Bruhat orders, fixed-point subgroups, and orbit closures

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Bruhat order on the symmetric group

- Let $G = \mathrm{GL}_n(\mathbb{C})$, $B \subseteq G$ the subgroup of upper-triangular invertible matrices. Recall the Bruhat decomposition
$$G = \coprod_{w \in \mathfrak{S}_n} BwB.$$
- Consider the flag variety $\mathcal{B} = G/B$. It is a projective algebraic variety. It follows from the Bruhat decomposition that the B-orbits in $\mathcal{B}$ are given by $C_w = BwB/B, \ w \in \mathfrak{S}_n$ ("Schubert cell").
- The Zarisky closure $X_w := \overline{C_w}$ is a union of B-orbits, and one defines the *strong Bruhat order* on $\mathfrak{S}_n$ by
$$x \leq w \Leftrightarrow C_x \subseteq X_w.$$
- Combinatorial criterion: $x \leq w$ iff every reduced expression of w in the simple transpositions has a subword which is a red. ex. for x, iff there is a red. ex. of w which has a red. ex. for x as a subword.

Bruhat order on $\mathfrak{S}_3$

Example: $W = \mathfrak{S}_3$, $S = \{s_1 = (1,2), s_2 = (2,3)\}$.

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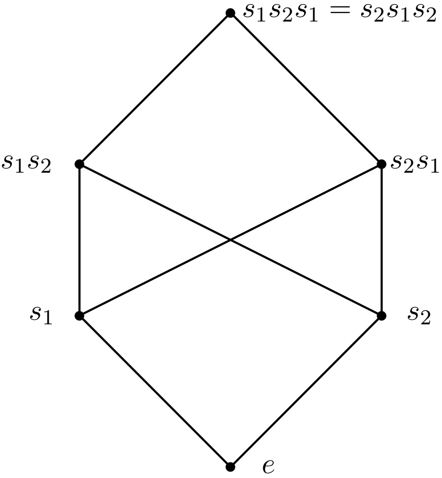


Coxeter groups

- (Coxeter matrix) Let S be a (finite) set, let $(m_{s,t})_{s,t\in S}$ be a matrix with $m_{s,t} \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$ such that
  - $m_{s,s} = 1$ for all $s \in S$,
  - $m_{s,t} = m_{t,s} \geq 2$ if $s \neq t$.
- (Coxeter group) Fix a Coxeter matrix $(m_{s,t})_{s,t\in S}$ and associate to it the group

  $$W = \langle s \in S \mid (st)^{m_{s,t}} = 1, \forall s,t \in S \text{ s.t. } m_{s,t} \neq \infty \rangle$$

  We may rewrite the presentation in the form

  $$\left\langle s \in S \;\middle|\; \begin{array}{l} s^2 = 1,\ \forall s \in S, \\ \underbrace{st\cdots}_{m_{s,t}\text{ letters}} = \underbrace{ts\cdots}_{m_{t,s}\text{ letters}},\ s \neq t \text{ and } m_{s,t} \neq \infty \end{array} \right\rangle$$

- Let $\ell : W \longrightarrow \mathbb{Z}_{\geq 0}$ denote the length function wrt S, and $T = \bigcup_{w\in W} wSw^{-1}$.
- Coxeter graph Γ_W: vertices are elements of S, with an edge whenever $m_{s,t} \neq 2$, labelled by $m_{s,t}$ if $m_{s,t} \geq 4$.

Generalizations of the Bruhat order

- (geometric setting) Replace G by any reductive group with a Borel subgroup B. Then one still has a generalized flag variety, Schubert cells parametrized by the Weyl group W (which is a Coxeter group), etc. More generally, consider the orbits of the action on $\mathcal{B} = G/B$ of closed subgroups $H \subseteq G$ acting with finitely many orbits ("spherical subgroups").
- (algebraic setting) Define the Bruhat order $\leq$ on an arbitrary Coxeter group W by the combinatorial criterion (key tool in Kazhdan-Lusztig theory):

Definition

Let (W, S) be a Coxeter group. Let $x, y \in W$. Define the *strong Bruhat order* on W by $x \leq y$ iff every reduced decomposition of y admits a subword which is a reduced expression of x.

Another example (parabolic quotients)

- Let G be a reductive group with Borel subgroup B. Let P be a parabolic subgroup of G (i.e., containing B). Since B acts on $\mathcal{B}$ with finitely orbits, P also acts with finitely many orbits. There is a subset J of the simple system S of W such that

$$G = \coprod_{w \in W^J} P w^{-1} B,$$

where

$$W^J = \{w \in W \mid \ell(ws) = \ell(w) + 1 \ \forall s \in J\}.$$

Moreover, the orbit closures are described by the restriction of the strong Bruhat order on W to W^J.

Parabolic quotients (general case)

- Let (W, S) be an arbitrary Coxeter system. Let $J \subseteq S$ and $W_J = \langle J \rangle$. Then (W_J, J) is again a Coxeter system, and

$$W^J := \{w \in W \mid \ell(ws) = \ell(w) + 1 \ \forall s \in J\}$$

is a set of representatives of W/W_J.

Algebraic properties of the (po)set W^J.

1. Every coset xW_J admits a unique element of minimal length x_0, and $xW_J \cap W^J = \{x_0\}$.
2. For every $y \in xW_J$, one has $x_0 \leq y$.
3. The poset $(W^J, \leq)$ is graded (Deodhar) by the restriction of the length function ℓ on W to W^J.

- The first and second properties above (and maybe the third one ?) generalize to a much larger family of "Coxeter subgroups" of Coxeter groups: the reflection subgroups of W (i.e., subgroups generated by a subset of T) (Dyer).

"Coxeter subgroups"

- There is no general theory of "Coxeter subgroups" of Coxeter groups.
- But there are several important families of subgroups of Coxeter groups admitting canonical structures of Coxeter groups. Reflection subgroups is an example of one such family generalizing the W_J's.
- The aim of the talk is to explore the analogues of properties (1) to (3) from the previous slide for another family of "Coxeter subgroups" of Coxeter groups also generalizing the W_J's, and relations to geometric settings.

Fixed-point subgroups

- Let (W, S) be an arbitrary Coxeter system (not necessarily finite). Let $L \subseteq S$, and let θ be an automorphism of diagram of L such that $\theta^2 = \mathrm{id}$.
- It is well-known (Steinberg, Hée, Mühlherr, Lusztig) that $W_L^\theta := \{w \in W_L \mid \theta(w) = w\}$ is again a Coxeter group. The set S_L^θ of generators of W_L^θ exactly contains the following elements:
  - Every $s \in L$ such that $\theta(s) = s$,
  - Whenever $\theta(s) \neq s$ with $s \in L$ and $\langle s, \theta(s) \rangle$ is finite, the longest element of the finite dihedral group $\langle s, \theta(s) \rangle$.
- Examples:
  1. W of type A_{2n-1}, with $\theta(s_i) = s_{2n-i}$. Then $S_L^\theta = \{s_1 s_{2n-1}, s_2 s_{2n-2}, \ldots, s_{n-1} s_{n+1}, s_n\}$, and (W_L^θ, S_L^θ) is of type B_n. It is *not* a refl. subgp of W.
  2. W of type A_3, $L = \{s_1, s_3\}$, θ permutes s_1 and s_3. Then $W_L^\theta = \langle s_1 s_3 \rangle$ has type A_1.
  3. If $\theta = \mathrm{id}$, then $W_L^\theta = W_L$, and we recover parabolic quotients.

Restriction of the Bruhat order to cosets modulo fixed-point subgroups

- Unlike for parabolic quotients, in general in xW_L^θ there are several elements of minimal length. We have

Theorem (Chapelier-G., 2023)

Let u, v be two elements of minimal length in xW_L^θ. Let $y \in W_L^\theta$ such that $uy = v$. Let $x_1 x_2 \cdots x_k$ be an S_L^θ-reduced expression of y. Then

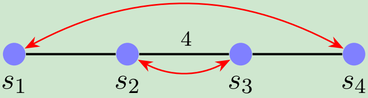
$$\ell(u) = \ell(ux_1) = \cdots = \ell(ux_1 x_2 \cdots x_{k-1}) = \ell(v).$$

In other (and weaker) words, there is a chain of elements of minimal length in the coset allowing one to pass from u to v, where at each step one just multiplies by an element of S_L^θ.

Restriction of the Bruhat order to cosets modulo fixed-point subgroups, II

Example

Let $W = F_4$, $L = S = \{s_1, s_2, s_3, s_4\}$ and θ be the unique nontrivial diagram automorphism of L:



We have $S_L^\theta = \{s_1 s_4, s_2 s_3 s_2 s_3\}$. Write $x = s_1 s_4$ and $y = s_2 s_3 s_2 s_3$. Then W_L^θ is a dihedral group of order 16. If $u = 42312342$ ($i \leftrightarrow s_i$) then there are 4 elements of minimal elements in uW_L^θ, given below:

$$42312342 \xleftrightarrow{x} 42312321 \xleftrightarrow{y} 43123121 \xleftrightarrow{x} 43123412$$

Example

Extremal case of a coset in $W = F_4$ with only minimal elements:

$$\begin{tikzcd}
 & 343231234312 \arrow[r, leftrightarrow, "x"] \arrow[dl, leftrightarrow, "y"'] & 432343123121 \arrow[r, leftrightarrow, "y"] & 432342312321 \arrow[r, leftrightarrow, "x"] & 234323123432 \arrow[dr, leftrightarrow, "y"] & \\
343231234123 \arrow[d, leftrightarrow, "x"'] & & & & & 234323123423 \arrow[d, leftrightarrow, "x"] \\
342312341231 \arrow[d, leftrightarrow, "y"'] & & & & & 423123431231 \arrow[d, leftrightarrow, "y"] \\
342312342312 \arrow[d, leftrightarrow, "x"'] & & & & & 423123432312 \arrow[d, leftrightarrow, "x"] \\
312343123121 \arrow[dr, leftrightarrow, "y"'] & & & & & 231234323121 \arrow[dl, leftrightarrow, "y"] \\
 & 312342312321 \arrow[r, leftrightarrow, "x"'] & 123423123432 \arrow[r, leftrightarrow, "y"'] & 123423123423 \arrow[r, leftrightarrow, "x"'] & 231234231231 &
\end{tikzcd}$$

Elements of minimal length are minimal for the restriction of the strong Bruhat order

Theorem (Chapelier-G., 2023)

Let $w \in W$. There is $u \in wW_L^\theta$ which is of minimal length in wW_L^θ and such that $u \leq w$.

- Since there are several elements of minimal length in a coset wW_L^θ in general, we cannot define a "Bruhat order" on W/W_L^θ by restricting the Bruhat order on W to elements of minimal length.
- Let H be a subgroup of W. Define a partial order on W/H by $uH \leq vH$ iff for all $w \in vH$, there is $w' \in uH$ such that $w' \leq w$.

A particular setting where one gets a grading

Suppose that L is of the form $I \cup K \cup J$, where I, J and K are disjoint and disconnected, and that there is a bijection $f : I \longrightarrow J$ extending to an isomorphism $W_I \cong W_J$. Define an automorphism of W_L by

- $\theta(s) = s$ if $s \in K$,
- $\theta(s) = f(s)$ if $s \in I$, $\theta(s) = f^{-1}(s)$ if $s \in J$.

Theorem (Chaput-Fresse-G., 2021)

Let L be as above. Then the poset $(W/W_L^\theta, \leq)$ is graded. The grading function is given by the restriction of the length function ℓ to elements of minimal length.

- Question: is $(W/W_L^\theta, \leq)$ graded in general ?

Link to a geometric setting

Let us now illustrate a particular case where $(W/W_L^\theta, \leq)$ describes an inclusion of orbit closures of a spherical subgroup.

- $e \in M_n(\mathbb{C})$ be a 2-nilpotent matrix of rank $r \leq \frac{n}{2}$, say
$$e = \begin{pmatrix} 0 & 0 & 1_r \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1}$$
so that the stabilizer $Z := Z_G(e)$, with $G = \mathrm{GL}_n(\mathbb{C})$, is given by
$$Z = \left\{ \begin{pmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & a \end{pmatrix} : a \in \mathrm{GL}_r(\mathbb{C}),\ b \in \mathrm{GL}_{n-r}(\mathbb{C}) \right\}.$$
- The action of Z on $\mathcal{B} = G/B$, equivalently of B on G/Z, is spherical, i.e., has finitely many orbits (Panyushev 1994).

Link to a geometric setting, II

Let W be of type A_{n-1}, $I = \{s_1, \dots, s_{r-1}\}$, $J = \{s_{n-r+1}, \dots, s_{n-1}\}$, $K = \{s_{r+1}, \dots, s_{n-r-1}\}$. Let $L = I \cup K \cup J$, with automorphism θ s.t. $\theta(s) = s$ for all $s \in K$, and $\theta(s_i) = s_{n-r+i}$ for all $i = 1, \dots, r-1$. Given $w \in W$, set $[w] = wW_L^\theta$.

Theorem (Boos-Reinecke 2012, Bender-Perrin 2018, Chaput-Fresse-G. 2021)

- *The Z-orbits on $\mathcal{B}$ are parametrized by W/W_L^θ, via $[w] \mapsto Zw^{-1}B/B =: \mathbb{O}_{[w]}$.*
- *One has*

$$\mathbb{O}_{[w]} \subseteq \overline{\mathbb{O}_{[w']}} \Leftrightarrow [w] \le [w'],$$

where $\le$ is the "Bruhat order" on W/W_L^θ (+ precise description of the covering relations)

- Question: what about other types ?

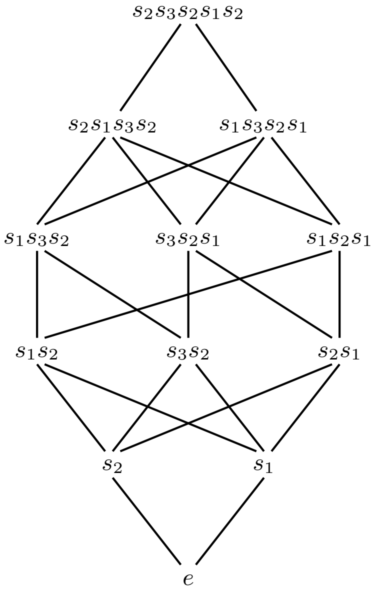


Figure: The poset $(W/W_L^\theta, \leq)$ in type A_3 with $I = \{s_1\}$, $J = \{s_3\}$, $K = \emptyset$. Here $W_L^\theta = \langle s_1 s_3 \rangle$ (type A_1).

Hecke algebras

- Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. Let $\mathcal{H}_W = \mathcal{H}(W, S)$ be the Iwahori-Hecke algebra attached to (W, S), that is, the associative unital $\mathcal{A}$-algebra with a presentation

$$\left\langle T_s, s \in S \;\middle|\; \begin{array}{c} \underbrace{T_s T_t \cdots}_{m_{s,t}} = \underbrace{T_t T_s \cdots}_{m_{t,s}}, \; s \neq t, m_{s,t} < \infty \\ T_s^2 = (v^{-2} - 1) T_s + v^{-2}, \forall s \in S \end{array} \right\rangle$$

Link to Hecke algebras

- Let W, L, θ as before. For $w \in W$ and $s \in S$, one can show that either $[sw] > [w]$, $[sw] = [w]$, or $[sw] < [w]$.

Proposition (Chaput-Fresse-G. 2023)

Let W be arbitrary (not nec. finite) and $L \subseteq S$ with an automorphism θ s. t. $\theta^2 = \mathrm{id}$. Assume that $S_L^\theta \cap T \subseteq S$. Let $\mathcal{A} = \mathbb{Z}[v^{\pm 1}]$ and $M = \bigoplus_{[w] \in W/W_L^\theta} \mathcal{A} m_{[w]}$. Let $s \in S$. Let $u \in \{-1, v^{-2}\}$. The formulas

$$T_s m_{[w]} = \begin{cases} v^{-2} m_{[sw]} + (v^{-2} - 1) m_{[w]} & \text{if } [sw] < [w], \\ m_{[sw]} & \text{if } [sw] > [w], \\ u m_{[w]} & \text{if } [sw] = [w]. \end{cases},$$

define an action of the Iwahori-Hecke algebra $\mathcal{H}_W$ on M.

- KL polynomials, Hecke category ?

Thank you for your attention!