup fixing $\mathrm{Ker}(s-1)$ pointwise (a parabolic subgroup). In the real case, one can show that B_W is isomorphic to the Artin group of W, and that p coincides with the canonical quotient map. Hence the notation is consistent.

The group B_W contains an important central element, which we denote by z_{B_W}. When W is irreducible, its center $Z(W)$ is cyclic of some order m, generated by $\zeta_m \mathrm{Id}$. In this setting, z_{B_W} is the homotopy class in X/W of the path $t \mapsto \exp(2\pi i t/m)x_0$, where $x_0 \in X$ is the chosen base-point. Finally, B_W also contains as remarkable elements the braided reflections associated to the reflections of W (see for instance [34] for their precise geometric definition).

2.5 Garside groups

Artin's braid group on n strands, that is, the Artin group $\mathcal{B}_n = B_{\mathfrak{S}_n}$ attached to the symmetric group, has long been known to have a solvable word problem. Already Artin gave a well-known solution to the word problem in 1925 [8] (made rigourous in 1948 [9]), using a faithful representation of $\mathcal{B}_n$ by automorphisms of the free group F_n. Recall that a finitely generated group defined by generators and relations is said to have a *solvable word problem* if there is an algorithm allowing one to determine in finite time whether a given word in the generators (and their inverses) represents the identity or not. Another fundamental question which one can ask for such a group is the conjugacy problem: is there an algorithm allowing one to determine in finite time if any two elements of the group are conjugate to each other?

For Artin's braid group $\mathcal{B}_n$, the conjugacy problem was solved by Garside in 1969 [70]; as a byproduct of his approach, a new solution to the word problem was also given (later improved by several authors), a new method to determine the center of $\mathcal{B}_n$, and developments of his approach also yielded new proofs that $\mathcal{B}_n$ is torsion-free, originally obtained by Fadell and Neuwirth [67] in 1962 as a corollary of a particular case of what became known as the $K(\pi, 1)$-conjecture for Artin groups (a still open problem in general). Garside's approach turned out to be an adapted framework to generalize many properties of $\mathcal{B}_n$ to Artin groups of spherical type, as observed by Brieskorn-Saito [30] and independently by Deligne [45] shortly after Garside's work appeared.

A central result on which Garside's approach is based is a proof of Theorem 2.3.4 above in the case of $\mathcal{B}_n$. He also showed that $\mathcal{B}_n$ is the group of fractions of $\mathcal{B}_n^+$, hence that every element of $\mathcal{B}_n$ can be written as a fraction in two elements of $\mathcal{B}_n^+$. Together with an algorithm to explicitly convert any word in the generating set of $\mathcal{B}_n$ into a fraction in two elements of $\mathcal{B}_n^+$, it solves the word problem in $\mathcal{B}_n$: if $\beta = xy^{-1}$ is an element of $\mathcal{B}_n$ with $x, y \in \mathcal{B}_n^+$, then $\beta = 1$ if and only if $x = y$ in $\mathcal{B}_n^+$ (because of the embedding $\mathcal{B}_n^+ \hookrightarrow \mathcal{B}_n$), and in $\mathcal{B}_n^+$ which has a homogeneous presentation, the word problem is trivial, hence one can check whether x equals

y or not in $\mathcal{B}_n^+$ in finite time. Garside's algorithm to convert any word into a fraction relies on the existence of a fundamental element $\Delta \in \mathcal{B}_n^+$ with several properties, which has a central power.

Dehornoy and Paris [44] axiomatized the properties used by Garside as follows[2]. Given a monoid M and two elements $a, b \in M$, we say that a is a *left-divisor* (respectively a *right-divisor*) of b is there exists $c \in M$ such that $ac = b$ (respectively $ca = b$). In this setting we say that b is a *right-multiple* (respectively a *left-multiple*) of a.

Definition 2.5.1 (Garside monoid and Garside group)**.** A *Garside monoid* is a pair (M, Δ) where M is a monoid and Δ is an element of M, satisfying the following five conditions:

1. M is left- and right-cancellative, that is, $(ab = ac \Rightarrow b = c, \forall a, b, c \in M)$ and $(ba = ca \Rightarrow b = c, \forall a, b, c \in M)$,
2. The divisibility in M is Noetherian, i.e., there exists a function $\lambda : M \to \mathbb{Z}_{\geq 0}$ such that
$$\forall a, b \in M, \lambda(ab) \geq \lambda(a) + \lambda(b) \text{ and } a \neq 1 \Rightarrow \lambda(a) \neq 0.$$
3. Any two elements in M admit a left- and right-lcm, and a left- and right-gcd,
4. The left- and right-divisors of the element Δ coincide and generate M,
5. the set of (left- or right-)divisors of Δ is finite.

If the five conditions above are satisfied, then M admits a group of fractions $G(M)$ in which it embeds. Such a group is a *Garside group*, and every presentation of M yields a presentation of G if viewed as a group presentation. If (M, Δ) satisfies the four first conditions, then it is said to be a *quasi-Garside monoid*, and its group of fractions is then a *quasi-Garside group*.

We often abuse notation and write M instead of (M, Δ). Given a group G, a Garside monoid (M, Δ) such that $G(M) \cong G$ is sometimes called a *Garside structure* for G.

Remark 2.5.2. The second condition implies that M has no nontrivial invertible element, and hence, using cancellativity, that the left- or right-divisibility relations define partial orders on M. It thus makes sense to talk about "lcm's" and "gcd's" as done in point 3 if the conditions in the first two points are satisfied.

Definition 2.5.3 (Simple elements)**.** In a Garside monoid (M, Δ), the set of left- (equivalently right-) divisors of Δ, often denoted $\mathrm{Div}(\Delta)$, is the set of *simple elements* of (M, Δ) (abusively of M).

It is difficult in general to check the five properties defining a Garside monoid, but this has strong consequences. Among other remarkable properties (see [43] for more on the topic), one has the following.

Proposition 2.5.4. *Every Garside group has a solvable word problem, and is torsion free.*

[2] The original definition was a bit more restrictive, but the definition of Garside monoid given here is the one used by most authors nowadays.

The following is one of the main examples of a Garside structure.

Theorem 2.5.5. *Let (W, S) be a finite Coxeter system, and let w_0 be the longest element of W. The pair (B_W^+, Δ), where Δ is the canonical positive lift of w_0 (see Definition 2.3.5), is a Garside monoid. In particular, every Artin group of spherical type is a Garside group.*

Definition 2.5.6 (Classical simple braids)**.** The simple elements of the above Garside structure turn out to be the canonical positive lifts $\mathbf{w}$ of elements w of W, also called *classical simple braids*.

The second example is an alternative Garside monoid for Artin groups of spherical type.

Definition 2.5.7 (Dual braid monoids)**.** Let (W, S) be a finite Coxeter system, with set of reflections T. Let c be a standard Coxeter element. Recall from Section 2.1.6 that every reflection $t \in T$ satisfies $t \leq_T c$. Consider a copy $T_c = \{t_c \mid t \in T\}$ of the set T and define a monoid B_c^* by

$$B_c^* = \langle\, T_c \mid t_c t'_c = t''_c t_c \text{ whenever } tt' = t''t \text{ and } tt' \leq_T c \,\rangle^+.$$

We call B_c^* the *dual braid monoid* attached to (W, S) and c.

Up to isomorphism, the monoid B_c^* does not depend on the choice of standard Coxeter element c (this follows easily from the fact that two standard Coxeter elements are conjugate and that T is stable under conjugation).

Theorem 2.5.8 (Bessis, 2003, [18])**.** *Let (W, S) be a finite Coxeter system, with $S = \{s_1, s_2, \ldots, s_n\}$, and let $c = s_1 s_2 \cdots s_n$. Set $\Delta := (s_1)_c(s_2)_c \cdots (s_n)_c$. The pair (B_c^*, Δ) is a Garside monoid, with corresponding Garside group $G(B_c^*)$ isomorphic to B_W. This isomorphism sends $(s_i)_c$ to $\mathbf{s}_i$ for all $i = 1, \ldots, n$ (the $(s_i)_c$'s do not generate B_c^* in general, but they generate $G(B_c^*)$).*

To prove Theorem 2.5.8, Bessis generalized a construction made by Birman, Ko and Lee [23] in type A_n, also interpreting it in the framework of Garside monoids and groups.

Definition 2.5.9 (Simple dual braids)**.** The simple elements of (B_c^*, Δ) are the *simple dual braids*.

We will say more about the set of simple elements of B_c^* using the approach of *interval groups* below.

Example 2.5.10. Let $W = \{s_1, s_2\}$ be of type A_2, with s_i identified with the simple transposition $(i, i+1)$. Let $c = s_1 s_2$. Write $t := s_1 s_2 s_1 = (1, 3)$. Since $\ell_T(c) = 2$, to find all the defining relations of B_c^*, we need to find all the products of two reflections which are equal to c. We have $s_1 s_2 = s_2 t = t s_1 = c$. Writing $x = (s_1)_c$, $y = (s_2)_c$, and $z = t_c$, we thus have

$$B_c^* = \langle\, x, y, z \mid xy = yz = zx \,\rangle^+.$$

Thanks to the above theorem, this is also a presentation of $\mathcal{B}_3$, if viewed as a group presentation. In terms of the classical generators of $\mathcal{B}_n$ we have $x = \sigma_1$, $y = \sigma_2$, $z = \sigma_1 \sigma_2 \sigma_1^{-1}$.

Checking that a pair (M, Δ) is a Garside monoid is a difficult task in general. Techniques were developed for instance in [44, 41, 43]. Often, one typically needs to enlarge a presentation by adding redundant relations to be able to apply such methods. In some contexts, the approach of so-called *interval groups* is helpful. Let us recall it here.

Let G be a group, and let A be a subset of G which generates G as a monoid. Consider the length function $\ell_A : G \longrightarrow \mathbb{Z}_{\geq 0}$ with respect to this set of generators. Define a partial order $\leq_A$ on G by setting $u \leq_A v \Leftrightarrow \ell_A(u) + \ell_A(u^{-1}v) + \ell_A(v)$, that is, if there is an A-reduced decomposition of v having an A-reduced decomposition of u as prefix. Similarly, define $u \leq'_A v \Leftrightarrow \ell_A(vu^{-1}) + \ell_A(u) = \ell_A(v)$.

Definition 2.5.11. In the above setting, an element $c \in G$ is *A-balanced* if

$$L(c) := \{u \in G \mid u \leq_A c\} = \{u \in G \mid u \leq'_A c\} =: R(c).$$

Let G, A be as above and let $c \in G$ be A-balanced. Denote by $[1, c]_A$ the set $L(c)$, which coincides with the set $R(c)$. The notation comes from the fact that $L(c)$ (or $R(c)$) is an interval in $(G, \leq_A)$ (or $(G, \leq'_A)$); they are nonisomorphic in general as posets. Consider a copy $X = \{\underline{u} \mid u \in [1, c]_A\}$ and define a monoid $M([1, c]_A)$ by generators and relations by

$$M([1, c]_A) := \langle\ \underline{u} \in X \mid \underline{u} \cdot \underline{v} = \underline{w} \text{ if } uv = w \text{ and } u \leq_A w\ \rangle.$$

Let $G([1, c]_A)$ denote the group with the same presentation. One has the following.

Theorem 2.5.12 (Bessis, 2004, [18, Theorem 0.5.2])**.** *Let $c \in G$ be A-balanced. Then $(L(c), \leq_A)$ is a lattice if and only if $(R(c), \leq'_A)$ is a lattice, and in this case, if $L(c) = R(c)$ is finite, then $M([1, c]_A)$ is a Garside monoid, with Garside element $\underline{c}$ and set of simples X. The corresponding Garside group $G([1, c]_A) = G(M([1, c]_A))$ is called the* interval (Garside) group *attached to the interval $[1, c]_A$. Without the finiteness assumption on $L(c)$, the conclusion remains true with "Garside" replaced by "quasi-Garside".*

This theorem is a tool to generate many Garside monoids. The classical and dual Garside structures on Artin groups of spherical type given in Theorems 2.5.5 and 2.5.8 above can be realized in this way. In fact, this is one way to show that B_W^+ and B_c^* are Garside monoids.

Theorem 2.5.13. *Let W be a finite Coxeter group with longest element w_0, and let $c \in W$ be a standard Coxeter element. Then*

$$B_W^+ \cong M([1, w_0]_S), \ \textit{and}\ B_c^* \cong M([1, c]_T).$$

As already mentioned, the set of simples of B_W^+ is the set of canonical positive lifts of elements of W, hence it is in bijection with W. In the dual setting, the set X of simples is in bijection with the interval $\mathrm{NC}(W, c) := [1, c]_T$ in the absolute order, which forms the *generalized noncrossing partitions*, counted by the generalized Catalan numbers. In type A_n with $c = s_1 s_2 \cdots s_n$, one get a bijection between elements from $\mathrm{NC}(W, c)$ and noncrossing partitions of the set $\{1, 2, \ldots, n+1\}$ by mapping each element $x \in \mathrm{NC}(W, c)$ to the partition given by the various supports of the cycles occurring in the cycle decomposition of x.

The simples form a generating set of a Garside monoid which is not minimal in general, but very useful for instance to solve the word problem, as there are normal forms for the elements of a Garside monoid and the corresponding Garside group which can be calculated very efficiently using this set of generators. A minimal generating set of a Garside monoid M is given by its set of *atoms*, that is, those elements x such that $x \neq 1$ and $x = ab \Rightarrow a = 1$ or $b = 1$. In B_W^+, the set of atoms is given by $\mathbf{S} = \{\mathbf{s} \mid s \in S\}$, and in B_c^*, it is given by the set $T_c = \{t_c \mid t \in T\}$.

Note that B_W viewed as Garside group of $M([1, c]_T)$ has more generators than viewed as Garside group of $M([1, w_0]_S)$, if we take as generators the sets of atoms of the corresponding monoids. One method to express the generators T_c of B_W in terms of the generators of the set $\mathbf{S}$ is to use the Hurwitz action (see Section 1.3.4 above). Roughly speaking, one can start from the decomposition

$$\underline{c} = (s_1)_c (s_2)_c \cdots (s_n)_c = \mathbf{s}_1 \mathbf{s}_2 \cdots \mathbf{s}_n,$$

representing it under the form $(\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n)$, and apply Hurwitz moves. For instance, one can "move $\mathbf{s}_2$ into the first position" without changing the product, that is

$$(\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n) \to (\mathbf{s}_2, \mathbf{s}_2^{-1}\mathbf{s}_1\mathbf{s}_2, \ldots, \mathbf{s}_n),$$

and $\mathbf{s}_2^{-1}\mathbf{s}_1\mathbf{s}_2$ then turns out to be the generator $(s_1 s_2 s_1)_c$, expressed in terms of the generators $\mathbf{S}$. It can be shown that every reflection in T can be obtained by applying such moves at the level of the Coxeter group, that is, on $(s_1, s_2, \ldots, s_n)$, and that this lifts at the level of B_W. See [18] or [53, Section 3.3] for more details.

2.6 Hecke algebras

2.6.1 Coxeter groups

Definition 2.6.1. Let (W, S) be an arbitrary Coxeter system and let $\mathcal{A} = \mathbb{Z}[v^{\pm 1}]$. The *Hecke algebra* (or *Iwahori-Hecke algebra*) H_W of W is the quotient of the group algebra $\mathcal{A}[B_W]$ by the relations

$$\mathbf{s}^2 = (v^{-2} - 1)\mathbf{s} + v^{-2}, \text{ for all } s \in S. \tag{2.6.1}$$

When specializing v to 1, we get the group algebra of W over $\mathbb{Z}$. In fact, one has a natural basis of H_W deforming the basis of $\mathbb{Z}[W]$ consisting of the group elements of W:

Proposition 2.6.2. *The algebra H_W is a free $\mathcal{A}$-module with basis $\{T_w\}_{w \in W}$, where T_w is the image of the canonical positive lift $\mathbf{w} \in B_W$ of w in H_W.*

The images of the generators $\mathbf{s}$ of B_W inside H_W will thus usually be denoted T_s. The basis $\{T_w\}_{w \in W}$ is usually called the *standard basis* of H_W. The elements T_s, $s \in S$ are all invertible because of the relation (2.6.1), hence every T_w is invertible. It is readily checked that the set $\{T_{w^{-1}}^{-1}\}_{w \in W}$ is also a basis of H_W, called the *costandard basis*.

To state some results in a cleaner way, it will be useful to renormalize the standard basis by setting $H_w := v^{\ell(w)} T_w$, for all $w \in W$.

There is a unique semilinear involution $\bar{\ }: H_W \to H_W$, called the *bar involution*, such that $\overline{v} = v^{-1}$, $\overline{T_w} = (T_{w^{-1}})^{-1}$. Kazhdan and Lusztig prove the following.

Theorem 2.6.3 (Kazhdan and Lusztig, 1979, [98, Theorem 1.1])**.** *Let (W, S) be an arbitrary Coxeter system and denote by $\leq$ the (strong) Bruhat order on W.*

1. *For any $w \in W$, there is a unique element $C'_w \in H_W$ such that $\overline{C'_w} = C'_w$ and $C'_w \in H_w + \sum_{y<w} v\mathbb{Z}[v]H_y$.*

2. *For any $w \in W$, there is a unique element $C_w \in H_W$ such that $\overline{C_w} = C_w$ and $C_w \in H_w + \sum_{y<w} v^{-1}\mathbb{Z}[v^{-1}]H_y$.*

It follows that $\{C_w\}_{w\in W}$ and $\{C'_w\}_{w\in W}$ are bases of H_W, called *canonical bases*. The coefficients of C'_w when expressed in the standard basis became known as *Kazhdan-Lusztig polynomials*. Kazhdan and Lusztig conjectured that these polynomials have nonnegative coefficients, which was proven in full generality by Elias and Williamson [65] as a corollary of Soergel's conjecture (see Subsection 2.7.1 below).

2.6.2 Complex reflection groups

Let W be a finite complex reflection group, and let B_W be its braid group. We recall from [34] the construction of the Hecke algebra H_W of W over some commutative ring $\mathbb{K}$. It is defined using parameters $u_{i,s} \in \mathbb{K}^\times$ for s running over the set of the distinguished reflections of W, where $0 \leq i < o(s)$ and $u_{i,s} = u_{i,t}$ when s, t belong to the same conjugacy class. Then H_W is the quotient of $\mathbb{K}[B_W]$ by the relations $\prod_{i=0}^{o(s)-1}(\sigma - u_{i,s}) = 0$ for every braided reflection σ associated to s – so that its most general definition ring is the ring of Laurent polynomials $\mathcal{A} = \mathbb{Z}[u_{i,s}^{\pm 1}]$.

If W is real, we essentially recover the definition from the previous section, except that we gave only one parameter in the real case (the definition given in the real case, where every reflection has order 2, corresponds to choices of parameters $u_{0,s} = -1$ and $u_{1,s} = v^{-2}$, for every distinguished reflection).

We have an analogue of Proposition 2.6.2, which is the so-called *Broué-Malle-Rouquier freeness conjecture* [34, Section 4.C], which is now a theorem, as a combination of the work of many authors.

Theorem 2.6.4 ([5, 6, 33, 107, 108, 112, 38, 111, 141])**.** *The algebra H_W is a free $\mathcal{A}$-module of rank $|W|$.*

2.7 Hecke categories and categorification of Artin groups

2.7.1 Soergel bimodules

Let (W, S) be a Coxeter system and V a reflection faithful representation of (W, S) over $\mathbb{R}$ in the sense of [137, Definition 1.5]. Let $R := \mathcal{O}(V) \cong S(V^*)$ be the coordinate ring of V. In particular R comes equipped with an action of W and a $\mathbb{Z}$-graduation with the convention that $\deg(V^*) = 2$.

Let $\mathcal{R}$ denote the category of $\mathbb{Z}$-graded $R \otimes_{\mathbb{R}} R$-modules which are finitely generated from the left and from the right (we call "left" action the action of $R \otimes_{\mathbb{R}} 1$ and "right" action the action of $1 \otimes_{\mathbb{R}} R$). The category $\mathcal{R}$ is a monoidal category via $\otimes_R$. It satisfies the Krull-Schmidt property (see [137, Remark 1.3]). For $M \in \mathcal{R}$ and $i \in \mathbb{Z}$ denote by M_i the homogeneous component of degree i of M. Given $M \in \mathcal{R}$ and $i \in \mathbb{Z}$, we denote by $M(i)$ the element of $\mathcal{R}$ equal to M as $R \otimes_{\mathbb{R}} R$-module but with graduation shifted by i, that is, such that $M(i)_j = M_{i+j}$ for all $j \in \mathbb{Z}$. We define the *graded rank* of free graded right R-module M as the element of $\mathbb{Z}[v^{\pm 1}]$ given by

$$\underline{\mathrm{rk}} M := \underline{\dim}(M/MR_{>0}),$$

where for a finite-dimensional $\mathbb{Z}$-graded vector space U we denote by $\underline{\dim}(U)$ the graded dimension $\sum_{i \in \mathbb{Z}} (\dim U_i) v^i \in \mathbb{Z}[v^{\pm 1}]$ of U and $R_{>0}$ denotes the ideal of polynomials without constant term. We denote by $\overline{\underline{\mathrm{rk}}} M$ the graded rank of M after substitution of v by v^{-1}.

Given $B, B' \in \mathcal{R}$, we denote by $\mathrm{Hom}(B, B')$ the morphisms in the category $\mathcal{R}$, that is, the homomorphisms of bimodules $B \to B'$ which are homogeneous of degree zero. We furthermore set

$$\mathrm{Hom}^\bullet(B, B') := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}(B, B'(i)).$$

Notice that it comes equipped with a structure of graded R-bimodule.

To every $s \in \mathcal{S}$ we associate the bimodule $B_s := R \otimes_{R^s} R(1) \in \mathcal{R}$, where $R^s \subseteq R$ is the graded subring of s-invariant functions. For $x \in W$, we denote by R_x the element of $\mathcal{R}$ equal to R as left R-module but with right action twisted by x, that is, $r \cdot r' = rx(r')$, for $r \in R_x$, $r' \in R$. Denote by $\langle \mathcal{R}, \otimes_R \rangle$ the split Grothendieck ring of $\mathcal{R}$, endowed with a $\mathbb{Z}[v^{\pm 1}]$-algebra structure via $v \cdot \langle M \rangle = \langle M(1) \rangle$ for $M \in \mathcal{R}$. Soergel showed the following, usually referred to as *Soergel's categorification theorem*.

Theorem 2.7.1 (Soergel, 2007, [137, Theorems 1.10 and 5.3])**.** *Let (W, S) be a Coxeter system.*

1. *There is a unique homomorphism of $\mathbb{Z}[v^{\pm 1}]$-algebras*

$$\mathcal{E} : H_W \to \langle \mathcal{R}, \otimes_R \rangle$$

such that $\mathcal{E}(v) = \langle R[1] \rangle$ and $\mathcal{E}(vT_s + v) = \langle B_s \rangle$ for every $s \in \mathcal{S}$.

2. *The homomorphism $\mathcal{E}$ has a left inverse*

$$\mathrm{ch} : \langle \mathcal{R}, \otimes_R \rangle \to H_W$$

given by $\mathrm{ch}(\langle B \rangle) = \sum_{x \in W} \overline{\underline{\mathrm{rk}}}(\mathrm{Hom}^\bullet(B, R_x)) T_x$, for $B \in \mathcal{R}$.

This theorem implies that H_W is isomorphic to the split Grothendieck ring $\langle \mathcal{B} \rangle$ of the additive monoidal category $\mathcal{B}$ generated by tensor products $B_s \otimes_R B_t \cdots \otimes_R B_u$ (called *Bott-Samelson bimodules*; here $st \cdots u$ is any finite word in the elements of S) and stable by direct sums, direct summands and graduation shifts (so that $\langle \mathcal{B} \rangle$ is a $\mathbb{Z}[v^{\pm 1}]$-algebra). By definition an object of $\mathcal{B}$ is a *Soergel bimodule*. Hence indecomposable Soergel bimodules are (shifts of) indecomposable direct summands of tensor products $B_s \otimes_R B_t \otimes_R \cdots \otimes_R B_u$. For simplicity we may write tensor products over R by juxtaposition. Given $B \in \mathcal{B}$, we will sometimes abuse notation and simply denote $\mathrm{ch}(\langle B \rangle)$ by $\langle B \rangle$.

Soergel shows that indecomposable Soergel bimodules are (up to shifts and isomorphism) indexed by elements of W (see [137, Theorem 6.16]). The indecomposable bimodule B_w indexed by $w \in W$ may be described as follows: let $st \cdots u$ be a reduced expression for w. Then there is a unique indecomposable direct summand B_w of $B_s B_t \cdots B_u$ which does not occur as a direct summand of a tensor product $B_{s_1} B_{s_2} \cdots B_{s_k}$ for $k < \ell(w)$. The Bott-Samelson bimodule $B_s B_t \cdots B_u$ depends on the reduced expression chosen for w, but it turns out that the direct summand B_w does not, up to isomorphism.

Soergel made the following conjecture, proven by Elias and Williamson [65]:

Theorem 2.7.2 (Soergel's conjecture, 2007, [137, Conjecture 1.13])**.** *For any $w \in W$, we have $\mathcal{E}(C'_w) = \langle B_w \rangle$.*

It has as an immediate corollary that the polynomials $h_{x,w}$ which are the coefficients of $H_x = v^{\ell(x)} T_x$ when expressing C'_w in the basis $\{H_x\}_{x \in W}$ have nonnegative coefficients, since

$$C'_w = \mathcal{E}^{-1}(\langle B_w \rangle) = \mathrm{ch}(\langle B_w \rangle) = \sum_{x \in W} \underline{\overline{\mathrm{rk}}}(\mathrm{Hom}^\bullet(B_w, R_x)) T_x,$$

and $\underline{\overline{\mathrm{rk}}}\mathrm{Hom}^\bullet(B_w, R_x)$ has nonnegative coefficients by definition of the graded rank. This was a major conjecture of Kazhdan and Lusztig [98].

2.7.2 Categorical braid group action on complexes of Soergel bimodules

We denote by $K^b(\mathcal{R})$ (resp. $K^b(\mathcal{B})$) the homotopy category of bounded complexes of bimodules in $\mathcal{R}$ (resp. $\mathcal{B}$). The monoidal structure on $\mathcal{R}$ and $\mathcal{B}$ induces a monoidal structure on the corresponding homotopy categories via the total tensor product of complexes, which we will simply denote by juxtaposition. Given a complex

$$C \ : \quad \ldots \to {}^{i-1}C \to {}^{i}C \to {}^{i+1}C \to \ldots \ \in K^b(\mathcal{R}),$$

We denote by $C[j]$, where $j \in \mathbb{Z}$, the complex C shifted in homological degree by j, that is, such that ${}^{i}C[j] = {}^{j+i}C$. Let us consider the following indecomposable complexes in $K^b(\mathcal{B})$:

$$F_s = 0 \longrightarrow B_s \xrightarrow{\mu_s} R(1) \longrightarrow 0,$$
$$E_s = 0 \longrightarrow R(-1) \xrightarrow{\eta_s} B_s \longrightarrow 0,$$

where B_s sits in both cases in homological degree zero. Here μ_s is the multiplication map $\mu_s(a \otimes b) = ab$, and $\eta_s(r) = \frac{1}{2}(r \otimes f_s + r f_s \otimes 1)$ for any $r \in R$, where f_s is a nonzero linear form vanishing on the hyperplane $H_s \subseteq V$ of s. The functors $F_s \otimes -$ and $E_s \otimes -$ define mutually inverse equivalences of $K^b(\mathcal{R})$, categorifying (a quotient of) the braid group of the Coxeter system in spherical type, see [133], [132]; Rouquier indeed proved that the F_s satisfy the braid relations and conjectured that these functors and their inverses categorify the whole braid group and not just a quotient. The conjecture holds in type A_n as a consequence of the work of Khovanov and Seidel [100], and was later proven by Jensen [97] in spherical type, following ideas from Brav and Thomas [29].

Given $\beta \in B_W$ represented by the word $\mathbf{s}_1^{\varepsilon_1}\mathbf{s}_2^{\varepsilon_2}\cdots\mathbf{s}_k^{\varepsilon_k}$, where $\varepsilon_i \in \{\pm 1\}$ for all i, denote by F_β the complex $D_{s_1} \otimes D_{s_2} \otimes \cdots \otimes D_{s_k}$, where $D_{s_i} = F_{s_i}$ if $\varepsilon_i = 1$ and $D_{s_i} = E_{s_i}$ if $\varepsilon_i = -1$. Note that this object depends on the chosen word, but two equivalent words yield complexes which are canonically isomorphic in $K^b(\mathcal{B})$ (see [133, Section 9.3.1]). We can replace F_β by a *minimal* complex in $K^b(\mathcal{B})$, that is, a complex obtained from F_β by removing all contractible direct summands (see [65, Section 6.1]; a minimal complex is isomorphic to the starting complex in $K^b(\mathcal{B})$ and any two miminal complexes turn out to be isomorphic *as complexes of bimodules*). We will denote by $F_\beta^{\min}$ the minimal Rouquier complex attached to the braid β. If $\beta = \mathbf{x}$ for some $x \in W$, we simply denote $F_\beta = F_{\mathbf{x}}$ by F_x.

Following Elias and Williamson [65], denote by $K^b(\mathcal{B})^{\geq 0}$ the full subcategory of $K^b(\mathcal{B})$ whose objects are those complexes with minimal complex F satisfying the following property: for any $i \in \mathbb{Z}$ such that ${}^iF$ is nonzero and any indecomposable summand B in ${}^iF$, there exists $x \in W$, $k \leq i$ such that $B \cong B_x(k)$. Similarly, denote by $K^b(\mathcal{B})^{\leq 0}$ the full subcategory of $K^b(\mathcal{B})$ whose objects are those complexes with minimal complex satisfying the following property: for any $i \in \mathbb{Z}$ such that ${}^iF$ is nonzero and any indecomposable summand B in ${}^iF$, there exists $x \in W$, $k \geq i$ such that $B \cong B_x(k)$. In other words, roughly speaking, $K^b(\mathcal{B})^{\geq 0}$ (resp. $K^b(\mathcal{B})^{\leq 0}$) consists of those complexes with minimal complex F having indecomposable Soergel bimodules with shifts at most (resp. at least) i occurring in homological degree i.

A complex $C \in K^b(\mathcal{B})$ is called *perverse* (or *linear*) if $C \in K^b(\mathcal{B})^{\geq 0} \cap K^b(\mathcal{B})^{\leq 0}$.

Chapter 3

Dual Coxeter systems and dual braid monoids

List of relevant publications

1. B. Baumeister and T. Gobet, *Simple dual braids, noncrossing partitions and Mikado braids of type D_n*, Bull. Lond. Math. Soc. **49** (2017), no.6, 1048-1065.

2. B. Baumeister, T. Gobet, K. Roberts, and P. Wegener, *On the Hurwitz action in finite Coxeter groups*, J. Group Theory **20** (2017), no.1, 103-131.

3. F. Digne and T. Gobet, *Dual braid monoids, Mikado braids and positivity in Hecke algebras*, Math. Z. **285** (2017), no. 1-2, 215-238.

4. T. Gobet, *On cycle decompositions in Coxeter groups*, Sém. Lothar. Combin. **78B** (2017), Art. 45, 12 pp.

5. T. Gobet, *Dual Garside structures and Coxeter sortable elements*, J. Comb. Algebra **4** (2020), no. 2, 167–213.

This chapter collects several works all realized between 2015 and 2018, which study dual Coxeter systems and Artin groups of spherical type. It is separated into two main sections.

The motivation for the study in Section 3.1 below was to understand certain positivity properties of images of simple elements of the dual Garside structures on Artin groups of spherical type inside Hecke or Temperley-Lieb type algebras. It naturally lead to the following fundamental question:

Question 3.0.1. How can one express the simple elements of dual braid monoids in terms of the classical Artin group generators?

Section 3.2 addresses a few questions on T-reduced expressions of elements of finite Coxeter groups. In that section, we present a characterization of elements having the "dual" Matsumoto property in terms of the Hurwitz action of Artin's braid group on the set of T-reduced expressions of the elements. We also show that these elements naturally admit a decomposition generalizing the cycle decomposition of the symmetric group.

3.1 Classical versus dual generators of Artin groups of spherical type

Recall that, for a spherical Coxeter system (W, S), there are (at least) two Garside structures on the corresponding Artin group B_W: the *classical one*, where the Garside monoid is B_W^+, and the *dual one*, where the monoid B_c^* depends on a choice c of standard Coxeter element in W, and has set T_c of generators in one-to-one correspondence with the reflections in W; these generators are subject to the dual braid relations, that is, the relations of the form $t_1t_2 = t_2t_3$ whenever $t_1, t_2, t_3 \in T$, $t_1 \neq t_2$, and $t_1t_2 \leq_T c$. Also recall that every element $w \in W$ admits a canonical positive lift $\mathbf{w} \in B_W^+ \subseteq B_W$, obtained by "lifting" any S-reduced expression of w to B_W^+.

3.1.1 Simple dual braids as quotients of positive simple braids

Let (W, S) be a Coxeter system. Let B_W be the attached Artin group.

Together with Digne, we proved the following result.

Theorem 3.1.1 (Digne-G., 2017, [53, Theorems 5.12, 6.6, 7.1]). Let W be a finite irreducible Coxeter group of type different from D_n, and let c be a standard Coxeter element in W. Let B_c^* be the corresponding dual braid monoid. Let u be a simple element of B_c^*. Then, inside B_W, the element u can be written in the form $\mathbf{x}^{-1}\mathbf{y}$, for some $x, y \in W$.

In types A_n and B_n, the proof uses a geometric interpretation of elements of the form $\mathbf{x}^{-1}\mathbf{y}$. We had no similar description for type D_n when we wrote the paper, and conjectured the result to also hold in type D_n:

Conjecture 3.1.2 (Digne-G., 2017, [53, Conjecture 8.7]). The result of Theorem 3.1.1 also holds if W has type D_n, and thus in every finite (not necessarily irreducible) Coxeter group W.

Licata and Queffelec [104], and independently Baumeister and myself [13], later proved this conjecture.

Theorem 3.1.3 (Baumeister-G., 2017, [13, Theorem 1.1], Licata-Queffelec, 2021, [104, Theorem 5.1]). Conjecture 3.1.2 is true.

Licata and Queffelec use categorical actions of (dual) Artin groups of spherical and simply-laced types; in particular, they also give new proofs of Theorem 3.1.1 in types A and E. In the joint work with Baumeister, we used an approach similar to the one in the joint work with Digne, based on geometric properties of braids. To be more precise, in [53], the proof of Theorem 3.1.1 in type A_n uses the following geometric characterization of braids of the form $\mathbf{x}^{-1}\mathbf{y}$, which motivated the name "Mikado braids" (see Figure 1.2 below for an example).

Proposition 3.1.4 (Digne-G., 2017, [53, Theorem 5.8])**.** Let $\mathcal{B}_n$ be Artin's n-strand braid group. Let $\beta \in \mathcal{B}_n$. The following are equivalent:

1. There are $x, y \in \mathfrak{S}_n$ such that $\beta = \mathbf{x}^{-1}\mathbf{y}$,
2. There are $x, y \in \mathfrak{S}_n$ such that $\beta = \mathbf{x}\mathbf{y}^{-1}$,
3. The braid β admits a braid diagram where one can inductively remove all strands by removing at each step a strand lying above all the others.
4. The braid β is f-realizable in the sense of Dehornoy [40].

We call a braid β satisfying any of the above equivalent conditions a *Mikado braid*.

Since Artin groups of type B_n can be realized inside Artin groups of type A_{2n-1}[1], one can derive similar pictural characterizations in type B_n. They are used to prove Theorem 3.1.1 in type B_n in [53]. In type D_n, in [13] we used a realization of the Artin group of type D_n as a subgroup of index two of a suitable quotient of an Artin group of type B_n, which also allows to get a geometrical description of elements of the form $\mathbf{x}^{-1}\mathbf{y}$. This is based on an observation of Allcock, which we reformulate algebraically in the proposition below.

Recall that, if W is a Coxeter group of type B_n with simple system $S = \{s_0, s_1, \dots, s_{n-1}\}$, where $s_1, s_2, \dots, s_{n-1}$ denote the generators of the standard parabolic subgroup of type A_{n-1}, then the Coxeter group of type D_n can be realized as a (non parabolic) reflection subgroup W' of W with simple system $\{s_0 s_1 s_0, s_1, s_2, \dots, s_{n-1}\}$.

Proposition 3.1.5 (Allcock, 2002, [4, Section 4])**.** *Let W be a Coxeter group of type B_n, with simple system as above. Let $\overline{B_W}$ be the quotient of the Artin group B_W by the relation $\mathbf{s_0}^2 = 1$. Then the Artin group of type D_n is isomorphic to the index-two subgroup of $\overline{B_W}$ generated by $\{\mathbf{s_0 s_1 s_0}, \mathbf{s_1}, \mathbf{s_2}, \dots, \mathbf{s_{n-1}}\}$.*

Note that the quotient map $B_W \longrightarrow W$ factors through $\overline{B_W}$. Denote by $\overline{\pi}$ the induced map from $\overline{B_W}$ to W. We then showed the following.

Proposition 3.1.6 (Baumeister-G., 2017, [13, Theorem 3.3])**.** Let W' be the Coxeter group of type D_n, viewed inside the Coxeter group W of type B_n. View $B_{W'}$ inside $\overline{B_W}$ as in Proposition 3.1.5. Then the elements of $B_{W'}$ which are of the form $\mathbf{x}^{-1}\mathbf{y}$ (for the type D_n Artin group structure) coincide with the elements $\beta \in \overline{B_W}$ such that the following two conditions are satisfied:

- β is the image of an element of the form $\mathbf{u}^{-1}\mathbf{v}$ in B_W (for the type B_n Artin group structure) under the quotient map $B_W \twoheadrightarrow \overline{B_W}$,
- $\overline{\pi}(\beta) \in W'$.

[1] In fact, if θ denotes the unique nontrivial automorphism of Coxeter group of a Coxeter group W of type A_{2n-1}, then W^θ, which as recalled in Section 2.1 is also a Coxeter group, happens to be of type B_n. Moreover, as shown in [115, Corollary 4.4], this can be lifted at the level of Artin monoids and groups: one has $B_{W^\theta}^{(+)} \subseteq B_W^{(+)}$.

Proposition 3.1.6 allows us to deduce geometric representations of elements of the form $\mathbf{x}^{-1}\mathbf{y}$ by using such representations in type B_n, but where one is also allowed to invert some of the crossings (because of the relation $\mathbf{s}_0^2 = 1$). We use this to deduce Conjecture 3.1.2 in type D_n.

Using results of Dyer and Lehrer [63], we deduce the following from Theorems 3.1.3 and 3.1.1.

Corollary 3.1.7. Let (W, S) be a Coxeter system of spherical type and let c be a standard Coxeter element in W. The image of any simple dual braid $u \in B_c^*$ inside the Hecke algebra H_W under the composition $B_c^* \subseteq B_W \longrightarrow H_W^\times$, when expressed in Kazhdan and Lusztig's basis $\{C_w\}_{w\in W}$, has coefficients which are polynomials with nonnegative coefficients.

In type A_n, the simple elements of B_c^* yield a basis of the Temperley-Lieb quotient TL_n of H_W, and positivity properties of this basis can also be derived under suitable conventions on the quotient map (see [53, Theorem 8.16]). Proving such positivity properties was the original motivation for the above theorems.

Note that it is clear from point 3 of Proposition 3.1.4 above that every braid β satisfying the condition in point 3 admits a braid diagram where any two strands cross at most once, and hence, that the length of β with respect to the generating set $\mathbf{S} \cup \mathbf{S}^{-1}$ of $\mathcal{B}_n$ is the same as the Coxeter length of $p(\beta)$ in $\mathfrak{S}_n$, where $p : \mathcal{B}_n \longrightarrow \mathfrak{S}_n$ is the quotient map. This property is not obvious at all if one starts from a braid of the form $\mathbf{x}^{-1}\mathbf{y}$, since concatenating a word for $\mathbf{x}^{-1}$ with a word for $\mathbf{y}$ yields a word which is much longer in general that the length of the image of $\mathbf{x}^{-1}\mathbf{y}$ in $\mathfrak{S}_n$. It is also easy to convince oneself that one can in fact "lift" *any* S-reduced expression of $p(\beta)$ in a word for β, where by "lifting" we mean replace every letter s_i in the S-reduced expression by either $\mathbf{s}_i$ or $\mathbf{s}_i^{-1}$.

In spherical type, the elements which can be written in the form $\mathbf{x}^{-1}\mathbf{y}$ coincide with the elements which can be written in the form $\mathbf{x}\mathbf{y}^{-1}$, but in general the two sets are distinct. Matthew Dyer informed me at the time we were writing [53] that he had an alternative definition for braids of the form $\mathbf{x}^{-1}\mathbf{y}$ using this philosophy of "lifting" reduced expressions, holding in an arbitrary Artin group. When B_W is finite, his definition precisely yields braids of the form $\mathbf{x}^{-1}\mathbf{y}$ (equivalently $\mathbf{x}\mathbf{y}^{-1}$), but it yields many more elements when B_W is attached to an infinite Coxeter group W, strictly containing both sets of elements of the form $\mathbf{x}^{-1}\mathbf{y}$ and elements of the form $\mathbf{x}\mathbf{y}^{-1}$. His definition is presented in the next subsection.

3.1.2 Generalized Mikado braids

The following construction is borrowed from [56, Lemma 9.1]. Let (W, S) be an arbitrary Coxeter system and let $A \subseteq T$ be biclosed. Let $x \in W$ and let $s_1 s_2 \cdots s_k$ be an S-reduced expression of x. Set

$$x_A := \mathbf{s}_1^{\varepsilon_1}\mathbf{s}_2^{\varepsilon_2}\cdots\mathbf{s}_k^{\varepsilon_k} \in B_W,$$

where for all $i = 1, \ldots, k$, we set $\varepsilon_i = \begin{cases} -1 & \text{if } s_k s_{k-1}\cdots s_i \cdots s_k \in A \\ 1 & \text{otherwise.} \end{cases}$

Lifted reduced expressions as above may be considered as reduced expressions associated to a biclosed set A[2].

[2]One can define an A-twisted length function $\ell_A : W \longrightarrow \mathbb{Z}$ as in [59] by $\ell_A(x) = \ell(x) - 2|N(x^{-1}) \cap A|$. We

Lemma 3.1.8 (Dyer, [56, Sections 9.1 and 9.4])**.** *Let $A \subseteq T$ be biclosed, let $x, y \in W$. Then*

1. *The element x_A is independent of the chosen reduced expression for x.*
2. *One has $(xy^{-1})_{N(y)} = \mathbf{x}\mathbf{y}^{-1}$.*
3. *One has $(x^{-1}y)_{T\setminus N(y^{-1})} = \mathbf{x}^{-1}\mathbf{y}$.*

In particular, one has $x_\emptyset = \mathbf{x}$ and $(x^{-1})_T = \mathbf{x}^{-1}$.

Definition 3.1.9 (Generalized Mikado braids)**.** We call a braid of the form x_A a *(generalized) Mikado braid*.

When W is finite, biclosed sets of reflections coincide with inversion sets (Lemma 2.1.17), hence in type A_n (more generally for finite W) we exactly get the braids of the form $\mathbf{x}\mathbf{y}^{-1}$ (or $\mathbf{x}^{-1}\mathbf{y}$, which is equivalent), but in general there are many more such braids, and they share or are expected to share the same kind of properties as the braids of the form $\mathbf{x}^{-1}\mathbf{y}$ and $\mathbf{x}\mathbf{y}^{-1}$ (for example positivity properties)–see Example 3.1.12 below. Also note that, when W is infinite, there are braids of the form $\mathbf{x}^{-1}\mathbf{y}$ which cannot be written in the form $\mathbf{u}\mathbf{v}^{-1}$, as the following example shows.

Example 3.1.10.

1. Let W be of type $\widetilde{A}_1$, that is, $W = \langle\ s, t \mid s^2 = t^2 = 1\ \rangle$. Then B_W is a free group on two generators $\mathbf{s}, \mathbf{t}$. It follows that $\mathbf{s}^{-1}\mathbf{t}$ cannot be written in the form $\mathbf{u}\mathbf{v}^{-1}$, with $u, v \in W$.
2. Let W be of type $\widetilde{A}_2$, with simple system $S = \{s, t, u\}$. We thus have $m_{q,r} = 3$ for all $q \neq r$, $q, r \in S$. Then setting $x = s$ and $y = tut = utu$, we have that $\mathbf{x}$ and $\mathbf{y}$ have no right common multiple in B_W^+ (this is a consequence of the fact that there is no $w \in W$ such that $x \leq_S w$ and $y \leq_S w$, as for every $a \in S$, the element w would have an S-reduced expression starting by a, hence by Proposition 2.1.18 the group W would be finite).

3.1.3 A closed formula for simple dual braids

Coming back to the matter of expressing a simple dual braid in spherical type as a quotient $\mathbf{x}^{-1}\mathbf{y}$, the works summarized in Section 3.1.1 give an algorithm to obtain a pair (x, y) such that the simple dual braid β we started with is equal to $\mathbf{x}^{-1}\mathbf{y}$, not a closed formula. Note that there is not a unique pair satisfying this property in general, but it is natural to wonder if a canonical one exists. This motivated further investigation on the subject. Note that, in [53], we gave a formula to express simple dual *atoms* in terms of the classical atoms, which is crucial in Licata and Queffelec's work [104]:

Proposition 3.1.11 (Digne-G., 2017, [53, Proposition 3.13])**.** Let (W, S) be a finite Coxeter system. Let $c = s_1 \dots s_n$ be a standard Coxeter element in W. Then, through the embedding of B_c^* into B_W, taking the index i in $\mathbf{s}_i$ modulo n, we have

$$T_c = \{\mathbf{s}_1\mathbf{s}_2 \dots \mathbf{s}_i\mathbf{s}_{i+1}\mathbf{s}_i^{-1}\mathbf{s}_{i-1}^{-1} \dots \mathbf{s}_1^{-1}, \mid 0 \leq i < 2|T|\}.$$

then have $\sum_{i=1}^k \varepsilon_i = \ell_A(x)$. One can thus consider "A-twisted reduced expressions" of elements of W, living in B_W instead of W. Note that, for $A = \emptyset$, we recover the classical length function ℓ, and $x_A = \mathbf{x}$, which is simply the canonical positive lift of x. We thus recover the S-reduced expressions in this case, but viewed inside B_W.

Note that the words appearing in the set above are not reduced in B_W with respect to $\mathbf{S}\cup\mathbf{S}^{-1}$ in general, and that there are repetitions.

Dyer's framework recalled in Subsection 3.1.2 seemed more appropriate to me to try to find a closed formula to express a simple dual braid as a quotient of two positive simple braids, having in mind potential generalizations for Artin groups of non-spherical types. Some Coxeter groups of affine type indeed admit a dual braid monoid which is quasi-Garside and a dual Artin group isomorphic to the classical one [51, 52], and the same holds for universal Coxeter systems [19] and Coxeter systems of rank three [47]. For such examples of Coxeter groups of non-spherical type for which there is a dual braid monoid which is quasi-Garside, one can find simple dual braids which are not of the form $\mathbf{x}^{-1}\mathbf{y}$ or $\mathbf{x}\mathbf{y}^{-1}$ in general, but which are of the form x_A, as the next example shows, and it is tempting to conjecture that simple dual braids are always generalized Mikado braids.

Example 3.1.12. Let $W=\langle\, s_1,s_2,s_3 \mid s_1^2=s_2^2=s_3^2=1\,\rangle$. Then B_W is a free group on three generators, and by Bessis [19], it admits a dual braid monoid which is quasi-Garside. Letting $c=s_2s_3s_1$, by Hurwitz moves we find

$$(\mathbf{s}_2,\mathbf{s}_3,\mathbf{s}_1)\to(\mathbf{s}_2\mathbf{s}_3\mathbf{s}_2^{-1},\mathbf{s}_2,\mathbf{s}_1)\to(\mathbf{s}_2\mathbf{s}_3\mathbf{s}_2^{-1},\mathbf{s}_1,\mathbf{s}_1^{-1}\mathbf{s}_2\mathbf{s}_1)\to(\mathbf{s}_1,\mathbf{s}_1^{-1}\mathbf{s}_2\mathbf{s}_3\mathbf{s}_2^{-1}\mathbf{s}_1,\mathbf{s}_1^{-1}\mathbf{s}_2\mathbf{s}_1),$$

hence $\beta:=\mathbf{s}_1^{-1}\mathbf{s}_2\mathbf{s}_3\mathbf{s}_2^{-1}\mathbf{s}_1$ is a generator of the dual braid monoid B_c^*.

It cannot be of the form $\mathbf{x}^{-1}\mathbf{y}$ or $\mathbf{x}\mathbf{y}^{-1}$ since B_W is free. Nevertheless, consider the hyperplane

$$\begin{aligned} H=\mathrm{span}(\alpha_{s_1s_2s_1},\alpha_{s_1s_2s_3s_2s_1s_2s_3s_2s_1}) &= \mathrm{span}(\alpha_2+2\alpha_1, 35\alpha_1+12\alpha_2+6\alpha_3)\\ &= \mathrm{span}(\alpha_2+2\alpha_1, 11\alpha_1+6\alpha_3). \end{aligned}$$

Then $\alpha_3\notin H$ and consider the closed half-space $H^+=H+\mathbb{R}_{\geq 0}\alpha_3$ containing α_3. Set $A:=H^+\cap\Phi^+$. Then A must be biclosed. Now we have

$$\alpha_1=\underbrace{(\alpha_1+\frac{6}{11}\alpha_3)}_{\in H}-\frac{6}{11}\alpha_3\notin A,\quad s_1(\alpha_2)=\alpha_{s_1s_2s_1}\in H\cap\Phi^+\subseteq A,$$

$$s_1s_2(\alpha_3)=6\alpha_1+2\alpha_2+\alpha_3=\underbrace{2(\alpha_2+2\alpha_1)+\frac{2}{11}(11\alpha_1+6\alpha_3)}_{\in H}-\frac{1}{11}\alpha_3\notin A,$$

$$s_1s_2s_3(\alpha_2)=10\alpha_1+3\alpha_2+2\alpha_3=\underbrace{3(\alpha_2+2\alpha_1)+\frac{4}{11}(11\alpha_1+6\alpha_3)}_{\in H}-\frac{2}{11}\alpha_3\notin A,$$

$$s_1s_2s_3s_2(\alpha_1)=\alpha_{s_1s_2s_3s_2s_1s_2s_3s_2s_1}\in H\cap\Phi^+\subseteq A.$$

We thus have that $\beta=(s_1s_2s_3s_2s_1)_A$.

In spherical type, I found a formula to express any simple element of a dual braid monoid in the form x_A, relying on Reading's c-sortable elements [126].

Let (W,S) be a Coxeter system. Fix an S-reduced expression $s_1s_2\cdots s_n$ of a standard Coxeter element c. Consider the semi-infinite word $c^\infty=s_1s_2\cdots s_n|s_1s_2\cdots s_n|s_1s_2\cdots s_n|\cdots$. Let $w\in W$. The *c-sorting word* for w is the lexicographically first S-reduced expression of w

appearing as a subword of c^∞. We write $w = w_1|w_2|\cdots|w_k$, where w_i is the subword of the c-sorting word for w coming from the i-th copy of c in c^∞ and k is maximal such that this subword is nonempty. Note that each w_i defines a subset of S, consisting of those letters in the word w_i. We abuse notation and also denote this set by w_i.

Definition 3.1.13 (Coxeter sortable elements)**.** We say that w is *c-sortable* if $w_k \subseteq w_{k-1} \subseteq \cdots \subseteq w_2 \subseteq w_1$. We denote by $\mathrm{Sort}_c(W)$ the set of c-sortable elements of W.

Example 3.1.14. Let W be of type A_3, $S = \{s_1, s_2, s_3\}$ with $s_1 s_3 = s_3 s_1$. Let $c = s_3 s_1 s_2$. Then $w = s_3 s_1 s_2 | s_3 s_1$ is the c-sorting word for w. We see that w is c-sortable, with $w_1 = s_3 s_1 s_2$, $w_2 = s_3 s_1 \subseteq w_1$. The element $s_2 s_1$ is not c-sortable since if it was, we would have $s_2 s_1 = w_1$, but in c the letter s_1 appears before s_2 and they do not commute.

Reading used these elements as an intermediate set to construct a bijection between noncrossing partitions and clusters; see [126, Theorem 6.1].

Theorem 3.1.15 (Reading, 2007, [126])**.** *Let (W, S) be a finite Coxeter system. There is an (explicit) bijection $\varphi_c : \mathrm{Sort}_c(W) \longrightarrow \mathrm{NC}(W, c)$ between the set $\mathrm{Sort}_c(W)$ of c-sortable elements and the set of c-noncrossing partitions.*

Theorem 3.1.16 (G., 2020, [78, Theorem 5.8])**.** Let (W, S) be a finite Coxeter system and let c be a standard Coxeter element in W. Let β be a simple element of B_c^*, and let x denote its image in W. Then $\beta = x_A$, where $A = N(\varphi_c^{-1}(x^{-1}c))$.

For $x \in \mathrm{NC}(W, c)$, the element $x^{-1}c \in \mathrm{NC}(W, c)$ is the *Kreweras complement* of x.

The above theorem gives a new proof that simple dual braids are generalized Mikado braids (using Lemma 3.1.8), and an explicit, uniform formula to express a simple dual braid β as a generalized Mikado braid, hence as a product of minimal length of the elements of $\mathbf{S} \cup \mathbf{S}^{-1}$. The proof is combinatorial, and takes advantage of the remarkable recursive properties of c-sortable elements from [126, Lemmas 2.4 and 2.5]. Unfortunately the proof still requires a technical lemma on c-sortable elements which I could only prove using a technical case-by-case analysis (see Lemma 3.1.18 below). To establish this lemma in [78], we give an explicit description of the inverse φ_c^{-1} of Reading's map φ_c in the classical types in terms of the combinatorial noncrossing partition models, which may be of independent interest.

3.1.4 Open problems

Given a Coxeter group W which is not necessarily of spherical type and a choice of standard Coxeter element c, one can still define a dual braid monoid B_c^* by dual braid relations and the corresponding dual Artin group, but

- There are examples where B_c^* is not quasi-Garside [113], because of the failure of the lattice property of $[1, c]_T$,
- It is not known in general if B_c^* is cancellative,
- It is not known in general if $G(B_c^*)$ is isomorphic to B_W.

Nevertheless, there are no known counterexamples to the last two properties which are conjectured to hold in general, and in some cases, the isomorphism between $G(B_c^*)$ and B_W could be established even without quasi-Garsideness [114, 120]. The last two properties are known to hold for all the Coxeter groups of affine type [114, 120], universal type [19], but also all Coxeter systems of rank 3 [47] (in those two last types we always have quasi-Garsideness).

Problem 3.1.17. Find a uniform proof of the fact that simple dual braids are generalized Mikado braids.

The formula given in [78] is uniform, but there is a technical lemma on c-sortable elements for which we only have a case-by-case proof:

Lemma 3.1.18 (G., 2020, [78, Lemma 4.6])**.** Let c be a standard Coxeter element. Let $x \in \mathrm{NC}(W, c)$, and let $y := x^{-1}c$. Let s be initial in c, that is, we have $\ell(sc) < \ell(c)$. Then $x^{-1}sx$ lies in $N(y)$ if and only if it lies in $N(\varphi_c^{-1}(y)$.

Finding a uniform proof of this lemma would thus be enough to solve Problem 3.1.17.

3.2 Hurwitz action in finite Coxeter groups

3.2.1 A characterization of Hurwitz transitivity

Let (W, S) be a Coxeter system with set of reflections T. Let $w \in W$ and consider the Hurwitz action of the braid group $\mathcal{B}_k$, where $k = \ell_T(w)$, on the set $\mathsf{Red}_T(w)$ of T-reduced expressions of w. Recall that the action of the standard Artin generator σ_i is given by

$$\sigma_i : (t_1, t_2, \ldots, t_k) \mapsto (t_1, \ldots, t_{i-1}, t_{i+1}, t_{i+1}t_it_{i+1}, t_{i+2}, \ldots, t_k).$$

The following property is an important ingredient in the construction of dual braid monoids [18]. It allows one to show that the natural map $B_W \longrightarrow G(B_c^*)$ mapping every standard generator $\mathbf{s}$ to s_c is surjective. Bessis works for finite W, but this property can be shown for an arbitrary Coxeter group W.

Recall that, for a standard Coxeter element c in an arbitrary Coxeter system, we have $\ell(c) = \ell_T(c) = n = |S|$ (this can be seen for instance using [60, Theorem 1.1]).

Theorem 3.2.1 (Igusa-Schiffler, 2010, [96, Theorem 1.4])**.** *Let c be a standard Coxeter element in an arbitrary Coxeter system (W, S) of rank n. Then the action of $\mathcal{B}_n$ on $\mathrm{Red}_T(c)$ is transitive.*[3]

From the point of view of dual Coxeter systems, Theorem 3.2.1 can be seen as a kind of *dual Matsumoto property.* Indeed, it says that any two T-reduced expressions of a Coxeter element c can be related by applying a sequence of dual braid relations. Unfortunately, replacing c by another element of W, the Hurwitz action is not transitive in general, as the following basic example shows.

[3] We give a few remarks about authorship of Theorem 3.2.1. An alternative proof of Theorem 3.2.1 later appeared in [12]. For finite Coxeter groups, Bessis shows it in [18, Proposition 1.6.1]. Still in the finite case, an earlier unpublished proof appeared in a letter of Deligne to Looijenga in 1974 [46], where Deligne explains that the proof was communicated to him by Tits and Zagier.

Example 3.2.2. Let W be of type B_2, with generators s, t. Let $w = w_0 = stst$ be the longest element of W. Then $\ell_T(w) = 2$, and there are two Hurwitz orbits with two reduced expressions each $(s, tst) \leftrightarrow (tst, s)$, $(t, sts) \leftrightarrow (sts, t)$.

Together with Baumeister, Roberts and Wegener, we obtained the following characterization for finite Coxeter groups. This is the main result of [14].

Theorem 3.2.3 (Baumeister-G.-Roberts-Wegener, 2017, [14, Theorem 1.1])**.** Let (W, S) be a finite Coxeter system. Let $w \in W$. The Hurwitz action on $\mathsf{Red}_T(w)$ is transitive if and only if there is a T-reduced expression of w such that the reflections in this expression generate a parabolic subgroup of W.

Our proof was case-by-case for one direction. Recently Wegener and Yahiatene found a uniform proof of this direction for finite Weyl groups [142, Theorem 1.4].

Definition 3.2.4. An element satisfying the equivalent assertions of Theorem 3.2.3 is called a *parabolic quasi-Coxeter element*. If the parabolic subgroup generated is the whole group W, then the element is said to be a *quasi-Coxeter element*.

Note that there are quasi-Coxeter elements which fail to be Coxeter elements, as the following example shows.

Example 3.2.5. Let W be of type D_4 with $S = \{s_0, s_1, s_2, s_3\}$, where s_2 commutes with no other simple reflection. Then the element $w = s_1(s_1 s_2 s_1)(s_2 s_0 s_2)s_3$ has reflection length 4, and $\{s_1, s_1 s_2 s_1, s_2 s_0 s_2, s_3\}$ generates W, hence w is a quasi-Coxeter element. One can check that w has no T-reduced expression which yields a simple system for D_4, hence it is not a Coxeter element.

3.2.2 Generalized cycle decomposition

In [77], I showed that parabolic quasi-Coxeter elements of finite Coxeter groups are precisely those elements admitting a generalization of the cycle decomposition in the symmetric group (in the symmetric group, every element is a parabolic quasi-Coxeter element).

Theorem 3.2.6 (G., 2017, [77, Theorem 1.3])**.** Let (W, S) be a finite Coxeter system. Let $w \in W$ be a parabolic quasi-Coxeter element. There exists a (unique up to the order of the factors) decomposition $w = x_1 x_2 \cdots x_m$, $x_i \in W$ such that

1. $x_i x_j = x_j x_i$ for all $i, j = 1, \dots, m$,
2. $\ell_T(w) = \ell_T(x_1) + \ell_T(x_2) + \cdots + \ell_T(x_m)$,
3. Each x_i is indecomposable, i.e., admits no nontrivial decomposition $x_i = uv$ where $u, v \in W$ with $uv = vu$ and $\ell_T(u) + \ell_T(v) = \ell_T(x_i)$.

Elements which fail to be parabolic quasi-Coxeter elements still admit a decomposition satisfying the above three properties, but it is not unique in general.

If W is infinite, one can still consider elements admitting a T-reduced expression which generates a parabolic subgroup, and obtain a version of the above result with a slightly different formulation for point 3 (see [77, Proposition 1.2]).

3.2.3 Open problems

The question of the transitivity of the Hurwitz action is of particular importance for the understanding of the dual braid monoid B_c^*. While Hurwitz transitivity on T-reduced expressions fails for arbitrary elements, it would be helpful to solve the following.

Problem 3.2.7. Let (W, S) be an arbitrary Coxeter system. Let c be a Coxeter element and let $w \in W$ such that $w \leq_T c$. Is the Hurwitz action transitive on $\mathsf{Red}_T(w)$?

For finite Coxeter groups, this holds true. To solve the above problem, it would be enough to solve the following one.

Problem 3.2.8. Let (W, S) be an arbitrary Coxeter system. Let $w \in W$ such that $w \leq_T c$. Let c be a Coxeter element and let $w \in W$ such that $w \leq_T c$. Let $W_w := \langle\, t \mid t \leq_T w \,\rangle$, which is a Coxeter group by 2.1.20. Is w a Coxeter element in W_w?

For finite Coxeter groups, see [18, Proposition 1.6.1], or [53, Corollary 3.6].

In the finite case, and still having in mind the construction of Garside interval groups, the following also seems to be open.

Problem 3.2.9. Let (W, S) be a (finite) Coxeter system. Can we characterize those $w \in W$ such that $[1, w]_T$ forms a lattice?

The known cases include:

- (Parabolic) Coxeter elements of finite Coxeter groups [18] (see also [127, 28], for uniform proofs of the lattice property),
- Elements $w \in W$ such that $\ell_T(w) = 3$ (see Theorem 5.2.4 below),

Chapter 4

Soergel bimodules

List of relevant publications

1. T. Gobet, *Twisted filtrations of Soergel bimodules and linear Rouquier complexes*, J. Algebra **484** (2017), 275–309.

2. T. Gobet and A.-L. Thiel, *On generalized categories of Soergel bimodules in type A2*, C. R. Math. Acad. Sci. Paris **356** (2018), no. 3, 258–263.

3. T. Gobet and A.-L. Thiel, *A Soergel-like category for complex reflection groups of rank one*, Math. Z. **295**, (2020), 643-665.

This chapter collects works realized between 2015 and 2018, all of which address properties of Soergel bimodules (see Subsection 2.7.1 above for a short introduction).

Section 4.1 below describes results from the first paper above, where generalizations of the nonnegativity of Kazhdan-Lusztig and inverse Kazhdan-Lusztig polynomials were obtained, as a corollary of Soergel's conjecture (Conjecture 2.7.2) established in [65]. These generalizations were conjectured by Dyer [55]. Subsection 4.1.1 explains how the generalization of the nonnegativity of ordinary Kazhdan-Lusztig polynomials can be obtained by using twisted filtrations of Soergel bimodules, where the "twisting" comes from a twisted Bruhat order. Subsection 4.1.2 explains how the nonnegativity of inverse Kazhdan-Lusztig polynomials is obtained, by proving suitable properties of the minimal Rouquier complexes of braids of the form $\mathbf{x}\mathbf{y}^{-1}$ or $\mathbf{x}^{-1}\mathbf{y}$ which already appeared in the previous chapter, and then taking the Euler-Poincaré characteristic of the complexes. It thus makes use of the categorical action of an arbitrary Artin group B_W on the bounded homotopy category of Soergel bimodules, considered by Rouquier [133, 132] (recalled in Subsection 2.7.2).

Section 4.2 describes analogues or extensions of categories of Soergel bimodules in two particular cases. The first case (corresponding to the second paper above) is in type A_2, where instead of taking one generating bimodule per simple reflection, we take one generating bimodule per reflection. The second one (corresponding to the last paper above) is an analogue of a category of Soergel bimodules for (finite) cyclic groups, viewed as finite complex reflection groups of rank 1.

4.1 Positivity properties

4.1.1 Twisted filtrations of Soergel bimodules

For an arbitrary Coxeter system (W, S) and a biclosed set of reflections $A \subseteq T$, recall the elements x_A defined in Subsection 3.1.2, which we call "generalized Mikado braids".

Lemma 4.1.1 (Twisted standard bases of Hecke algebras)**.** *Let (W, S) be a Coxeter system with Iwahori-Hecke algebra H_W. Let $A \subseteq T$ be biclosed, and let $T_{x,A} := \varphi(x_A)$, where $\varphi : B_W \longrightarrow H_W^\times$ is the group homomorphism mapping* $\mathbf{s}$ *to T_s, $\forall s \in S$. Then $\{T_{x,A}\}_{x\in W}$ is a basis of H_W.*

Proof. By definition of x_A, expanding $T_{x,A}$ in the standard basis $\{T_w\}_{w\in W}$ yields a linear combination of the form $v^{n_x} T_x + \sum_{y<x} \alpha_y T_y$, since $T_s^{-1} = v^2 T_s + v^2 - 1$ for all $s \in S$. Here $\leq$ denotes the strong Bruhat order on W. This implies that there is an upper-triangular matrix with invertible coefficients on the diagonal allowing one to pass from $\{T_x\}_{x\in W}$ to $\{T_{x,A}\}_{x\in W}$, hence that the latter is a basis. □

Example 4.1.2. By Lemma 3.1.8, we have the following:

- The basis $\{T_{x,\emptyset}\}_{x\in W}$ is the standard basis,
- The basis $\{T_{x,T}\}_{x\in W}$ is the costandard basis $\{T_x^{-1}\}_{x\in W}$,
- The basis $\{T_{x,N(y)}\}_{x\in W}$ is the basis $\{T_x T_y^{-1}\}_{x\in W}$,
- The basis $\{T_{x,T\setminus N(y)}\}_{x\in W}$ is the basis $\{T_x^{-1} T_y\}_{x\in W}$.

In his thesis, Dyer conjectured the following.

Conjecture 4.1.3 (Dyer, 1987, [55, §7.16])**.** *Let (W, S) be a Coxeter system. The following properties are verified*

$$C'_x T_y \in \sum_{z\in W} \mathbb{Z}_{\geq 0}[v^{\pm 1}] T_z, \text{ for all } x, y \in W \tag{P_1}$$

$$T_x^{-1} T_y \in \sum_{z\in W} \mathbb{Z}_{\geq 0}[v^{\pm 1}] C_z, \text{ for all } x, y \in W, \tag{P_2}$$

$$C'_x C'_y \in \sum_{z\in W} \mathbb{Z}_{\geq 0}[v^{\pm 1}] C'_z, \text{ for all } x, y \in W, \tag{P_3}$$

$$C'_x C_y \in \sum_{z\in W} \mathbb{Z}_{\geq 0}[v^{\pm 1}] C_z, \text{ for all } x, y \in W. \tag{P_4}$$

Note that (P_1) at $y = 1$ is nothing but the positivity of ordinary Kazhdan-Lusztig polynomials. It was known for finite and affine Weyl groups [99] when Dyer formulated the conjectures above, and more generally for crystallographic groups [105, §3.2.1(a)]. One interprets the coefficients as Poincaré polynomials of a localization of the intersection cohomology of a Schubert variety at a Schubert cell in this case. Similarly (P_3) was known for finite Weyl groups [138, Corollaire 2.14], and more generally crystallographic groups [105, §3.2.1], also via geometric methods.

Dyer also showed the following.

Proposition 4.1.4 (Dyer, 1987, [55, Prop. 7.17]). *If (W, S) is finite, then (P_1) is equivalent to (P_2), and (P_3) is equivalent to (P_4).*

He also showed that all four properties are verified for universal Coxeter systems, using combinatorial arguments:

Theorem 4.1.5 (Dyer, 1987, [55, Chapters 8 and 9]). *Conjecture 4.1.3 is verified for universal Coxeter systems.*

Dyer and Lehrer then showed using geometric techniques that (P_1) (and hence (P_2)) holds true in any finite Weyl group:

Theorem 4.1.6 (Dyer-Lehrer, 1990, [63, Theorem 2.8]). *(P_1) is true for all finite Weyl groups.*

Also note that Grojnowski and Haiman generalized (P_1) to Weyl groups of symmetrizable Kac-Moody algebras, also using geometric techniques [90].

Combining the above results, one gets in particular that all four properties are valid for finite Weyl groups and universel Coxeter systems. For dihedral groups, they also hold true and are straightforward to check.

The general case remained mysterious for many years, due the lack of geometric tools in the case of arbitrary Coxeter groups (intersection cohomology of Schubert varieties).

Then Soergel's approach [136, 137] by means of a monoidal category of graded bimodules over a polynomial ring defined using only a Coxeter group together with a representation fulfilling certain properties gave new hope for a general proof (see Section 2.7.1 above). Soergel's Conjecture 2.7.2 was proven by Elias and Williamson [65].

Theorem 4.1.7 (Elias-Williamson, 2014, [65]). *Soergel's conjecture is true. As a corollary, (P_1) at $y = 1$ and (P_3) are true in an arbitrary Coxeter system (W, S). Moreover, (P_2) at $x = 1$ is true in an arbitrary Coxeter system (W, S).*

The positivity of Kazhdan-Lusztig polynomials is obtained, following the approach developed by Soergel, by interpreting the Kazhdan-Lusztig polynomials as graded multiplies of certain filtrations of indecomposable Soergel bimodules. Namely, thanks to Soergel's Conjecture 2.7.2, the indecomposable Soergel bimodule B_w corresponds, under the isomorphism of Theorem 2.7.1, to the element C'_w of the canonical basis. As shown by Soergel, for any choice of total order refining the Bruhat order, the bimodule B_w admits a canonical filtration by bimodules of the form $\{R_x\}_{x \in W}$ [137], the *support filtration.* Up to some technical renormalizations, the polynomial $h_{x,w}$ is given by the graded multiplicity of R_x in this filtration. The multiplicities turn out to be independent of the chosen total order refining the Bruhat order.

Property (P_3) is also immediate from Soergel's conjecture, obtained by decomposing a bimodule of the form $B_x \otimes_R B_y$ into a direct sum of indecomposables, all of which are isomorphic to (shifts of) B_z's.

Property (P_2) at $x = 1$, sometimes called *inverse Kazhdan-Lusztig positivity*, is obtained in a less direct corollary of Soergel's conjecture in [65, Section 6] by taking the Euler-Poincaré characteristic of a complex of Soergel bimodules categorifying the element T_y in the bounded homotopy category of Soergel bimodules, following the framework developed by Rouquier to categorify Artin groups (see Section 2.7.2).

It is tempting to rewrite (P_1) as

$$C'_x \in \sum_{z \in W} \mathbb{Z}_{\geq 0}[v^{\pm 1}] T_z T_y^{-1}, \text{ for all } x, y \in W, \tag{P'_1}$$

since $C'_x T_y$ is not the class of an object of Soergel's category (except for $y = 1$). We thus replace the standard basis by the basis $\{T_x T_y^{-1}\}_{x \in W}$, and this suggests to "twist" Soergel's support filtrations by considering total orders refining a *twisted* Bruhat order, defined by

$$x \leq_y z \Leftrightarrow xy \leq zy. \tag{4.1.1}$$

Note that y^{-1} is the unique minimal element for this order. For $x \in W$ and $t \in T$, one has

$$\begin{aligned} x <_y xt &\Leftrightarrow xy < xty \Leftrightarrow y^{-1}ty \notin N(y^{-1}x^{-1}) \Leftrightarrow y^{-1}ty \notin (N(y^{-1}) + y^{-1}N(x^{-1})y) \\ &\Leftrightarrow t \notin (N(y) + N(x^{-1})). \end{aligned}$$

Following [59], the equivalence $x <_y xt \Leftrightarrow t \notin (N(y) + N(x^{-1}))$ suggests to generalize these orders, by replacing $N(y)$ by a biclosed set of roots A, and defining $x <_A xt$ if and only if $t \notin (A + N(x^{-1}))$, and then taking the transitive closure of the relation $<_A$. We denote such a preorder by $\leq_A$. Note that one could technically take any set of positive roots A, but it is not clear that $\leq_A$[1] defines a partial order. Letting

$$\ell_A : W \longrightarrow \mathbb{Z}, w \mapsto \ell(w) - 2|N(w^{-1}) \cap A|,$$

we have the following theorem of Edgar.

Theorem 4.1.8 (Edgar, 2007, [64, Theorem 2.3])**.** *Let $A \subseteq T$. The following are equivalent*

1. *$\leq_A$ is a partial order,*
2. *A is biclosed,*
3. *$\ell_A(xt) < \ell_A(x)$ for all $x \in W$, $t \in (N(x^{-1}) + A)$.*

Example 4.1.9. While it is clear from (4.1.1) that the poset $(W, \leq_y)$ is isomorphic to $(W, \leq)$, the situation can be quite different in the more general setting involving biclosed sets of roots. As an easy example, let $W = \langle\, s, t \,\rangle$ be an infinite dihedral group with $S = \{s, t\}$. Then the set $A := \{\alpha_r \mid r \in \{s, sts, ststs, \dots\}\}$ is biclosed, but neither finite nor cofinite. One has

$$\dots \; <_A tsts <_A sts <_A ts <_A s <_A e <_A t <_A st <_A tst <_A stst <_A \dots,$$

that is, $\leq_A$ is a total order in that case.

This suggests to consider an even more general version of (P'_1) involving the bases from Lemma 4.1.1, given by

$$C'_x \in \sum_{z \in W} \mathbb{Z}_{\geq 0}[v^{\pm 1}] T_{z,A}, \text{ for all } x \in W \text{ and } A \subseteq T \text{ biclosed.} \tag{P_1^{gen}}$$

[1]This partial order is different from the partial order also denoted $\leq_A$ from Section 2.5.

As a corollary of Elias and Williamson's work [65], one can obtain (P_1^{gen}) (and thus (P_1'), which is equivalent to (P_1')), by generalizing Soergel's support filtrations: instead of taking linear extensions of $\leq$, one takes linear extensions of $\leq_A$, and one can show that support filtrations still exist. This is shown in [75, Section 4.2], and one then shows the following, where for $B \in \mathcal{B}$, we denote by $[B : \Delta_x^A(i)]_A$ the graded-multiplicity of $\Delta_x^A := R_x(-\ell_A(x))$ in an A-twisted support filtration of B.

Theorem 4.1.10 (G., 2017, [75, Theorem 4.10])**.** Let (W, S) be an arbitrary Coxeter system. Let $w \in W$ and $A \subseteq T$ be biclosed. Write $C'_w = \sum_{x \in W} h^A_{x,w} T_{x,A}$. Then

$$h^A_{x,w} = \sum_{i \in \mathbb{Z}} [B_w : \Delta_x^A(i)]_A v^{i+\ell_A(x)}.$$

In particular the generalized Kazhdan-Lusztig polynomials $h^A_{x,w} \in \mathbb{Z}[v, v^{-1}]$ have nonnegative coefficients, and (P_1^{gen}) is true for arbitrary Coxeter systems.

As pointed out to me by Matthew Dyer, one can even derive the more general property that

$$C'_w T_{y,A} \in \sum_{x \in W} \mathbb{Z}_{\geq 0}[v^{\pm 1}] T_{x,A}, \text{ for all } w, y \in W \text{ and biclosed } A \subseteq T,$$

see [75, Corollary 4.16].

4.1.2 Linearity of Rouquier complexes

The aim of this subsection is to explain how to derive (P_2) from the results in [65] in the case of an arbitrary Coxeter system, that is,

$$T_x^{-1} T_y \in \sum_{z \in W} \mathbb{Z}_{\geq 0}[v^{\pm 1}] C_z, \text{ for all } x, y \in W.$$

A first observation is that, if W is infinite, there are in general elements of the form $T_x^{-1} T_y$ which cannot be written under the form $T_u T_v^{-1}$ (see Example 3.1.10 above[2]). Nevertheless, it is clear that it suffices to obtain (4.1.2) above to obtain the positivity of the expansion of elements of the form $T_u T_v^{-1}$, since it suffices to apply the bar involution to (4.1.2) to obtain

$$T_{x^{-1}} T_{y^{-1}}^{-1} \in \sum_{z \in W} \mathbb{Z}_{\geq 0}[v^{\pm 1}] C_z, \text{ for all } x, y \in W. \qquad (\overline{P_2})$$

In fact, in order to get a positivity statement that would be the exact "inverse" statement of the positivity statement obtained in Theorem 4.1.10, namely (P_1^{gen}), it would be natural to generalize the second property to

$$T_{x,A} \in \sum_{z \in W} \mathbb{Z}_{\geq 0}[v^{\pm 1}] C_z, \text{ for all } x \in W \text{ and } A \subseteq T \text{ biclosed.} \qquad (P_2^{\mathrm{gen}})$$

[2] Example 3.1.10 shows such a phenomenon inside B_W, not inside H_W. Nevertheless, the group homomorphism $B_W \longrightarrow H_W^\times$ is conjectured to always be injective, and is known to be injective for universal Coxeter groups [102], hence this applies at least to Example 3.1.10(1).

Unfortunately, we were not able to establish a statement as general as (P_2^{gen}), but we can derive (P_2) (and thus $(\overline{P_2})$ as well) by adapting the techniques developed in Elias and Williamson [65] to show inverse Kazhdan-Lusztig positivity, that is, (P_2) at $x = 1$.

A difficulty encountered here is that there is no object in $\mathcal{B}$ categorifying $T_{x,A}$ (except for $x = 1$), hence one cannot expect to obtain (P_2) in the same spirit as (P_1^{gen}) by means of filtrations on an object in $\mathcal{B}$. But since the $T_{x,A}$'s are in the image of the map $B_W \twoheadrightarrow H_W^\times$, Rouquier's framework briefly recalled in Subsection 2.7.2 appears as the natural tool to consider here. The crucial property at the categorified level from what one can derive (P_2) is the following.

Theorem 4.1.11 (G., 2017, [75, § 6])**.** Let (W, S) be arbitrary and let $\beta = \mathbf{x}^{-1}\mathbf{y}$. Then

1. We have $F_\beta \in K^{\geq 0}(\mathcal{B}) \cap K^{\leq 0}(\mathcal{B})$, that is, the minimal Rouquier complex of β is *perverse* (or *linear*).
2. The i-th cohomological degree ${}^iF_\beta^{\min}$ of $F_\beta^{\min}$ is a Soergel bimodule whose indecomposable summands all have the form $B_v(i)$, where $\ell(v) - \ell(p(\beta))$ and i have the same parity.

Point (2) shows that for a fixed $v \in W$, the B_v's occurring as indecomposable summands in the cohomological degrees of $F_\beta^{\min}$ are either all in odd degrees, or all in even degrees. This is crucial to prove (P_2), which is obtained by taking the Euler-Poincaré caracteristic of $F_\beta^{\min}$. One thus gets an alternating sum by expressing $T_x^{-1}T_y$ in the basis $\{C'_w\}_{w\in W}$ which is the one corresponding to indecomposable Soergel bimodules. Expressing it in $\{C_w\}_{w\in W}$ yields (P_2):

Corollary 4.1.12 (G., 2017, [75, §6.1])**.** Property (P_2) is true in an arbitrary Coxeter system.

Together with results from Section 4.1.1, this shows that (P_1) to (P_3) are verified for arbitrary (W, S). We do not know how to prove (P_4) in general, and list it as Problem 4.1.16 below.

4.1.3 Open problems

A first problem to settle would be to have the inverse positivity in general, that is, to prove (P_2^{gen}) for arbitrary Coxeter systems.

Problem 4.1.13. Let (W, S) be an arbitrary Coxeter system. Show that $T_{x,A} \in \sum_{z\in W} \mathbb{Z}_{\geq 0}[v^{\pm 1}]C_z$, for all $x \in W$ and all biclosed $A \subseteq T$.

A more precise program to achieve this is given in [75, Conjecture 6.10]. More than the positivity results for themselves, it is often the property obtained at the categorical level that is of interest. Indeed, as explained in the previous section, property (P_2) is obtained by taking the Euler-Poincaré characteristic of the Rouquier complex $F_\beta^{\min}$ of the braid $\beta = \mathbf{x}^{-1}\mathbf{y}$ (or $\mathbf{x}\mathbf{y}^{-1}$), and the positivity is obtained as a corollary of the fact that $F_\beta^{\min}$ is *perverse* or *linear*: the grading shift of any indecomposable bimodule occurring in the complex coincides with its cohomological degree. Several conjectures can be made in this direction.

Conjecture 4.1.14. *If a braid β has its Rouquier complex F_β minimal, then it is a generalized Mikado braid.*

This would be very strong, as it would imply the (conjectured) faithfulness of Rouquier's categorical action of B_W on the bounded homotopy category of Soergel bimodules. Indeed, let β be a braid such that the action of $F_\beta \otimes (-)$ on $K^b(\mathcal{B})$ is trivial. Then its minimal Rouquier complex is the base ring R concentrated in homological degree 0. It is linear, hence β is Mikado. But the only Mikado braid acting trivially is the trivial braid (this can be seen already at the decategorified level, as every nontrivial Mikado braid has a nontrivial image in H_W.)

Another indication that this is extremely hard is that one has algorithms to determine the minimal Rouquier complex of a given braid, hence a proof of the faithfulness would imply that the word problem in an arbitrary Artin group is solvable, another very hard open problem.

One could also conjecture the more extrem property below.

Conjecture 4.1.15. *Assume that a braid $\beta \in B_W$ has a positive expansion on the basis $\{C_w\}_{w\in W}$ of H_W. Then β is a generalized Mikado braid.*

This would imply faithfulness *at the decategorified level*, i.e., this would imply that the group homomorphism $B_W \longrightarrow H_W^\times$ is injective (which is open for $W = \mathfrak{S}_n$, $n \geq 4$, and when $n = 4$, it is equivalent to the faithfulness of the Burau representation at $n = 4$, one of the biggest open problems in quantum topology...). Injectivity is known only for dihedral and universal Coxeter systems, thanks to work of Lehrer and Xi [102].

Note that, considering the action of B_W for a spherical, simply-laced Coxeter system W on the bounded homotopy category of projective modules over the zigzag algebra, Licata and Queffelec have shown that in this setting, a braid is Mikado if and only if its braid complex is linear [104]. This does not imply Conjecture 4.1.14 in these specific cases, as the triangulated category that they consider is only a quotient of Rouquier's category.

Resuming with positivity properties, the following seems to remain to be shown.

Problem 4.1.16. Show (P_4) for an arbitrary Coxeter system.

4.2 Extended and generalized Soergel categories in some particular cases

4.2.1 Type A_2

The aim of this subsection is to describe an extended category of Soergel bimodules in type A_2, where one generator per reflection is taken.

Specifically, given a Coxeter system (W, S) with set of reflections T, one can define the R-bimodule $B_t := R \otimes_{R^t} R$, where $R^t = \{r \in R \mid t(r) = r\}$, for every $t \in T$, not only for $t \in S$. Note that for a non-simple reflection $t \in T\backslash\{S\}$, this B_t is *not* the same as the Soergel bimodule indexed by t, also denoted B_t. We will not make use of the latter in this section.

Consider the monoidal, additive, graded, Karoubian category $\mathcal{B}^T$ generated by the B_t, $t \in T$. Then since $S \subseteq T$, it contains Soergel's category $\mathcal{B}$ as a full subcategory. One would like to understand the category $\mathcal{B}^T$.

In type A_2, this was achieved in joint work with A.-L. Thiel. Denoting by $A(W_{A_2})$ the split Grothendieck ring of the category $\mathcal{B}^T$, we have the following. Denote by t_1, t_2, t_3 any enumeration of the three reflections in A_2.

Theorem 4.2.1 (G.-Thiel, 2018, [86, Proposition 3.1 and Theorem 4.1])**.** We have

1. Up to isomorphism and grading shifts, there are 20 indecomposable objects in the category $\mathcal{B}^T$; they are given by $\mathcal{O}(A)$, where $\mathcal{O}(A)$ denotes the algebra of regular functions on the union of twisted graphs $\bigcup_{x\in A}\{(xv, v) \mid v \in V\}$, and A is either $\{1\}$ or a nonempty set stable by left multiplication by a reflection. The algebra $A(W_{A_2})$ is thus a free $\mathcal{A}$-module of rank 20.

2. The algebra $A(W_{A_2})$ admits a presentation as $\mathcal{A}$-algebra with generators C_i, $i = 1, 2, 3$ and relations

   (a) $C_i^2 = (v + v^{-1})C_i, \ \forall i = 1, 2, 3,$

   (b) $C_iC_jC_i + C_j = C_i + C_jC_iC_j, \ \forall i \neq j, i, j \in \{1, 2, 3\},$

   (c) $C_iC_jC_i = C_iC_kC_i, \ \text{if } \{i, j, k\} = \{1, 2, 3\},$

   (d) $C_iC_jC_kC_i = C_iC_kC_jC_i, \ \text{if } \{i, j, k\} = \{1, 2, 3\}.$

   For all i we have $C_i = \langle B_{t_i}\rangle$ and $\langle R(1)\rangle = v$.

Remark 4.2.2. The two relations (a) and (b) above are the defining relations, in the Kazhdan-Lusztig generators, of the affine Hecke algebra of type $\widetilde{A}_2$. The algebra $A(W_{A_2})$ is thus a quotient of $H_{\widetilde{A}_2}$.

One can also consider the category $\mathcal{B}^{\text{ext}}$ generated, in the same sense as above, by the $B_s, s \in S$, and $R_w, w \in W$, where we recall that R_w is the R-bimodule equal to R as a left R-module, but with right operation twisted by w (also isomorphic to $\mathcal{O}(\{w\})$). This category in fact contains $\mathcal{B}^T$ as a full subcategory, since whenever $t = wsw^{-1}$, $w \in W$, $s \in S$, one has

$$B_t \cong R_w \otimes_R B_s \otimes_R R_{w^{-1}} \in \mathcal{B}^{\text{ext}}.$$

One checks that, in type A_2, there are (up to grading shifts) 5 more indecomposable bimodules in $\mathcal{B}^{\text{ext}}$ than in $\mathcal{B}^T$, given by the R_w for $w \neq 1$ [86, Lemma 3.3]. The split Grothendieck ring of $\mathcal{B}^{\text{ext}}$ is thus an $\mathcal{A}$-algebra which is a free $\mathcal{A}$-module of rank 25. It is easy to derive a presentation of this algebra which we denote $A(W_{A_2})^{\text{ext}}$ from the results in [86].

Proposition 4.2.3 (G.-Thiel, 2018, [86])**.** The algebra $A(W_{A_2})^{\text{ext}}$ admits a presentation with generators C_i, D_i, $i = 1, 2, 3$, and relations

1. $C_i^2 = (v + v^{-1})C_i, \ \forall i = 1, 2, 3,$

2. $C_iC_jC_i + C_j = C_i + C_jC_iC_j, \ \forall i \neq j,$

3. $C_iD_j = D_jC_k, \ \text{if } \{i, j, k\} = \{1, 2, 3\}$, and $C_iD_i = C_i = D_iC_i$ for all $i = 1, 2, 3$.

4. The D_i's satisfy the defining relations of W in the generating set T, where D_i corresponds to t_i.

For all i we have $C_i = \langle B_{t_i}\rangle$, $D_i = \langle R_{t_i}\rangle$, and $\langle R(1)\rangle = v$.

What makes the type A_2 easily accessible is that every indecomposable object in either category $\mathcal{B}^T$ or category $\mathcal{B}^{\text{ext}}$ has the form $\mathcal{O}(A)$ for some $A \subseteq W$ (in particular, every indecomposable object is cyclic as an R-bimodule). This fails for types B_2 and A_3 (see [86, Section 5]), and we were not able to give a presentation of the split Grothendieck group of either the category $\mathcal{B}^{\text{ext}}$ or $\mathcal{B}^T$ for irreducible W of type different from A_1 or A_2. It is not even clear that the algebras obtained as split Grothendieck rings in other cases are of finite rank as $\mathcal{A}$-modules.

4.2.2 A Soergel-like category for cyclic groups

Let W be a finite complex reflection group. Most of the definitions given to build Soergel bimodules in the real case can still be carried out for complex reflection groups. For instance, given $t \in \text{Ref}(W)$, we can still consider the graded R-bimodule $R \otimes_{R^t} R$, where $R = S(V^*)$. It is nevertheless unclear that this is the right object to associate to a reflection of W, and that the corresponding Soergel-like category is well-behaved.

First of all, if one wishes to define a category of Soergel bimodules for finite complex reflection groups, it would be natural to expect recovering the category $\mathcal{B}$ when the group is real. On one hand, taking the whole set $\text{Ref}(W)$ as set of generators would yield, in type A_2, the category $\mathcal{B}^T$ described in the previous section, which is too big. On the other hand, there are no natural choices of "simple systems" in general for finite complex reflection groups, and one could also argue that if one wishes to see A_2 as a reflection group, the category $\mathcal{B}^T$ is more natural than the category $\mathcal{B}$.

Secondly, the definition of the Soergel bimodule attached to a reflection t as $R \otimes_{R^t} R$ does not seem to be the right object when t has order greater than two. Indeed, in this case one can show that $R \otimes_{R^t} R \cong \mathcal{O}(\langle t \rangle)$, and as a consequence, the tensor product of $R \otimes_{R^t} R$ with itself decomposes as a direct sum of $|\langle t \rangle|$ (shifted) copies of $R \otimes_{R^t} R$. Hence the monoidal category generated by such a bimodule has a Grothendieck ring which is a free $\mathcal{A}$-module of rank two, while one would expect to have at least $|\langle t \rangle|$ nonisomorphic indecomposable objects (up to shift) in the category.

In the real case, one has $B_t = R \otimes_{R^t} R \cong \mathcal{O}(\{1, t\})$, but as noticed above, when t has order greater than 2 the last isomorphism does not hold anymore. It seems more adapted to define B_t as $\mathcal{O}(\{1, t\})$ in the complex case, based on the observation from the previous paragraph on the rank of the split Grothendieck ring of the generated category.

We consider such a situation for a cyclic group. Hence let $W = \langle s \rangle$ be a finite complex reflection group of rank 1. Let d be the order of s. Note that every s^k with $1 \leq k \leq d-1$ is a reflection of W, but we will only consider one generator $B_s := \mathcal{O}(\{1, s\})$, the one corresponding to the unique distinguished reflection. Let $\mathcal{B}$ be the additive, graded, monoidal, Karoubian category generated by such a B_s.

We say that a subset of W of the form $\{s^i, s^{i+1}, \ldots, s^{i+j}\}$ where $0 \leq j \leq d-1$ and $0 \leq j \leq d-1$ is *cyclically connected*.

Theorem 4.2.4 (G.-Thiel, 2020, [87, Theorem 1.1])**.** Let W and $\mathcal{B}$ be as above. The indecomposable objects in $\mathcal{B}$ are, up to isomorphism and grading shifts, given by the bimodules $\mathcal{O}(A)$, where A runs over the set of cyclically connected subsets of W. In particular, the split Grothendieck ring $A_W := \langle \mathcal{B} \rangle$ is an $\mathcal{A}$–algebra which is a free $\mathcal{A}$–module of rank $d(d-1)+1$.

Theorem 4.2.5 (G.-Thiel, 2020, [87, Theorem 1.1])**.** The algebra A_W has a presentation with generators s, $C_1, \cdots, C_{d-1}$ and relations

$$\begin{cases} s^d = 1, \\ C_i C_j = C_j C_i \ \forall i, j \text{ and } sC_i = C_i s \ \forall i, \\ C_1 C_i = C_{i+1} + sC_{i-1} \ \forall i = 1, \dots, d-2, \\ C_1 C_{d-1} = (v + v^{-1})\, C_{d-1}, \\ sC_{d-1} = C_{d-1}, \end{cases}$$

with the convention that $C_0 := 1$. In particular A_W is commutative, and has a subalgebra isomorphic to the group algebra of W, with generator abusively denoted s above.

Theorem 4.2.6 (G.-Thiel, 2020, [87, Theorem 1.2])**.** The algebra $A_W^{\mathbb{C}}$ defined by the presentation from Theorem 4.2.5 but over the complex numbers is generically semisimple. More precisely, if $v + v^{-1} \neq 2\cos\left(\frac{k\pi}{d}\right)$ for all $k = 1, \dots, d-1$, then $A_W^{\mathbb{C}}$ is semisimple.

Note that specializing s to 1 yields the Hecke algebra of the complex reflection group W for a suitable choice of parameters.

4.2.3 Open problems

It is natural to try to extend the constructions made in Subsections 4.2.1 and 4.2.2 to other families of groups.

Problem 4.2.7. Can one describe the category $\mathcal{B}^T$ for other finite irreducible Coxeter groups W? If not, for finite Weyl groups W, can we at least describe the intermediate category $\mathcal{B}^i$ which is an extension of Soergel's category, where one adds a generating bimodule B_{t_0} attached to the reflection t_0 corresponding to the highest root[3]?

Problem 4.2.8. Can one describe the category $\mathcal{B}_W$ from Subsection 4.2.2 for other families of complex reflection groups?

Problem 4.2.9. Can one obtain a presentation by generators and relations of the monoidal category $\mathcal{B}$ from Theorem 4.2.4, in the spirit of [66]?

In type A_2, the algebra $A(W)$ from Theorem 4.2.1 turns out to be a quotient of a "virtual Hecke algebra" defined using the virtual braid group. The virtual braid group has recently been generalized to all Artin groups by Bellingeri, Paris and Thiel [16]. This suggests the following.

Problem 4.2.10. For a Coxeter group W, is the split Grothendieck ring $A(W)$ from Subsection 4.2.1 a quotient of a virtual Hecke algebra? Can this help to understand $A(W)$ in a more general setting than just A_2?

[3] In type A_2, we have $\mathcal{B}^i = \mathcal{B}^T$, and in type B_2 it can be shown that $\mathcal{B}^i$ has 20 indecomposable objects up to isomorphism and grading shifts.

Chapter 5

Reflection subgroups: structure, normalizer, braid group, Hecke algebra

List of relevant publications

1. T. Gobet, A. Henderson, and I. Marin, *Braid groups of normalizers of reflection subgroups*, Ann. Inst. Fourier **71** (2021), no. 6, 2273-2304.
2. T. Gobet and I. Marin, *Hecke Algebras of Normalizers of Parabolic Subgroups*, Algebr. Represent. Theor. **26** (2023), 1609-1639.
3. T. Gobet, *On maximal dihedral reflection subgroups and generalized noncrossing partitions*, preprint (2023), `https://arxiv.org/abs/2307.16791`.

This chapter collects works realized between 2019 and 2023. The first two papers above (whose results are described in Section 5.1 below) began during the visit of Ivan Marin when I was a postdoc in Sydney in February 2019, while the last one (whose results are presented in Section 5.2 below) grew up partly with an observation on rank three noncrossing partition lattices which I had also made in 2019, and discussions with Jean-Yves Hée in 2021.

Section 5.1 explores the structure of braid groups and Hecke algebras of normalizers of (full) reflection subgroups, as defined by Ivan Marin [110, Sections 2.2 and 2.3] for finite complex reflection groups. Several definitions also make sense for arbitrary Coxeter groups. In Subsection 5.1.1, corresponding to the first paper above, we show that braid groups of normalizers of reflection subgroups of *finite* Coxeter groups are semidirect products, and deduce the construction of a standard basis for the Hecke algebra of the normalizer of the reflection subgroup. In Subsection 5.1.2, corresponding (mostly) to the second paper above, the situation is investigated in the complex case. Most of the results holding in the real case do not hold anymore, but in the case of *parabolic* subgroups of finite complex reflection groups, we show that if the Hecke algebra is defined over a large enough field, then one still has a semidirect (or crossed) product decomposition of the Hecke algebra of the normalizer, even in cases where the braid group does not satisfy the semidirect product decomposition that holds in the real case. For the infinite family and some exceptional groups, explicit conditions on the field to guarantee such a semidirect product decomposition of the Hecke algebra of the normalizer are computed.

In Section 5.2, we give a new proof of a theorem of Dyer stating that in an arbitrary Coxeter group, every pair of distinct reflections lies in a unique maximal dihedral reflection subgroup of the ambiant group. We deduce a new proof of the lattice property of generalized noncrossing partitions in Coxeter groups of rank three, recently proven by Delucchi-Paolini-Salvetti [47].

5.1 Braid groups and Hecke algebras of normalizers of reflection subgroups

There has been a lot of results about normalizers of parabolic subgroups of Coxeter groups. In the case of a finite Coxeter system (W, S) and a a subset $J \subseteq S$, Howlett [94] showed that the normalizer $N_W(W_J)$ of the standard parabolic subgroup W_J is of the form $W_J \rtimes U_J$ for some subgroup U_J that has the form $W' \rtimes \Gamma$, where W' is again a Coxeter group and Ω is a group of automorphisms of diagram of W'. The group W' is sometimes called the *reflection* part of the normalizer, while Γ is its *non-reflection part*. The complement U_J is obtained as the stabilizer of a set of roots. Brink and Howlett [31] removed the assumption that (W, S) is finite and gave a presentation by generators and relations of a certain groupoid attached to $J \subseteq S$, in which U_J is realized as the group of endomorphisms of an object. Borcherds used the properties of the normalizer to calculate automorphisms groups of some K_3 surfaces and Lorentzian lattices [26].

The fact that parabolic subgroups of finite Coxeter groups always admit a complement insider their normalizers can easily be generalized to reflection subgroups of arbitrary Coxeter groups (see Lemma 5.1.1 below). For finite complex reflection groups, the situation is more intricate, as there are examples of reflection subgroups which do not admit a complement inside their normalizers. Nevertheless, it was shown by Muraleedaran and Taylor [118] that *parabolic* subgroups of finite complex reflection groups always admit a complement inside their normalizer.

Marin [110, Sections 2.2 and 2.3.1] defined a braid group and a Hecke algebra of the normalizer of a full reflection subgroup of a finite complex reflection group. Such algebras turn out to be related to an algebra $\mathcal{C}_W$ previously introduced also by Marin [109] and generalizing the algebra of "braids and ties" of Arcadi and Jujumaya [3], in the sense that $\mathcal{C}_W$ is Morita equivalent to a direct sum of Hecke algebras of normalizers of reflection subgroups.

Let us recall the construction. Let $\mathcal{H}$ denote the collection of reflecting hyperplanes of a finite complex reflection group W. Consider the quotient map $\pi : B := B_W \longrightarrow W$. Let W_0 be a reflection subgroup of W and let $N_0 := N_W(W_0)$. Consider the subgroup $\widehat{B_0} := \pi^{-1}(N_0) \subseteq B$. Note that $P_W \subseteq \widehat{B_0}$. Let $\mathcal{H}_0 \subseteq \mathcal{H}$ be the set of hyperplanes associated to the reflections of W_0. Consider the (normal) subgroup $K_0 \subseteq P_W$ generated by all the meridians around hyperplanes that are not in the set $\mathcal{H}_0$. Letting $\beta \in \widehat{B_0}$, since $\pi(\beta)$ normalizes W_0, we have $\beta m \beta^{-1} \in K_0$ for every generator m of K_0, hence K_0 is still normal in $\widehat{B_0}$. Define the *braid group of N_0* as the quotient $\widetilde{B_0}$ of $\widehat{B_0}$ by K_0. Note that $\pi_{|_{\widehat{B_0}}}$ induces a surjection $\widetilde{\pi_0} : \widetilde{B_0} \twoheadrightarrow N_0$.

One thus gets the following commutative diagram, where the rows are exact

$$
\begin{tikzcd}
1 \arrow[r] & \pi^{-1}(W_0)/K_0 \arrow[r] \arrow[d, "\pi_0"] & \widetilde{B_0} \arrow[r] \arrow[d, "\widetilde{\pi}_0"] & N_0/W_0 \arrow[r] \arrow[d, "\mathrm{Id}"] & 1 \\
1 \arrow[r] & W_0 \arrow[r] & N_0 \arrow[r] & N_0/W_0 \arrow[r] & 1
\end{tikzcd}
\tag{5.1.1}
$$

Note that a splitting of the short exact sequence above automatically implies a splitting of the bottom short exact sequence.

The above defined groups and short exact sequences can also be defined when W is a not necessarily finite Coxeter group, and W_0 is a reflection subgroup of W: one defines K_0 as the (normal) subgroup of the pure braid group P_W generated by those elements of the form $\beta\mathbf{s}^2\beta^{-1}$, where $\mathbf{s}$ is a standard generator of B_W such that $p(\beta\mathbf{s}\beta^{-1})$ is not in W_0, and $\beta \in B_W$.

It is natural to wonder, in thoses cases where the bottom short exact sequence splits, if there is a splitting of the top short exact sequence.

Returning to the case of a finite complex reflection group, assume that W_0 is a *full* reflection subgroup of a finite complex reflection group W, in the sense that for every reflection $r \in W_0$, any reflection r' sharing the same hyperplane as r is also in W_0 (this holds for instance if W_0 is a parabolic subgroup of W). Let $\Bbbk$ be a field containing the ring of Laurent polynomials $\mathbb{Z}[u_{s,i}^{\pm 1}]$, where s runs over the set of distinguished reflections of W and $i \in \{0, \dots, o(s)-1\}$, with the convention that $u_{s,i} = u_{wsw^{-1},i}$ for all $w \in N_0$. By definition, the *Hecke algebra $\widetilde{H}_0$ of the normalizer N_0* as defined by Marin in [110, Section 2.3.1] is the quotient of the group algebra $\Bbbk[\widehat{B}_0]$ by two types of relations:

- The relations $\sigma^{m_H} = 1$, for every braided reflection σ associated to a hyperplane $H \in \mathcal{H} \setminus \mathcal{H}_0$. Here m_H is the order of the pointwise stabilizer of H in W.
- The defining relations of the Hecke algebra H_0 of W_0 on the braided reflections σ with respect to the hyperplanes H that lie in $\mathcal{H}_0$, that is, the relations $\prod_{i=0}^{m_H-1}(\sigma - u_{s,i}) = 0$, where s denotes the distinguished reflection with hyperplane H.

5.1.1 Braid groups and Hecke algebras of normalizers of reflection subgroups: finite Coxeter groups

The following is an easy generalization of Howlett's construction [94] of complements for parabolic subgroups inside their normalizers.

Lemma 5.1.1 (G.-Henderson-Marin, 2021, [88, Lemma 3.3])**.** Let (W, S) be an arbitrary Coxeter system. Let $W_0 \subseteq W$ be a reflection subgroup. Let

$$U_0 := \{w \in N_0 \mid N(w) \cap W_0 = \emptyset\}.$$

Then U_0 is a subgroup of N_0 which is complementary to W_0. That is, we have a semidirect product decomposition $N_0 = W_0 \rtimes U_0$, and the bottom short exact sequence in (5.1.1) splits. Moreover, the conjugation action of U_0 on W_0 preserves the canonical Coxeter generating set $\chi(W_0)$ of W_0.

Define a set-theoretic map $\psi : N_0 \longrightarrow \widetilde{B_0}$ by $w \mapsto \mathbf{w}K_0$, where $\mathbf{w}$ denotes the (image of the) positive lift of w in B_W (in $\widehat{B_0}$). This is *not* a group homomorphism in general, but it has the following properties.

Proposition 5.1.2 (G.-Henderson-Marin, 2021, [88, Proposition 3.10 and Corollary 3.11])**.** Let (W, S) be a Coxeter system and $W_0 \subseteq W$ be a reflection subgroup.

1. Let $w_1, w_2 \in N_0$ such that $N(w_1^{-1}) \cap N(w_2) \cap W_0 = \emptyset$. Then
$$\psi(w_1 w_2) = \psi(w_1)\psi(w_2).$$
In particular, the restriction $\psi : U_0 \hookrightarrow \widetilde{B_0}$ is a group homomorphism,
2. The restriction $\psi : W_0 \hookrightarrow \pi^{-1}(W_0)/K_0$ satisfies $\psi(w_1 w_2) = \psi(w_1)\psi(w_2)$ whenever the product $w_1 w_2$ is S_0-reduced inside (W_0, S_0),
3. Denoting by B_0 the Artin group of (W_0, S_0), there is a unique group homomorphism $\widetilde{\psi} : B_0 \longrightarrow \pi^{-1}(W_0)/K_0$ such that $\widetilde{\psi}(\beta) = \psi(w)$ for any $w \in W_0$ with positive lift β in B_0.

We then showed the following.

Theorem 5.1.3 (G.-Henderson-Marin, 2021, [88, Theorem 3.13 and Proposition 3.16])**.** Let (W, S) be a Coxeter system and $W_0 \subseteq W$ be a reflection subgroup.

1. If the homomorphism $\widetilde{\psi}$ from Proposition 5.1.2 (3) is an isomorphism, then the top short exact sequence in 5.1.1 splits. After identification of N_0/W_0 with U_0 (see Lemma 5.1.1), the splitting map $U_0 \hookrightarrow \widetilde{B_0}$ is given by the restriction of ψ, which is a homomorphism by Proposition 5.1.2 (1). We thus have
$$\widetilde{B_0} = B_0 \rtimes \psi(U_0).$$
Moreover, the conjugation action of $\psi(U_0)$ on the set Σ_0 of Artin generators of B_0 preserves Σ_0, and for $u \in U_0$ the action of $\psi(u)$ on Σ_0 is the same as the action of u on S_0.
2. The homomorphism $\widetilde{\psi}$ from Proposition 5.1.2 (3) is an isomorphism whenever W is finite and W_0 is an arbitrary reflection subgroup. In particular, the conclusion of point (1) above always holds when W is finite.

Remark 5.1.4. The first point above is easily deduced from Proposition 5.1.2, which is proven using only the combinatorics of words in Coxeter groups, and properties of inversion sets. We provide an alternative proof of (1) in Section 4 of [88], using a groupoid description of normalizers, in the spirit of Brink and Howlett's groupoid [31] (but unlike in [31], these groupoids are defined for arbitrary reflection subgroups, not just parabolic ones, and are bigger than the ones from [31] in the parabolic case).

Let us discuss an example illustrating the above results and properties.

Example 5.1.5. Let W be a finite Coxeter group of type B_n, and W_0 be a reflection subgroup of W of type D_n. In terms of the generating set $\{s_0, s_1, \dots, s_{n-1}\}$ of W, where $s_0 s_1$ has order 4 and $S' := \{s_1, \cdots, s_{n-1}\}$ generates a type A_{n-1} standard parabolic subgroup, the Coxeter generating set of W_0 is given by $S_0 = S' \cup \{s_0 s_1 s_0\}$. Then since W_0 has index two in W, it is normal, hence $N_0 = W$. Hence $U_0 = \{w \in W \mid N(w) \cap W_0 = \emptyset\}$, and U_0 is thus given in this case by the set of elements of minimal length in the two cosets modulo W_0; as there is always a unique such element per coset, that is, we get $U_0 = \{1, s_0\}$. We see that the conjugation action of U_0 on S_0 indeed preserves S_0, as predicted by Lemma 5.1.1.

Now we have $\widehat{B_0} = \pi^{-1}(W) = B_W$, and $\widetilde{B_0}$ is the quotient of B_W by the subgroup normally generated by $\mathbf{s}_0^2$, since every reflection of W which is not in W_0 is a conjugate of s_0. We thus have

$$\widetilde{B_0} = B_0 \rtimes \mathbb{Z}/2\mathbb{Z},$$

and we recover the well-known fact that the Artin group of type D_n can be realized as an index two subgroup of the quotient of the Artin group of type B_n by the relation $\mathbf{s}_0^2 = 1$. We thus recover Proposition 3.1.5 from Chapter 3.1.5.

Continue to assume that W is a finite Coxeter group. The semidirect product decomposition of $\widetilde{B_0}$ induces a semidirect (or crossed) product decomposition of the corresponding Hecke algebra $\widetilde{H}_0$, which allows us to write down a standard basis for that algebra. Note that in the real case, every reflection subgroup W_0 of W is full. Recall that for $w \in W_0$, the image of the positive lift $\psi(w) \in B_0$ in the Hecke algebra H_0 is written T_w, and the elements $\{T_w\}_{w \in W_0}$ form the standard basis of H_0. We simply extend this notation to $w \in N_0$, writing T_w for the image in $\widetilde{H_0}$ of $\psi(w) \in \widetilde{B_0}$.

Theorem 5.1.6 (G.-Henderson-Marin, 2021, [88, Theorem 3.19])**.** The elements $\{T_w\}_{w \in N_0}$ form a basis of $\widetilde{H_0}$. The subset $\{T_w\}_{w \in W_0}$ spans a subalgebra which can be identified with H_0 with its standard basis. The subset $\{T_u\}_{u \in U_0}$ spans a subalgebra which can be identified with the group algebra $\Bbbk[U_0]$ with its basis given by the elements of U_0. Multiplication induces a $\Bbbk$-module isomorphism $H_0 \otimes_{\Bbbk} \Bbbk[U_0] \cong \widetilde{H_0}$, and we have

$$T_w T_u = T_{wu} = T_u T_{u^{-1}wu} \text{ for } w \in W_0, u \in U_0.$$

Example 5.1.7. Continuing Example 5.1.5, the above theorem allows one to realize the Iwahori-Hecke algebra of type D_n inside a Iwahori-Hecke algebra of type B_n with unequal parameters, the parameter corresponding to the conjugacy class of s_0 being specialized at 1. Here as well, this phenomenon has been well-known to representation theorists for a long time (see for instance [93, Section 2.3]).

5.1.2 Hecke algebras of normalizers of parabolic subgroups: complex reflection groups

In the case of finite complex reflection groups, the short exact sequence at the bottom of 5.1.1 does not split in general, as the following elementary example shows. It also shows that a splitting of the bottom short exact sequence does not necessarily imply a splitting at the top.

Example 5.1.8. Let $W = \langle s \rangle$ be cyclic of order d, and let $W_0 = \langle s^e \rangle$, where $1 < e < d$ is a divisor of d. Then W_0 is a non-parabolic reflection subgroup of W. If $\gcd(e, d/e) > 1$, then there is no complement to W_0 inside W. Without assuming $\gcd(e, d/e) > 1$, we have $\widetilde{B_0} = B = \langle \sigma \rangle$, while $\pi^{-1}(W_0)/K_0 = \langle \sigma^e \rangle$ (we have $\mathcal{H} = \mathcal{H}_0$, hence $K_0 = \{1\}$). The top short exact sequence in 5.1.1 is thus given by

$$1 \longrightarrow \mathbb{Z} \overset{z \mapsto ez}{\longrightarrow} \mathbb{Z} \longrightarrow \mathbb{Z}/e\mathbb{Z} \longrightarrow 1,$$

which does not split.

The above example may lead one to think that the problem here comes from the fact that there are reflections of order greater than 2 in W, but counterexamples to the splitting of the top short exact sequence can be found as well in 2-reflection groups (that is, finite complex reflection groups generated by reflections of order 2). See [88, Section 6.2] for more examples, and for the discussion of an obstruction to the splitting of the top short exact sequence when the bottom short exact sequence splits coming from the center $Z(B)$ of B.

On the other hand, as shown by Muraleedaran and Taylor [118], when W_0 is a *parabolic* subgroup of a complex reflection group, then the bottom short exact sequence always splits, and there are examples where the top short exact sequence also splits:

Proposition 5.1.9 (G.-Henderson-Marin-G., 2021, [88, Proposition 5.1])**.** Let $W = G(d, 1, n)$ and $W_0 = G(d, 1, k)$, where $1 \leq k \leq n-1$. Then $N_0 = W_0 \times U_0$, where U_0 is isomorphic to $G(d, 1, n-k)$, and there is a direct product decomposition $\widetilde{B_0} = B_0 \times U_0$.

Unfortunately, for other parabolic subgroups of $G(d, 1, n)$, the situation can be very different, as the following example shows.

Example 5.1.10 ([88, Example 6.6])**.** Let $W = G(3, 1, 2)$, and let W_0 be a rank-1 parabolic subgroup generated by a reflection of order 2. Then the top short exact sequence in (5.1.1) does not split.

Nevertheless, with a suitable choice of parameters defining the Hecke algebra, and a sufficiently large field of definition $\mathbb{K}$, we can show that we still have a semidirect (or crossed) product decomposition at the level of *Hecke algebras* (as obtained in Theorem 5.1.6 in the real case).

More precisely, let W_0 be a parabolic subgroup of W. Let $\Bbbk$ be the ring of Laurent polynomials $\mathbb{Z}[u_{s,i}^{\pm 1}]$, where s runs over the set of distinguished reflections of W and $i \in \{0, \ldots, o(s)-1\}$, with the convention that $u_{s,i} = u_{wsw^{-1},i}$ for all $w \in N_0$. In this section we consider the generic case, that is, the case where $\mathbb{K}$ is a field containing $\Bbbk$. In particular $\mathbb{K}$ has characteristic 0.

Theorem 5.1.11 (G.-Marin, 2022, [89, Theorem 1.1])**.** Let $W_0 \subseteq W$ be a parabolic subgroup of a finite complex reflection group W. If the defining parameters of H_0 are generic and $\mathbb{K}$ is a sufficiently large field of characteristic zero, then

$$\widetilde{H_0} \cong H_0 \rtimes N_0.$$

The question is thus quite different than in the real case treated in Section 5.1. Namely, one would like to determine explicit, algebraic conditions on the field $\mathbb{K}$ and the parameters $u_{s,i} \in \mathbb{K}$ to ensure the semidirect (or crossed) product decomposition of $\widetilde{H}_0$.

For the infinite series, we have the following conditions.

Theorem 5.1.12 (G.-Marin, 2022, [89, Theorem 1.2])**.** Let $W = G(de, e, n)$ and

$$W_0 = G(de, e, n_0) \times \prod_{k=1}^{n} G(1, 1, k)^{b_k},$$

where $n_0 + \sum_{i=1}^{n} b_i = n$. Then $\widetilde{H}_0 \cong N_0 \ltimes H_0$ as soon as, whenever $b_k \neq 0$,

- there exists $T_k \in \mathbb{K}[X]$ such that $T_k(\Delta(k))^{-de} = \Delta(k)^2$, where the equality holds inside the Iwahori-Hecke algebra H_k of type A_{k-1} and $\Delta(k)$ denotes the image inside H_k of the (classical) Garside element of the k-strand braid group $\mathcal{B}_k$, and
- if moreover $e \neq 1$ and $n_0 \geq 1$, there exists $T_{0,k} \in \mathbb{K}[X]$ such that $T_{0,k}(\sigma)^{de} = \sigma^{kd}$ whenever σ is a braided reflection associated to the hyperplane $\{z_1 = 0\}$.

In particular the second condition is empty when $d = 1$, since $\{z_1 = 0\}$ is not a reflecting hyperplane in that case.

Moreover, conditions on the defining parameters of the Hecke algebra ensuring the existence of such polynomials T_k and $T_{0,k}$ are provided (see [89, Lemmatas 2.8 and 2.9]).

For the exceptional types, the results may be summarized as follows.

Theorem 5.1.13 (G.-Marin, 2022, [89, Theorem 5.1])**.** Let W be an irreducible complex reflection group of exceptional type, and W_0 a proper parabolic subgroup of maximal rank. Let z_{B_0} be the canonical positive central element of B_0. Except for two exceptions in ranks 3 and 5, if there exists $T \in \mathbb{K}[X]$ such that the equality $T(z_{B_0})^{|Z(W)|} = z_{B_0}^{-|Z(W_0)|}$ holds true inside H_0, then $\widetilde{H_0} \cong N_0 \ltimes H_0$.

Some of the remaining exceptional types are also treated in [89, Section 6].

5.1.3 Open problems

It is natural to wonder if, in the case of an infinite Coxeter system (W, S) and an arbitrary reflection subgroup W_0, we still have the semidirect product decomposition $\widetilde{B_0} = B_0 \rtimes \psi(U_0)$, since by Lemma 5.1.1 the short exact sequence at the bottom of (5.1.1) always splits. By Theorem 5.1.3 (1), in order to establish the semidirect product decomposition, one must check that the map $\widetilde{\psi}$ from Proposition 5.1.2 (3) is an isomorphism.

Problem 5.1.14. In the case of an infinite Coxeter system (W, S) and an arbitrary reflection subgroup $W_0 \subseteq W$, is $\widetilde{\psi}$ an isomorphism?

The proof given in [88, Section 3.4] of this property in the case where W is finite uses the realization of B_W as the fundamental group of regular orbits of the complexified geometric representation $V_{\mathbb{C}}$ of W.

What can at least be said when S is finite and W_0 is parabolic is that $\widetilde{\psi}$ is injective (see [88, Remark 3.18]).

5.2 Dihedral reflection subgroups of Coxeter systems

Recall that in the study of dual Coxeter systems and dual Artin groups, the poset $[1, c]_T$ of generalized noncrossing partitions plays a key role (see Subsection 2.1.6 above). Proving the lattice property of the interval $[1, c]_T$ for large families of Coxeter groups is a hard problem in general.

The aim of this section is to present a few results which led to a new proof of the lattice property $[1, c]_T$ for all Coxeter groups of rank 3. The first proof of this result was given by Delucchi-Paolini-Salvetti [47, Theorem 4.2], using factorizations of the Coxeter element as products of isometries of the hyperbolic plane $\mathbb{H}^2$. The proof that we provide here only uses the combinatorics of words in Coxeter systems, without any use of geometry or root systems.

It is based on Dyer's theorem that any two distinct reflections t, t' in an arbitrary Coxeter group W always lie in a unique maximal dihedral reflection subgroup of W (see Theorem 5.2.3 below). We begin by providing a new proof of this result, which does not use geometry or root systems, in Subsection 5.2.1 below. The proof of the lattice property of $[1, c]_T$ for Coxeter groups of rank three is then derived in Subsection 5.2.2, as a corollary of a more general result.

5.2.1 New proof of Dyer's Theorem on maximal dihedral reflection subgroups

The proof is based on the following two propositions, which were suggested to me by Jean-Yves Hée during discussions on dual Coxeter systems in 2021. The proofs are given in [83], where the proof of Proposition 5.2.2 is Hée's proof, while the origonal proof of Proposition 5.2.1 given to me by Hée was using root systems.

Proposition 5.2.1 (G., 2023, [83, Proposition 1.3])**.** Let (W, S) be a Coxeter system. Suppose that there is $w \in W$, $w \neq 1$ such that, for every $s \in S$, there is $s' \in T$ such that $w = ss'$. Then $|S| = 2$.

Proposition 5.2.2 (G., 2023, [83, Proposition 1.4])**.** Let (W, S) be a Coxeter system. Let w in W be a product of two distinct reflections of W. Consider the set

$$R_w := \{t \in T \mid \text{There exists } t' \in T \text{ such that } w = tt'\}.$$

Then setting $W_w := \langle R_w \rangle$, we have $w \in W_w$, and there exists a subset $S' \subseteq R_w$ such that (W_w, S') is a Coxeter system with set of reflections R_w. Moreover, for any subset $S' \subseteq R_w$ such that (W_w, S') is a Coxeter system, we have $|S'| = 2$.

From the above two propositions, it is not difficult to deduce a new proof of Dyer's theorem:

Theorem 5.2.3 (Dyer, 1987, [55, Corollary 3.18], see also [58, Remark 3.2], and [62])**.** *Let t, t' be two distinct reflections in an arbitrary Coxeter system (W, S). Then there is a unique maximal dihedral reflection subgroup $W(t, t')$ of W containing both t and t'.*

In fact, we show that the maximal dihedral reflection subgroup is given by the dihedral reflection subgroup W_w of Proposition 5.2.2, where $w = tt'$.

Proof. Set $w := tt'$ and consider the dihedral reflection subgroup W_w from Proposition 5.2.2. Since $tt' = t'(t'tt')$ we have that $t, t' \in W_w$. We claim that $W(t, t') := W_w$ is the unique maximal dihedral reflection subgroup of W containing both t and t'.

Assume that W' is a dihedral reflection subgroup of W containing both t and t'. Then W' contains w. Since w is a product of two reflections and W' is dihedral, then for any reflection r of W' we have that rw is a reflection of W' (hence of W). Hence $r \in R_w \subseteq W_w$, and thus W_w contains every reflection of W'. It follows that $W' \subseteq W_w$, which concludes the proof. ☐

5.2.2 Lattice property of intervals of rank three in noncrossing partition posets

The results from the previous section allow one to deduce the following result, which is a generalization of [47, Theorem 4.2].

Theorem 5.2.4 (G., 2023, [83, Theorem 2.2])**.** Let (W, S) be a Coxeter system. Let $u, v \in W$ such that $u \leq_T v$ and $\ell_T(v) = \ell_T(u) + 3$. Then the interval $[u, v]_T$ is a lattice. In particular, for every standard Coxeter element c in a Coxeter system (W, S) of rank three, the poset $[1, c]_T$ of generalized noncrossing partitions is a lattice.

Sketch of the proof. The interval $[u, v]_T$ is isomorphic as a poset to $[1, u^{-1}v]_T$. We can thus assume that $u = 1$, and $\ell_T(v) = 3$. To conclude the proof, it suffices to show the following: let $t \neq t' \in T$, $w, w' \in [1, v]_T$ with $\ell_T(w) = \ell_T(w') = 2$, $t, t' \leq_T w$, and $t, t' \leq_T w'$. Then $w = w'$. Indeed, since $\ell_T(v) = 3$, any obstruction to the lattice property must come from a so-called "bowtie" [113, Definition 1.5], that is in our setting, a pair (t_1, t_2) of distinct elements of reflection length 1 together with a pair (w_1, w_2) of distinct elements of reflection length 2, such that $t_i \leq_T w_j$, for all i and j.

Under the above assumptions, one then shows that w_1, w_2 lie in the same maximal dihedral reflection subgroup W' of W. If $w_1 \neq w_2$, letting $q_1, q_2 \in T$ such that $w_i q_i = v$, one then has that $q_1 q_2 = w_1^{-1} w_2 \in W'$, hence that $q_1 q_2 \in W'$. One then shows that q_1 and q_2 themselves lie in W', hence that v lies in the dihedral reflection subgroup W', hence that v is a product of at most two reflections, contradicting $\ell_T(v) = 3$. ☐

We deduce the following corollary.

Corollary 5.2.5 (G., 2023, [83, Corollary 2.3])**.** Let (W, S) be a Coxeter system and $w \in W$ such that $\ell_T(w) = 3$. Then the interval group $G([1, w]_T)$ is a quasi-Garside group.

Remark 5.2.6. For some finite Coxeter groups W of rank three and quasi-Coxeter elements w of W which fail to be Coxeter elements (this happens for instance in H_3), the lattice property of $[1, w]_T$ was established by computer on a case-by-case basis in [15], where the corresponding interval groups are also studied.

5.3 Open problems

The following is an extremely hard question in the field.

Problem 5.3.1. For which Coxeter systems (W, S) and which choices of standard Coxeter elements c does the interval $[1, c]_T$ form a lattice?

There is a full answer to this question in finite, affine, universal, and rank three Coxeter systems. Note that in the affine type, the lattice property does not always hold (for instance in type $\widetilde{A}_n$, it depends on the choice of standard Coxeter element [51]). Apart from those cases, this question appears to be completely open. We do not have any precise conjecture to formulate.

Problem 5.3.2. Let (W, S) be an arbitrary Coxeter system and $w \in W$ such that $\ell_T(w) = 3$. What are the properties of the interval quasi-Garside group $G([1, c]_T)$?

Chapter 6

On Garside structures for torus knot groups

List of relevant publications

1. T. Gobet, *On some torus knot groups and submonoids of the braid groups*, J. Algebra **607** (2022), Part B, 260-289.
2. T. Gobet, *Addendum to "On some torus knot groups and submonoids of the braid groups" [J. Algebra 607 (Part B) (2022) 260–289]*, J. Algebra **633** (2023), 666–667.
3. T. Gobet, *Toric reflection groups*, *Journal of the Australian Mathematical Society*, online first (2023).
4. T. Gobet, *A new Garside structure on torus knot groups and some complex braid groups*, to appear in *Journal of Knot Theory and its Ramifications* (2023).
5. T. Gobet and B. Rognerud, *Odd and even Fibonacci lattices arising from a Garside monoid*, preprint (2023). `https://arxiv.org/abs/2301.00744`.

In this section, we present several results related to Garside structures for torus knot groups, and study the structure and properties of some natural quotients of some of these Garside structures. These results were obtained between 2020 and 2022, and began with attempts to understand the exotic Garside structure $\langle\, a, b \mid aba = b^2 \,\rangle$ on the 3-strand braid groups $\mathcal{B}_3$, and the submonoid $\mathcal{M}_n$ of $\mathcal{B}_n$ generated by $\sigma_1, \sigma_1\sigma_2, \ldots, \sigma_1\sigma_2\cdots\sigma_{n-1}$ (see [43, Chapter IX, Question 30]).

Given $n, m \geq 2$ with n and m coprime, the (n,m)-torus knot group is the fundamental group of the complement of the torus knot $T_{n,m}$ in S^3. Note that $G(n,m) \cong G(m,n)$, since the two knots $T_{n,m}$ and $T_{m,n}$ are isotopic.

In some particular cases, these groups are complex braid groups: $G(2,3)$ is isomorphic to the 3-strand braid group $\mathcal{B}_3$, while $G(3,4)$, $G(3,5)$ are the complex braid groups of G_{12}, respectively G_{22} (see [10]), and $G(2,m)$ is isomorphic to the Artin group of dihedral type.

The first two sections below correspond to results from the first, second, fourth and fifth papers above. Section 6.1 below introduces a new Garside structure for torus knot groups, generalizing the aforementioned exotic Garside structure on $\mathcal{B}_3$. Section 6.2 explains the link between the newly introduced Garside structure in the specific case where $m = n+1$ and the aforementioned motivating problem on the submonoid $\mathcal{M}_n$ of $\mathcal{B}_n$.

Since, as mentioned above, some particular quotients of torus knot groups happen to be complex reflection groups, we investigate in Section 6.3 below the structure and properties of analogous quotients of torus knot groups. This corresponds to the third paper above. We show that these quotients behave in a certain sense like "(infinite) complex reflection groups whose braid groups are torus knot groups", show that their center is cyclic, and classify them.

6.1 New Garside structures generalizing an exotic Garside structure on $\mathcal{B}_3$

6.1.1 Torus knot groups

Let $n, m \geq 2$, with n and m coprime. The *(n,m)-torus knot group* $G(n,m)$ is the knot group of the torus knot $T_{n,m}$, that is, the fundamental group of the complement of $T_{n,m}$ in S^3 (see [131, Chapter 3] for more on torus knots and their groups). As $T_{n,m}$ and $T_{m,n}$ are isotopic, we have $G(n,m) \cong G(m,n)$. There is a well-known presentation of $G(n,m)$ given by

$$\langle\, x, y \mid x^n = y^m \,\rangle. \tag{6.1.1}$$

It was shown by Schreier [134] that the center of $G(n,m)$ is infinite cyclic, generated by $x^n = y^m$. Another presentation of $G(n,m)$ is given by

$$\langle\, x_1, x_2, \ldots, x_n \mid \underbrace{x_1 x_2 \cdots}_{m \text{ factors}} = \underbrace{x_2 x_3 \cdots}_{m \text{ factors}} = \cdots = \underbrace{x_n x_1 \cdots}_{m \text{ factors}} \,\rangle, \tag{6.1.2}$$

where indices are taken modulo n if $n < m$. An isomorphism is given by $x \mapsto x_1 x_2 \cdots x_m$, $y \mapsto x_1 x_2 \cdots x_n$. Since $G(n,m) \cong G(m,n)$, a third presentation is given by

$$\langle\, y_1, y_2, \ldots, y_m \mid \underbrace{y_1 y_2 \cdots}_{n \text{ factors}} = \underbrace{y_2 y_3 \cdots}_{n \text{ factors}} = \cdots = \underbrace{y_m y_1 \cdots}_{n \text{ factors}} \,\rangle. \tag{6.1.3}$$

For $n = 2$ and $m = 3$, Presentation 6.1.2 is nothing but Artin's presentation of the 3-strand braid group $\mathcal{B}_3$, while Presentation 6.1.3 is its Birman-Ko-Lee ([23]) or dual ([18]) presentation.

Note that, on the algebraic side, it is not obvious that Presentations 6.1.1, 6.1.2, and 6.1.3 define isomorphic groups. The link between these presentations is given in the following Lemma, which is straightforward to check (see also [69, Corollary 2.21] for an isomorphism for a more general family of groups).

Lemma 6.1.1. *Assume that $n < m$.*

1. *The map*

$$y_1 \mapsto x_1, \text{ and for } 2 \leq i \leq m, y_i \mapsto x_n^{-1} x_{n-1}^{-1} \cdots x_{n+3-i}^{-1} x_{n+2-i} x_{n+3-i} \cdots x_n$$

   (where indices in the x_i's are taken modulo n) defines an isomorphism between the group with presentation 6.1.3 and the group with presentation 6.1.2.

2. The map

$$x \mapsto x_1 x_2 \cdots x_m,\ y \mapsto x_1 x_2 \cdots x_n$$

(where indices in the x_i's are taken modulo n) defines an isomorphism between the group with presentation 6.1.1 and the group with presentation 6.1.2.

Torus knot groups are examples of groups which possess many non-isomorphic Garside monoids. Indeed, it was shown by Dehornoy and Paris that Presentations 6.1.1, 6.1.2 and 6.1.3 define Garside monoids (see [44, Examples 4 and 5]). Picantin [124] gave Garside presentations for all torus link groups, yielding in the particular case of knots a Garside presentation which is similar to 6.1.2, but distinct in general.

Let us now assume that n, m are two integers such that $2 \leq n$, $n < m$ and n and m are coprime. Let $m = qn + r$ be the Euclidean division of m by n ; in particular $0 < r < n$. Consider the monoid $\mathcal{M}(n, m)$ defined by the following presentation.

$$\mathcal{M}(n,m) = \left\langle \omega_1, \omega_2, \ldots, \omega_n \;\middle|\; \begin{array}{c} \omega_r \omega_n^q \omega_{i-r} = \omega_i \omega_n^q \text{ if } r < i \leq n, \\ \omega_r \omega_n^q \omega_{n+i-r} = \omega_i \omega_n^{q+1} \text{ if } 1 \leq i < r. \end{array} \right\rangle. \tag{6.1.4}$$

We will denote by $\mathcal{G}(n, m)$ the group defined by the same presentation.

Theorem 6.1.2 (G., 2022, [82, Theorem 6.4, Propositions 4.12 and 5.10])**.** Let $\mathcal{M}(n, m)$ with its presentation given in (6.1.4). We set $\Delta := \omega_n^m$, omitting the dependency on n and m.

1. The pair $(\mathcal{M}(n, m), \Delta)$ is a Garside monoid with Garside group $\mathcal{G}(n, m)$ isomorphic to the torus knot group $G(n, m)$.

2. The Garside element Δ is a generator of the center of $G(n, m)$.

3. The right-lcm of the generators ω_i, $1 \leq i \leq n$ of $\mathcal{M}(n, m)$ is given by ω_n^{m-q}.

4. The left-lcm of the generators ω_i, $1 \leq i \leq n$ of $\mathcal{M}(n, m)$ is given by

$$\begin{cases} \omega_{n-r} \omega_n^{q(n-2)+r-1} & \text{if } 2r > n, \\ \omega_n^{m-q} & \text{otherwise.} \end{cases}$$

Note that for $n = 2$ and $m = 3$, we recover the exotic monoid $\langle\, a, b \mid aba = b^2 \,\rangle$ on $\mathcal{B}_3$.

Remark 6.1.3. The Garside structure obtained in 6.1.2 is particularly complicated compared for instance to the Garside structure $\langle\, x, y \mid x^n = y^m \,\rangle$ for $G(n, m)$. The motivation for studying a generalization of the exotic structure $\langle\, a, b \mid aba = b^2 \,\rangle$ on $\mathcal{B}_3$ was an attempt for a better understanding of the submonoid $\mathcal{M}_n$ of $\mathcal{B}_n$ generated by $\sigma_1, \sigma_1\sigma_2, \ldots, \sigma_1\sigma_2\cdots\sigma_{n-1}$, which is related to the Garside structure obtained above in the case $G(n-1, n)$. In this particular case, the Garside structure from Theorem 6.1.2 was already studied in an earlier paper of ours in honour of Patrick Dehornoy [79, Theorem 1.2]. We explain the relationship with the n-strand braid group in the Section 6.2 below. In this case the theorem is much easier to establish (right-cancellativity is particularly technical in the general case).

Example 6.1.4. We list several non-isomorphic Garside monoids for $G(3,4)$, which is also isomorphic to the braid group of the exceptional complex reflection group G_{12} (see [10, Theorem 1.2 (viii)]). Presentations 6.1.1 to 6.1.3 respectively yield the Garside presentations

$$\langle\, x, y \mid x^2 = y^3 \,\rangle, \left\langle\, x_1, x_2, x_3 \;\middle|\; \begin{array}{c} x_1x_2x_3x_1 \\ = x_2x_3x_1x_2 \\ = x_3x_1x_2x_3 \end{array} \right\rangle, \left\langle\, y_1, y_2, y_3, y_4 \;\middle|\; \begin{array}{c} y_1y_2y_3 = y_2y_3y_4 \\ = y_3y_4y_1 = y_4y_1y_2 \end{array} \right\rangle,$$

where the Garside element is given respectively by x^2, $x_1x_2x_3x_1$, and $y_1y_2y_3$. Picantin's presentation [124, Lemma 3.2] yields the presentation

$$\langle\, \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1\sigma_2\sigma_3\sigma_1 = \sigma_2\sigma_1\sigma_2\sigma_3,\ \sigma_3\sigma_1\sigma_2\sigma_3 = \sigma_1\sigma_2\sigma_3\sigma_2 \,\rangle,$$

which also appears in work of Bessis-Bonnafé-Rouquier [20, §4.6]. The Garside element is given by $(\sigma_1\sigma_2\sigma_3)^4$. Note that Picantin also exhibited alternative Garside monoids for $G(3,4)$, including the monoid $\langle\, x, y \mid xyxyxyx = y^2 \,\rangle$ which has Garside element y^3, and even an infinite family of Garside monoids (see [124, Example 2.7 and Remark 5.2]). The presentation coming from Theorem 6.1.2 is given by the following presentation, with Garside element ω_3^4:

$$\langle\, \omega_1, \omega_2, \omega_3 \mid \omega_1\omega_3\omega_1 = \omega_2\omega_3,\ \omega_1\omega_3\omega_2 = \omega_3^2 \,\rangle.$$

Theorem 6.1.2 does not yield an explicit description of the lattice of simples, and not even a formula for the number of simples. For the monoid $\mathcal{M}(n-1,n)$, we have the following.

Theorem 6.1.5 (Rognerud-G., 2023; [85, Theorem 3.12 and Corollary 4.1])**.** Let $n \geq 3$. Then

- The number of simples in $\mathcal{M}(n-1,n)$ is equal to F_{2n-2}, where $F_0, F_1, F_2, \ldots$ denotes the Fibonacci sequence $1, 2, 3, 5, 8, \ldots$.
- The set of words for the Garside element ω_{n-1}^n in $\mathcal{M}(n-1,n)$ is in bijection with the set of Schroeder trees on n leaves. Its cardinality is given recursively by $S(n)$, where

$$S(1) = S(2) = 1, S(n) = \frac{3(2n-3)S(n-1) - (n-3)S(n-2)}{n} \text{ for } n \geq 3.$$

6.1.2 Other analogous Garside structures

Theorem 6.1.2 suggests that the correct framework to generalize the exotic Garside structure $\langle\, a, b \mid aba = b^2 \,\rangle$ is torus knot groups rather than braid groups (see also the results in Section 6.2 below). Nevertheless, let us first point out that there is another possible generalization of this Garside structure, which already appeared in work of Dehornoy and Paris.

Proposition 6.1.6 (Dehornoy-Paris, 1996, [44, Example 2])**.** *Let $p > q \geq 1$. The monoid $\langle\, a, b \mid b^p = ab^qa \,\rangle$ is a Garside monoid.*

The corresponding Garside group is isomorphic to $\langle\, x, y \mid x^2 = y^{p+q} \,\rangle$ via $x \mapsto ab^q$, $y \mapsto b$, which is a torus knot group or Artin group of dihedral type $I_2(p+q)$ when $p+q$ is odd.

As mentioned in Example 6.1.4, the complex braid group of G_{12} is isomorphic to $G(3,4)$ and hence admits the presentation

$$B_{G_{12}} \cong \langle\, x_1, x_2, x_3 \mid x_1x_2x_3x_1 = x_2x_3x_1x_2 = x_3x_1x_2x_3 \,\rangle.$$

In terms of complex braid groups, the generators x_i are braided reflections. An isomorphism with the presentation of Theorem 6.1.2 is given by $\omega_1 \mapsto x_1$, $\omega_2 \mapsto x_3x_1$, $\omega_3 \mapsto x_2x_3x_1$.

The complex braid group $B_{G_{13}}$ of G_{13} admits a similar presentation with generators also given by braided reflections (see [10, 34]):

$$B_{G_{13}} \cong \left\langle\, x_1, x_2, x_3 \;\middle|\; \begin{array}{c} x_1x_2x_3x_1x_2 = x_2x_3x_1x_2x_3, \\ x_3x_1x_2x_3 = x_1x_2x_3x_1 \end{array} \,\right\rangle.$$

As noticed by Picantin [123, Exemple 13], this presentation is Garside. A Garside structure similar to the one constructed in this paper for torus knot groups can be constructed for G_{13}. Namely, the assignment $\omega_1 \mapsto x_1$, $\omega_2 \mapsto x_2x_1$, $\omega_3 \mapsto x_2x_3x_1$ yields the presentation

$$B_{G_{13}} \cong \left\langle\, \omega_1, \omega_2, \omega_3 \;\middle|\; \begin{array}{c} \omega_2\omega_3\omega_1 = \omega_3^2 \\ \omega_1\omega_3\omega_2 = \omega_3^2 \end{array} \,\right\rangle. \tag{6.1.5}$$

Proposition 6.1.8 below states that this presentation is Garside, as part of a larger family of Garside presentations. To be more precise, the complex braid group $B_{G_{13}}$ is isomorphic to the Artin group of type $I_2(6)$ (see [10, Theorem 1.2 (iii)]). Recall that the standard presentation of $B_{I_2(6)}$ is given by

$$B_{I_2(6)} \cong \langle\, \sigma, \tau \mid \sigma\tau\sigma\tau\sigma\tau = \tau\sigma\tau\sigma\tau\sigma \,\rangle, \tag{6.1.6}$$

and one can check directly that an isomorphism is given by $\sigma \mapsto (x_1x_2x_3x_1)^{-1}$, $\tau \mapsto x_1$ (with inverse $x_1 \mapsto \tau$, $x_2 \mapsto (\sigma\tau\sigma\tau\sigma\tau)^{-1}\sigma^2$, $x_3 \mapsto \sigma^{-1}\tau\sigma$). Hence one passes from Presentation 6.1.5 to Presentation 6.1.6 by $\omega_1 \mapsto \tau$, $\omega_2 \mapsto \sigma\tau^{-1}\sigma^{-1}\tau^{-1}\sigma^{-1}$, $\omega_3 \mapsto \tau^{-1}\sigma^{-1}$. Somewhat surprisingly, one can generalize Presentation 6.1.5 to a Garside presentation for all dihedral Artin groups of even type. It is done as follows.

Let $n \geq 1$. Consider the monoid presentation

$$\langle\, \tau_1, \tau_2, \rho \mid \tau_1\rho\tau_2 = \rho^2, \tau_2\rho^n\tau_1 = \rho^{n+1} \,\rangle. \tag{6.1.7}$$

Note that for $n = 1$, this is the same as Presentation 6.1.5 above.

Remark 6.1.7. Presentation 6.1.7 still makes sense for $n = 0$, but in that case ρ is not an atom of the corresponding monoid as $\rho = \tau_2\tau_1$. In this case the obtained monoid is nothing but the Artin monoid of type $B_2 = I_2(4)$. The monoid defined by Presentation 6.1.7 is shown in Proposition 6.1.8 below to be a Garside monoid with corresponding Garside group isomorphic to the Artin group of dihedral type $I_2(2n+4)$, but we distinguish the case $n \geq 1$ from the case $n = 0$ which is not new and where the number of atoms differs.

Recall that the Artin group of dihedral type $I_2(4+2n)$ ($n \geq 0$) has standard presentation

$$B_{I_2(4+2n)} = \langle\, \sigma, \tau \mid (\sigma\tau)^{n+2} = (\tau\sigma)^{n+2} \,\rangle. \tag{6.1.8}$$

Proposition 6.1.8 (G., 2022, [82, Lemma 7.2 and Proposition 7.6])**.** Let $n \geq 1$.

1. The group defined by Presentation 6.1.7 is isomorphic to the Artin group of type $I_2(4+2n)$ via $\tau_1 \mapsto \tau$, $\tau_2 \mapsto \sigma\tau^{-1}\sigma^{-1}\tau^{-1}\sigma^{-1}$, $\rho \mapsto \tau^{-1}\sigma^{-1}$. The inverse map is given by $\tau \mapsto \tau_1$, $\sigma \mapsto (\tau_1\rho)^{-1}$.

2. The monoid defined by Presentation 6.1.7 is a Garside monoid, with (central) Garside element $\Delta = \rho^{n+2}$. The corresponding Garside group is isomorphic to the Artin group of type $I_2(4+2n)$ (which is also isomorphic to the complex braid group of G_{13} when $n = 1$).

6.2 Connection to the n-strand braid group

The Garside structures from Section 6.1 were discovered when considering the following question of Dehornoy-Digne-Godelle-Krammer-Michel [43, Chapter IX, Question 30]:

Question 6.2.1. When $n \geq 4$, does the submonoid $\mathcal{M}_n$ of $\mathcal{B}_n$ generated by $\sigma_1, \sigma_1\sigma_2, \ldots, \sigma_1\sigma_2\cdots\sigma_{n-1}$ admit a finite presentation? Is it a Garside monoid?

It is not hard to convince oneself that $\mathcal{M}_n$, $n \geq 4$ cannot be Garside, by finding elements which have distinct "smallest" common multiples. But $\mathcal{M}_n$ is the image of the Garside monoid $\mathcal{M}(n-1,n)$ from Theorem 6.1.2, via a quotient map $G(n-1,n) \twoheadrightarrow \mathcal{B}_n$[1]:

Proposition 6.2.2 (G., 2022, [79, Theorem 1.2(3) and Proposition 4.3])**.** Let $n \geq 2$.

1. The map $\omega_i \mapsto \sigma_1\cdots\sigma_i$ $(1 \leq i \leq n-1)$ defines a surjective map $G(n-1,n) \twoheadrightarrow \mathcal{B}_n$, which is an isomorphism for $n = 2$ and $n = 3$.

2. The monoid $\mathcal{M}_n$ is the image of the Garside monoid $\mathcal{M}(n-1,n)$ under this quotient map. It is *not* a Garside monoid.

Proposition 6.2.2 could be an explanation of why the Garside structure $\langle\, a, b \mid aba = b^2 \,\rangle$ on $\mathcal{B}_3$ does not seem to generalize to $\mathcal{B}_n$, $n \geq 4$: it seems that the suitable framework for a generalization is torus knot groups rather that Artin's braid groups, but that $\mathcal{M}_n$ is close to being Garside since it is the projection of a Garside monoid. Concerning the question of whether $\mathcal{M}_n$ admits a finite presentation or not, we made in [79] the following conjecture, generalizing a conjecture made by Dehornoy for $n = 4$ (see [42, Question 3.8 and the paragraph after it]):

Conjecture 6.2.3 (G., 2022, [79, Conjecture 1.4])**.** Let $n \geq 4$. The monoid $\mathcal{M}_n$ admits a presentation with generators $\omega_1, \ldots, \omega_{n-1}$, and relations

$$\omega_1\omega_j\omega_i = \omega_{i+1}\omega_j, \quad \forall 1 \leq i < j \leq n-1.$$

Unfortunately, we noticed recently that Dehornoy's conjecture and our generalization are false for $n \geq 4$. This was published in an addendum [80] to the article [79].

[1] This map is an isomorphism when $n = 2$ or 3.

Proposition 6.2.4 (G., 2023, [80, Proposition 0.3])**.** Let $n \geq 4$. In the monoid defined by the presentation from Conjecture 6.2.3, we have $\omega_1\omega_3\omega_1\omega_3^2 = \omega_1\omega_2\omega_1\omega_3^2\omega_1$ but $\omega_3\omega_1\omega_3^2 \neq \omega_2\omega_1\omega_3^2\omega_1$, hence this monoid is not left-cancellative. Thus Conjecture 6.2.3 is false.

The question of whether $\mathcal{M}_n$ is finitely presented or not remains open.

6.3 Reflection-like quotients of torus knot groups

The Garside structures of Theorem 6.1.2 were first discovered in the case where $m = n + 1$ in [79]. In this case, as recalled in the previous section, the group $G(n-1, n)$ is an extension of $\mathcal{B}_n$, and I had not noticed at first that the group $\mathcal{G}(n-1, n)$ with the same presentation as $\mathcal{M}(n-1, n)$ was isomorphic to a torus knot group (this was pointed out to me by a referee of the paper [79]). Nevertheless, I had noticed that the quotient of $\mathcal{G}(3, 4)$ by the relation $\omega_1^2 = 1$ was isomorphic to the complex reflection group G_{12}, and that the quotient of $\mathcal{G}(4, 5)$ by $\omega_1^2 = 1$ was infinite. In the case $n = 3$, we have $\mathcal{G}(2, 3) \cong \mathcal{B}_3$, and ω_1 corresponds to σ_1, hence taking the quotient of $\mathcal{G}(3, 2)$ by $\omega_1^2 = 1$ yields the symmetric group. The natural question was thus to try to understand if the quotient of $\mathcal{G}(4, 5)$ (and more generally $\mathcal{G}(n-1, n)$, and then $\mathcal{G}(n, m)$...) by the relation $\omega_1^2 = 1$ (or a suitable generalization in the general case) had a natural structure of "(infinite) complex reflection group". This also raises the question of whether torus knot groups are the "braid groups" of some reflection group in some reasonable sense.

Every complex braid group appearing in the previous sections is attached to a complex reflection group W of rank two, i.e., such that $W \subseteq \mathrm{GL}_2(\mathbb{C})$ (as reflection group). Achar and Aubert [2] introduced a family of in general infinite groups, with the property that a group is a finite group in their family if and only if it is a finite complex reflection group of rank 2. This seemed to be a framework to consider for the above-mentioned problems.

Rather than the presentations (6.1.4), the suitable presentations to consider for this problem are (6.1.2) and (6.1.3). Indeed, in all cases where such a group is a complex braid group, the generators in these presentations are given by braided reflections, and the reflection group is obtained as a quotient by adding torsion on these generators. The quotients that one wishes to study as suggested above are the following groups.

Definition 6.3.1. Let $n, m \geq 2$ with $n < m$ and n, m pairwise coprime, and let $k \geq 2$. Define

$$W(k, n, m) := \left\langle \; x_1, x_2, \ldots, x_n \;\middle|\; \begin{array}{c} x_i^k = 1 \text{ for } i = 1, \ldots, n, \\ \underbrace{x_1 x_2 \cdots}_{m \text{ factors}} = \underbrace{x_2 x_3 \cdots}_{m \text{ factors}} = \cdots = \underbrace{x_n x_1 \cdots}_{m \text{ factors}} \end{array} \right\rangle, \tag{6.3.1}$$

which we call a *toric reflection group*.

6.3.1 J-groups and toric reflection groups

Let $a, b, c \geq 2$. Let $J\begin{pmatrix} a & b & c \end{pmatrix}$ be the group defined by the presentation

$$J\begin{pmatrix} a & b & c \end{pmatrix} := \langle \; s, t, u \mid s^a = t^b = u^c = 1, \; stu = tus = ust \; \rangle.$$

Let a', b', c' be three pairwise coprime positive integers[2] such that k' divides k for all $k \in \{a, b, c\}$. Let $J\begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix}$ be the normal closure in $J\begin{pmatrix} a & b & c \\ & & \end{pmatrix}$ of the elements $s^{a'}, t^{b'}$ and $u^{c'}$. These groups were defined by Achar and Aubert in [2], and are called *J-groups*. Note that $J\begin{pmatrix} a & b & c \\ 1 & 1 & 1 \end{pmatrix} = J\begin{pmatrix} a & b & c \\ & & \end{pmatrix}$, hence $J\begin{pmatrix} a & b & c \\ & & \end{pmatrix}$ is itself a J-group, and we call it the *parent* J-group of $J\begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix}$; in general, and also for other J-groups, parameters equal to 1 will be omitted in the second row. Achar and Aubert's main result is the following.

Theorem 6.3.2 (Achar-Aubert, 2008, [2, Theorem 1.2])**.** *Let G be a group. Then G is a finite J-group if and only if G is a finite complex reflection group of rank two. Moreover, for such a group, the parameters a, b, c, a', b', c' are uniquely determined up to permutation of the columns.*

This result is somehow reminiscent of Coxeter's theorem that a Coxeter group is finite if and only if it is a finite real reflection group.

Remark 6.3.3. Achar and Aubert's definition of J-groups does not yield a presentation of a J-group in general.

Achar and Aubert also showed [2, Proposition 4.2] that every J-group has a natural representation as a reflection group over $\mathbb{C}^2$ (but this representation is not faithful in general when the J-group is infinite, see [81, Example 2.20]).

Our first main result on toric reflection groups is that they are J-groups. Note that, as a byproduct, this also gives a presentation by generators and relations for a family of J-groups.

Theorem 6.3.4 (G., 2021, [81, Theorem 2.12])**.** Let $k, n, m \geq 2$ with n, m coprime (we do not necessarily assume $n < m$ here), and let $H = J\begin{pmatrix} k & n & m \\ & n & m \end{pmatrix} \trianglelefteq J\begin{pmatrix} k & n & m \\ & & \end{pmatrix} = G$. Then H has a presentation with generators $x_1, x_2, \ldots, x_n$ and relations (indices are taken modulo n)

$$\begin{aligned} &x_i^k = 1, \ \forall i = 1, \ldots, n, \\ &x_1 x_2 \cdots x_m = x_i x_{i+1} \cdots x_{i+m-1}, \ \forall i = 2, \ldots, n. \end{aligned}$$

If $n < m$ we therefore have $W(k, n, m) \cong H$. In terms of the generators of G we have $x_i = t^{i-1} s t^{-i+1}$ for all $i = 1, \ldots, n$.

This theorem also motivates the use of the terminology "toric reflection group", since by Theorem 6.3.2 Achar and Aubert's J-groups are generalizations of complex reflection groups of rank two in some sense. Using Achar and Aubert's classification [2, Theorem 1.3], we deduce the following.

[2]The coprimality assumtion is needed for the uniqueness part of Theorem 6.3.2, but the definition still makes sense without such an assumption.

Corollary 6.3.5 (G., 2021, [81, Example 2.14])**.** A toric reflection group $W(k, n, m)$ is finite if and only if

$$(k, n, m) \in \{((3, 2, 3), (4, 2, 3), (5, 2, 3), (3, 2, 5), (2, 3, 4), (2, 3, 5), (2, 2, \ell) \text{ with odd } \ell\}.$$

Such groups are precisely the finite complex reflection groups of rank 2 with a single orbit of reflection hyperplanes, namely

$$G_4, G_8, G_{16}, G_{20}, G_{12}, G_{22}, G(\ell, \ell, 2) = I_2(\ell) \text{ with odd } \ell.$$

In all these cases, the torus knot group $G(n, m)$ is isomorphic to the complex braid group of the reflection group.

6.3.2 Center of toric reflection groups

Let k, n, m be as above with $k, n, m \geq 2$, $n < m$, and n and m coprime. Let $W_{k,n,m}$ denote the Coxeter group of rank three defined by

$$W_{k,n,m} = \left\langle r_1, r_2, r_3 \;\middle|\; \begin{array}{c} r_1^2 = r_2^2 = r_3^2 = 1, \\ (r_1 r_2)^k = (r_2 r_3)^n = (r_3 r_1)^m = 1 \end{array} \right\rangle. \tag{6.3.2}$$

Let $W^+_{k,n,m}$ be its alternating subgroup, *i.e.*, the kernel of the map $W_{k,n,m} \longrightarrow \mathbb{Z}/2\mathbb{Z}$, $r_i \mapsto 1$, equivalently the subgroup of elements of even length. Let $c = (x_1 x_2 \cdots x_n)^m \in W(k, n, m)$, which is easily seen to be central in $W(k, n, m)$.

Theorem 6.3.6 (G., 2021, [81, Theorems 1.4 and 3.3])**.** The group $W(k, n, m)$ is a central extension of $W^+_{k,n,m}$ by the subgroup $\langle c \rangle$. That is, we have a short exact sequence

$$1 \longrightarrow \langle c \rangle \longrightarrow W(k, n, m) \longrightarrow W^+_{k,n,m} \longrightarrow 1.$$

In most cases, the group $W_{k,n,m}$ is infinite and irreducible. The determination of the center of a toric reflection group requires the following result.

Proposition 6.3.7 (G., 2021, [81, Proposition 1.5 and 3.1])**.** Let (W, S) be a Coxeter system of rank at least three. Let W^+ be the alternating subgroup of W. Then the center $Z(W^+)$ of W^+ is included in the center of W. In particular, if (W, S) is infinite, irreducible and of rank at least three, then $Z(W^+)$ is trivial.

Note that similar results also exist, see for instance [122, Proposition 6.4] or [125], where it is shown that in an irreducible, infinite and non-affine Coxeter group, every subgroup of finite index has a trivial center.

Together with Theorem 6.3.6 and a case-by-case check in the cases where $W_{k,n,m}$ is finite, Proposition 6.3.7) yields the following.

Theorem 6.3.8 (G., 2021, [81, Corollary 1.6 and Theorem 3.3])**.** The center of $W(k, n, m)$ is cyclic, generated by c.

6.3.3 Classification and "braid group"

In the case where a toric reflection group $W(k,n,m)$ is finite, the reflections are precisely the conjugates of the nontrivial powers of the x_i's. In the case where the group is infinite, these elements still act by reflections in Achar and Aubert's representation. We thus denote by R the set of conjugates of nontrivial powers of the x_i's inside $W(k,n,m)$, and call them the *reflections* of $W(k,n,m)$. We say that two toric reflection groups W, W' with respective sets of reflections R, R' are *reflection-isomorphic*, written $W \cong_{\text{ref}} W'$, if there is a group isomorphism $\varphi : W \longrightarrow W'$ such that $\varphi(R) = R'$.

Theorem 6.3.9 (G., 2021, [81, Theorem 1.2])**.** Let $k, k', n, n', m, m' \geq 2$ with $n < m$, $n' < m'$, n and m coprime, and n' and m' coprime. Then

$$W(k,n,m) \cong_{\text{ref}} W(k',n',m') \Leftrightarrow k = k', n = n', \text{ and } m = m'.$$

This theorem is important because it shows that the group $G(n,m)$, which surjects onto $W(k,n,m)$, can be entirely recovered from the reflection group structure of $W(k,n,m)$, and can thus be called the "braid group" of $W(k,n,m)$ without ambiguity. Note that, in the cases where $W(k,n,m)$ is finite, it is isomorphic to the complex braid group of $W(k,n,m)$ (see Corollary 6.3.5 above). We thus have, in the spirit of Achar and Aubert's Theorem 6.3.2:

Corollary 6.3.10 (G., 2021, [81, Corollary 1.3])**.** To each toric reflection group $W(k,n,m)$, one can associate a group

$$B(W(k,n,m)) := G(n,m),$$

which is a Garside group, surjects onto $W(k,n,m)$, and coincides with the complex braid group of $W(k,n,m)$ whenever $W(k,n,m)$ is finite.

6.4 Open problems

The works from the previous sections leave many open problems.

Problem 6.4.1. Can we find a presentation for every J-group?

Problem 6.4.2. Can we define a "braid group" for every J-group? Is it a Garside group? Is it the fundamental group of the complement of a link?

Problem 6.4.3. Do toric reflection groups have a solvable word problem?

Problem 6.4.4. Can one realize torus knot groups in a nontrivial way as interval groups, for instance using toric reflection groups?

Problem 6.4.5. Do toric reflection groups have faithful linear representations as complex reflection groups?

Problem 6.4.6. Do torus knot groups admit linear representations similar to those known for braid groups? For instance, can we define a Burau representation? Is it faithful?

Problems 6.4.1 to 6.4.3 are part of the PhD project of my student Igor Haladjian.

Chapter 7

Bruhat order on quotients

List of relevant publications

1. N. Chapelier-Laget and T. Gobet, *Elements of minimal length and Bruhat order on fixed-point cosets of Coxeter groups*, preprint (2023). `https://arxiv.org/abs/2311.06827`.

2. P.-E. Chaput, L. Fresse, and T. Gobet, *Parametrization, structure and Bruhat order of certain spherical quotients*, Represent. Theory **25** (2021), 935-974.

This chapter collects works realized between 2017 and 2023.

The (strong) Bruhat order $\leq$ recalled in Section 2.1.23 above can be defined on an arbitrary Coxeter system. In the case where W is a finite Weyl group attached to a connected reductive group G with Borel subgroup B, one has as a consequence of the Bruhat decomposition that the B-orbits on the flag variety $\mathcal{B} = G/B$ are given by the Schubert cells $C_w := BwB/B$, $w \in W$. The Bruhat order describes the inclusion of the orbit closures $X_w := \overline{C_w}$ for the Zariski topology, that is, for $u, v \in W$ we have

$$u \leq v \Leftrightarrow C_u \subseteq X_v.$$

More generally, one can consider the action of closed subgroups $H \subseteq G$ which act on $\mathcal{B}$ with finitely many orbits. It is then natural to

- Determine a parametrization of the H-orbits on $\mathcal{B}$, equivalently, of the B-orbits on G/H,
- Wonder if there is a combinatorial description of the inclusions of orbit closures, for instance in terms of a subset or quotient of the Weyl group W of G.

There are several situations where an answer to the above question is positive. One of the most basic situations generalizing the one presented above is as follows. One can consider the action of a standard parabolic subgroup P of G (that is, a subgroup containing B). Then P still acts on $\mathcal{B}$ with finitely many orbits, and there is a subset J of the simple system S of W such that

$$\mathcal{B} = \coprod_{w \in {}^J W} PwB/B,$$

where ${}^JW := \{w \in W \mid \ell(sw) > \ell(w),\ \forall s \in S\}$. The inclusion or orbit closures is given by the restriction of the (strong) Bruhat order $\leq$ to ${}^JW$. In the case where $P = B$, we have $J = \emptyset$, and ${}^JW = W$. The set ${}^JW$ has many remarkable algebraic and combinatorial properties, some of which we list below:

1. It is a set of representatives of the right cosets modulo W_J, where $W_J = \langle s \in J \rangle \subseteq W$ (recall that (W_J, J) is again a Coxeter system).
2. It follows from 1 that every $w \in W$ admits a unique decomposition $w = w_J {}^Jw$ with $w_J \in W_J$ and ${}^Jw \in^J W$. Moreover, for such a decomposition, we have $\ell(w) = \ell(w_J) + \ell({}^Jw)$.
3. Every coset W_Jw admits a unique element of minimal lenth, given by ${}^Jw$. Combined with 2, we get that every element in a coset is greater than or equal to the minimal element for the left weak order (hence for the strong Bruhat order).
4. The restriction of the strong Bruhat order to ${}^JW$ yields a poset $({}^JW, \leq)$ which is graded by the restriction of the length function on W.
5. The definition of ${}^JW$ and the properties 1 to 4 above can be given and hold true for an arbitrary Coxeter system (W, S).

See for instance [24, Section 2.4]. The grading on $({}^JW, \leq)$ in the general case was observed by Deodhar [48, Corollary 3.8].

At the purely algebraic level of Coxeter groups, there are many possible generalizations of the structures appearing above. For instance, one can wonder what happens if one replaces W_J, which is a very particular example of reflection subgroup of W, by a an arbitrary reflection subgroup W' of W.

Dyer has shown [58, Theorem 1.4] that every coset xW' still has a unique element of minimal length, and that the graph induced on xW' by the Bruhat graph on W is isomorphic to the Bruhat graph on W'.

One could also replace W_J by other subgroups admitting a canonical structure of Coxeter group, such as subgroups of the form W_L^θ, where W_L is a standard parabolic subgroup of W and θ is an automorphism of Coxeter group of (W_L, L) (see Section 2.1.4 above). We propose such an approach in Section 7.3 below for θ such that $\theta^2 = \mathrm{Id}$, which corresponds to the first paper above. Of course, it is also a natural question to wonder if such generalizations of the strong Bruhat order describe inclusion of orbit closures in certain situations. The motivation for considering subgroups such as W_L^θ's comes from such a specific situation corresponding to the earlier second paper above, which we present in Section 7.2 below.

7.1 Bruhat order on quotients

Let (W, S) be an arbitrary Coxeter system and W' be a subgroup of W also admitting a structure of Coxeter group. Note that, unlike in the case where W' is a standard parabolic subgroup W_J (or even a reflection subgroup W'), in the case where W' is of the form W_L^θ, there is no reason for a given coset xW_L^θ to have a unique element of minimal length; we give an easy example of this phenomenon.

Example 7.1.1. Let (W, S) be of type A_3 with $s_i = (i, i+1)$ for all $i = 1, 2, 3$, let $L = \{s_1, s_3\}$, and let θ be the automorphism of Coxeter group of (W_L, L) exchanging s_1 and s_3. Then W_L^θ is of type A_1, with generator $s_1 s_3$. The coset $s_1 W_L^\theta$ has two elements of minimal length, given by s_1 and s_3.

We thus define an analogue of the restriction of the Bruhat order to ${}^J W$ or to the set of elements of minimal length in cosets modulo a reflection subgroup W' as follows. In fact, such a definition can be given for an arbitrary subgroup H of a Coxeter group W, with H not necessarily admitting a structure of Coxeter group.

By "Bruhat order" we will always mean "strong Bruhat order".

Definition 7.1.2. Let (W, S) be a Coxeter system. Let $H \subseteq W$ be a subgroup. We define a Bruhat-like order on W/H, which we still call the *Bruhat order* on W/H, induced by the Bruhat order on W by

$$xH \leq yH \Leftrightarrow \forall v \in yH, \exists u \in xH \text{ such that } u \leq v.$$

Note that for $H = \{1\}$ we recover the Bruhat order on W. For $H = W_J$ where $J \subseteq S$, we recover the restriction of the strong Bruhat order on W to $W^J := \{w \in W \mid \ell(ws) > \ell(w), \forall s \in S\}$. For a reflection subgroup W' of W we recover the restriction of the Bruhat order to the set of elements x which are of minimal length in their coset xW'.

7.2 A Bruhat-like order coming from 2-nilpotent matrices

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Let $G = \mathrm{GL}_n(\mathbb{K})$ and let $e \in \mathfrak{gl}_n(\mathbb{K})$ be a 2-nilpotent matrix of rank r. We thus have $r \leq \frac{n}{2}$. Let $Z := \{g \in \mathrm{GL}_n(\mathbb{K}) \mid ge = eg\}$. It was shown by Panuyshev [119] that the action of Z on $\mathcal{B} = \mathrm{GL}_n(\mathbb{K})/B$ has finitely many orbits. Here B denotes the subgroup of upper-triangular invertible matrices of G. Up to conjugating e by a suitable matrix, we may assume that

$$e = \begin{pmatrix} 0 & 0 & \mathrm{Id}_r \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so that the centralizer Z is given by

$$Z = \left\{ \begin{pmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & a \end{pmatrix} : a \in \mathrm{GL}_r(\mathbb{C}),\ b \in \mathrm{GL}_{n-r}(\mathbb{C}) \right\}.$$

Let W be the Weyl group of type A_{n-1}, $I = \{s_1, \ldots, s_{r-1}\}$, $J = \{s_{n-r+1}, \ldots, s_{n-1}\}$, $K = \{s_{r+1}, \ldots, s_{n-r-1}\}$, where s_i denotes the simple transposition $(i, i+1)$. Let $L = I \cup K \cup J$, and let θ be the unique automorphism of Coxeter group of (W_L, L) swapping I and J and acting as the identity on K, i.e., defined by $\theta(s) = s$ for all $s \in K$, and $\theta(s_i) = s_{n-r+i}$ for all $i = 1, \ldots, r-1$. Then W_L^θ is again a Coxeter system (see Section 2.1.4), with Coxeter generating set S_L^θ given by $\{s\theta(s) \mid s \in I\} \cup K$.

Given $w \in W$, set $[w] := wW_L^\theta$. We have the following.

Theorem 7.2.1 (Boos-Reineke, 2012, [25, Theorem 4.3], Bender-Perrin, 2019, [17, Lemma 7.3.1], Chaput-Fresse-G., 2021, [37, Theorems 7.2 and 9.1])**.** We have

1. The Z-orbits on $\mathcal{B}$ are parametrized by W/W_L^θ, via $[w] \mapsto Zw^{-1}B/B =: \mathbb{O}_{[w]}$.

2. One has
$$\mathbb{O}_{[w]} \subseteq \overline{\mathbb{O}_{[w']}} \Leftrightarrow [w] \leq [w'],$$
where $\leq$ is the Bruhat order on W/W_L^θ from Definition 7.

Remark 7.2.2. A parametrization of the Z-orbits on $\mathcal{B}$ was first given by Boos and Reineke [25, Theorem 4.3]. In fact, they consider B-orbits on G/Z, but there is an obvious bijection between B-orbits on G/Z and Z-orbits on G/B, which preserves orbit closures since the projection maps from G to G/B or G/Z are open. Boos and Reineke's parametrization, obtained using representations of quivers, is in terms of oriented link patterns. A parametrization with the above parametrizing set was given by Bender and Perrin [17, Lemma 7.3.1], where they also claim to describe orbit closures with a criterion that is equivalent to the one stated above, but the proof seems incomplete (see the discussion in [37, Section 9]). We reproved the criterion in point 2 above using Boos and Reineke's parametrization, and obtained a parametrization as in point 1 which in fact applies to a more general family of orbits. Also, using Theorem 7.2.4 below, we give a precise description of the covering relations, which is only partial in [25].

Example 7.2.3. In type A_3 with $r = 2$, hence $I = \{s_1\}$, $J = \{s_3\}$, $K = \emptyset$, the order $\leq$ on W/W_L^θ is given in Figure 7.1.

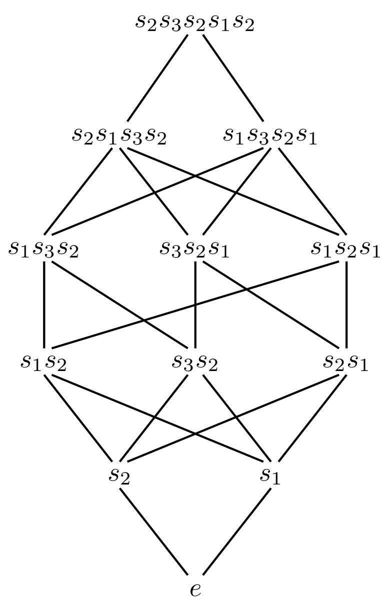


Figure 7.1: The order $\leq$ on W/W_L^θ in type A_3 with $I = \{s_1\}$, $J = \{s_3\}$, $K = \emptyset$, $\theta : s_1 \leftrightarrow s_3$.

Theorem 7.2.1 is a motivation for a deeper study of the cosets xW_L^θ and the Bruhat order on W/W_L^θ in the more general setting of an arbitrary Coxeter system, which is initiated with the results of the next section.

Let (W, S) be a Coxeter system, $L \subseteq S$, and θ be an automorphism of Coxeter group of (W_L, L) such that $\theta^2 = \mathrm{Id}$. Given $w \in W$, let $\mathrm{Min}(w)$ denote the set of elements of minimal length in wW_L^θ. Let

$$\mathcal{M} := \bigcup_{w \in W} \mathrm{Min}(w).$$

We denote by $\prec$ a covering relation either the poset $(W, \leq)$ or the poset $(W/W_L^\theta, \leq)$ (we use the same notations $\leq, \prec$ for both the posets $(W, \leq)$ and $(W/W_L^\theta, \leq)$, in general it will be clear from the context and notation which poset we are considering).

Let us finish the section with a result allowing one to get a precise description of the covering relations in the poset appearing in point 2 of Theorem 7.2.1 above, from what one can deduce that $(W/W_L^\theta, \leq)$ is graded in this specific case by the restriction of the length function on W to elements of minimal length in their cosets. This can be achieved combinatorially, in the following more general framework than the one from Theorem 7.2.1.

Theorem 7.2.4 (Chaput-Fresse-G., 2021, [37, Theorem 8.16])**.** Let (W, S) be a Coxeter system. Let $L \subseteq S$ be of the form $I \cup K \cup J$, where I, J and K are disjoint and disconnected, and there is a bijection f between I and J inducing a group isomorphism $W_I \cong W_J$. Let θ be the automorphism of Coxeter group of (W_L, L) defined by $\theta(s) = f(s)$ for all $s \in I$, $\theta(s) = f^{-1}(s)$ for all $s \in J$, and $\theta(s) = s$ for all $s \in K$. Let $w, w' \in \mathcal{M}$. The following are equivalent:

1. We have $[w'] \prec [w]$,
2. There are $u \in \mathrm{Min}(w)$ and $u' \in \mathrm{Min}(w')$ such that $u' \prec u$,
3. There is $u' \in \mathrm{Min}(w')$ such that $u' \prec w$.
4. For all $u \in \mathrm{Min}(w)$, there is $u' \in \mathrm{Min}(w')$ such that $u' \prec u$.

Corollary 7.2.5 (Chaput-Fresse-G., 2021, [37, Corollary 8.17])**.** With the assumptions of Theorem 7.2.4, the poset $(W/W_L^\theta, \leq)$ is graded, where the rank function ρ is given by

$$\rho([w]) := \min\{\ell(x) \mid x \in wW_L^\theta\}.$$

7.3 Properties of cosets and quotients modulo fixed-point subgroups

To establish Theorem 7.2.4 in [37], two particular results on cosets xW_L^θ are needed. The first one is a statement about elements of minimal length in cosets, and the second one is a generalization of the property that elements of minimal length in cosets xW_J are minimal for the restriction of the Bruhat order (the second part of property 3 in the list given at the beginning of this chapter). They are proven in [37, Lemmatas 8.7 and 8.8] for those L satisfying the assumptions of Theorem 7.2.4, and together with N. Chapelier [36] we later proved them for an arbitrary Coxeter system (W, S) and an arbitrary $L \subseteq S$ with automorphism θ of (W_L, L) of order at most two [36]. We present the general version of these properties as Theorems 7.3.2 and 7.3.3 below.

The setting here is the following. Let (W, S) be an arbitrary Coxeter system, and let $L \subseteq S$. Let θ be au automorphism of Coxeter group of (W_L, L) such that $\theta^2 = \mathrm{Id}$. We use the same notations as in the previous section.

A key ingredient in the proofs of Theorems 7.3.2 and 7.3.3 below is the following proposition.

Proposition 7.3.1 (Chapelier-G., 2023, [36, Proposition 1.2])**.** Let $w \in W$ and let $u \in wW_L^\theta \cap \mathcal{M}$. Let $z \in W_L^\theta$ such that $w = uz$ and let $x_1 x_2 \cdots x_k$ be an S_L^θ-reduced expression of z. For all $i = 0, \ldots, k-1$, exactly one of the following two situations happens

- either $\ell(ux_1 \cdots x_i) = \ell(ux_1 \cdots x_{i+1})$,
- or $ux_1 \cdots x_i < ux_1 \cdots x_{i+1}$.

In particular, if x_{i+1} is a reflection of W, then we are in the second situation since multiplying by a reflection changes the parity of length, while the first situation can only happen if x_{i+1} is not a reflection of W.

The two main results are then the following.

Theorem 7.3.2 (Chapelier-G., 2023, [36, Theorem 1.1])**.** Let u, v be two elements of minimal length in xW_L^θ. Let $y \in W_L^\theta$ such that $uy = v$. Let $x_1 x_2 \cdots x_k$ be an S_L^θ-reduced expression of y. Then

$$\ell(u) = \ell(ux_1) = \ell(ux_1x_2) = \cdots = \ell(ux_1x_2 \cdots x_{k-1}) = \ell(v).$$

In other (and weaker) words, there is a chain of elements of minimal length in the coset allowing one to pass from u to v, where at each step one just multiplies by an element of S_L^θ on the right.

Theorem 7.3.3 (Chapelier-G., 2023, [36, Theorem 1.3])**.** Let $w \in W$. There is $u \in wW_L^\theta \cap \mathcal{M}$ such that $u \leq w$.

Remark 7.3.4. Note that the conclusion of Theorem 7.3.3 does not hold in general without the assumption that $\theta^2 = \mathrm{Id}$. As a counterexample, consider a Coxeter system (W, S) of type D_4, with $S = \{s_0, s_1, s_2, s_3\}$, where s_0 is the simple reflection commuting with no other simple reflection. Let θ be an automorphism of (W_L, L) acting as a 3-cycle on $L := \{s_1, s_2, s_3\}$. Then W_L^θ has type A_1, with generator $s_1 s_2 s_3$. The coset $s_1 W_L^\theta$ has two elements s_1 and $s_2 s_3$, hence $s_2 s_3 W_L^\theta \cap \mathcal{M} = \{s_1\}$, but $s_1 \not\leq s_2 s_3$.

7.4 Open problems

Together with Chaput and Fresse, we are working on several questions related to the constructions made in this chapter. We list some of the questions we are interested in below, which are under current investigation.

Problem 7.4.1. Is the poset $(W_L^\theta, \leq)$ graded in general? (Corollary 7.2.5 only covers subsets $L \subseteq S$ and automorphisms of a certain form).

Problem 7.4.2. Can one define analogues of parabolic Kazhdan-Lusztig polynomials for the "Coxeter subgroups" W_L^θ? If the answer is positive, is there a natural categorical framework (like a Hecke category) where these polynomials have a natural interpretation?

Problem 7.4.3. For which W and which L, θ does the Bruhat order on W/W_L^θ from Definition 7.1.2 describe a situation of orbit closures inclusion, in the same way as in Theorem 7.2.1?

Problem 7.4.4. What is the most general class of "Coxeter subgroups" of a Coxeter system for which one can state and prove results such as Theorems 7.3.2 and 7.3.3?

Bibliography

[1] P. Abramenko and K.S. Brown, *Buildings. Theory and applications*, Graduate Texts in Mathematics 248, Springer, New York, 2008.

[2] P. Achar and A.-M. Aubert, *On rank 2 complex reflection groups*, Comm. Algebra **36** (2008), no. 6, 2092–2132.

[3] F. Aicardi and J. Juyumaya, *Markov trace on the algebra of braids and ties*, Mosc. Math. J. **16** (2016), no. 3, 397–431.

[4] D. Allcock, Braid pictures for Artin groups, *Trans. Amer. Math. Soc.* **354** (2002), 3455-3474.

[5] S. Ariki, *Representation theory of a Hecke algebra of* $G(r,p,n)$, J. Algebra **177** (1995), 164-185.

[6] S. Ariki, K. Koike, *A Hecke algebra of* $(\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n$ *and construction of its irreducible representations*, Adv. Math. **106** (1994), 216-243.

[7] D. Armstrong, *Generalized noncrossing partitions and combinatorics of Coxeter groups*, Mem. Amer. Math. Soc. **202** (2009), no. 949, x+159 pp.

[8] E. Artin, *Theorie der Zöpfe*, Abh. Math. Sem. Univ. Hamburg **4** (1925), no.1, 47–72.

[9] E. Artin, *Theory of braids*, Ann. of Math. (2) **48** (1947), 101–126.

[10] E. Bannai, *Fundamental groups of the spaces of regular orbits of the finite unitary reflection groups of dimension 2*, J. Math. Soc. Japan **28** (1976), no. 3, 447–454.

[11] G.T. Barkley and D.E. Speyer, *Combinatorial descriptions of biclosed sets in affine type*, preprint (2022), `https://arxiv.org/abs/2207.05998`.

[12] B. Baumeister, M. Dyer, C. Stump, P. Wegener, *A note on the transitive Hurwitz action on decompositions of parabolic Coxeter elements*, *Proc. Amer. Math. Soc. Ser. B* **1** (2014), 149-154.

[13] B. Baumeister and T. Gobet, *Simple dual braids, noncrossing partitions and Mikado braids of type* D_n, Bull. Lond. Math. Soc. **49** (2017), no.6, 1048-1065.

[14] B. Baumeister, T. Gobet, K. Roberts, P. Wegener, *On the Hurwitz action in finite Coxeter groups*, J. Group Theory **20** (2017), no.1, 103-131.

[15] B. Baumeister, D.F. Holt, G. Neaime, and S. Rees, *Interval groups related to finite Coxeter groups, Part II*, to appear in Trans. LMS.

[16] P. Bellingeri, L. Paris, and A.-L. Thiel, *Virtual Artin groups*, Proc. Lond. Math. Soc. **126**, Issue 1, 192-215.

[17] M. Bender and N. Perrin, *Singularities of closures of B-conjugacy classes of nilpotent elements of height 2*, Transform. Groups **24**(2019), no.3, 741–768.

[18] D. Bessis, *The dual braid monoid*, Ann. Sci. École Normale Supérieure **36** (2003), 647–683.

[19] D. Bessis, *A dual braid monoid for the free group*, J. Algebra **302** (2006), 275-309.

[20] D. Bessis, C. Bonnafé, and R. Rouquier, *Quotients et extensions de groupes de réflexions*, Math. Annalen **323** (2002), no. 3, 405-436.

[21] P. Biane, *Some properties of crossings and partitions*, Discrete Math. **175** (1997), no. 1-3, 41-53.

[22] J. Birman and T. Brendle, *Braids: a survey*, Handbook of Knot Theory, 19-103, Elsevier B.V., Amsterdam, 2005.

[23] J. Birman, K.H. Ko, and S.J. Lee, *A New Approach to the Word and Conjugacy Problems in the Braid Groups*, Adv. in Math. **139** (1998), 322–353.

[24] A. Björner and F. Brenti, *Combinatorics of Coxeter groups*, Graduate Text in Math. Springer, 2005.

[25] M. Boos and M. Reineke, *B-orbits of 2-nilpotent matrices and generalizations*. In: Highlights in Lie algebraic methods, pp. 147–166, Progr. Math., vol. 295, Birkhäuser/Springer, New York, 2012.

[26] R. Borcherds, *Coxeter groups, Lorentzian lattices, and $K3$ surfaces*, Internat. Math. Res. Notices **19** (1998), 1011–1031.

[27] N. Bourbaki, *Groupes et algèbres de Lie*, Éléments de mathématique, Chapitres IV-VI, Paris, Hermann, 1968.

[28] T. Brady and C. Watt, *Non-crossing partition lattices in finite real reflection groups*, Trans. Amer. Math. Soc. **360** (2008), no. 4, 1983-2005.

[29] C. Brav and H. Thomas, *Braid groups and Kleinian singularities*, Math. Ann. **351** (2011), no. 4, 1005–1017.

[30] E. Brieskorn and K. Saito, *Artin-Gruppen und Coxeter-Gruppen*, Invent. Math. **17** (1972), 245-271.

[31] B. Brink and R.B. Howlett, *Normalizers of parabolic subgroups in Coxeter groups*, Invent. Math. **136** (1999), 323-351.

[32] M. Broué, *Introduction to complex reflection groups and their braid groups*, Lecture Notes in Math., 1988 Springer-Verlag, Berlin, 2010, xii+138 pp.

[33] M. Broué, G. Malle, *Zyklotomische Heckealgebren*, in *Représentations unipotentes génériques et blocs des groupes réductifs finis*, Astérisque **212** (1993), 119-189.

[34] M. Broué, G. Malle, and R. Rouquier, *Complex reflection groups, braid groups, Hecke algebras*, J. Reine Angew. Math. **500** (1998), 127-190.

[35] R.W. Carter, *Conjugacy Classes in the Weyl group*, Compositio Math. **25** (1972), 1-59.

[36] N. Chapelier-Laget and T. Gobet, *Elements of minimal length and Bruhat order on fixed-point cosets of Coxeter groups*, preprint (2023). `https://arxiv.org/abs/2311.06827`.

[37] P.-E. Chaput, L. Fresse, and T. Gobet, *Parametrization, structure and Bruhat order of certain spherical quotients*, Represent. Theory **25** (2021), 935-974.

[38] E. Chavli, *The BMR freeness conjecture for the tetrahedral and octahedral family*, Comm. Algebra **46** (2018), 386-464.

[39] H.S.M. Coxeter, *Factor groups of the braid group*, Proceedings of the Fourth Can. Math. Cong., Banff 1957, University of Toronto (1959), 95–122.

[40] P. Dehornoy, *Three-dimensional realizations of braids*, J. London Math. Soc. **60** (1999), 108–132.

[41] P. Dehornoy, *Groupes de Garside*, Ann. Sci. École Norm. Sup. (4) **35** (2002), no. 2, 267-306.

[42] P. Dehornoy, *The subword reversing method*, Internat. J. Algebra Comput. **21** (2011), no. 1-2, 71-118.

[43] P. Dehornoy, F. Digne, D. Krammer, E. Godelle, and J. Michel. *Foundations of Garside theory*, Tracts in Mathematics **22**, Europ. Math. Soc. 2015.

[44] P. Dehornoy and L. Paris, *Gaussian groups and Garside groups, two generalisations of Artin groups*, Proc. London Math. Soc. (3) **79** (1999), no. 3, 569-604.

[45] P. Deligne, *Les immeubles des groupes de tresses généralisés*, Invent. Math. **17** (1972), 273-302.

[46] P. Deligne, Letter to E. Looijenga, 1974.
`http://homepage.rub.de/christian.stump/Deligne_Looijenga_Letter_09-03-1974.pdf`.

[47] E. Delucchi, G. Paolini, and M. Salvetti, *Dual structures on Coxeter and Artin groups of rank three*, to appear in Geometry & Topology.

[48] V.V. Deodhar, *Some characterizations of Bruhat ordering on a Coxeter group and determination of the relative Möbius function*, Invent. Math. **39** (1977), no. 2, 187–198.

[49] V.V. Deodhar, *On some geometric aspects of Bruhat orderings. II. The parabolic analogue of Kazhdan-Lusztig polynomials*, J. Algebra **111** (1987), no.2, 483–506.

[50] V.V. Deodhar, *A note on subgroups generated by reflections in Coxeter groups*, Arch. Math. (Basel) **53** (1989), no. 6, 543–546.

[51] F. Digne, *Présentations duales des groupes de tresses de type affine $\tilde{A}$, Dual presentations of braid groups of affine type $\tilde{A}$*, Comment. Math. Helv. **81** (2006), no. 1, 23-47.

[52] F. Digne, *A Garside presentation for Artin groups of type $\tilde{C}_n$*, Ann. Inst. Fourier **62** (2012), no. 2, 641-666.

[53] F. Digne and T. Gobet, *Dual braid monoids, Mikado braids and positivity in Hecke algebras*, Math. Z. **285** (2017), no. 1-2, 215-238.

[54] K. Duszenko, *Reflection length in non-affine Coxeter groups*, Bull. Lond. Math. Soc. **44** (2012), no. 3, 571-577.

[55] M.J. Dyer, *Hecke algebras and reflections in Coxeter groups*, Ph. D. Thesis, University of Sydney (1987).

[56] M.J. Dyer, *Modules for the dual nil Hecke ring*, preprint. `http://www3.nd.edu/~dyer/papers/nilhecke.pdf`.

[57] M.J. Dyer, *Reflection subgroups of Coxeter systems*, J. of Algebra **135** (1990), Issue 1, 57-73.

[58] M.J. Dyer, *On the Bruhat graph of a Coxeter system*, Compositio Math. **78** (1991), no. 2, 185-191.

[59] M.J. Dyer, *Hecke algebras and shellings of Bruhat intervals II: Twisted Bruhat orders*, in *Kazhdan-Lusztig theory and related topics*, Contemp. Math. **139** (1992), AMS, 141-165.

[60] M.J. Dyer, *On minimal lengths of expressions of Coxeter group elements as products of reflections*, Proc. Amer. Math. Soc. **129** (2001), no. 9, 2591-2595.

[61] M.J. Dyer, *On the Weak Order of Coxeter Groups*, Canadian Journal of Mathematics **71** (2019), 2, 299-336.

[62] M.J. Dyer, *n-low elements and maximal rank k reflection subgroups of Coxeter groups*, J. Algebra **607** (2022), part A, 139–180.

[63] M.J. Dyer and G.I. Lehrer, *On positivity in Hecke algebras*, Geom. Ded. **25** (1990), 115-125.

[64] T. Edgar, *Sets of reflections defining twisted Bruhat orders*, J. of Algebraic Combinatorics **26** (2007), no. 3, 357-362.

[65] B. Elias and G. Williamson, *The Hodge theory of Soergel bimodules*, Ann. of Math. **180** (2014), 1089-1136.

[66] B. Elias and G. Williamson, *Soergel calculus*, Represent. Theory **20** (2016), 295–374.

[67] E. Fadell and L. Neuwirth, *Configuration spaces*, Math. Scand. **10** (1962), 111–118.

[68] D. Garber, *Braid group cryptography*, Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap., 19 World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2010, 329–403.

[69] O. Garnier, *Uniqueness of roots up to conjugacy in circular and hosohedral-type Garside groups*, preprint (2023). `https://arxiv.org/abs/2307.07809`.

[70] F.A. Garside, *The braid group and other groups*, Quart. J. Math. Oxford Ser. **20** (1969), no. 2, 235–254.

[71] M. Geck and G. Pfeiffer, *Characters of finite Coxeter groups and Iwahori-Hecke algebras*, London Mathematical Society Monographs, New Series, 21. The Clarendon Press, Oxford University Press, New York, (2000), 446 pp.

[72] T. Gobet, *Bases of Temperley-Lieb algebras*, PhD thesis, Université de Picardie, 2014. `https://www.idpoisson.fr/gobet/wp-content/uploads/sites/56/2023/08/these.pdf`.

[73] T. Gobet, *Categorification of the Temperley-Lieb algebra by bimodules*, J. Algebra **419** (2014), 277-317.

[74] T. Gobet, *Noncrossing partitions, fully commutative elements and bases of the Temperley-Lieb algebra*, J. Knot Theory Ramifications **25** (2016), no. 6, 27 pp.

[75] T. Gobet, *Twisted filtrations of Soergel bimodules and linear Rouquier complexes*, J. Algebra **484** (2017), 275–309.

[76] T. Gobet, *Coxeter-Catalan combinatorics and Temperley-Lieb algebras*, Algebr. Represent. Theory **24** (2021), 169-201.

[77] T. Gobet, *On cycle decompositions in Coxeter groups*, Sém. Lothar. Combin. **78B** (2017), Art. 45, 12 pp.

[78] T. Gobet, *Dual Garside structures and Coxeter sortable elements*, J. Comb. Algebra **4** (2020), no. 2, 167–213.

[79] T. Gobet, *On some torus knot groups and submonoids of the braid groups*, J. Algebra **607** (2022), Part B, 260-289.

[80] T. Gobet, *Addendum to "On some torus knot groups and submonoids of the braid groups" [J. Algebra 607 (Part B) (2022) 260–289]*, J. Algebra **633** (2023), 666–667.

[81] T. Gobet, *Toric reflection groups*, *Journal of the Australian Mathematical Society*, online first (2023).

[82] T. Gobet, *A new Garside structure on torus knot groups and some complex braid groups*, to appear in *Journal of Knot Theory and its Ramifications* (2023).

[83] T. Gobet, *On maximal dihedral reflection subgroups and generalized noncrossing partitions*, preprint (2023). `https://arxiv.org/abs/2307.16791`.

[84] T. Gobet and N. Williams, *Noncrossing partitions and Bruhat order*, European J. Combin. **53** (2016), 8-34.

[85] T. Gobet and B. Rognerud, *Odd and even Fibonacci lattices arising from a Garside monoid*, preprint 2023. `https://arxiv.org/abs/2301.00744`.

[86] T. Gobet and A.-L. Thiel, *On generalized categories of Soergel bimodules in type $A2$*, C. R. Math. Acad. Sci. Paris **356** (2018), no. 3, 258–263.

[87] T. Gobet and A.-L. Thiel, *A Soergel-like category for complex reflection groups of rank one*, Math. Z. **295**, (2020), 643-665.

[88] T. Gobet, A. Henderson, and I. Marin, *Braid groups of normalizers of reflection subgroups*, Ann. Inst. Fourier **71** (2021), no. 6, 2273-2304.

[89] T. Gobet and I. Marin, *Hecke Algebras of Normalizers of Parabolic Subgroups*, Algebr. Represent. Theor. **26** (2023), 1609-1639.

[90] I. Grojnowski and M. Haiman, *Affine Hecke algebras and positivity of LLT and Macdonald polynomials*, preprint (2007).

[91] J.-Y. Hée, *Systèmes de racines sur un anneau commutatif totalement ordonné*, Geom. Dedicata **37** (1991), 65-102.

[92] D. Hernandez and B. Leclerc, *Cluster algebras and quantum affine algebras*, Duke Math. J. **154** (2010), no. 2, 265–341.

[93] P.N. Hoefsmit, *Representations of Hecke Algebras of Finit Groups with BN-Pairs of Classical type*, PhD thesis, University of British Columbia, August 1974.

[94] R.B. Howlett, *Normalizers of Parabolic Subgroups of Reflection Groups*, J. of the London Math. Soc. **21** (1980), 62-80.

[95] J. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Stud. Adv. Math., 29, Cambridge University Press, Cambridge, 1990. xii+204 pp.

[96] K. Igusa and R. Schiffler, *Exceptional sequences and clusters*, J. Algebra **323** (2010), no. 8, 2183–2202.

[97] L.T. Jensen, *The 2-braid group and Garside normal form*, Math. Z. **286** (2017), no. 1-2, 491–520.

[98] D. Kazhdan and G. Lusztig, *Representations of Coxeter Groups and Hecke Algebras*, Invent. Math. **53** (1979), 165-184.

[99] D. Kazhdan and G. Lusztig, *Schubert varieties and Poincaré duality*, Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), Proc. Sympos. Pure Math., XXXVI (1980), Amer. Math. Soc., Providence, R.I., 185-203.

[100] M. Khovanov, P. Seidel, *Quivers, Floer cohomology, and braid group actions*, J. Amer. Math. Soc. **15** (2002), no. 1, 203-271.

[101] G.I. Lehrer and D.E. Taylor, *Unitary reflection groups*, Austral. Math. Soc. Lect. Ser., 20, Cambridge University Press, Cambridge, 2009, viii+294 pp.

[102] G.I. Lehrer, N. Xi, *On the injectivity of the braid group in the Hecke algebra*, Bull. Austral. Math. Soc. **64** (2001), no. 3, 487-493.

[103] J.B. Lewis, J. McCammond, T.K. Petersen, P. Schwer, *Computing reflection length in an affine Coxeter group*, Trans. Amer. Math. Soc. **371** (2019), no. 6, 4097–4127.

[104] A. Licata and H. Queffelec, *Braid groups of type ADE, Garside monoids, and the categorified root lattice*, Ann. Sci. Éc. Norm. Supér. (4) **54** (2021), no. 2, 503–548.

[105] G. Lusztig, *Cells in affine Weyl groups*, Adv. Stud. Pure Math. **6** (1985), 255-287, in "Algebraic groups and related topics".

[106] G. Lusztig, *Hecke algebras with unequal parameters*, CRM Monographs Ser. **18**, AMS, Providence, RI, 2003.

[107] I. Marin, *The cubic Hecke algebra on at most 5 strands*, J. Pure Appl. Algebra **216** (2012), 2754-2782.

[108] I. Marin, *The freeness conjecture for Hecke algebras of complex reflection groups, and the case of the Hessian group G_{26}*, J. Pure Appl. Algebra **218** (2014), 704-720.

[109] I. Marin, *Artin groups and Yokonuma–Hecke algebras*, Int. Math. Res. Not. IMRN **2018** (2018), 4022–4062.

[110] I. Marin, *Lattice extensions of Hecke algebras*, J. Algebra **503** (2018), 104-120.

[111] I. Marin, *Proof of the BMR freeness conjecture for G_{20} and G_{21}*, J. Symbolic Computations **92** (2019), 1-14.

[112] I. Marin, G. Pfeiffer, *The BMR freeness conjecture for the 2-reflection groups*, Math. of Computation **86** (2017), 2005-2023.

[113] J. McCammond, *Dual euclidean Artin groups and the failure of the lattice property*, J. Algebra **437** (2015), 308-343.

[114] J. McCammond and R. Sulway, *Artin groups of Euclidean type*, Invent. Math. **210** (2017), no.1, 231–282.

[115] J. Michel, *A note on words in braid monoids*, J. Algebra **215** (1999), no. 1, 366–377.

[116] P. Moszkowski, *Addendum to: "Involutions and reflection subgroups of finite Coxeter groups"*, European J. Combin. **30** (2009), no.8, 1913–1918.

[117] B. Mühlherr, *Coxeter groups in Coxeter groups*, Finite geometry and combinatorics (Deinze, 1992), 277–287, London Math. Soc. Lecture Note Ser., 191, Cambridge Univ. Press, Cambridge, 1993.

[118] K. Muraleedaran and D.E. Taylor, *Normalisers of parabolic subgroups in finite unitary reflection groups*, J. Algebra **504** (2018), 479-505.

[119] D. I. Panyushev, *Complexity and nilpotent orbits*, Manuscripta Math. **83** (1994), no. 3–4, 223–237.

[120] G. Paolini and M. Salvetti, *Proof of the $K(\pi,1)$ conjecture for affine Artin groups*, Invent. Math. **224** (2021), no. 2, 487–572.

[121] L. Paris, *Artin monoids inject in their groups*, Commentarii Mathematici Helvetici **77** (2002), 609–637.

[122] L. Paris, *Irreducible Coxeter groups*, Internat. J. Algebra Comput. **17** (2007), no. 3, 427–447.

[123] M. Picantin, *Petits groupes gaussiens*, PhD Thesis, Université de Caen, 2000. `https://www.irif.fr/~picantin/papers/phd.pdf`.

[124] M. Picantin, *Automatic structures for torus link groups*, J. Knot Theory Ramifications **12** (2003), no. 6, 833-866.

[125] D. Qi, *On irreducible, infinite, nonaffine Coxeter groups*, Fund. Math. **193** (2007), no. 1, 79–93.

[126] N. Reading, *Clusters, Coxeter-sortable elements and noncrossing partitions*, Trans. Amer. Math. Soc. **359** (2007), no. 12, 5931-5958.

[127] N. Reading, *Noncrossing partitions and the shard intersection order*, J. Algebraic Combin. **33** (2011), no. 4, 483–530.

[128] N. Reading, D.E. Speyer, *Sortable elements in infinite Coxeter groups*, Trans. Amer. Math. Soc. **363** (2011), no. 2, 699-761.

[129] V. Reiner, V. Ripoll, and C. Stump, *On non-conjugate Coxeter elements in well-generated reflection groups*, Math. Z. **285** (2017), no. 3-4, 1041–1062.

[130] S. Riche, *La théorie de Hodge des bimodules de Soergel, d'après Soergel et Elias-Williamson*, Astérisque **414** (2019), Séminaire Bourbaki Exp. No. 1139, 125–165.

[131] D. Rolfsen, *Knots and links*, Mathematics Lecture Series, No. 7. Publish or Perish, Inc., Berkeley, Calif., 1976. 439 pp.

[132] R. Rouquier, *Categorification of braid groups*, preprint (2004). http://arxiv.org/abs/math/0409593.

[133] R. Rouquier, *Categorification of $\mathfrak{sl}_2$ and braid groups*, Trends in representation theory of algebras and related topics, Contemp. Math. **406** (2006), Amer. Math. Soc., Providence, RI, 137-167.

[134] O. Schreier, *Über die Gruppen $A^aB^b = 1$*, Abh. Math. Sem. Hamburg **3** (1923), 167–169.

[135] G.C. Shephard and J.A. Todd, *Finite unitary reflection groups*, Canad. J. Math. **6** (1954), 274–304.

[136] W. Soergel, *The combinatorics of Harish-Chandra bimodules*, J. Reine Angew. Math. **429** (1992), 49-74.

[137] W. Soergel, *Kazhdan-Lusztig polynomials and indecomposable bimodules over polynomial rings*, J. Inst. Math. Jussieu **6** (2007), 501-525.

[138] T.A. Springer, *Quelques applications de la cohomologie d'intersection*, Astérisque, **92–93** Société Mathématique de France, Paris, 1982, pp. 249–273.

[139] R. Steinberg, *Differential equations invariant under finite reflection groups*, Trans. Amer. Math. Soc. **112** (1964), 392–400.

[140] R. Steinberg, *Lectures on Chevalley groups*, Univ. Lecture Ser., 66, American Mathematical Society, Providence, RI, 2016, xi+160 pp.

[141] S. Tsuchioka, *BMR freeness for icosahedral family*, Exp. Math. **29** (2020), 234-245.

[142] P. Wegener and S. Yahiatene, *Reflection factorizations and quasi-Coxeter elements*, J. Comb. Algebra **7** (2023), no.1, 127–157.

[143] M.G. Zinno, *A Temperley-Lieb basis coming from the braid group*, J. Knot Theory Ramifications **11** (2002), 575-599.