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Bases des algèbres de Temperley-Lieb

Bases of Temperley-Lieb algebras

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A mes parents.

Le temps nous est compté
Mais pas toujours autant qu'on le croit.

Contents

Introduction

L'étude qui suit est constituée de deux parties essentiellement indépendantes, traitant toutes deux des algèbres dites de Temperley-Lieb.

Fixons un entier strictement positif n et un paramètre δ. L'algèbre de Temperley-Lieb $\mathrm{TL}_n(\delta)$ est l'algèbre associative unitaire sur $\mathbb{Z}[\delta]$ engendrée par n générateurs $b_1, \ldots, b_n$ soumis aux relations suivantes:

$$b_j b_i b_j = b_j \text{ si } |i-j| = 1, \tag{1}$$

$$b_i b_j = b_j b_i \text{ si } |i-j| > 1, \tag{2}$$

$$b_i^2 = \delta b_i. \tag{3}$$

Elle fut introduite par Temperley et Lieb (voir [38]) dans le cadre des modèles de percolation. Dans un contexte plus topologique, elle a été utilisée par Jones (voir [27]) en raison de ses liens étroits avec certains invariants de noeuds. Dans ce cas-là, on la définit sur un anneau de polynômes de Laurent $\mathbb{Z}[v, v^{-1}]$ et l'on donne la même présentation par générateurs et relations que celle donnée ci-dessus, en remplaçant δ par $v+v^{-1}$. L'algèbre de Temperley-Lieb telle que considérée par Lieb, Temperley puis Jones possède une réalisation sous la forme d'algèbre dite "de diagrammes".

Bien qu'il n'existe pas réellement de définition formelle d'algèbre de diagramme, cela signifique qu'elle possède une base formée de diagrammes, de telle sorte que la multiplication soit définie sur la base par une concaténation de diagrammes (plus une règle de réduction pour certains diagrammes, faisant intervenir le paramètre de l'anneau des coefficients et correspondant à la relation quadratique ci-dessus). Dans la version utilisée par Jones (c'est-à-dire lorsqu'elle est définie sur un anneau de polynômes de Laurent), on peut la réaliser comme quotient de l'algèbre de Iwahori-Hecke $\mathcal{H}$ de type A_n. Il existe des analogues d'algèbres de Temperley-Lieb hors du type A mais aucune définition générale satisfaisante. Par exemple, pour le type B, il existe deux définitions non équivalentes, l'une due à Graham (voir [22]) et l'autre à Tom Dieck (voir [39]). Dans ce travail, nous nous intéresserons exclusivement aux algèbres de Temperley-Lieb de type A.

La motivation initiale à ce travail consistait à chercher à mieux comprendre la base de l'algèbre de Temperley-Lieb donnée par les travaux de Zinno (voir [42]) ainsi que ses liens avec la base des diagrammes. Il existe un homomorphisme multiplicatif du groupe de tresses à $n+1$ brins dans l'algèbre de Temperley-Lieb. Dans l'article mentionné, Zinno démontre que l'image par cet homomorphisme de l'ensemble des *facteurs canoniques* du groupe de tresses fournit une base de l'algèbre de Temperley-Lieb. Ces facteurs canoniques, en bijection canonique avec les *partitions non croisées*, constituent un ensemble d'éléments particuliers du *monoïde de Birman-Ko-Lee* (voir [4]), un monoïde infini s'injectant dans le groupe de tresses et généralisé plus tard en *monoïde dual* par Bessis (voir [2]): la généralisation est double en ce sens que d'une part le monoïde de Birman-Ko-Lee dépend implicitement d'un choix d'orientation du diagramme de Dynkin (ce qui est équivalent à un choix d'élément de Coxeter) et d'autre part qu'il est propre au type A; ainsi, le monoïde dual dépend d'un type ainsi que d'un choix d'élément de Coxeter. Notons qu'une première génélisation partielle avait auparavant été donnée par Bessis, Digne et Michel ([3]). Le monoïde dual, tout comme le monoïde positif de tresses, possède une structure dite *de Garside* au sens de Dehornoy et Paris (voir [12]).

Tout monoïde dual a pour groupe de fractions le groupe de tresses du système de Coxeter correspondant et s'injecte dans celui-ci. Dans la terminologie de Dehornoy et Paris, les facteurs canoniques sont appelés éléments *simples*. Le cas considéré par Zinno correspond à l'élément de Coxeter $c = s_n s_{n-1} \cdots s_1$. Il a été démontré par Vincenti (voir [41]) que si l'on considère n'importe quel monoïde dual de type A_n, l'image des éléments simples de ce monoïde fournit toujours une base de l'algèbre de Temperley-Lieb correspondante, généralisant le Théorème de Zinno. On obtient ainsi une famille de bases, une pour chaque monoïde dual de type A_n, c'est-à-dire une par choix d'élément de Coxeter ou d'orientation du diagramme de Dynkin.

Pour démontrer que l'image de l'ensemble des éléments simples dans l'algèbre de Temperley-Lieb forme une base, Zinno démontre qu'il existe une matrice triangulaire supérieure avec coefficient inversible sur la diagonale permettant de passer de cet ensemble à la base des diagrammes. Pour ce faire, il exhibe une bijection entre les deux ensembles indexant les bases respectives, à savoir les partitions non croisées et les éléments totalement commutatifs du groupe symétrique. Ces deux ensembles, dénombrés par les nombres de Catalan, possèdent de nombreuses propriétés combinatoires. Zinno ordonne ensuite totalement l'ensemble des partitions non croisées de façon judicieuse et démontre qu'en munissant l'ensemble des éléments totalement commutatifs de l'ordre induit par sa bijection, la matrice susmentionnée est triangulaire supérieure avec coefficient inversible explicitement déterminé sur la diagonale. Notons également que le résultat de Zinno a été démontré de façon directe par Lee et Lee (voir [31]), mais que cette preuve ne donne pas la triangularité de la matrice de changement de base. La généralisation de Vincenti s'inspire de l'approche de Lee et Lee.

Il a été noté par Digne dans des travaux non publiés que dans de petits cas, la matrice de changement de base entre la base de Zinno et celle des diagrammes restait triangulaire supérieure à coefficient inversible sur la diagonale (pour des ordres convenables sur les bases) si l'on fait varier l'élément de Coxeter. De plus, Digne a constaté que les coefficients de la matrice de changement de base faisaient apparaître des phénomènes de positivité (avec une preuve de la positivité pour $c = s_1 s_2 \cdots s_n$ utilisant des résultats de positivité dans l'algèbre de Hecke de Dyer et Lehrer, [14]). Ceci suggère les différentes questions suivantes:

- Ces différents phénomènes peuvent-ils être expliqués par une bonne catégorification de l'algèbre de Temperley-Lieb, qui mettrait à la fois en évidence la base des diagrammes et celle de Zinno?

- Que peut-on dire des coefficients de la matrice de changement de base? Possèdent-ils une interprétation géométrique? Existe-t-il des formules fermées pour les déterminer?

- Que dire de l'ordre dont on munit les ensembles indexant les bases et qui donne la triangularité? Quelles sont ses propriétés?

- Que se passe-t-il lorsqu'on change d'élément de Coxeter? Peut-on démontrer la triangularité pour tout choix d'élément de Coxeter? Si oui, quels sont les ordres à considérer sur les partitions non croisées associées à un élément de Coxeter arbitraire pour obtenir la triangularité?

La première partie du travail qui suit est le fruit de tentatives de réponse à la première question et propose une catégorification des algèbres de Temperley-Lieb. Il existe plusieurs catégorifications des algèbres de Temperley-Lieb, par exemple par Bernstein, Frenkel et Khovanov (voir [1]) où plusieurs conjectures démontrées plus tard par Stroppel (voir [37], où l'algèbre de Temperley-Lieb est également considérée dans d'autres types) sont émises; il s'agit de constructions utilisant des foncteurs projectifs sur des blocs principaux de versions paraboliques de la catégorie $\mathcal{O}$ de Bernstein, Gelfand et Gelfand. Elias (voir [16]) donne également une catégorification de l'algèbre de Temperley-Lieb obtenue comme quotient de la catégorie des bimodules de Soergel (qui catégorifie l'algèbre de Hecke d'un système de Coxeter de rang fini), plus précisément d'une version diagrammatique de celle-ci due à Elias et Khovanov (voir [17]). On perd ainsi la structure de bimodule sur les objets (mais pas sur les morphismes). On se propose ici de fournir une catégorification directe de l'algèbre de Temperley-Lieb par des analogues de bimodules de Soergel. Le contenu de cette partie, à l'exception de la section 2.4, a fait l'objet d'une publication (voir [20]).

La seconde partie, plus combinatoire, est motivée par les trois derniers points ci-dessus.

Comme exemple, nous donnons l'expression de la base des diagrammes dans celle de Zinno ci-dessous pour le type A_3; notons $\mathcal{W}_f$ l'ensemble des éléments totalement commutatifs du groupe symétrique. On a

$$\begin{aligned}\mathcal{W}_f &= \{e, s_1, s_2, s_3, s_1s_2, s_1s_3, s_2s_3, s_1s_2s_3, s_2s_1, s_3s_2, s_2s_1s_3,\\ &\quad s_3s_1s_2, s_3s_2s_1, s_2s_1s_3s_2\}.\end{aligned}$$

Notons $\mathcal{P}_c$ l'ensemble des partitions non croisées associées à l'élément de Coxeter $c = s_1s_2s_3$. On a

$$\begin{aligned}\mathcal{P}_c &= \{e, s_1, s_2, s_3, s_1s_2, s_1s_3, s_2s_3, s_1s_2s_3, s_2s_1s_2, s_3s_2s_3, s_2s_1s_2s_3,\\ &\quad s_1s_3s_2s_3, s_3s_2s_1s_2s_3, s_2s_3s_2s_1s_2s_3\}.\end{aligned}$$

Nous noterons $\{b_w\}_{w\in\mathcal{W}_f}$ la base des diagrammes et $\{Z_x\}_{x\in\mathcal{P}_c}$ la base de Zinno. Nous exprimons ci-dessous les éléments b_w comme combinaisons linéaires de $\{Z_x\}$, donnés par GAP; on obtient ainsi la matrice inverse de celle considérée par Digne. Un tel choix de matrice à considérer provient en partie du fait que la base des diagrammes est la projection de la base canonique C'_w de Kazhdan-Lusztig de l'algèbre de Hecke qui se catégorifie plus aisément que d'autres bases de l'algèbre de Hecke. La base $\{b_w\}$ apparaît donc comme la base naturelle de l'algèbre de Temperley-Lieb à catégorifier.

$b_e = Z_e,$
$b_{s_1} = -Z_{s_1} + v^{-1}Z_e,$
$b_{s_2} = -Z_{s_2} + v^{-1}Z_e,$
$b_{s_3} = -Z_{s_3} + v^{-1}Z_e,$
$b_{s_1s_2} = Z_{s_1s_2} - v^{-1}(Z_{s_1} + Z_{s_2}) + v^{-2}Z_e,$
$b_{s_1s_3} = Z_{s_1s_3} - v^{-1}(Z_{s_1} + Z_{s_3}) + v^{-2}Z_e,$
$b_{s_2s_3} = Z_{s_2s_3} - v^{-1}(Z_{s_2} + Z_{s_3}) + v^{-2}Z_e,$
$b_{s_2s_1} = vZ_{s_2s_1s_2} - v^2Z_{s_1s_2},$
$b_{s_3s_2} = vZ_{s_3s_2s_3} - v^2Z_{s_2s_3},$
$b_{s_1s_2s_3} = -Z_{s_1s_2s_3} + v^{-1}(Z_{s_1s_2} + Z_{s_1s_3} + Z_{s_2s_3}) - v^{-2}(Z_{s_1} + Z_{s_2} + Z_{s_3}) + v^{-3}Z_e,$
$b_{s_2s_1s_3} = -vZ_{s_2s_1s_2s_3} + v^2Z_{s_1s_2s_3} + Z_{s_2s_1s_2} - vZ_{s_1s_2},$
$b_{s_1s_3s_2} = -vZ_{s_1s_3s_2s_3} + v^2Z_{s_1s_2s_3} + Z_{s_3s_2s_3} - vZ_{s_2s_3},$
$b_{s_3s_2s_1} = -v^2Z_{s_3s_2s_1s_2s_3} + v^3(Z_{s_2s_1s_2s_3} + Z_{s_1s_3s_2s_3}) - v^4Z_{s_1s_2s_3} + vZ_{s_1s_3} - (Z_{s_1} + Z_{s_3}) + v^{-1}Z_e,$
$b_{s_2s_1s_3s_2} = Z_{s_2s_3s_2s_1s_2s_3} - v^{-1}Z_{s_3s_2s_1s_2s_3} - v^{-1}Z_{s_2} + v^{-2}Z_e.$

Si l'on munit $\mathcal{P}_c$ de l'ordre total donné par $x < y$ si x intervient avant y dans l'ensemble $\mathcal{P}_c$ tel qu'écrit plus haut et que l'on fait de même pour $\mathcal{W}_f$, la matrice M permettant de passer de $\{Z_x\}_{x\in\mathcal{P}_c}$ à $\{b_w\}_{w\in\mathcal{W}_f}$ est ainsi donnée par la matrice triangulaire supérieure suivante:

$$\begin{pmatrix}
1 & v^{-1} & v^{-1} & v^{-1} & v^{-2} & v^{-2} & v^{-2} & v^{-3} & 0 & 0 & 0 & 0 & v^{-1} & v^{-2} \\
0 & -1 & 0 & 0 & -v^{-1} & -v^{-1} & 0 & -v^{-2} & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & -v^{-1} & 0 & -v^{-1} & -v^{-2} & 0 & 0 & 0 & 0 & 0 & -v^{-1} \\
0 & 0 & 0 & -1 & 0 & -v^{-1} & -v^{-1} & -v^{-2} & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & v^{-1} & -v^2 & 0 & -v & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & v^{-1} & 0 & 0 & 0 & 0 & v & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & v^{-1} & 0 & -v^{-2} & 0 & -v & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & v^2 & v^2 & -v^4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -v & 0 & v^3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -v & v^3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -v^2 & -v^{-1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

On pourrait se demander si les coefficients des Z_x dans une expansion linéaire d'un élément b_w sont toujours des monômes. Ceci est faux en général comme le montrent des calculs effectués avec GAP en type A_5. Toutefois certains coefficients, incluant les coefficients diagonaux, sont toujours des monômes comme nous le verrons dans le chapitre 3 (Théorème 3.3.28).

Première partie: Catégorification de l'algèbre de Temperley-Lieb par des analogues de bimodules de Soergel

On réalise l'algèbre de Temperley-Lieb de type A_n comme groupe de Grothendieck scindé d'une catégorie additive graduée munie d'une opération conférant à ce groupe de Grothendieck une structure additionnelle d'anneau. En d'autres termes, on construit un isomorphisme d'anneaux (et même de $\mathbb{Z}[v, v^{-1}]$-algèbres en donnant une interprétation du paramètre v dans la catégorie en question) entre l'algèbre de Temperley-Lieb et l'anneau de Grothendieck susmentionné (Théorème 2.3.20). Ce théorème est un analogue pour le cas Temperley-Lieb du Théorème dit "de catégorification de Soergel" (voir le Théorème 1.3.2) qui catégorie l'algèbre de Hecke d'un système de Coxeter. Les bimodules élémentaires à considérer pour une telle construction sont suggérés par Ben Elias (voir [16], Remark 3.22). Les relations qui sont catégorifiées sont les relations (1) à (3) ci-dessus où l'on a posé $\delta = 1 + v^{-2}$. Or la valeur du paramètre δ permettant de réaliser l'algèbre de Temperley-Lieb comme un quotient d'algèbre de Hecke (et donc de l'algèbre de groupe du groupe de tresses sur $\mathbb{Z}[v, v^{-1}]$) est $\delta = v + v^{-1}$. On n'obtient donc pas d'homomorphisme du groupe de tresses dans l'algèbre donnée par la catégorification et il n'est donc pas possible d'y définir la base de Zinno (ceci est expliqué de façon détaillée dans la Remarque 2.3.1).

Expliquons le procédé utilisé pour la construction de notre catégorie. On considère la représentation géométrique V d'un système de Coxeter $(\mathcal{W}, \mathcal{S})$ de type A_n sur un corps k de caractéristique nulle. On note Z la sous-variété obtenue comme réunion des *droites de Weyl*, c'est-à-dire, des droites obtenues comme intersections d'hyperplans de réflexions. On a une opération de $\mathcal{W}$ sur Z. Les bimodules considérés sont des bimodules gradués sur la k-algèbre des fonctions régulières sur Z que l'on note $\bar{R}$.

On définit alors des bimodules B_i, briques élémentaires correspondant aux générateurs b_i de l'algèbre de Temperley-Lieb. On définit un produit noté $\star$ de bimodules en utilisant les variétés correspondant aux annulateurs de bimodules (Sous-section 2.2.4). La définition du produit fournit une opération qui n'est ni associative ni distributive en général. Toutefois, cette opération devient associative (à isomorphisme près) lorsqu'on se resteint à une famille de bimodules qui sont les bimodules obtenus comme produits $\star$ successifs de B_i pour divers i (Théorème 2.2.15). En tant que modules à gauche ou à droite, ces produits successifs sont libres en tant que modules sur le quotient de $\bar{R}$ par l'annulateur à gauche ou à droite du bimodule; un tel quotient est isomorphe à un anneau de fonctions régulières sur une réunion de droites de Weyl dépendant de la suite d'indices du produit. Les bimodules B_i et le produit $\star$ satisfont les relations de Temperley-Lieb (Théorème 2.3.3). Toutefois, l'associativité partielle ne résout pas a priori les différents problèmes, dans la mesure où l'on souhaite avoir associativité sur une famille plus grande de bimodules, ainsi que distributivité. On démontre que cette famille n'est constituée que de bimodules indécomposables ou de sommes de décalés du même module indécomposable (Théorème 2.3.19). On peut grâce à ce résultat étendre le produit $\star$ à la catégorie additive engendrée par cette famille de bimodules et de leurs décalés en prolongeant l'opération $\star$ par bilinéarité. Ceci fournit un produit bien défini à isomorphisme près, qui munit par conséquent le groupe de Grothendieck scindé de la catégorie d'une structure d'anneau et même d'algèbre en interprétant le paramètre v comme un décalage. On démontre qu'il existe un isomorphisme d'algèbres entre l'algèbre de Temperley-Lieb et l'anneau de Grothendieck susmentionné et que l'image de la base dite des diagrammes de l'algèbre coïncide avec l'ensemble des classes de bimodules indécomposables de la catégorie (Théorème 2.3.20). Pour ce faire, on utilise le fait que l'information diagrammatique est contenue de façon surprenante dans les annulateurs à gauche et à droite de bimodules indécomposables (Proposition 2.3.5).

Un ingrédient essentiel (qui permet d'étendre le produit $\star$ par bilinéarité et de démontrer le Théorème de catégorification) est la preuve de l'indécomposabilité des bimodules correspondant à la base des diagrammes. Les bimodules en question sont positivement gradués et on démontre leur indécomposabilité en prouvant qu'ils sont engendrés par un élément dans leur composante homogène de degré nul, laquelle est de dimension égale à un. Cette propriété de cyclicité, fausse dans la catégorie de Soergel en général, peut laisser supposer que ces bimodules ne sont autre que des algèbres de fonctions régulières sur des sous-schémas appropriés de $Z \times Z$. On démontre cette propriété pour certains bimodules indécomposables de la catégorie (Proposition 2.4.4).

Seconde partie: Autour des bases de Zinno

Cette seconde partie, à dimension plus combinatoire, présente différents résultats relatifs aux bases de Zinno des algèbres de Temperley-Lieb et à leur lien avec la base dite des diagrammes ou de Kazhdan-Lusztig.

La première constatation repose sur le fait que la façon dont ordonne Zinno sa base, autrement dit dont il ordonne les partitions non croisées qui indexent cette base, est par la longueur d'un mot privilégié représentant l'élément simple dans le groupe de tresses; il s'avère que cette longueur n'est autre que la longueur de Coxeter de la partition non croisée correspondante vue comme élément du groupe symétrique. On constate que plusieurs résultats de Zinno utilisant cet ordre peuvent être raffinés à l'ordre de Bruhat (ce qui est plus intéressant que la longueur de Coxeter si l'on cherche à comprendre les coefficients de la matrice de changement de base, par exemple lorsqu'ils sont non nuls). Il paraît donc naturel de faire d'une part une étude plus approfondie de l'ordre de Bruhat sur les partitions non croisées. D'autre part, la seule information sur les coefficients présente dans la littérature est une détermination explicite par Zinno du coefficient sur la diagonale de la matrice de changement de base, lorsque les deux bases sont ordonnées de façon à avoir la triangularité, c'est-à-dire par la longueur de Coxeter sur les partitions non croisées et par l'ordre sur les éléments totalement commutatifs induit par la bijection de Zinno. La présente étude propose ainsi les résultats suivants:

- Des bijections entre partitions non croisées associées à un élément de Coxeter arbitraire et éléments totalement commutatifs, qui généralisent celle donnée par Zinno. On a de plus une description des bijections inverses (section 3.2),

- Une nouvelle base de l'algèbre de Temperley-Lieb, faisant intervenir l'ordre de Bruhat sur les partitions non croisées (section 3.3.2),

- Dans le cas où $c = s_1 s_2 \cdots s_n$, une condition nécessaire mais non suffisante (Corollaire 3.3.26) pour qu'un coefficient de la matrice de changement de base soit non nul ainsi que la détermination explicite de certains coefficients (Théorème 3.3.28); pour ce faire on utilise la base du point précédent,

- Une étude approfondie de l'ordre de Bruhat sur les partitions non croisées associées à l'élément de Coxeter c (Sections 3.4 et 3.5). On donne un critère combinatoire (Théorème 3.5.6) permettant de déterminer si une partition non croisée x est plus petite qu'une partition non croisée y pour l'ordre de Bruhat en introduisant des vecteurs d'entiers avec certaines conditions de parité. Ce critère permet entre autres de démontrer la propriété surprenante de treillis de l'ensemble des partitions non croisées ordonnées par l'ordre de Bruhat (Théorème 3.6.1): il s'avère que la structure de treillis obtenue est étonnamment la même que celle provenant des idéaux dans le poset des racines positives ou encore du poset des chemins de Dyck pour l'inclusion (Section 3.6). Toutefois, lorsqu'on change d'élément de Coxeter, la propriété de treillis n'est plus vraie (pour l'ordre de Bruhat); on explique alors quel ordre à considérer, impliquant les vecteurs susmentionnés, pour obtenir la même structure de treillis que pour c (Section 3.7). On donne un procédé permettant de se ramener au cas où l'élément de Coxeter est c, ce qui permet d'utiliser le critère pour caractériser cet ordre pour des partitions non croisées correspondant à un choix d'élément de Coxeter arbitraire (qui n'est pas en général l'ordre de Bruhat si l'élément de Coxeter est différent de c),

- En utilisant les ordres spéciaux sur les partitions non croisées mentionnés au point précédent et les bijections du premier point, une démonstration de la triangularité de la matrice de changement de base entre la base de Zinno généralisée à un élément de Coxeter arbitraire et la base des diagrammes (Section 3.8, Théorème 3.8.28). En particulier, on peut facilement donner une nouvelle preuve plus générale d'un ingrédient essentiel à la preuve de la triangularité (Théorème 3.8.26) en utilisant le critère du point précédent.

Chapter 1

Préliminaires

1.1 Eléments totalement commutatifs

Soit $(\mathcal{W}, \mathcal{S})$ un système de Coxeter.

Définition 1.1.1. *Un élément $w \in \mathcal{W}$ est dit* totalement commutatif *(en anglais* fully commutative*) si l'on peut passer de toute expression $\mathcal{S}$-réduite de w à toute autre en utilisant uniquement des relations de la forme $st = ts$, où $s, t \in \mathcal{S}$.*

Dans le type A qui nous intéresse dans ce document, il existe différentes caractérisations des éléments totalement commutatifs dont nous présentons certaines dans la proposition ci-dessous:

Proposition 1.1.2. *Soit $(\mathcal{W}, \mathcal{S})$ un système de Coxeter de type A_n, où $\mathcal{W}$ est identifié avec le groupe symétrique $\mathfrak{S}_{n+1}$ et $\mathcal{S} = \{s_i = (i, i+1)\}_{i=1}^{n}$. Soit $w \in \mathcal{W}$. Les conditions suivantes sont équivalentes:*

1. *L'élément w est totalement commutatif,*

2. *Si $s_{i_1} \cdots s_{i_k}$ est une expression $\mathcal{S}$-réduite de w, alors pour tout $i = 1, \ldots, n$,*

$$n_i(w) := |\{j \mid i_j = i\}|$$

   est indépendant de l'expression $\mathcal{S}$-réduite choisie,

3. *L'élément w possède une expression $\mathcal{S}$-réduite de la forme*

$$(s_{i_\ell} s_{i_\ell - 1} \cdots s_{j_\ell})(s_{i_{\ell-1}} s_{i_{\ell-1}-1} \cdots s_{j_{\ell-1}}) \cdots (s_{i_1} s_{i_1 - 1} \cdots s_{j_1})$$

   où tous les indices appartiennent à l'ensemble $\{1, \ldots, n\}$, $i_\ell < i_{\ell-1} < \cdots < i_1$, $j_\ell < j_{\ell-1} < \cdots < j_1$ et $j_m \leq i_m$ pour $m = 1, \ldots, \ell$.

4. *Si $s_{i_1} \cdots s_{i_k}$ est une expression $\mathcal{S}$-réduite de w avec $s_{i_j} = s_i = s_{i_d}$, $j < d$ et $s_{i_k} \neq s_i$ pour tout $j < k < d$, alors $(s_{i_{j+1}}, \ldots, s_{i_{d-1}})$ possède exactement une composante égale à s_{i+1} et exactement une autre égale à s_{i-1}.*

Proof. L'équivalence entre les deux premières assertions est évidente et vraie pour tout système de Coxeter n'ayant pas d'entrée paire différente de 2 dans leur matrice de Coxeter. Les deux dernières conditions sont souvent considérées dans le cas des mots dits réduits de l'algèbre de Temperley-Lieb, mais les résultats sont vrais au niveau du groupe de Coxeter et les preuves s'adaptent facilement; pour une équivalence entre 1 et 3, voir par exemple ([21], §2.8); l'existence de telles expressions réduites dans le cas Temperley-Lieb avaient été notée par Jones dans ([27], §3.5); pour une équivalence entre 1 et 4, on peut adapter la preuve de ([42], Theorem 1). □

On trouvera d'autres caractérisations et propriétés par exemple dans les travaux de Stembridge (voir [35], [36]) qui fit les premières études systématiques d'éléments totalement commutatifs.

1.2 Algèbres de Hecke et de Temperley-Lieb

1.2.1 Algèbres de Hecke

Soit $(\mathcal{W}, \mathcal{S})$ un système de Coxeter avec matrice de Coxeter $(m_{s,t})_{s,t \in \mathcal{S}}$. Notons $\ell_{\mathcal{S}} : \mathcal{W} \to \mathbb{Z}_{\geq 0}$ la longueur de Coxeter, i.e., pour $w \in \mathcal{W}$, $\ell_{\mathcal{S}}(w)$ est le nombre minimal possible de facteurs dans une expression de w comme produit d'éléments de $\mathcal{S}$.

L'*algèbre de Hecke* $\mathcal{H} = \mathcal{H}(\mathcal{W}, \mathcal{S})$ de $(\mathcal{W}, \mathcal{S})$ est l'algèbre associative unitaire sur l'anneau $\mathbb{Z}[v, v^{-1}]$ des polyômes de Laurent à coefficients dans $\mathbb{Z}$ ayant pour générateurs T_s, $s \in \mathcal{S}$ et relations

$$\underbrace{T_s T_t \cdots}_{m_{s,t} \text{ facteurs}} = \underbrace{T_t T_s \cdots}_{m_{s,t} \text{ facteurs}} \quad \text{si } m_{s,t} \text{ est fini} \quad \text{(Relations de tresses)},$$

$$T_s^2 = (v^{-2} - 1)T_s + v^{-2} \quad \text{(Relations quadratiques)}.$$

Il s'agit d'une déformation de l'algèbre de groupe de $(\mathcal{W}, \mathcal{S})$ sur $\mathbb{Z}$. L'algèbre $\mathcal{H}$ possède une base $\{T_w\}_{w \in \mathcal{W}}$ dite *standard* où T_w est défini de la façon suivante: soit $s_1 \cdots s_k$ une expression $\mathcal{S}$-réduite de w. Les relations de tresses étant satisfaites dans $\mathcal{H}$, le produit

$$T_{s_1} T_{s_2} \cdots T_{s_k}$$

est indépendant de l'expression $\mathcal{S}$-réduite de w choisie et on le dénote par conséquent par T_w. Si $k = 0$, c'est-à-dire si $w = e$, on pose $T_e = 1$.

Il existe d'autres bases intervenant si l'on s'intéresse aux représentations des groupes de Coxeter et de leurs algèbres de Hecke communément appelées *bases canoniques* ou *bases de Kazhdan-Lusztig*. Considérons l'unique involution $\iota : \mathcal{H} \to \mathcal{H}$ satisfaisant $\iota(v) = v^{-1}$, $\iota(T_w) = (T_{w^{-1}})^{-1}$. On a le Théorème suivant de Kazhdan et Lusztig:

Théorème 1.2.1 ([29], Theorem 1.1). *Pour tout $w \in \mathcal{W}$, il existe un unique élément $C'_w \in \mathcal{H}$ satisfaisant $\iota(C'_w) = C'_w$ et*

$$C'_w \in v^{\ell_{\mathcal{S}}(w)} T_w + \sum_{y \in \mathcal{W}} v\mathbb{Z}[v] v^{\ell_{\mathcal{S}}(y)} T_y.$$

Ceci fournit une base $\{C'_w\}_{w \in \mathcal{W}}$ de $\mathcal{H}$. Il existe une involution $j : \mathcal{H} \to \mathcal{H}$ définie par $j(v) = v^{-1}$, $j(T_w) = v^{2\ell_{\mathcal{S}}(w)}(-1)^{\ell_{\mathcal{S}}(w)} T_w$. Il existe une seconde base $\{C_w\}_{w \in \mathcal{W}}$ définie par un théorème similaire à celui donné ci-dessus et reliée à la base $\{C_w\}_{w \in \mathcal{W}}$ par la relation $C_w = j((-1)^{\ell_{\mathcal{S}(w)}} C'_w)$ (voir [29], remarques suivant Theorem 1.1).

1.2.2 Algèbres de Temperley-Lieb

L'*algèbre de Temperley-Lieb* $\mathrm{TL}_n(\delta)$ est l'algèbre associative unitaire sur $\mathbb{Z}[\delta]$, où δ est un paramètre, engendrée par n générateurs $b_1, \ldots, b_n$ soumis aux relations suivantes:

$$\begin{aligned} b_j b_i b_j &= b_j \text{ si } |i-j| = 1, \\ b_i b_j &= b_j b_i \text{ si } |i-j| > 1, \\ b_i^2 &= \delta b_i. \end{aligned}$$

Elle fut introduite par Temperley et Lieb ([38]). Dans un cadre plus topologique, elle fut utilisée par Jones ([27]) dans le contexte des invariants de noeuds. Dans ce cas-là, on la définit sur un anneau de polynômes de Laurent $\mathbb{Z}[v, v^{-1}]$ et l'on donne la même présentation par générateurs et relations que celle donnée ci-dessus, en remplaçant δ par $v + v^{-1}$. On peut ainsi la réaliser comme un quotient de l'algèbre de Hecke $\mathcal{H}(\mathcal{W}, \mathcal{S})$ de type A_n de deux façons différentes (voir par exemple [25], §2.3 et Remark 2.4)

$$\theta : \mathcal{H}(\mathcal{W}, \mathcal{S}) \to \mathrm{TL}_n(v + v^{-1}), T_{s_i} \mapsto v^{-1} b_i - 1,$$

$$\theta' : \mathcal{H}(\mathcal{W}, \mathcal{S}) \to \mathrm{TL}_n(v + v^{-1}), T_{s_i} \mapsto v^{-2} - v^{-1} b_i.$$

Le noyau de θ est l'idéal bilatère engendré par les éléments de la forme

$$\sum_{w \in \langle s_i, s_{i+1} \rangle} T_w$$

pour $i = 1, \dots, n-1$. Le noyau de θ' est l'idéal bilatère engendré par les éléments de la forme

$$\sum_{w \in \langle s_i, s_{i+1} \rangle} (-1)^{\ell_{\mathcal{S}}(w)} v^{2\ell_{\mathcal{S}}(w)} T_w$$

pour $i = 1, \dots, n-1$. L'algèbre $\mathcal{H}$ possède un unique automorphisme involutif semilinéaire $j_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$ tel que $j_{\mathcal{H}}(T_{s_i}) = -v^2 T_{s_i}$ et $j_{\mathcal{H}}(v) = v^{-1}$. L'algèbre de Temperley-Lieb possède un unique automorphisme involutif semilinéaire j_{TL_n} tel que $j_{\mathrm{TL}_n}(b_i) = b_i$ et $j_{\mathrm{TL}_n}(v) = v^{-1}$. On a $\theta' = j_{\mathrm{TL}_n} \circ \theta \circ j_{\mathcal{H}}$.

L'algèbre de Temperley-Lieb possède une base indexée par les éléments totalements commutatifs du groupe symétrique $\mathfrak{S}_{n+1}$ que l'on notera $\{b_w\}_{w \in \mathcal{W}_f}$. Elle est construite de la façon suivante: pour $i = 1, \dots, n$, on pose $b_{s_i} := b_i$. Pour $w \in \mathcal{W}_f$, on choisit une décomposition $\mathcal{S}$-réduite $s_{i_1} \cdots s_{i_k}$ de w et on considère le produit

$$b_{i_1} \cdots b_{i_k} \in \mathrm{TL}_n.$$

On peut montrer qu'un tel produit ne dépend pas du choix d'expression $\mathcal{S}$-réduite choisie et on le note par conséquent b_w. En particulier, b_w possède une unique expression sous la forme

$$(b_{i_\ell} b_{i_\ell - 1} \cdots b_{j_\ell})(b_{i_{\ell-1}} b_{i_{\ell-1}-1} \cdots b_{j_{\ell-1}}) \cdots (b_{i_1} b_{i_1 - 1} \cdots b_{j_1})$$

où tous les indices sont dans $\{1, \ldots, n\}$, $i_\ell < i_{\ell-1} < \cdots < i_1$, $j_\ell < j_{\ell-1} < \cdots < j_1$ et $j_m \leq i_m$ pour $m = 1, \ldots, \ell$, héritée de l'expression $\mathcal{S}$-réduite

$$(s_{i_\ell} s_{i_\ell - 1} \cdots s_{j_\ell})(s_{i_{\ell-1}} s_{i_{\ell-1}-1} \cdots s_{j_{\ell-1}}) \cdots (s_{i_1} s_{i_1 - 1} \cdots s_{j_1})$$

de w donnée par le point 3. de la Proposition 1.1.2.
On a le résultat suivant

Théorème 1.2.2 (Fan, Green, [19]). *La base $\{b_w\}_{\mathcal{W}_f}$ est la projection sous θ de la base canonique $\{C'_w\}_{w \in \mathcal{W}}$, i.e., $\theta(C'_w) = b_w$ si $w \in \mathcal{W}_f$ tandis que $\theta(C'_w) = 0$ si $w \notin \mathcal{W}_f$.*

Corollaire 1.2.3. *La base $\{b_w\}_{\mathcal{W}_f}$ est à signature près la projection sous θ' de la base canonique $\{C_w\}_{w \in \mathcal{W}}$, i.e., $\theta'(C_w) = (-1)^{\ell s(w)} b_w$ si $w \in \mathcal{W}_f$ tandis que $\theta'(C_w) = 0$ si $w \notin \mathcal{W}_f$.*

Démonstration. Les bases C_w et C'_w sont reliées par la relation $C_w = (-1)^{\ell s(w)} j_{\mathcal{H}}(C'_w)$. En utilisant l'égalité $\theta' = j_{\mathrm{TL}_n} \circ \theta \circ j_{\mathcal{H}}$ on a le résultat. □

La base $\{b_w\}_{w \in \mathcal{W}_f}$ sera appelée *base de Kazhdan-Lusztig* ou *base des diagrammes*, cette dernière terminologie provenant du fait que lorsque l'algèbre est réalisée comme une algèbre de diagrammes, la base constituée des diagrammes correspond à la base $\{b_w\}_{w \in \mathcal{W}_f}$. La version diagrammatique de TL_n sera rappelée en sous-section 2.3.1.

1.3 Bimodules sur des algèbres $\mathbb{Z}$-graduées

1.3.1 Modules sur des algèbres $\mathbb{Z}$-graduées

Soit A une k-algèbre commutative $\mathbb{Z}$-graduée avec $A_0 = k$, $A_i = 0$ si $i < 0$. On note $A - \mathrm{Mod}_{\mathbb{Z}}$ la catégorie des A-modules gradués et $A - \mathrm{mod}_{\mathbb{Z}}$ la catégorie des A-modules gradués de type fini, c'est-à-dire que si $M \in A - \mathrm{Mod}_{\mathbb{Z}}$ ou $A - \mathrm{mod}_{\mathbb{Z}}$, M possède une décomposition en somme directe de sous-k-modules

$$M = \bigoplus_{i \in \mathbb{Z}} M_i$$

de telle sorte que pour tous $i, j \in \mathbb{Z}$, on ait

$$A_i \cdot M_j \subset M_{i+j}.$$

Pour $M \in A - \mathrm{Mod}_{\mathbb{Z}}$ ou $A - \mathrm{mod}_{\mathbb{Z}}$, on note $M[i]$ le A-module gradué égal à M en tant que A-module et muni de l'unique graduation telle que $(M[i])_k = M_{i+k}$.

Remarque 1.3.1. Dans le cas d'une k-algèbre $\mathbb{Z}$-graduée A finiment engendrée avec graduation positive et $A_0 = k$, l'anneau d'endomorphismes d'un A-module gradué M de type fini est de dimension finie. Par conséquent les modules indécomposables ont anneau d'endomorphisme local et la propriété de Krull-Schmidt est vérifiée dans la catégorie $A - \mathrm{mod}_{\mathbb{Z}}$.

1.3.2 Bimodules sur des algèbres $\mathbb{Z}$-graduées

Avec les mêmes hypothèses sur A que dans la sous-section précédente, on note $A - \mathrm{mod}_{\mathbb{Z}} - A$ la catégorie des A-bimodules $\mathbb{Z}$-gradués de type fini avec même opération de k des deux côtés, i.e., on exige que les objets de cette catégorie soient des $A \otimes_k A$-bimodules gradués de type fini où $A \otimes_k A$ est muni de la $\mathbb{Z}$-graduation évidente. Pour $M, N \in A - \mathrm{mod}_{\mathbb{Z}} - A$ et $\varphi : M \to N$ un homomorphisme de $A \otimes_k A$-bimodules, on dit que φ est *homogène* de degré i si $\varphi(M_j) \subset N_{i+j}$, $\forall j \in \mathbb{Z}$. On note $\mathrm{Hom}(M, N)_i$ l'ensemble des homomorphismes homogènes de degré i. On pose

$$\mathrm{Hom}(M, N) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}(M, N)_i.$$

On munit $\mathrm{Hom}(M, N)$ d'une structure de $A \otimes_k A$-module gradué via les opérations à gauche et à droite provenant des opérations à gauche et à droite sur M et N, c'est-à-dire, pour $a, a' \in A$, $f \in \mathrm{Hom}(M, N)$, on pose

$$afa'(v) = f(ava') = af(v)a'.$$

1.3.3 Bimodules de Soergel

Les bimodules de Soergel fournissent une catégorification de la base canonique C'_w de Kazhdan-Lusztig (voir le Théorème 1.2.1) et interviennent dans le cadre des conjectures de Kazhdan-Lusztig.

Soit $(\mathcal{W}, \mathcal{S})$ un système de Coxeter de rang fini. Soit V une représentation *réflexion-fidèle* de $(\mathcal{W}, S)$, c'est-à-dire une représentation fidèle sur un corps k de caractéristique différente de deux telle que si $x \in \mathcal{W}$,

$$\mathrm{codim}(V^x) = 1 \text{ si et seulement si } x \in \mathcal{T}.$$

Cette définition est introduite dans ([34], Definition 1.5) où il est démontré que tout système de Coxeter de rang fini possède une représentation réflexion-fidèle ([34], Proposition 2.1).

Dans le cas où k est infini, on note $R = \mathcal{O}(V) \cong S(V^*)$ la k-algèbre des fonctions polynomiales de V. On munit un tel anneau d'une graduation positive en plaçant le corps en degré zéro et V^* en degré 2. En particulier, une telle convention implique que $R_i = 0$ pour i impair. On note $\mathcal{R}$ la catégorie des R-bimodules $\mathbb{Z}$-gradués avec même opération de k des deux côtés et de type fini en tant que $R \otimes_k 1$-modules ou $1 \otimes_k R$-modules. Une telle catégorie, munie du produit $\otimes_R$, est monoïdale. Puisque $(\mathcal{W}, \mathcal{S})$ agit sur V, il en résulte une opération de $(\mathcal{W}, \mathcal{S})$ sur R par

$$(w \cdot f)(v) = f(w^{-1} \cdot v),\ w \in \mathcal{W}, f \in R, v \in V.$$

Pour toute réflexion $s \in T$, on note $R^s \subset R$ le sous-anneau des fonctions s-invariantes. Pour tout $x \in \mathcal{W}$, on note R_x l'élément de $\mathcal{R}$ égal à R en tant que $R \otimes_k 1$-module, avec opération à droite tordue par x, c'est-à-dire,

$$(f \cdot r)(v) = f(v)r(xv),\ f \in R_x, r \in R, v \in V.$$

Soit $U = \oplus_{i \in \mathbb{Z}} U_i$ un espace vectoriel $\mathbb{Z}$-gradué de dimension finie. La *dimension graduée* de U est l'élément de $\mathbb{Z}[v, v^{-1}]$ défini par

$$\underline{\mathrm{dim}}(U) = \sum_{i \in \mathbb{Z}} (\mathrm{dim} U_i) v^{-i}.$$

Si M est un R-module à droite, $\mathbb{Z}$-gradué et de type fini, on définit son *rang gradué* par l'élément de $\mathbb{Z}[v, v^{-1}]$ donné par

$$\underline{\mathrm{rk}} M = \underline{\mathrm{dim}}(M / M R_{>0}).$$

On note $\overline{\underline{\mathrm{rk}}} M$ le rang gradué de M où l'on a substitué v par v^{-1}. On note $\langle \mathcal{R}, \otimes_R \rangle$ l'anneau de Grothendieck scindé de la catégorie $\mathcal{R}$.

Théorème 1.3.2 ([33], Theorems 1, 2; [34], Theorems 1.10, 5.3)**.** *Notons $\mathcal{H}$ l'algèbre de Hecke de $(\mathcal{W}, \mathcal{S})$. Soit V une représentation réflexion-fidèle de $(\mathcal{W}, \mathcal{S})$ et R la k-algèbre des fonctions régulières sur V.*

1. *Il existe un unique homomorphisme de* $\mathbb{Z}[v, v^{-1}]$*-algèbres*

$$\mathcal{E} : \mathcal{H} \to \langle \mathcal{R}, \otimes_R \rangle$$

tel que $\mathcal{E}(v) = \langle R[1] \rangle$ *et pour tout* $s \in \mathcal{S}$*, on ait* $\mathcal{E}(T_s + 1) = \langle R \otimes_{R^s} R \rangle$.

2. *L'application* $\mathcal{E}$ *possède un inverse à gauche*

$$\mathrm{ch} : \langle \mathcal{R}, \otimes_R \rangle \to \mathcal{H}$$

donné par $\mathrm{ch}(B) = \sum_{x \in \mathcal{W}} \overline{\underline{\mathrm{rk}}}(\mathrm{Hom}(B, R_x))T_x$.

On peut définir une sous-catégorie de $\mathcal{R}$ constituée des bimodules dits *spéciaux*, c'est-à-dire, des bimodules dont la classe est dans l'image de $\mathcal{E}$. La conjecture de Soergel stipule que les classes des objets indécomposables de la catégorie des bimodules spéciaux sont exactement les images des éléments de la base de Kazhdan-Lusztig $\{C'_w\}_{w \in \mathcal{W}}$ par l'homomorphisme $\mathcal{E}$. Elle a été établie en toute généralité pour les système de Coxeter de rang fini par Elias et Williamson ([18]). Des preuves dans certains cas particuliers avaient été auparavant données par Soergel et Härterich (voir [34], Remarks 1.14, 1.16 et 1.17 pour plus de détails). Cette conjecture implique entre autres la positivité des polynômes de Kazhdan-Lusztig et donne une preuve algébrique de la conjecture de Kazhdan-Lusztig sur les multiplicités de facteurs de composition de modules de Verma.

1.4 Partitions non croisées et monoïde dual

1.4.1 Monoïde dual

La référence pour cette section est principalement [2]. Soit $(\mathcal{W}, \mathcal{S})$ un système de Coxeter fini. On note $B = B(\mathcal{W}, \mathcal{S})$ le groupe de tresses correspondant engendré par une copie $\mathbf{S}$ de $\mathcal{S}$ et $B^+ = B^+(\mathcal{W}, \mathcal{S})$ le monoïde de tresses positif. On note $\mathbf{s} \in \mathbf{S}$ l'élément correspondant à $s \in \mathcal{S}$ et $p : B \to \mathcal{W}$ la surjection canonique définie par $p(\mathbf{s}) = s$. Un *élément de Coxeter* dans $\mathcal{W}$ est un produit de tous les éléments de $\mathcal{S}$. Soit $\mathcal{T} = \bigcup_{w \in \mathcal{W}} w\mathcal{S}w^{-1}$ l'ensemble des réflexions de $\mathcal{W}$ et $\ell_{\mathcal{T}} : \mathcal{W} \to \mathbb{Z}_{\geq 0}$ la *longueur de réflexion* ou *longueur absolue* définie de la façon suivante: pour $w \in \mathcal{W}$, $\ell_{\mathcal{T}}(w)$ est le nombre miminal de facteurs dans une expression de w comme produit de réflexions. Il existe un ordre partiel sur $\mathcal{W}$ noté $<_{\mathcal{T}}$ et défini par $u <_{\mathcal{T}} v$ si

$$\ell_{\mathcal{T}}(u) + \ell_{\mathcal{T}}(u^{-1}v) = \ell_{\mathcal{T}}(v).$$

Malheureusement, $(\mathcal{W}, <_\mathcal{T})$ ne forme pas un treillis. En revanche, on peut démontrer que $\mathcal{P}_c := \{w \in \mathcal{W} \mid w <_\mathcal{T} c\}$ où c est un élément de Coxeter forme un treillis; il s'agit d'un résultat difficile à démontrer. Des preuves pour certains types en ont été données par Brady et Watt ([8]) et Bessis-Digne-Michel ([3]). La première preuve complète au cas par cas est due à Bessis dans [2]. Des preuves uniformes ont ensuite été données par Brady et Watt [9], Ingalls et Thomas [26].

On définit le *monoïde de tresses dual* ou simplement *monoïde dual* B_c^* associé au triplet $(\mathcal{W}, \mathcal{T}, c)$ comme le monoïde engendré par une copie $\{i_c(t) \mid t \in \mathcal{T}\}$ de $\mathcal{T}$ avec relations

$$i_c(t)i_c(t') = i_c(tt't)i_c(t), \text{ si } tt' \in \mathcal{P}_c.$$

Les relations ci-dessus sont appelées *relations de tresses duales*. Le groupe de tresses B est isomorphe au groupe de fractions de B_c^* et on a une injection $B_c^* \hookrightarrow B$. Dans [13], on démontre que $i_c \circ p$ fournit une bijection entre les éléments de B qui appartiennent à la première composante d'un élément de l'orbite de Hurwitz du relevé $\mathbf{c}$ de c dans B (obtenu en remplaçant chaque réflexion simple de c par le générateur du groupe de tresses correspondant) et l'ensemble des *atomes* $\{i_c(t) \mid t \in \mathcal{T}\}$ de B_c^*. La bijection inverse permet d'obtenir l'injection de B_c^* en tant que sous-monoïde de B contenant B^+. Ceci avait déjà été démontré auparavant dans [2] pour les éléments de Coxeter définis par une paire chromatique. Pour $x \in \mathcal{P}_c$ et $t_1 \cdots t_k$ une décomposition $\mathcal{T}$-réduite de x, le produit

$$i_c(t_1) \cdots i_c(t_k)$$

dans B_c^* est indépendant du choix d'expression $\mathcal{T}$-réduite (il s'agit d'une conséquence des relations de tresses duales). On le note par conséquent $i_c(x)$. Les éléments de la forme $i_c(x)$ pour $x \in \mathcal{P}_c$ sont les éléments dits *simples* de B_c^*.

1.4.2 Partitions non croisées classiques

Dans [3] and [2], il est démontré que le poset $(\mathcal{P}_c, <_\mathcal{T})$ est isomorphe au treillis des *partitions non croisées* introduites par Kreweras (voir [30]). La partition correspondant à $x \in \mathcal{P}_c$ est donnée par les supports des différents cycles intervenant dans la décomposition de x vu comme permutation du groupe symétrique $\mathfrak{S}_{n+1}$ en produit de cycles à supports disjoints (voir la sous-section suivante et la figure 1.1 pour une illustration); en fait, ceci est vrai si l'élément de Coxeter c est égal à $s_1 s_2 \cdots s_n$ (ou pour c^{-1} en renversant le sens des indices) mais si c' est un autre élément de Coxeter, on a des isomorphismes de treillis $\mathcal{P}_c \to \mathcal{P}_{c'}$. Dans le chapitre 3, nous verrons qu'il existe également une géométrie pour un élément de Coxeter arbitraire

c'; ce qui change est simplement l'ordre des entiers indexant les différents points du cercle. Nous identifierons systématiquement $\mathcal{P}_c$ avec le treillis des partitions non croisées. Les partitions non croisées ont cardinal égal au *nombre de Catalan* C_{n+1}, c'est-à-dire, $|\mathcal{P}_c| = C_{n+1}$ où

$$C_{n+1} := \frac{1}{n+2}\binom{2(n+1)}{n+1}.$$

1.4.3 Représentation géométrique des partitions non croisées classiques et des éléments simples

Comme mentionné plus haut, il existe une représentation géométrique de toute partition non croisée $x \in \mathcal{P}_c$ par des unions disjointes de polygones ayant leurs sommets parmi un ensemble de $n+1$ points situés sur un cercle et indexés par les entiers $1, 2, \ldots, n+1$ disposés par ordre croissant dans le sens des aiguilles d'une montre, comme dans la figure 1.1. On considérera qu'une arête est un polygone, mais qu'un point isolé n'est pas un polygone. Chaque polygone correspond à un cycle dans la décomposition de $x \in \mathcal{P}_c \subset \mathfrak{S}_{n+1}$ en produit de cycles à supports disjoints. Tout polygone P intervenant dans la représentation géométrique de $x \in \mathcal{P}_c$ sera

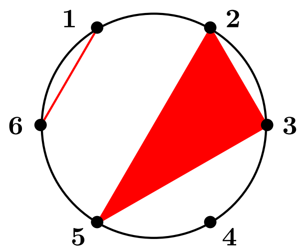

FIG. 1.1: Représentation géométrique de la partitions non croisée $x = (1,6)(2,3,5)$. Ici on a $n = 5$.

noté en utilisant la suite ordonnée des entiers indexant ses somments: on écrira $P = [i_1 i_2 \cdots i_k]$ où $i_1 < i_2 < \cdots < i_k$ et i_j sont les indices des sommets de P, avec $i_j < i_{j+1}$. Le cycle correspondant à P est alors $(i_1, i_2, \ldots, i_k)$. Dans l'exemple donné en figure 1.1, on a deux polygones $P_1 = [235]$ et $P_2 = [16]$.

Définition 1.4.1. *On dira que i_1 est un indice* initial *et que i_k est un indice* terminal *de P ou de x.*

Par conséquent, une arête reliant les points indexés par j et k respectivement, où $j, k \in \{1, \ldots, n+1\}$, correspond à la transposition (j, k). Un triangle ayant ses sommets indexés par les trois entiers $j < k < m$ correspond au cycle $(j, k, m) =$

$(j,k)(k,m)$. A chaque tel triangle correspond une relation de tresse duale (écrite ici dans le groupe de Coxeter):

$$(j,k)(k,m) = (k,m)(m,j) = (m,j)(j,k).$$

Le triplet ordonné $((j,k),(k,m),(m,j))$ est souvent dit *admissible*. Etant donné que $(j,k,m) <_{\mathcal{T}} s_1 s_2 \cdots s_n)$ on obtient l'une des relations de tresses duales du monoïde dual associé à l'élément de Coxeter $c = s_1 s_2 \cdots s_n$:

$$i_c((j,k))i_c((k,m)) = i_c((k,m))i_c((m,j)) = i_c((m,j))i_c((j,k))$$

et toute relation de tresse duale est obtenue de cette façon.

Plus généralement, tout polygone convexe à k arêtes ayant ses sommets parmi les $n+1$ points est la représentation géométrique d'une partition non croisée qui est un cycle, obtenu à partir des transpositions de la façon suivante: il suffit de choisir $k-1$ arêtes du polygone (rappelons qu'une arête correspond à une transposition) et d'effectuer le produit des réflexions correspondantes dans le sens des aiguilles d'une montre, en commençant par la réflexion correspondant à l'arête se situant juste après celle qui a été retirée (dans le sens des aiguilles d'une montre). Ceci est une conséquence des relations duales et puisque celles-ci sont aussi vraies dans le monoïde dual, on peut par conséquent utiliser la géométrie et les règles associées ci-dessus lorsqu'on est amené à travailler avec les éléments simples du monoïde dual.

Chapter 2

A categorification of the diagram basis of the Temperley-Lieb algebra by analogues of Soergel bimodules

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The Coxeter systems $(\mathcal{W}, \mathcal{S})$ considered in this chapter will always be of type A_n (with $n \geq 2$) unless otherwise specified, identifying $\mathcal{W}$ with the symmetric group $\mathfrak{S}_{n+1}$ on $n+1$ letters and $\mathcal{S}$ with the set of simple transpositions $s_i = (i, i+1)$ for all $i = 1, \ldots, n$. We will denote by $\mathcal{T}$ the set of reflections and by V the geometric representation of $(\mathcal{W}, \mathcal{S})$ over a field k of characteristic zero. For $t \in \mathcal{T}$ we denote by H_t the reflecting hyperplane of t.

2.1 Combinatorics of Weyl lines

2.1.1 Weyl lines

The representation V is reflection faithful in the sense of ([34], Definition 1.5) as a consequence of ([34], Proposition 2.1). We recall the definition of a *Weyl line* from ([16], Definition 3.18):

Definition 2.1.1. *A* Weyl line *is a subspace of V of dimension one that is the intersection of reflection hyperplanes. A Weyl line is* transverse *to some reflection $t \in \mathcal{T}$ if it is not contained in H_t.*

We denote by Z the union of all the Weyl lines in V, which is a $\mathcal{W}$-stable subvariety of V. We write V_t for the union of the Weyl lines transverse to $t \in \mathcal{T}$ viewed as a subvariety of $Z \subset V$; if the reflection is simple, that is, if $t = s_i$, we will often write V_i to mean V_{s_i}.

Lemma 2.1.2. *There exists a $\mathcal{W}$-equivariant bijection*

$$\left\{ \text{ Weyl lines in } V \text{ } \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{partitions of } \{1, \dots, n+1\} \text{ into} \\ \text{two nonempty subsets} \end{array} \right\},$$

which to any Weyl line $L = \bigcap_{i=1}^{n-1} H_{t_i}$, where H_{t_i} is the reflection hyperplane of $t_i \in \mathcal{T}$, associates the partition given by the supports of the cycles of the decomposition of $t_1 \cdots t_{n-1}$ into a product of cycles with disjoint support (which turns out to be a partition into two subsets as the proof will show).

Proof. One has to show that the map defined above is well-defined. Suppose $L = \bigcap_{i=1}^{n-1} H_{t_i}$ is a Weyl line in V. The product $w = t_1 \cdots t_{n-1}$ has reflection length equal to $n-1$ since L has dimension one (the set of roots of the t_i consists of linearly independent vectors, which implies that $t_1 \cdots t_{n-1}$ is a reduced $\mathcal{T}$-decomposition for w ; the parabolic subgroup generated by the t_i is equal to the subgroup of elements of $\mathcal{W}$ fixing L; see [10], section 2). Now the reflection length of an element of the symmetric group $\mathfrak{S}_{n+1}$ is equal to $n+1$ minus the number of cycles occurring in the decomposition into disjoint cycles. This forces w as element of $\mathfrak{S}_{n+1}$ to fix at most one letter. If it fixes exactly one letter j, suppose that L is written as another intersection of reflecting hyperplanes $\bigcap_{i=1}^{n-1} H_{t_i'}$. Then all the t_i' fix L and hence have to be in the parabolic subgroup of $\mathcal{W}$ generated by the t_i. Hence all the t_i' have to fix the letter j and one gets the same partition of $n+1$ into two sets as before.

If no letter is fixed, write $S_1 \dot\cup S_2$ for the disjoint union of the supports of the two cycles. If L is written $\bigcap_{i=1}^{n-1} H_{t_i'}$, then every t_j' has to be in the parabolic subgroup generated by the t_i and since it is a conjugate of some t_i it will either fix S_1 or fix S_2. Hence we obtain the same partition into two sets as before.

Now for each partition $S_1 \dot\cup S_2$ of $\{1, \ldots, n+1\}$ write a corresponding n-cycle if either S_1 or S_2 has cardinality one or write a corresponding product of two cycles if both have cardinality more than one and decompose them in the obvious way as products of $n-1$ reflections. This proves that the map is surjective. Now if $L \neq L'$ are two different Weyl lines, then one can find some reflecting hyperplane $L' \subset H_s$ such that $L \cap H_s = 0$. Then s cannot be in the parabolic subgroup of elements fixing L and hence L and L' will not yield the same partition. □

Remark 2.1.3. In fact, Weyl lines are in bijection with rank $n-1$ parabolic subgroups (that is, maximal parabolic subgroups, not necessarily standard);it is a general fact for reflection groups that there is a $\mathcal{W}$-equivariant bijection between subspaces of dimension one obtained by intersections of reflecting hyperplanes and maximal parabolic subgroups, given by

$$L \mapsto \mathrm{Fix}_{\mathcal{W}}(L) := \{w \in \mathcal{W} \mid w(v) = v \ \forall v \in L\}.$$

It will be convenient to give another description of the Weyl lines by considering V as a subspace of k^{n+1}. In that setting we have

$$V = \left\{ (x_1, \ldots, x_{n+1}) \in k^{n+1} \mid \sum_{i=1}^{n+1} x_i = 0 \right\},$$

$$Z = \{(x_1, \ldots, x_{n+1}) \in V \mid |\{x_1, \ldots, x_{n+1}\}| \leq 2\}.$$

Now given any partition P of the form $S_1 \dot\cup S_2 = \{1, \ldots, n+1\}$ with $S_1, S_2 \neq \emptyset$, the corresponding Weyl line Z_P is obtained by

$$Z_P = \{(x_1, \ldots, x_{n+1}) \mid x_i = x_j \text{ if either } \{i,j\} \subset S_1 \text{ or } \{i,j\} \subset S_2\}.$$

Any reflection $t \in \mathcal{T}$ corresponds to a subset $\{i,j\} \subset \{1, \ldots, n+1\}$ given by its support. We have

$$V_t \backslash \{0\} = \{x \in Z \mid x_i \neq x_j\} \text{ whereas } H_t = \{x \in V \mid x_i = x_j\}.$$

Two reflections $t, t' \in \mathcal{T}$ commute with each other if and only if the intersection of the two corresponding subsets is empty.

Lemma 2.1.4. *Let $t, t', t'' \in \mathcal{T}$ be three distinct reflections each commuting with neither of the others. In particular $t'tt' = tt't = t''$. Then*

$$V_t \cap V_{t'} = V_t \cap H_{t''} = V_{t'} \cap H_{t''}.$$

In particular $V_t \cap (V_{t'} \cup V_{t''}) = V_t$.

Proof. We write $t = (i,j), t' = (j,k), t'' = (i,k)$. One has

$$V_t \cap V_{t'} = \{x \in Z \mid x_i \neq x_j \neq x_k\}.$$

But since Z contains all the vectors with at most two different entries, the set above is also equal to

$$\{x \in Z \mid x_j \neq x_i = x_k\} = V_t \cap H_{t''} = V_{t'} \cap H_{t''}.$$

□

2.1.2 Noncrossing and dense sets of reflections

Notation. For $i \leq j$ two integers we write $[i,j]$ for the set $\{i, i+1, \ldots, j-1, j\}$.

Definition 2.1.5. *Two indices i, j in $[1,n]$ are* distant *if* $|i - j| > 1$.

Notation. Given any closed subset $W \subset Z$ we write sW for the closed subset $\{s(w) \mid w \in W\}$.

To any sequence $i_1 \cdots i_m$ with $i_j \in \{1, \ldots, n\}$ of length at least one, we associate the variety $W(i_1 \cdots i_m)$ built inductively by setting $W(i) = V_i$ and

$$W(i_1 \cdots i_m) = V_{i_1} \cap (W(i_2 \cdots i_m) \cup s_{i_1} W(i_2 \cdots i_m)).$$

These varieties will play a key role later on. We write $\mathcal{V}_n$ for the family of varieties obtained in this way.

Example 2.1.6 For i and j with $|j - i| > 1$, one has $W(ij) = V_i \cap V_j = W(ji)$.

Example 2.1.7 We have $W(i(i \pm 1)) = V_i \cap (V_{i\pm1} \cup s_i V_{i\pm1}) = V_i$ by Lemma 2.1.4.

Example 2.1.8 One has $W(i(i \pm 1)i) = V_i = W(i)$.

One should see an analogy between example 2.1.6 and the Temperley-Lieb relation $b_i b_j = b_j b_i$ as well as between example 2.1.8 and the Temperley-Lieb relation $b_i b_{i\pm1} b_i = b_i$. We will make this analogy into an explicit link below.

Lemma 2.1.9. *Let $j \leq m \leq i$. Then $W(m(m-1) \cdots j) = V_m$ and $W(m(m+1) \cdots i) = V_m$.*

Proof. We prove the first equality by induction on $m-j$, the second being similar. If $m-j=0$ then $W_m = V_m$ by definition. Suppose $m-j>0$. Then

$$W(m(m-1)\cdots j) = V_m \cap (W((m-1)\cdots j) \cup s_m W((m-1)\cdots j))$$

and $W((m-1)\cdots j) = V_{m-1}$ by induction. Example 2.1.7 concludes. □

Notation. Let $s \in \mathcal{T}$ and $W \subset Z$ a closed subset. For short, we write $s \cdot W$ for the variety $V_s \cap (W \cup sW)$. If $s = s_i \in \mathcal{S}$ we even write $i \cdot W$ for the variety $s_i \cdot W = V_i \cap (W \cup s_i W)$. More generally given any sequence $i_1 \cdots i_k$ of indices in $\{1, \ldots, n\}$, we write $i_1 \cdots i_k \cdot W$ for the variety $s_{i_1} \cdot (s_{i_2} \cdot (\cdots (s_{i_k} \cdot W) \cdots))$. It is not a group action but an action of the Kauffman monoid $\mathcal{K}_n$ as explained below. Notice that if $W = W(i_1 \cdots i_k)$,

$$s_j \cdot W = W(j i_1 \cdots i_k).$$

The Kauffman monoid $\mathcal{K}_n$ has $n+1$ generators $b_1, \ldots, b_n, \delta$ and relations

$$\begin{aligned} b_j b_i b_j &= b_j \text{ if } |i-j| = 1, \\ b_i b_j &= b_j b_i \text{ if } |i-j| > 1, \\ b_i^2 &= \delta b_i = b_i \delta. \end{aligned}$$

One checks easily that the assignments $b_i \cdot W := s_i \cdot W = V_i \cap (W \cup s_i W)$, $\delta \cdot W = W$ for $W \in \mathcal{V}_n$ define an operation of $\mathcal{K}_n$ on $\mathcal{V}_n$.

Lemma 2.1.10. *Suppose that $Q \subset \mathcal{T}$ is a set of pairwise commuting reflections. Let $s \in \mathcal{T}$. Set $W := \bigcap_{t \in Q} V_t$. Then $W \neq 0$ and $s \cdot W = \bigcap_{t \in Q'} V_t$ where*

$$Q' = \begin{cases} Q \cup \{s\} & \textit{if } st = ts \textit{ for each } t \in Q \\ (Q \backslash t) \cup \{s\} & \textit{if } \exists! t \in Q \textit{ such that } st \neq ts \\ (Q \backslash \{t, t'\}) \cup \{s, tt'st't\} & \textit{if } \exists t \neq t' \in Q \textit{ such that } st \neq ts, st' \neq t's. \end{cases}$$

and Q' is also a set of pairwise commuting reflections, in particular $s \cdot W \neq 0$.

Proof. Notice that

$$W = \{x \in Z \mid x_i \neq x_j \text{ if } (i,j) \in Q\}.$$

In particular $W \neq 0$ since different reflections in Q correspond to disjoint subsets of $\{1, \ldots, n+1\}$. The fact that Q' is a set of pairwise commuting reflections is obvious in the first two cases and is a straightforward computation in the third case, viewing the reflections as transpositions.

Recall that for $s, t \in \mathcal{T}$, $sV_t = V_{sts}$. If $s \in \mathcal{T}$ commutes with every $t \in Q$ then $s \cdot W = \left(\bigcap_{t \in Q} V_t\right) \cap V_s$ since $sV_t = V_{sts} = V_t$ whenever s and t commute.

If $st \neq ts$ for some $t \in T_W$ but s commutes with any $t' \in Q$ with $t' \neq t$, then

$$s \cdot W = \left(\bigcap_{r \in Q \setminus t} V_r\right) \cap V_s \cap (V_t \cup V_{sts})$$

As we have seen in Lemma 2.1.4 we have $V_s \cap (V_t \cup V_{sts}) = V_s$ hence

$$s \cdot W = \bigcap_{r \in (Q \setminus t) \cup \{s\}} V_r,$$

The remaining case is the case where s does not commute with exactly two reflections $t, t' \in Q$. In that case one has

$$s \cdot W = \left(\bigcap_{r \in Q \setminus \{t, t'\}} V_r\right) \cap V_s \cap ((V_t \cap V_{t'}) \cup (V_{sts} \cap V_{st's})).$$

We claim that

$$V_s \cap ((V_t \cap V_{t'}) \cup (V_{sts} \cap V_{st's})) = V_s \cap V_{tst'st},$$

which concludes. Indeed, by Lemma 2.1.4 we have

$$\begin{aligned} V_s \cap V_t \cap V_{t'} &= V_t \cap V_{t'} \cap H_{st's} = V_{t'} \cap (V_t \cap H_{st's}) \\ &= V_{t'} \cap V_t \cap V_{tst'st}. \end{aligned}$$

Similarly, $V_s \cap V_{sts} \cap V_{st's} = V_{sts} \cap V_{st's} \cap V_{tst'st}$. Conversely, since $V_s \cap V_{tst'st}$ is not equal to zero consider a Weyl line $L \subset V_s \cap V_{tst'st}$. If $L \not\subset H_t$, then $L \subset H_{st's}$ and hence $L \not\subset H_{t'}$ since $L \not\subset H_s$. Similarly if $L \subset V_s \cap V_{tst'st}$ and $L \subset H_t$ then $L \not\subset H_{st's}$ (since $L \not\subset H_{tst'st}$) and $L \not\subset H_{sts}$ (since $L \not\subset H_s$). □

Proposition 2.1.11. *Let $W \in \mathcal{V}_n$. Then $W \neq \{0\}$ and there exists a unique set $T_W \subset \mathcal{T}$ with $tt' = t't$ for each $t, t' \in T_W$ such that $W = \bigcap_{t \in T_W} V_t$. Moreover,*

$$T_W = \{s \in \mathcal{T} \mid W \subset V_s\}.$$

Proof. Existence is shown using induction on the length of a sequence associated to a variety in $\mathcal{V}_n$. If $W \in \mathcal{V}_n$ is obtained from a sequence of length one, then $W = V_j$ for some j and $W \neq 0$. Now assume the result holds for each variety in $\mathcal{V}_n$ obtained from a sequence of length less than or equal to m, and suppose $W \in \mathcal{V}_n$ is obtained from a sequence of length equal to $m+1$. Then by definition $W = s \cdot W'$ where $s \in \mathcal{S}$ and $W' \in \mathcal{V}_n$ is obtained from a sequence of length equal to m. By induction $W' = \bigcap_{t \in T_{W'}} V_t$ with $T_{W'} \subset \mathcal{T}$ a set of pairwise commuting reflections. Thanks to Lemma 2.1.10 we have $W = \bigcap_{t \in Q'} V_t$ with $Q' \subset \mathcal{T}$ a set of pairwise commuting reflections and $W \neq \{0\}$.

For uniqueness, notice that for any set $Q \subset \mathcal{T}$ of pairwise commuting reflections, the set

$$W_Q \backslash \{0\} = \{x \in Z \mid (i,j) \in Q \Rightarrow x_i \neq x_j\}$$

determines Q since $(i,j) \notin Q$ if and only if there exists $x \in W_Q \backslash \{0\}$ with $x_i = x_j$. But $W = W_{T_W}$ which concludes. It also proves the second claim since $W = W_{T_W} \subset V_s$ where $s = (i,j)$ implies that $x_i \neq x_j$ for any $x \in W \backslash \{0\}$. □

The following consequence will be crucial further:

Corollary 2.1.12. *Let $W \in \mathcal{V}_n$, $i \in \{1, \ldots, n\}$. Then $V_i \cap W \neq \{0\}$. Moreover, the following are equivalent:*

1. *The variety W is s_i-invariant,*
2. *The variety $W \cap V_i$ is s_i-invariant,*
3. *For each $t \in T$ such that $ts_i \neq s_i t$, $(W \cap V_i) \cap V_t \neq 0$.*

Proof. Thanks to Proposition 2.1.11, $W = \bigcap_{t \in T_W} V_t$, where $T_W \subset \mathcal{T}$ is a set of pairwise commuting reflections. Hence we can find a partition $S_1 \dot\cup S_2 = \{1, \ldots, n+1\}$ such that $i \in S_1$, $i+1 \in S_2$ and each $t \in T_W$ can be written as a transposition (j,k) with $j \in S_1$ and $k \in S_2$. Thanks to Lemma 2.1.2 this gives us a corresponding Weyl line included in $W \cap V_i$, hence $W \cap V_i \neq \{0\}$. If W is s_i-invariant then so is $W \cap V_i$. Now assume that $W \cap V_i$ is s_i-invariant and suppose that $W \cap V_i \subset H_t$ for some reflection t which does not commute with s_i. One then gets $W \cap V_i \subset H_{s_i t s_i}$ by s_i-invariance and hence also $W \cap V_i \subset H_i$ by t-invariance which force $W \cap V_i = \{0\}$. Now assume that W is not s_i-invariant. It implies that there exists $t' \in T_W$ such that $t's_i \neq s_i t'$. Since $W \subset V_{t'}$ by Proposition 2.1.11, we have that $V_i \cap W \subset V_i \cap V_{t'} \subset H_{s_i t' s_i}$ by Lemma 2.1.4, and $t = s_i t' s_i$ does not commute with s_i since t' does not. □

Definition 2.1.13. *A set $Q \subset \mathcal{T}$ of pairwise commuting reflections is* noncrossing *if after identification with a set of transpositions of the isomorphic symmetric group, it contains no pair of transpositions (i,j) and (k,l) with $i < k < j < l$.*

If we draw $n+1$ points on a circle and label each of them with an index between 1 and $n+1$, starting by 1 at some point and writing the increasing indices in clockwise order, and represent a transposition by a line segment between the two indices it exchanges, a set $Q \subset W$ of reflections is noncrossing if and only if any two segments in the corresponding circle never cross each other. Equivalently, one can draw a line with $n+1$ points labeled with integers between 1 and $n+1$, starting on the left with the point labeled with 1. One then represents a transposition by an arc between the two numbers it exchanges. The arcs are considered up to isotopy. In that setting, a set of reflections is noncrossing if and only if there is a way of writing the arcs such that any two arcs associated to different reflections from our set never cross. This last way of representing noncrossing sets will turn out to be the most convenient one.

Definition 2.1.14. *Let $Q \subset \mathcal{W}$. The* support *of Q, written* $\mathrm{supp}(Q)$*, is the union of the supports of its elements viewed as elements of the symmetric group. A set $Q \subset \mathcal{T}$ of pairwise commuting reflections is* dense *if it is noncrossing and if there exists an integer $k > 0$ and integers $0 < m_1 < j_1 < m_2 < j_2 < \cdots < m_k < j_k \leq n+1$ such that $(m_q, j_q) \in Q$ and $\mathrm{supp}(Q) = \bigcup_{q=1}^{k} [m_q, m_q+1, \ldots, j_q]$. This forces in particular $j_q - m_q$ to be odd for each q since Q is noncrossing and $(m_q, j_q) \in Q$. A subset of* $\mathrm{supp}(Q)$ *of the form $[m_q, m_q+1, \ldots, j_q]$ as above will be called a* block *of Q.*

Remark 2.1.15. The reason we choose to call a set $Q \subset \mathcal{T}$ with the properties above a dense set is the following: given any reflection $t = (i,j) \in Q$ with $i < j$, any number k with $i < k < j$ must lie in the support of a reflection $t' \in Q$;this is a maximality property.

Lemma 2.1.16. *Let $W \in \mathcal{V}_n$. Then T_W is noncrossing.*

Proof. Again, we use induction on the length of the sequence defining W. If such a sequence has length one the result is clear. Let $W = s \cdot W'$ and suppose that $Q = T_{W'}$ is noncrossing, then $Q' = T_{s \cdot W'}$ is also noncrossing using the formulas from Lemma 2.1.10 (it is obvious in the two first cases and clear for the last one if we represent Q and Q' as arcs joining points on a line).

□

Notation. If $W \in \mathcal{V}_n$ is associated to a sequence $i_1 \cdots i_k$ we will often write $T(i_1 \cdots i_k)$ instead of T_W for convenience. Notice that using Lemma 2.1.10 together with Proposition 2.1.11 one can inductively compute the variety and the corresponding dense set associated to a sequence.

Theorem 2.1.17. *Let $W \in \mathcal{V}_n$. Then T_W is dense. Conversely, any dense subset $Q \subset \mathcal{T}$ is equal to a $T_{V'}$ for some variety $V' \in \mathcal{V}_n$. In formulas,*

$$\{T_W \mid W \in \mathcal{V}_n\} = \{Q \subset \mathcal{T} \mid Q \text{ is dense}\}.$$

Proof. Thanks to Lemma 2.1.16, T_W is noncrossing for any $W \in \mathcal{V}_n$. If W is associated to a sequence of length one then T_W contains only one simple reflection, hence is dense. It suffices then to show that the rules from Lemma 2.1.10 preserve dense sets, which is clear for the first two rules and easy for the last one if we represent the reflections as arcs joining points on a line.

Conversely suppose that Q is dense, in particular $\mathrm{supp}(Q) = \bigcup_{q=1}^{k}[m_q, j_q]$, with $j_q - m_q$ odd for each $1 \leq q \leq k$. Consider the set of simple reflections $\bigcup_{q=1}^{k}\{s_{m_q}, s_{m_q+2}, \dots, s_{j_q-1}\}$ and rewrite this union as $\{s_{k_1}, \dots, s_{k_{n(Q)}}\}$ with $k_i < k_j$ if $i < j$. Notice that this is a set of pairwise commuting reflections. We will show by induction on the size of the biggest block of Q that there exists a sequence $\mathrm{seq} = n_1 n_2 \cdots n_\ell$ with $n_i \in \bigcup_{q=1}^{k}[m_q, j_q - 1]$ for each $1 \leq i \leq \ell$ such that $Q = T_W$ where W is associated to the sequence

$$\mathrm{seq} k_1 k_2 \cdots k_{n(Q)}$$

obtained by concatenation of the sequence seq and the sequence $k_1 k_2 \cdots k_{n(Q)}$. Firstly we suppose that the size of the biggest block is equal to one. Then each block has size one, in other words, $j_q = m_q + 1$ for each q and there is only one corresponding dense set Q: the set of reflections $\{s_{k_1}, s_{k_2}, \dots, s_{k_{n(Q)}}\}$. One then has $Q = T_W$ with W associated to the sequence $k_1 k_2 \cdots k_{n(Q)}$ (see example 2.1.6).

Now suppose that the biggest block $\mathcal{B}_i = [m_i, j_i]$ of Q has size bigger than one. It suffices to show the induction hypothesis for the set Q_i of reflections in Q supported in $\mathcal{B}_i$, i.e., that Q_i is equal to T_W for some W associated to a sequence $\mathrm{s(i)} = \mathrm{seq}_i m_i (m_i+2) \cdots (j_i - 1)$ where seq_i is a sequence with all indices in $[m_i, j_i - 1]$: if this holds, one associates to each block $\mathcal{B}_q$ of Q the variety $W_{\mathrm{s(q)}}$ such that $T_{W_{\mathrm{s(q)}}}$ is equal to the set Q_q of reflections in Q supported in $\mathcal{B}_q$ (this is possible since we show it for the biggest block(s) and the result holds by induction for blocks of smaller size); but then if $q \neq q'$ the reflections in Q_q commute with the reflections in $Q_{q'}$ since they are supported in $[m_q, j_q]$ and $[m_{q'}, j_{q'}]$ respectively which are disjoint.

Hence one gets

$$\begin{aligned} Q &= \bigcup_{q=1}^{k} T(\mathrm{s}(q)) \\ &= T(\mathrm{s}(1)\cdots\mathrm{s}(k)) \\ &= T(\mathrm{seq}_1 m_1(m_1+2)\cdots(j_1-1)\cdots\mathrm{seq}_k m_k(m_k+2))\cdots(j_k-1) \\ &= T(\mathrm{seq}_1\cdots\mathrm{seq}_k \underbrace{m_1(m_1+2)\cdots(j_1-1)\cdots m_k(m_k+2)\cdots(j_k-1)}_{=k_1k_2\cdots k_{n(Q)}}), \end{aligned}$$

where second and last equalities hold since the indices in $\mathrm{s}(i)$ are distant from the indices $\mathrm{s}(i')$ whenever $i \neq i'$ (if two sequences x and y are such that any index in x is distant from any index in y then it is a consequence of Lemma 2.1.10 that $T(\mathrm{xy}) = T(\mathrm{yx}) = T(\mathrm{x}) \cup T(\mathrm{y})$).

Therefore we have to show that a dense set Q having only one block $[k_1, k_{n(Q)}+1]$ is equal to T_W for $W \in \mathcal{V}_n$ associated to a sequence obtained by concatenating a sequence with indices in $[k_1, k_{n(Q)}]$ to the left of the sequence $k_1\cdots k_{n(Q)}$; since Q has a single block we have $k_{j+1} = k_j+2$ for each $k = 1, \dots, n(Q)-1$. We first show that we can concatenate a sequence to the left of $k_1\cdots k_{n(Q)}$ to obtain a corresponding variety W' such that $T_{W'} = Q'$ contains exactly the reflection $(k_1, k_{n(Q)}+1)$ and all the simple reflections s_{k_1+1}, s_{k_2+1},$\dots$,$s_{k_{n(Q)-1}+1}$. Then we will build W from W' by induction; see figure 2.1 for an illustration of this process. Using Lemma 2.1.10 inductively we get that $T_{W_{(k_i+1)\cdots(k_{n(Q)-1}+1)k_1\cdots k_{n(Q)}}}$ is equal to the set

$$\{s_{k_1}, s_{k_2}, \dots, s_{k_{i-1}}, (k_i, k_{n(Q)}+1), s_{k_i+1}, s_{k_{i+1}+1}, \dots, s_{k_{n(Q)-1}+1}\},$$

hence $Q' = T_{W'}$ where W' is associated to the sequence

$$(k_1+1)(k_2+1)\cdots(k_{n(Q)-1}+1)k_1\cdots k_{n(Q)}.$$

Now $Q'' := Q\backslash\{(k_1, k_{n(Q)}+1)\}$ is dense since Q is dense and $\mathrm{supp}(Q'') = [k_1+1, k_{n(Q)}]$; hence all blocks of Q'' have a size smaller than $k_{n(Q)}+1-k_1$. By induction, Q'' is therefore equal to $T_{W''}$ for W'' associated to a sequence $\mathrm{seq}(k_1+1)(k_2+1)\cdots(k_{n(Q)-1}+1)$ for some sequence seq having all its indices lying in $\{k_1+1, \dots, k_{n(Q)-1}+1\}$. But then $s = (k_1, k_{n(Q)}+1)$ commutes with any reflection

s_ℓ where ℓ is an index in seq, hence one has

$$\begin{aligned}
W_{\mathrm{seq}(k_1+1)\cdots(k_{n(Q)-1}+1)k_1\cdots k_{n(Q)}} &= \mathrm{seq}\cdot W_{(k_1+1)\cdots(k_{n(Q)-1}+1)k_1\cdots k_{n(Q)}} = \mathrm{seq}\cdot W' \\
&= \mathrm{seq}\cdot\left(\bigcap_{t\in Q'} V_t\right) \\
&= \mathrm{seq}\cdot\left(V_s \cap \bigcap_{i=1}^{n(Q)-1} V_{s_{k_i+1}}\right) \\
&= \mathrm{seq}\cdot\left(V_s \cap W_{(k_1+1)\cdots(k_{n(Q)-1}+1)}\right) \\
&= V_s \cap \left(\mathrm{seq}\cdot W_{(k_1+1)\cdots(k_{n(Q)-1}+1)}\right) \\
&= V_s \cap \left(\bigcap_{t\in Q''} V_t\right) = \bigcap_{t\in Q} V_t,
\end{aligned}$$

and the sequence $\mathrm{seq}(k_1+1)(k_2+1)\cdots(k_{n(Q)-1}+1)$ has all its indices lying in $[k_1, k_{n(Q)-1}+1] \subset [k_1, k_{n(Q)}]$. □

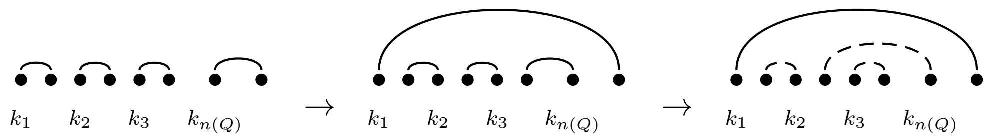


FIG. 2.1: Illustration of the process used in the proof of Theorem 2.1.17 to build a dense set Q having a single block of maximal size from the sequence $k_1\cdots k_{n(Q)}$ with $n(Q)=4$. On the left is the dense subset associated to this sequence; in the middle is the dense set Q' associated to the sequence $(k_1+1)\cdots(k_{n(Q)-1}+1)k_1\cdots k_{n(Q)}$; on the right is the set Q. The reflections of the dense set Q'' are drawn in dashed on the rightmost picture.

2.2 Quasi-coherent sheaves on Weyl lines

2.2.1 Regular functions on Weyl lines

Let R be the algebra of regular functions on V and $\bar{R}$ be the algebra of regular functions on Z. Notice that $R \twoheadrightarrow \bar{R}$. In ([16], Proposition 3.24) it is shown that $I(Z)$ is generated in degree 3. For each subset $J \subset \mathcal{T}$, we write R_J for the algebra of regular functions on the union of Weyl lines transverse to every element in J. We will write R_i instead of $R_{\{s_i\}}$ where $s_i \in \mathcal{S}$, $R_{i,j}$ instead of $R_{\{s_i,s_j\}}$, etc.

We denote by f_k the element of R or $\bar{R}$ which is the equation of the reflecting hyperplane H_{s_k}. We will often abuse notation and write f_i for $f_i|_X$ where X is a subvariety of Z.

Let $t \in \mathcal{T}$. If $X \subset V$ is a t-stable Zariski-closed subset, then t induces a map $\mathcal{O}(X) \to \mathcal{O}(X)$ and one has a decomposition into eigenspaces $\mathcal{O}(X) = \mathcal{O}(X)^t \oplus \mathcal{O}(X)^t f_t$ where f_t is an equation of the reflecting hyperplane H_t. If moreover no irreducible component of X lies in H_t, then the Demazure operator $\partial_t : R \to R, f \mapsto (2f_t)^{-1}(f - tf)$ induces a map $\mathcal{O}(X) \to \mathcal{O}(X)$ and as R^t-modules $\mathcal{O}(X)^t \xrightarrow{\sim} \mathcal{O}(X)^t f_t$ where the isomorphism is given by multiplication by f_t and its inverse by the restriction of ∂_t.

The operation of $\mathcal{W}$ on $R \cong k[f_1, \cdots f_n]$ or $\bar{R}$ is the following: if suffices to give the operation of a simple reflection. For $s_i \in \mathcal{S}$ we have

$$s_i \cdot f_i = -f_i,$$

$$s_i \cdot f_{i\pm1} = f_i + f_{i\pm1},$$

$$s_i \cdot f_j = f_j \text{ if } i \text{ and } j \text{ are distant.}$$

In particular if X is a closed subset that is s_i-invariant and i and j are distant, then $f_j \in \mathcal{O}(X)^{s_i}$.

Remark 2.2.1. A consequence of Corollary 2.1.12 which will be crucial further is the following : suppose $W \cap V_i$ is not s_i-invariant. Then $W \cap V_i \subset H_t$ for some $t \in T$ such that $ts_i \neq s_i t$. Then $t = (i, k)$ or $(i+1, k)$ for some $k \neq i, i+1$, say $t = (i+1, k)$. Suppose $k < i$. In H_t one has

$$f_k + f_{k+1} + \cdots + f_i = 0,$$

hence

$$f_i = -2f_k - \cdots - 2f_{i-1} - f_i.$$

Viewing the right hand side in R_i one sees that it lies in $R_i^{s_i}$ since $f_j \in R_i^{s_i}$ if $|i - j| > 2$ and $2f_{i-1} + f_i \in R_i^{s_i}$. One can do the same for the other cases (the case where $k > i + 1$ and the cases where $t = (i, k)$). Since $R_i = R_i^{s_i} \oplus R_i^{s_i} f_i$ one has that $R_i^{s_i} \twoheadrightarrow \mathcal{O}(W \cap V_i)$. In other words, when choosing a function f in R_i such that $f|_{W \cap V_i}$ is equal to a given $g \in \mathcal{O}(W \cap V_i)$, one can always suppose that f is s_i-invariant.

2.2.2 Gradings

The Temperley-Lieb algebra will be realized via $\bar{R} \otimes_k \bar{R}$-modules. In order to interpret the parameter v in a categorification of the Temperley-Lieb algebra, the bimodules we will consider need to be $\mathbb{Z}$-graded, and multiplication by v will correspond to a shift in graduation. If A, B are two $\mathbb{Z}$-graded k-algebras, we write $A - \text{mod} - B$ for the category of $A \otimes_k B^{\text{op}}$-modules (that we will call "(A, B)-bimodules") and $A - \text{mod}_{\mathbb{Z}} - B$ for the category of $\mathbb{Z}$-graded $A \otimes_k B^{\text{op}}$-modules (that we will call "graded (A, B)-bimodules") with morphisms the bimodule morphisms that are homogeneous of degree zero. In all the cases we will consider in this document, A and B will be commutative graded k-algebras, hence both operations give left or right-module structures. However, to distinguish the operations for example in case $A = B$, we will always refer to the operation of A as the "left" operation and the operation of B as the "right" operation on an (A, B)-bimodule M.

If $M \in A - \text{mod}_{\mathbb{Z}} - B$, recall from subsection 1.3.1 that we write $M[k]$ for the bimodule equal to M in $A - \text{mod} - B$ but with graduation shifted by k, that is, $(M[k])_i = M_{i+k}$.

The algebra R of regular functions on V is naturally graded ; we use the convention that it is nonnegatively graded with $R_2 = V^*$. Now $I(Z)$ is the intersection of the ideals of all the Weyl lines and the ideal of a line is homogeneous; hence $I(Z)$ is also homogeneous and $\bar{R}$ inherits a $\mathbb{Z}$-grading from R. From now on the word "graded" will always mean "$\mathbb{Z}$-graded".

Remark 2.2.2. Let A, B, C be $\mathbb{Z}$-graded rings, let $M \in A - \text{mod}_{\mathbb{Z}} - B$ and $N \in B - \text{mod}_{\mathbb{Z}} - C$. There is a unique $\mathbb{Z}$-grading on $M \otimes_B N$ such that

$$m \in M_i, n \in N_j \Rightarrow m \otimes n \in (M \otimes_B N)_{i+j}.$$

Lemma 2.2.3. *Let A, B, C be (graded) rings, $f : C \to A$ a morphism of (graded) rings, M a (graded) module in $B - \text{mod} - C$. Let $I \subset A$ be a (homogeneous) ideal which annihilates $M \otimes_C A$ on the right. Then one has an isomorphism in $B - \text{mod} - C$, resp. $B - \text{mod}_{\mathbb{Z}} - C$*

$$M \otimes_C A \cong M \otimes_C (A/I).$$

Proof. By right exactness of the tensor product we have a surjection

$$M \otimes_C A \twoheadrightarrow M \otimes_C (A/I).$$

Since by assumption the map $M \otimes_C I \to M \otimes_C A$ induced by the injection $I \hookrightarrow A$

is the zero map, the surjection above has an inverse. □

Lemma 2.2.4. *Let $W \in \mathcal{V}_n$. Then $\mathcal{O}(W)$ is graded.*

Proof. Since W is a union of Weyl lines its vanishing ideal is homogeneous as it is an intersection of ideals of lines, which are known to be homogeneous. □

Remark 2.2.5. Putting 2.2.2, 2.2.3 and 2.2.4 together we have the following: if $M \in \bar{R}-\mathrm{mod}_{\mathbb{Z}}-\bar{R}$, $W \in \mathcal{V}_n$ and if the right operation of $\bar{R}$ on M factors through $\mathcal{O}(V_M)$ where $V_M \in \mathcal{V}_n$ (in other words, M can be viewed as an object in $\bar{R}-\mathrm{mod}-\mathcal{O}(V_M)$), then M lies in $\bar{R}-\mathrm{mod}_{\mathbb{Z}}-\mathcal{O}(V_M)$ and

$$B := M \otimes_{\mathcal{O}(V_M)} \mathcal{O}(V_M \cap W)$$

lies in $\bar{R}-\mathrm{mod}_{\mathbb{Z}}-\bar{R}$.

Lemma 2.2.6. *Let $i \in \{1, \dots, n\}$. The bimodule $B_i := R_i \otimes_{R_i^{s_i}} R_i$ is graded. It is free of rank 2 as left R_i-module and as right R_i-module.*

Proof. Since s_i preserves the degrees, $R_i^{s_i}$ is a graded subring of R_i and so R_i lies in $R_i - \mathrm{mod}_{\mathbb{Z}} - R_i^{s_i}$ and in $R_i^{s_i} - \mathrm{mod}_{\mathbb{Z}} - R_i$. By Remark 2.2.2 we obtain that B_i is graded.

The fact that the bimodule B_i is free as left R_i-module and as right R_i-module is a direct consequence of the decomposition into eigenspaces $R_i = R_i^{s_i} \oplus R_i^{s_i} f_i$. □

The bimodules B_i as defined in the lemma above are the equivalent of the *Soergel bimodules* $R \otimes_{R^s} R$ used in [34] to categorify the Kazhdan-Lusztig basis of the Hecke algebra of an arbitrary Coxeter system of finite rank.

2.2.3 Elementary bimodules

Lemma 2.2.7. *The ring $R_{i,i+1}$ of regular functions on $V_i \cap V_{i+1}$ is a free $R_i^{s_i}$-module of rank 1 and a free $R_{i+1}^{s_{i+1}}$-module of rank 1.*

Proof. Since V_i is s_i-stable we have a decomposition $R_i = R_i^{s_i} \oplus R_i^{s_i} f_i$. Since $R_i \twoheadrightarrow R_{i,i+1}$ if follows that $R_{i,i+1}$ is generated by 1 and f_i as an $R_i^{s_i}$-module. Thanks to Lemma 2.1.4, $V_i \cap V_{i+1} \subset H_{s_i s_{i+1} s_i}$ and hence $f_i + f_{i+1} = 0$ in $R_{i,i+1}$. It follows that the element $2f_{i+1} + f_i \in R_i^{s_i}$ applied on $1 \in R_{i,i+1}$ yields $-f_i$ and hence that $R_{i,i+1}$ is generated as an $R_i^{s_i}$-module by 1. It remains to show that if $f \in R_i^{s_i}$, $f \cdot 1 = f|_{V_i \cap V_{i+1}} = 0$ implies $f = 0$. Since f is s_i-invariant it is enough to show that

$$(V_i \cap V_{i+1}) \cup s_i(V_i \cap V_{i+1}) = V_i.$$

But this holds thanks to example 2.1.7. The proof of the second statement is similar. □

Corollary 2.2.8. *As a left $R_i^{s_i}$-module, $R_{i,i+1} \otimes_{R_{i+1}^{s_{i+1}}} R_{i+1}$ is free of rank 2. Similarly as a right $R_{i+1}^{s_{i+1}}$-module, $R_i \otimes_{R_i^{s_i}} R_{i,i+1}$ is free of rank 2.*

Proof. Thanks to Lemma 2.2.7, $R_{i,i+1} \cong R_i^{s_i}$ as a left $R_i^{s_i}$-module. Since $R_{i+1} = R_{i+1}^{s_{i+1}} \oplus R_{i+1}^{s_{i+1}} f_{i+1}$, the claim follows. □

Corollary 2.2.9. *The bimodule $B_{i,i+1} := R_i \otimes_{R_i^{s_i}} R_{i,i+1} \otimes_{R_{i+1}^{s_{i+1}}} R_{i+1}$ which lies in $R_i - \mathrm{mod}_{\mathbb{Z}} - R_{i+1}$ is free of rank 2 in $R_i - \mathrm{mod}$ and free of rank 2 in $\mathrm{mod} - R_{i+1}$. In particular, if we view $B_{i,i+1}$ in $\bar{R} - \mathrm{mod}_{\mathbb{Z}} - \bar{R}$, then the left annihilator of $B_{i,i+1}$ in $\bar{R}$ is the ideal of functions vanishing on V_i and its right annihilator is the ideal of functions vanishing on V_{i+1}.*

Remark 2.2.10. Notice that a basis of $B_{i,i+1}$ as a left R_i-module is given by $\{1 \otimes 1 \otimes 1, 1 \otimes 1 \otimes f_{i+1}\}$. A basis as right R_{i+1}-module is given by $\{1 \otimes 1 \otimes 1, f_i \otimes 1 \otimes 1\}$.

We now study bimodules $B_{i,j}$ as defined in Corollary 2.2.9 but for $|i-j| > 1$. Notice that $R_{i,j} \otimes_{R_j^{s_j}} R_j$ is free as left $R_{i,j}$-module since $R_j = R_j^{s_j} \oplus R_j^{s_j} f_j$.

Lemma 2.2.11. *Any function $f \in R_j$ which vanishes on $V_i \cap V_j$ acts on $M := R_{i,j} \otimes_{R_j^{s_j}} R_j$ on the right by zero. In other words, the right operation of R_j on M gives rise to a right $R_{i,j}$-module structure on M. Moreover, M is free of rank 2 as a right $R_{i,j}$-module.*

Proof. Decompose f as $r + r' f_j$ with $r, r' \in R_j^{s_j}$. By assumption one has

$$r|_{V_i \cap V_j} + r'|_{V_i \cap V_j} f_j|_{V_i \cap V_j} = 0.$$

Now since $|i-j| > 1$, $V_i \cap V_j$ is s_j-stable, giving rise to a natural operation of s_j on $R_{i,j}$. Applying s_j to the above equation one gets

$$r|_{V_i \cap V_j} - r'|_{V_i \cap V_j} f_j|_{V_i \cap V_j} = 0,$$

which implies that $r|_{V_i \cap V_j} = 0$ and $r'|_{V_i \cap V_j} f_j|_{V_i \cap V_j} = 0$. Since $f_j(v) \neq 0$ for $v \in V_i \cap V_j - \{0\}$, this forces $r'|_{V_i \cap V_j} = 0$. Hence if $v \otimes w \in R_{i,j} \otimes_{R_j^{s_j}} R_j$, one gets

$$(v \otimes w) \cdot f = vr|_{V_i \cap V_j} \otimes w + vr'|_{V_i \cap V_j} \otimes w f_j = 0.$$

To see that M is free on the right over $R_{i,j}$, one first uses Lemma 2.2.3 to get an isomorphism $M \cong R_{i,j} \otimes_{R_j^{s_j}} R_{i,j}$ and then concludes by using the decomposition

$R_{i,j} = R_{i,j}^{s_j} \oplus R_{i,j}^{s_j} f_j$ which holds since $V_i \cap V_j$ is s_j-invariant and has no irreducible component included in H_{s_j}. ☐

Proposition 2.2.12. *The bimodule* $B_{i,j} := R_i \otimes_{R_i^{s_i}} R_{i,j} \otimes_{R_j^{s_j}} R_j$ *(which lies in* $R_{i,j} - \mathrm{mod}_{\mathbb{Z}} - R_{i,j}$ *thanks to the preceding lemma) is free of rank 4 as left* $R_{i,j}$*-module and as right* $R_{i,j}$*-module. In particular the left annihilator of* $B_{i,j}$ *is equal to its right annihilator and is the ideal of functions vanishing on* $V_i \cap V_j$.

Proof. As a left R_i-module, $B_{i,j}$ is generated by $t_1 := 1 \otimes 1 \otimes 1$, $t_2 := 1 \otimes 1 \otimes f_j$, $t_3 := 1 \otimes f_i \otimes f_j$ and $t_4 := 1 \otimes f_i \otimes 1$. Let us show that it is a basis of $B_{i,j}$ over $R_{i,j}$. Consider elements $a_k \in R_i$, $k = 1, 2, 3, 4$ and write them as $a_k = r_k + r'_k f_i$ with $r_k, r'_k \in R_i^{s_i}$, $k = 1, \ldots, 4$, and suppose $\sum_{i=1}^4 a_k \cdot t_k = 0$. One gets

$$\begin{aligned}&1 \otimes (r_2 + r_3 f_i)|_{V_i \cap V_j} \otimes f_j + 1 \otimes (r_1 + r_4 f_i)|_{V_i \cap V_j} \otimes 1\\ &+ f_i \otimes (r'_2 + r'_3 f_i)|_{V_i \cap V_j} \otimes f_j + f_i \otimes (r'_1 + r'_4 f_i)|_{V_i \cap V_j} \otimes 1\\ &= 0.\end{aligned}$$

Now since $N := R_i \otimes_{R_i^{s_i}} R_{i,j}$ is free as a right $R_{i,j}$-module and $M := R_{i,j} \otimes_{R_j^{s_j}} R_j$ is free as a left $R_{i,j}$-module, $B_{i,j} \cong N \otimes_{R_{i,j}} M$ is free for the induced structure of $R_{i,j}$-module (which is not the same as its left or right $R_{i,j}$-module structure!), and a basis is given by $1 \otimes 1 \otimes 1$, $f_i \otimes 1 \otimes 1$, $1 \otimes 1 \otimes f_j$ and $f_i \otimes 1 \otimes f_j$. This implies that

$$0 = (r_2 + r_3 f_i)|_{V_i \cap V_j} = (r_1 + r_4 f_i)|_{V_i \cap V_j} = (r'_2 + r'_3 f_i)|_{V_i \cap V_j} = (r'_1 + r'_4 f_i)|_{V_i \cap V_j}.$$

Now the same argument as in the proof of the preceding lemma (applying s_i this time) gives that $r_k|_{V_i \cap V_j} = 0 = r'_k|_{V_i \cap V_j}$, hence that $a_k|_{V_i \cap V_j} = 0$ for all k, which concludes. ☐

2.2.4 A product of bimodules

Given two bimodules $B, B' \in \bar{R} - \mathrm{mod}_{\mathbb{Z}} - \bar{R}$, one defines a bimodule $B \star B'$ in the following way : let I_B^R be the right annihilator of B and $I_{B'}^L$ the left annihilator of B', and write V_B^R, $V_{B'}^L$ for the corresponding closed subvarieties of Z. Then set

$$B \star B' := B \otimes_{\bar{R}} \mathcal{O}(V_B^R \cap V_{B'}^L) \otimes_{\bar{R}} B'.$$

We will omit the exponents L and R when no confusion is possible. Thanks to Remark 2.2.5, such a bimodule lies in $\bar{R} - \mathrm{mod}_{\mathbb{Z}} - \bar{R}$ in case all the varieties occurring in its definition are union of Weyl lines. Note the following:

- If B, B' have trivial right, respectively left annihilators (for example if they are free as right, resp. left $\bar{R}$-modules), then $B \star B' = B \otimes_{\bar{R}} B'$,

- One has $\bar{R} \star B \cong B \cong B \star \bar{R}$.

- In all the cases we will consider further, we will always have the equalities $I_B^R = I(V_B^R)$ and $I_{B'}^L = I(V_{B'}^L)$. We will therefore often write the $\star$-product as

$$B \otimes_{\mathcal{O}(V_B^R)} \mathcal{O}(V_B^R \cap V_{B'}^L) \otimes_{\mathcal{O}(V_{B'}^L)} B'.$$

Recall the bimodule $B_i := R_i \otimes_{R_i^{s_i}} R_i$ with $i \in \{1, \dots, n\}$ from Lemma 2.2.6.

Lemma 2.2.13. *Let M be a right $\bar{R}$-module which is free of rank m over $\mathcal{O}(V_M)$ for $V_M \in \mathcal{V}_n$. The right annihilator of $M \star B_i$ is the ideal of the variety*

$$V_i \cap (V_M \cup s_i V_M).$$

Moreover $M \star B_i$ is free as a right $\mathcal{O}(V_i \cap (V_M \cup s_i V_M))$-module, of rank m if V_M is not s_i-invariant and of rank $2m$ if V_M is s_i-invariant. The same statement holds for the left annihilator of a bimodule $B_i \star M$ in case M is a left $\bar{R}$-module which is free over $\mathcal{O}(V_M)$.

Proof. Let $f \in \bar{R}$ and annihilate $M \star B_i$ on the right. One can suppose $f \in R_i$. Write $f = r + r' f_i$ with $r, r' \in R_i^{s_i}$. We can suppose $M \star B_i \cong \mathcal{O}(V_M \cap V_i) \otimes_{R_i^{s_i}} R_i$ (as a right $\bar{R}$-module) since M is free as a right $\mathcal{O}(V_M)$-module.

Now if $f = r + r' f_i$, $r, r' \in R_i^{s_i}$ annihilates $M \star B_i$, in particular it annihilates $1 \otimes 1$. Hence one has

$$r|_{V_M \cap V_i} \otimes 1 + r'|_{V_M \cap V_i} \otimes f_i = 0.$$

This forces $r|_{V_M \cap V_i} = 0 = r'|_{V_M \cap V_i}$ (because $\mathcal{O}(V_M \cap V_i) \otimes_{R_i^{s_i}} R_i$ is free as a module over $\mathcal{O}(V_M \cap V_i)$ for the obvious operation). Since r, r' are s_i-invariant, this forces them to be zero on $V_i \cap (V_M \cup s_i V_M)$, and the same holds for f. Conversely if $f \in R_i$ is zero on $V_i \cap (V_M \cup s_i V_M)$, then write $f = r + r' f_i$ with r, r' invariant under s_i. This forces r, r' to be zero on $V_i \cap (V_M \cup s_i V_M)$ and in particular on $V_i \cap V_M$.

We now prove the freeness; firstly we suppose that $V_M \cap V_i$ is s_i-invariant ; hence $\mathcal{O}(V_M \cap V_i) = \mathcal{O}(V_M \cap V_i)^{s_i} \oplus \mathcal{O}(V_M \cap V_i)^{s_i} f_i$. It follows that $\mathcal{O}(V_M \cap V_i) \otimes_{R_i^{s_i}} R_i$ is generated as a right $\bar{R}$-module by $1 \otimes 1$ and $f_i \otimes 1$. Let $r, r' \in R_i$ be such that

$$1 \otimes r + f_i \otimes r' = 0.$$

Write $r = r_1 + r_2 f_i$ and $r' = r'_1 + r'_2 f_i$ with $r_j, r'_j \in R_i^{s_i}$ and get

$$(r_1 + r'_1 f_i)|_{V_M \cap V_i} \otimes 1 + (r_2 + r'_2 f_i)|_{V_M \cap V_i} \otimes f_i = 0.$$

This implies that $(r_1 + r'_1 f_i)|_{V_M \cap V_i} = 0 = (r_2 + r'_2 f_i)|_{V_M \cap V_i}$ and by s_i-invariance one gets $r'_j|_{V_M \cap V_i} = 0 = r_j|_{V_M \cap V_i}$ for $j = 1, 2$. Hence $M \star B_i$ is free on the right over $\mathcal{O}(V_M \cap V_i)$, of rank 2.

We now suppose that $V_i \cap V_M$ is not s_i-invariant. Remark 2.2.1 implies that

$$R_i^{s_i} \twoheadrightarrow \mathcal{O}(V_M \cap V_i).$$

Hence as a right $\mathcal{O}(V_i \cap (V_M \cup s_i V_M))$-module, $\mathcal{O}(V_M \cap V_i) \otimes_{R_i^{s_i}} R_i$ is generated by $1 \otimes 1$. We have to show that if $f \in R_i$, the equality $1 \otimes f = 0$ implies that $f|_{V_i \cap (V_M \cup s_i V_M)} = 0$. Write $f = r + r' f_i$ with $r, r' \in R_i^{s_i}$. This implies that $r'|_{V_M \cap V_i} = 0 = r|_{V_M \cap V_i}$. Now since r', r are s_i-invariant one concludes that they also vanish on $V_i \cap (V_M \cup s_i V_M)$ and the same holds for f. $\square$

The lemma above will allow us to use induction.

2.2.5 Associativity

Unfortunately, the product defined in the previous section is not associative for arbitrary bimodules B, B'. However, as we will see in this section, it will be associative when restricted to a suitable $\star$-stable family of bimodules, exactly the bimodules occurring up to isomorphism by considering successive $\star$-products of the bimodules B_i, $i \in \{1, \ldots, n\}$.

A first step towards a proof of the associativity of the product $\star$ is the following: If $M, N \in \bar{R} - \mathrm{mod}_{\mathbb{Z}} - \bar{R}$ with M having $I(V_M)$ as right annihilator and N having $I(V_N)$ as left annihilator, then

$$(M \star B_i) \star N \cong M \star (B_i \star N) \tag{2.1}$$

provided V_N, V_M lie in a certain family of subvarieties of Z; thanks to Lemma 2.2.13 the good family to choose is $\mathcal{V}_n$. The idea will be then to show associativity of the $\star$ product for products of three of the bimodules B_i and then use this previous result to generalize to arbitrary products of the B_i.

Let us rewrite equation 2.1. We suppose that M is free on the right over $\mathcal{O}(V_M)$ and that N is free on the left over $\mathcal{O}(V_N)$. Set $W_{i,M} := V_i \cap (V_M \cup s_i V_M)$, $W_{i,N} := V_i \cap (V_N \cup s_i V_N)$. By definition of the $\star$ product together with Lemma 2.2.13 the left hand side of 2.1 can be rewritten up to isomorphism as

$$(M \otimes_{\mathcal{O}(V_M)} \mathcal{O}(V_M \cap V_i) \otimes_{R_i} (R_i \otimes_{R_i^{s_i}} R_i)) \otimes_{\mathcal{O}(W_{i,M})} \mathcal{O}(W_{i,M} \cap V_N) \otimes_{\mathcal{O}(V_N)} N,$$

or shorter

$$(M \otimes_{\mathcal{O}(V_M)} \mathcal{O}(V_M \cap V_i) \otimes_{R_i^{s_i}} R_i) \otimes_{\mathcal{O}(W_{i,M})} \mathcal{O}(W_{i,M} \cap V_N) \otimes_{\mathcal{O}(V_N)} N.$$

Now using Lemmas 2.2.13 and 2.2.3 we can rewrite this as

$$(M \otimes_{\mathcal{O}(V_M)} \mathcal{O}(V_M \cap V_i) \otimes_{R_i^{s_i}} \mathcal{O}(W_{i,M})) \otimes_{\mathcal{O}(W_{i,M})} \mathcal{O}(W_{i,M} \cap V_N) \otimes_{\mathcal{O}(V_N)} N.$$

or shorter

$$M \otimes_{\mathcal{O}(V_M)} \mathcal{O}(V_M \cap V_i) \otimes_{R_i^{s_i}} \mathcal{O}(W_{i,M} \cap V_N) \otimes_{\mathcal{O}(V_N)} N.$$

Doing the same reductions for the right hand side one gets

$$M \otimes_{\mathcal{O}(V_M)} \mathcal{O}(V_M \cap W_{i,N}) \otimes_{R_i^{s_i}} \mathcal{O}(V_i \cap V_N) \otimes_{\mathcal{O}(V_N)} N.$$

Now our job is to show that these two bimodules are isomorphic in $\bar{R} - \mathrm{mod}_{\mathbb{Z}} - \bar{R}$. It is therefore enough to show that

$$\mathcal{O}(V_M \cap V_i) \otimes_{R_i^{s_i}} \mathcal{O}(W_{i,M} \cap V_N) \cong \mathcal{O}(V_M \cap W_{i,N}) \otimes_{R_i^{s_i}} \mathcal{O}(V_i \cap V_N),$$

where the isomorphism holds in $\mathcal{O}(V_M) - \mathrm{mod}_{\mathbb{Z}} - \mathcal{O}(V_N)$.

Proposition 2.2.14. *One has*

$$\mathcal{O}(V_M \cap V_i) \otimes_{R_i^{s_i}} \mathcal{O}(W_{i,M} \cap V_N) \cong \mathcal{O}(V_M \cap W_{i,N}) \otimes_{R_i^{s_i}} \mathcal{O}(V_i \cap V_N),$$

as graded $(\mathcal{O}(V_M), \mathcal{O}(V_N))$*-bimodules.*

Proof. The strategy is to find the left and right annihilators and then use Lemma 2.2.3. We first suppose $V_N \cap V_i$ is s_i-invariant. Hence $W_{i,M} \cap V_N$ is s_i-invariant. Let $g \in \mathcal{O}(V_N \cap V_i)$ be such that $g|_{V_N \cap W_{i,M}} = 0$. Choose $h \in R_i$, $h = r + r' f_i$ with $r, r' \in R_i^{s_i}$ such that $h|_{V_N \cap V_i} = g$. Since $V_N \cap W_{i,M}$ is s_i-invariant one has that $r'|_{V_N \cap W_{i,M}} = 0 = r|_{V_N \cap W_{i,M}}$, hence also $r'|_{V_M \cap W_{i,M}} = 0 = r|_{V_M \cap W_{i,M}}$ since in our case $V_M \cap W_{i,N} \hookrightarrow V_N \cap W_{i,M}$ (because of s_i-invariance of $V_N \cap V_i$). We have shown that an element $g \in \mathcal{O}(V_N \cap V_i)$ which vanishes on $V_N \cap W_{i,M}$ kills $\mathcal{O}(V_M \cap W_{i,N}) \otimes_{R_i^{s_i}} \mathcal{O}(V_i \cap V_N)$ on the right, hence by Lemma 2.2.3, the right hand side is isomorphic to

$$\mathcal{O}(V_M \cap W_{i,N}) \otimes_{R_i^{s_i}} \mathcal{O}(V_N \cap W_{i,M}).$$

Now if $V_M \cap V_i$ is s_i-invariant one uses the same argument for the left hand side for the left operation and this left hand side is isomorphic to

$$\mathcal{O}(V_M \cap W_{i,N}) \otimes_{R_i^{s_i}} \mathcal{O}(V_N \cap W_{i,M}),$$

which concludes.

Now suppose that $V_M \cap V_i$ is not s_i-invariant. Consider $g \in \mathcal{O}(V_M \cap V_i)$ vanishing on $X := V_M \cap W_{i,N}$. By assumption V_M lies in $\mathcal{V}_n$ and thanks to Remark 2.2.1, one can choose $h \in R_i^{s_i}$ such that $h|_{V_M \cap V_i} = g$. In particular $h|_X = 0$. Now since h is s_i-invariant it has to vanish on $X \cup s_i X$. But $V_N \cap W_{i,M} \hookrightarrow X \cup s_i X$. Hence h, whence g kills $\mathcal{O}(V_M \cap V_i) \otimes_{R_i^{s_i}} \mathcal{O}(W_{i,M} \cap V_N)$ on the left, and this bimodule is hence isomorphic to

$$\mathcal{O}(V_M \cap W_{i,N}) \otimes_{R_i^{s_i}} \mathcal{O}(V_N \cap W_{i,M})$$

thanks to Lemma 2.2.3. The case where $V_N \cap V_i$ is not s_i-invariant but $V_M \cap V_i$ is is symmetric; in case none of them is s_i-invariant, the argument given above (choose a preimage h which is invariant and then restrict) can still be given, for the left as well as for the right operation, since it makes no use of the fact that the variety on the other side is s_i-invariant or not. □

We define bimodules associated to finite sequences of integers in $[1, n]$. If the sequence has length 1, containing a single index j, the corresponding bimodule is B_j. Let $i_1, \ldots, i_k \in [1, n]$. Define the bimodule associated to the sequence seq $= i_k i_{k-1} \cdots i_1$ by setting $B(\emptyset) = \bar{R}$, $B(\text{seq}) = B_{i_k} \star B(i_{k-1} \cdots i_1)$. A bimodule B will be said to be *associated* to such a sequence if it is obtained from $B_{i_k}, \ldots, B_{i_1}$ by doing a product in this order but with a possibly different choice of brackets from the one we made for $B(\text{seq})$. For example, $(B_{i_4} \star B_{i_3}) \star (B_{i_2} \star B_{i_1})$ and $B_{i_4} \star ((B_{i_3} \star B_{i_2}) \star B_{i_1})$ are associated to the same sequence $i_4 i_3 i_2 i_1$.

Theorem 2.2.15. *Let $i_k \cdots i_1$ be a sequence of indices in $\{1, \ldots, n\}$.*

1. *Two bimodules associated to this sequence are isomorphic in $\bar{R} - \text{mod}_{\mathbb{Z}} - \bar{R}$.*
2. *The bimodule $B(i_k \cdots i_1)$ is free on the left on $\mathcal{O}(W(i_k \cdots i_1))$ and free on the right on $\mathcal{O}(W(i_1 \cdots i_k))$.*

Proof. Notice that the first property is trivial if $k < 3$. In that case, the second property is a consequence of 2.2.9, 2.2.12 since in the notations borrowed from there, one has $B_i \star B_{i\pm1} \cong B_{i,i\pm1}$ and $B_i \star B_j \cong B_{i,j}$ for $|i-j|>1$. But the bimodule $B_{i,i\pm1}$ was shown to be free of rank 2 as left $R_i = \mathcal{O}(V_i)$-module and by definition $W(i) = V_i$. It was also shown to be free as right $R_{i\pm1} = \mathcal{O}(V_{i\pm1})$-module and $W(i\pm1) = V_{i\pm1}$. The bimodule $B_{i,j}$ was shown to be free of rank 4 as left (or right) $R_{i,j} = \mathcal{O}(V_i \cap V_j)$-module and we know from example 2.1.6 that $W(ij) = W(ji) = V_i \cap V_j$.

In case $k \geq 3$, both properties are proved simultaneously by induction on the number of elementary bimodules B_i occurring in the product. If our bimodule is a product of three of the B_i, say $(B_i \star B_j) \star B_k$, then associativity is immediate by Proposition 2.2.14 and the arguments above it: one has

$$(B_i \star B_j) \star B_k \cong B_i \star (B_j \star B_k),$$

and both of theses bimodules are free as left $\mathcal{O}(W(ijk))$-modules and as right $\mathcal{O}(W(kji))$-modules thanks to Corollary 2.2.9, Proposition 2.2.12 and Lemma 2.2.13. Now suppose the result holds for any product of at most $m-1$ of the B_i's. Consider a sequence $i_1, \ldots, i_m \in \{1, \ldots, n\}$. By induction it is enough to show that

$$(B_{i_1} \star \cdots \star B_{i_j}) \star (B_{i_{j+1}} \star \cdots \star B_{i_m}) \cong (B_{i_1} \star \cdots \star B_{i_k}) \star (B_{i_{k+1}} \star \cdots \star B_{i_m}),$$

with $k \neq j$, where by induction the products $B_{i_1} \star \cdots \star B_{i_j}$, $B_{i_{j+1}} \star \cdots \star B_{i_m}$, $B_{i_1} \star \cdots \star B_{i_k}$ and $B_{i_{k+1}} \star \cdots \star B_{i_m}$ are well defined up to isomorphism (they can be written without brackets) and free over the varieties associated to their sequences (on the left over $\mathcal{O}(W(i_1 \cdots i_j))$ and on the right over $\mathcal{O}(W(i_j \cdots i_1))$ for the first one, etc.). One just has to apply successively Proposition 2.2.14 to move B_j's from one bracket to the other one. In particular both of our bimodules are isomorphic to

$$B_{i_1} \star (B_{i_2} \star \cdots \star B_{i_k}) \text{ and } (B_{i_1} \star \cdots \star B_{i_{k-1}}) \star B_{i_k},$$

which are free by induction together with Lemma 2.2.13. In particular this lemma tells us that the left annihilator is $I(W(i_1 \cdots i_k))$ and the right one is $I(W(i_k \cdots i_1))$.

□

2.3 Realization of the Temperley-Lieb algebra

2.3.1 The Temperley-Lieb algebra

Let τ be a parameter. The definition we use of the Temperley-Lieb algebra in this chapter is the following: the Temperley-Lieb algebra TL_n is the $\mathbb{Z}[\tau, \tau^{-1}]$-algebra generated by elements $b_{s_i} = b_i$ for $i = 1, \ldots, n$ with relations

$$\begin{aligned} b_j b_i b_j &= b_j \text{ if } |i-j| = 1, \\ b_i b_j &= b_j b_i \text{ if } |i-j| > 1, \\ b_i^2 &= (1+\tau^{-2}) b_i. \end{aligned}$$

Remark 2.3.1. In subsection 1.2.2, we defined TL_n with a parameter v instead of τ, the last relation being replaced by $b_i^2 = (v + v^{-1})b_i$, which allows TL_n to be realized as a quotient of the Hecke algebra $\mathcal{H}$ of type A_n. The reason for choosing another parameter τ is that the bimodules B_i defined before will satisfy the above relations where the multiplication in TL_n corresponds to the $\star$-product, the sum to direct sums of bimodules and the parameter τ to a shift of graduation. In the case of Soergel bimodules categorifying the Hecke algebra (see subsection 1.3.3), one defines the analogue of our bimodule B_i by $S_i' := R \otimes_{R^{s_i}} R$; it turns out that the relation $S_i' \otimes_R S_i' \cong S_i' \oplus S_i'[-2]$ is satisfied but one then sets $S_i := S_i'[1]$ and the relation becomes $S_i \otimes_R S_i \cong S_i[1] \oplus S_i[-1]$. The parameter v is then interpreted as a shift and such a relation corresponds to the relation $C_{s_i}'^2 = (v+v^{-1})C_{s_i}'$ which holds in $\mathcal{H}$, C_{s_i}' being the element of the Kazhdan-Lusztig basis $\{C_w'\}$ (see Theorem 1.2.1) indexed by the simple reflection s_i. In our case shifting the bimodules B_i as in Soergel's work is a priori not possible since the first relation defining TL_n is not homogeneous. As a consequence, the algebra we will categorifiy is unfortunately not isomorphic to the same algebra but with the parameter $1+\tau^{-2}$ in the quadratic relation replaced by $v + v^{-1}$, and the algebra above is not a quotient of $\mathcal{H}$. In particular, with the parameter $1 + \tau^{-2}$, we do not have an interesting homomorphism from the group algebra of the braid group over $\mathbb{Z}[\tau, \tau^{-1}]$ to TL_n. We therefore cannot define Zinno basis in this version of the Temperley-Lieb algebra.

Recall that the set of fully commutative elements of the symmetric group is denoted by $\mathcal{W}_f$. Now if $(\mathcal{W}, \mathcal{S})$ is of type A_n and $w \in \mathcal{W}_f$ and $s_{i_1} \cdots s_{i_k}$ is an $\mathcal{S}$-reduced expression for w, one can show that the element $b_w := b_{s_{i_1}} \cdots b_{s_{i_k}} \in \mathrm{TL}_n$ is independent of the choice of the $\mathcal{S}$-reduced expression for w and that the set $\{b_w\}_{w \in \mathcal{W}_f}$ is a basis of TL_n as a $\mathbb{Z}[\tau, \tau^{-1}]$-module.

Definition 2.3.2. *The basis* $\{b_w\}_{w \in \mathcal{W}_f}$ *of* TL_n *is the* Kazhdan-Lusztig basis *or* diagram basis *of* TL_n.

The first name is due to the following fact; if we define TL_n with a parameter v instead of τ as mentioned in Remark 2.3.1, then it is a quotient of the Hecke algebra $\mathcal{H}$ of type A_n, as explained in subsection 1.2.2. If $w \in \mathcal{W}_f$, the image via the quotient map θ of the element C_w' of the Kazhdan-Lusztig basis of $\mathcal{H}$ is equal

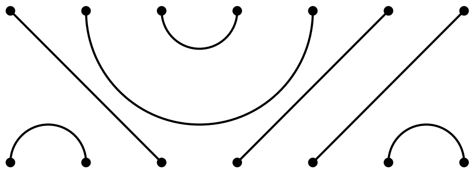


FIG. 2.2: A diagram representing an element b_w of the diagram basis of the Temperley-Lieb algebra. Here $w = s_3s_2s_1s_4s_5s_6$.

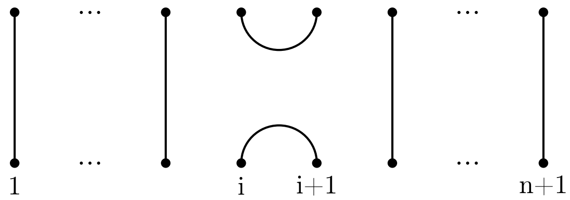


FIG. 2.3: Planar diagram corresponding to the element b_i.

to b_w and any element C'_x for $x \notin \mathcal{W}_f$ is sent to zero (see subsection 1.2.2 for more details).

The second name is due to the fact that the basis $\{b_w\}_{w\in\mathcal{W}_f}$ has a well-known interpretation by planar diagrams. Draw a sequence of $n+1$ points on a line and another one under the first one. Draw arcs between any two points of the two sequences (the two points of an arc can belong to the same sequence) such that each point occurs in exactly one arc and such that two distinct arcs never cross to obtain a diagram like the one given in figure 3.8; we always consider such diagrams up to isotopy. Elements of the Temperley-Lieb algebra are $\mathbb{Z}[\tau,\tau^{-1}]$-linear combinations of such diagrams, where the element $b_i = b_{s_i}$ is given by the diagram in figure 2.3. Multiplication of two planar diagrams is then given by concatenating the diagrams ; if circles occur in the resulting diagram, we remove them and multiply the diagram by $(1+\tau^{-2})^k$ where k is the number of circles. The diagram algebra over $\mathbb{Z}[\tau,\tau^{-1}]$ obtained in this way turns out to be isomorphic to TL_n.

2.3.2 Temperley-Lieb relations

The aim of this section is to prove that the bimodules B_i together with the $\star$-product from the previous section satisfy the Temperley-Lieb relations, i.e.,

$$\begin{aligned} B_j \star B_i \star B_j &\cong B_j \text{ if } |i-j| = 1,\\ B_i \star B_j &\cong B_j \star B_i \text{ if } |i-j| > 1,\\ B_i \star B_i &\cong B_i \oplus B_i[-2], \end{aligned}$$

where all the isomorphisms hold in $\bar{R} - \mathrm{mod}_{\mathbb{Z}} - \bar{R}$.

Theorem 2.3.3. *The bimodules B_i satisfy the Temperley-Lieb relations.*

Proof. We write $R_i := \mathcal{O}(V_i)$, $R_{i,j} := \mathcal{O}(V_i \cap V_j)$. For the first relation, assume that $j = i+1$, the other case being similar. The left hand side of the first relation which is isomorphic to $(B_j \star B_i) \star B_j$ can be rewritten (up to isomorphism) thanks to Corollary 2.2.9 as

$$(R_{i+1} \otimes_{R_{i+1}^{s_{i+1}}} R_{i+1} \otimes_{R_{i+1}} R_{i,i+1} \otimes_{R_i} R_i \otimes_{R_i^{s_i}} R_i) \otimes_{R_i} R_{i,i+1} \otimes_{R_{i+1}} (R_{i+1} \otimes_{R_{i+1}^{s_{i+1}}} R_{i+1}),$$

which is isomorphic to

$$R_{i+1} \otimes_{R_{i+1}^{s_{i+1}}} R_{i,i+1} \otimes_{R_i^{s_i}} R_{i,i+1} \otimes_{R_{i+1}^{s_{i+1}}} R_{i+1}.$$

Hence it suffices to show that $R_{i,i+1} \otimes_{R_i^{s_i}} R_{i,i+1} \cong R_{i+1}^{s_{i+1}}$ as graded $(R_{i+1}^{s_{i+1}}, R_{i+1}^{s_{i+1}})$-bimodule. We know from Lemma 2.2.7 that $R_{i+1}^{s_{i+1}}$ is isomorphic to $R_{i,i+1}$ as $R_{i+1}^{s_{i+1}}$-module; since the left and right operations are the same this is even an isomorphism of $(R_{i+1}^{s_{i+1}}, R_{i+1}^{s_{i+1}})$-bimodule. We define a map

$$\varphi : R_{i,i+1} \otimes_{R_i^{s_i}} R_{i,i+1} \to R_{i,i+1}$$

$$a \otimes b \mapsto ab.$$

This clearly defines a morphism of bimodules. Define a map

$$\psi : R_{i,i+1} \to R_{i,i+1} \otimes_{R_i^{s_i}} R_{i,i+1}$$

$$c \mapsto c \otimes 1.$$

One checks using Lemma 2.2.7 that this defines a morphism of bimodules which is an inverse to φ. Hence the first Temperley-Lieb relation holds.

For the second relation, using Proposition 2.2.12 together with Lemma 2.2.3, it is enough to show that

$$R_{i,j} \otimes_{R_i^{s_i}} R_{i,j} \otimes_{R_j^{s_j}} R_{i,j} \cong R_{i,j} \otimes_{R_j^{s_j}} R_{i,j} \otimes_{R_i^{s_i}} R_{i,j}$$

as graded $(R_{i,j}, R_{i,j})$-bimodules. Let $m, n, q \in R_{i,j}$. Since $V_i \cap V_j$ is s_i-invariant one has that $R_{i,j} = R_{i,j}^{s_i} \oplus R_{i,j}^{s_i} f_i$; write $n = r + r' f_i$ with $r, r' \in R_{i,j}^{s_i}$. Define a map

$$\varphi : R_{i,j} \otimes_{R_i^{s_i}} R_{i,j} \otimes_{R_j^{s_j}} R_{i,j} \to R_{i,j} \otimes_{R_j^{s_j}} R_{i,j} \otimes_{R_i^{s_i}} \otimes R_{i,j}$$

$$m \otimes n \otimes q \mapsto mr \otimes 1 \otimes q + mr' \otimes 1 \otimes f_i q.$$

It is routine to check that such a map is well-defined and that it is a morphism of graded bimodules. By permuting the indices i and j one also gets a map ψ in the other direction and one shows that ψ is an inverse of φ.

For the third relation one has to show that

$$R_i \otimes_{R_i^{s_i}} R_i \otimes_{R_i^{s_i}} R_i \cong (R_i \otimes_{R_i^{s_i}} R_i) \oplus (R_i \otimes_{R_i^{s_i}} R_i)[-2].$$

Now $R_i = R_i^{s_i} \oplus R_i^{s_i} f_i$ and since no irreducible component of V_i is included in H_{s_i} one has an $R_i^{s_i}$-(bi)module isomorphism $R_i^{s_i} f_i \cong R_i^{s_i}[-2]$ given by the restriction of the Demazure operator ∂_{s_i} (which has in this case multiplication by f_i as inverse). Hence $R_i \cong R_i^{s_i} \oplus R_i^{s_i}[-2]$ as graded $(R_i^{s_i}, R_i^{s_i})$-bimodule and one gets the claim by decomposing in such a way the R_i in the middle of the tensor product on the left hand side above. □

Definition 2.3.4. *Let $w \in \mathcal{W}_f$. Let $s_{i_1} \cdots s_{i_k}$ be an $\mathcal{S}$-reduced expression for w. We consider the bimodule*

$$B(i_1 \cdots i_k) := B_{i_1} \star \cdots \star B_{i_k} \in \bar{R} - \mathrm{mod}_{\mathbb{Z}} - \bar{R}.$$

Notice that we abuse notation since $B_{i_1} \star \cdots \star B_{i_k}$ is not a single bimodule but various isomorphic bimodules (one for each choice of brackets). Since the bimodules B_i satisfy the Temperley-Lieb relations, this bimodule is independent up to isomorphism of the choice of an $\mathcal{S}$-reduced expression for w and we denote by B_w any bimodule isomorphic to it in $\bar{R} - \mathrm{mod}_{\mathbb{Z}} - \bar{R}$. Such a bimodule B_w will be called fully commutative.

2.3.3 Link with dense sets of reflections

For each fully commutative element $w \in \mathcal{W}_f$, one can consider the dense sets $T(i_1 \cdots i_k)$ and $T(i_k \cdots i_1)$ where $s_{i_1} \cdots s_{i_k}$ is a reduced expression for w ; such sets characterize the varieties whose ideals are the left and right annihilators in $\bar{R}$ of the bimodule $B_{i_1} * \cdots * B_{i_k}$. We have another way of associating a pair of dense sets to w:

Notation. Let $w \in \mathcal{W}_f$ and consider the planar diagram corresponding to the element $b_w \in \mathrm{TL}_n$; if we remove the lines joining a point in the sequence at the top of the diagram to a point in the sequence at the bottom, we obtain a dense set at the top of the diagram that we write $Q(i_1 \cdots i_k)$. We also obtain a dense set at the bottom that we can write $Q(i_k \cdots i_1)$ since it is equal to the dense set obtained at the top of the diagram of $b_{w^{-1}}$ after applying the same process of removing lines going from the top to the bottom of the diagram (notice that w^{-1} lies in $\mathcal{W}_f$ if and only if w does). The left and right dense sets associated to any fully commutative element in type A_4 are drawn in figure 2.4.

Proposition 2.3.5. *Let $w \in \mathcal{W}_f$ and suppose that $s_{i_1} \cdots s_{i_k}$ is an $\mathcal{S}$-reduced expression of w. Then*

$$T(i_1 \cdots i_k) = Q(i_1 \cdots i_k).$$

Proof. We argue by induction on k; if $k = 1$, then $T(i_1) = \{s_{i_1}\}$ and the dense set at the top of the diagram corresponding to b_{i_1} contains only the reflection s_{i_1} (see figure 2.3). We suppose that the result holds for a sequence of length at most $k-1$. By induction, $T(i_2 \cdots i_k) = Q(i_2 \cdots i_k)$ and it suffices to show that the same three rules as in Lemma 2.1.10 hold when passing from $Q(i_2 \cdots i_k)$ to $Q(i_1 \cdots i_k)$. If s_{i_1} commutes with any reflection in $Q(i_2 \cdots i_k)$, then the dense set at the top of b_w is $Q(i_2 \cdots i_k) \cup \{s_{i_1}\}$. If s_{i_1} commutes with any element of $Q(i_2 \cdots i_k)$ except t then t will become a line from the top to the bottom of the diagram associated to b_w when collapsing the diagrams for b_{i_1} and $b_{s_{i_1}w}$ and hence t disappears from $Q(i_2 \cdots i_k)$, s_{i_1} is added and all other reflections are unchanged, hence $Q(i_1 \cdots i_k) = (Q(i_2 \cdots i_k) \backslash t) \cup \{s_{i_1}\}$. If s_{i_1} commutes with any element of $Q(i_2 \cdots i_k)$ except two distinct reflections $(j_1, i_1), (i_1+1, j_2) \in Q(i_2 \cdots i_k)$ with $|\{i_1, i_1+1, j_1, j_2\}| = 4$, one sees by drawing the situation that when concatenating the diagram associated to b_{i_1} to the one associated to $b_{s_{i_2} \cdots s_{i_k}}$, no line from the top to the bottom of the diagram corresponding to b_w is added, that the simple reflection s_{i_1} which lies at the bottom of the diagram corresponding to b_{i_1} will join the index i_1 to the index i_1+1, removing the above two reflections $(j_1, i_1), (i_1+1, j_2)$ to replace them by (j_1, j_2), that of course the simple reflection s_{i_1} coming from the top of the diagram of b_{i_1} is added and that all other reflections in $Q(i_2 \cdots i_k)$ stay unchanged, hence

$$Q(i_1 \cdots i_k) = \big(Q(i_2 \cdots i_k) \backslash \{(j_1, i_1), (i_1+1, j_2)\}\big) \cup \{s_{i_1}, (j_1, j_2)\}.$$

We deduce from Lemma 2.1.10 that $T(i_1 \cdots i_k) = Q(i_1 \cdots i_k)$. □

Corollary 2.3.6. *The bimodules B_w for $w \in \mathcal{W}_f$ are pairwise non-isomorphic in $\bar{R} - \mathrm{mod} - \bar{R}$ (hence in $\bar{R} - \mathrm{mod}_{\mathbb{Z}} - \bar{R}$).*

Proof. If $w \in \mathcal{W}_f$ and $s_{i_1} \cdots s_{i_k}$ is an $\mathcal{S}$-reduced expression of w, then the planar diagram corresponding to the element $b_w \in \mathrm{TL}_n$ is entirely determined by the two dense sets obtained by removing the lines going from the top to the bottom of the

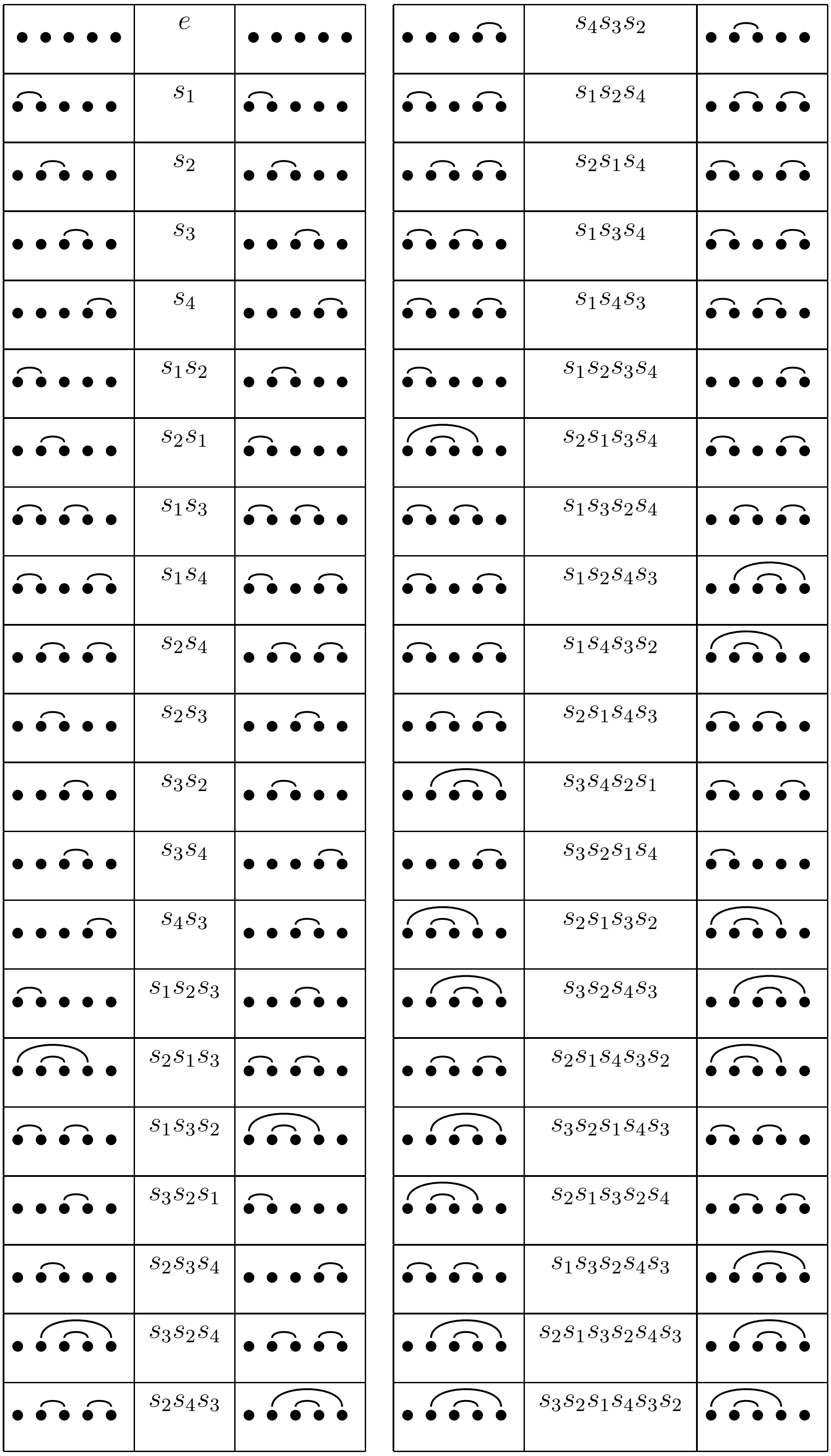


FIG. 2.4: Left and right dense sets of reflections for any fully commutative element in type A_4.

diagram, that is the pair $(Q(i_1 \cdots i_k), Q(i_k \cdots i_1))$, since the lines in the diagram must be noncrossing. Hence two distinct fully commutative elements $w, w' \in \mathcal{W}_f$ will have distincts such pairs. Using Proposition 2.3.5, the corresponding fully commutative bimodules B_w and $B_{w'}$ will then have distinct left annihilators or distinct right annihilators (by uniqueness of a dense set associated to a variety in $\mathcal{V}_n$, see Proposition 2.1.11), hence will be non-isomorphic as $(\bar{R}, \bar{R})$-bimodules. □

Example 2.3.7 Let $n = 2$. It follows from Corollary 2.2.9 that $B_1 \star B_2$ is free as a left R_1-module and free as a right R_2-module. Hence its left dense set is equal to $\{s_1\}$ and its right dense set to $\{s_2\}$. It corresponds to the dense sets obtained from the Temperley-Lieb diagram of $b_1 b_2$ given in figure 2.5.

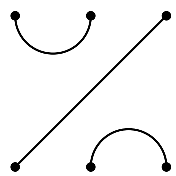


FIG. 2.5: Temperley-Lieb diagram corresponding to the element $b_1 b_2$.

Proposition 2.3.8. *Let $w \in \mathcal{W}_f$. Let $s_{i_1} \cdots s_{i_k}$ be an $\mathcal{S}$-reduced expression of w. The rank of B_w as left $\mathcal{O}(W(i_1 \cdots i_k))$-module is equal to $2^{|T(i_1 \cdots i_k)|}$ and its rank as right $\mathcal{O}(W(i_k \cdots i_1))$-module is equal to $2^{|T(i_k \cdots i_1)|}$. Moreover,*

$$|T(i_1 \cdots i_k)| = |T(i_k \cdots i_1)|.$$

Proof. The last equality is an immediate consequence of 2.3.5 since $|T(i_1 \cdots i_k)|$ is just the numbers of arcs at the top of the planar diagram corresponding to b_w which is equal to the number of arcs at the bottom of the diagram given by $|T(i_k \cdots i_1)|$.

The first property is shown by induction on $k = \ell_{\mathcal{S}}(w)$ using Lemma 2.2.13. Notice that it suffices to prove the statement for the left module structure. If the length of w is equal to 1, then $w = s_i$ for some $i \in \{1, \ldots, n\}$ whence $B_w = B_i$. In that case we know from Lemma 2.2.6 that B_i is free of rank 2 as left $\mathcal{O}(V_i)$-module and that $T(i) = \{s_i\}$. Now assume that the statement holds for w of length at most $k-1$. In particular, $B(i_2 \cdots i_k) = B_{s_{i_1} w}$ is a free $\mathcal{O}(W(i_2 \cdots i_k))$-module of rank $2^{|T(i_2 \cdots i_k)|}$. Now if $W(i_2 \cdots i_k)$ is not s_{i_1}-invariant, we know from Lemma 2.2.13 that B_w is a free $\mathcal{O}(W(i_1 \cdots i_k))$-module of rank $\mathrm{rk}(B(i_2 \cdots i_k))$. Hence it remains to show that $|T(i_1 \cdots i_k)| = |T(i_2 \cdots i_k)|$. But in case $W(i_2 \cdots i_k) = \bigcap_{t \in T(i_2 \cdots i_k)} V_t$ is not s_{i_1}-invariant, we are in one of the last two cases of Lemma 2.1.10 where the cardinality of the dense set is unchanged, giving the equality. In case $W(i_2 \cdots i_k)$ is s_{i_1}-invariant, it means that s_{i_1} commutes with any reflection in $T(i_2 \cdots i_k)$. It means that either $s_{i_1} \in T(i_2 \cdots i_k)$ or that the support of $T(i_2 \cdots i_k)$ and that of s_{i_1} are disjoint. In the first case, thanks to Proposition 2.3.5, it means that s_{i_1} lies in $Q(i_2 \cdots i_k)$, hence that the planar diagram corresponding to $b_{s_{i_2} \cdots s_{i_k}}$ has an arc joining i to $i+1$ at the top. In that case, one shows easily that s_{i_1} is a left descent of $s_{i_2} \cdots s_{i_k}$, hence that $s_{i_1} \cdots s_{i_k}$ is not reduced, a contradiction. In the latter case, we have that $T(i_1 \cdots i_k) = \{s_{i_1}\} \cup T(i_2 \cdots i_k)$ by the first case of Lemma 2.1.10 whence $|T(i_1 \cdots i_k)| = |T(i_2 \cdots i_k)| + 1$. By Lemma 2.2.13, we have that

$$\mathrm{rk}(B_w) = 2 \cdot \mathrm{rk}(B(i_2 \cdots i_k)) = 2 \cdot 2^{|T(i_2 \cdots i_k)|} = 2^{|T(i_1 \cdots i_k)|}.$$

□

2.3.4 Indecomposability of fully commutative bimodules

The next step is to prove indecomposability of $\star$-products of the bimodules B_i corresponding to elements of the Kazhdan-Lusztig basis of the Temperley-Lieb algebra, i.e., fully commutative bimodules B_w. Let $w \in \mathcal{W}_f$. Then $b_w \in \mathrm{TL}_n$ can be written uniquely as a product

$$(b_{i_k} b_{i_k - 1} \cdots b_{j_k})(b_{i_{k-1}} b_{i_{k-1}-1} \cdots b_{j_{k-1}}) \cdots (b_{i_1} b_{i_1 - 1} \cdots b_{j_1})$$

with all indices in $\{1, \dots, n\}$ and $i_k < i_{k-1} < \cdots < i_1$, $j_k < j_{k-1} < \cdots < j_1$ and $j_m \leq i_m$ for each $m = 1, \dots, k$ (see [28], §5.7; we have reversed the indices $1, \dots, k$ since it will be more convenient for the inductions we will use later).

Since the bimodules B_i satisfy the Temperley-Lieb relations any fully commutative bimodule can written in the form

$$(B_{i_k} B_{i_k - 1} \cdots B_{j_k})(B_{i_{k-1}} B_{i_{k-1}-1} \cdots B_{j_{k-1}}) \cdots (B_{i_1} B_{i_1 - 1} \cdots B_{j_1}),$$

where the products are $\star$-products.

Definition 2.3.9. *We say that B_w is* associated *to the corresponding sequence*

$$i_k \cdots j_k i_{k-1} \cdots j_{k-1} \cdots i_1 \cdots j_1.$$

The integer k is the rank *of the sequence. A fully commutative bimodule is* intertwined *if for each $1 < m \leq k$, the set $[i_m, j_m]$ contains both the indices $i_1 - 2(m-1)$ and $i_1 - 2(m-1) + 1$.*

Example 2.3.10 In case $n \geq 9$, the bimodule associated to the sequence

$$(1)(432)(654)(7)(98)$$

is not intertwined. Bimodules associated to the sequences

$$(3\underline{21})(\underline{43})(7\underline{65}4)(\underline{87}6)(9),\ (\underline{21})(\underline{43})(\underline{65})(\underline{87})(98),\ (543\underline{21})(65\underline{43})(7\underline{65})(\underline{87})(9)$$

are intertwined ; here $i_1 = 9$ and the indices of the form $i_1 - 2(m-1)$ and $i_1 - 2(m-1) + 1$ from the definition are underlined.

Lemma 2.3.11. *Let B be a fully commutative bimodule. If B is associated to a sequence of rank 1, then B is indecomposable (as graded bimodule).*

Proof. We write $i(i-1)\cdots j$ for the sequence associated to our bimodule, $i - j \geq 0$. We therefore have

$$B \cong B_i \star B_{i-1} \star \cdots \star B_j.$$

One has $W_{m(m-1)\cdots j} = V_m$ and $W_{m(m+1)\cdots i} = V_m$ for each $m \in [j, i]$ thanks to Lemma 2.1.9. As a consequence with any choice of brackets for computing the above product one gets that B is isomorphic to

$$B_i \otimes_{R_i} R_{i,i-1} \otimes_{R_{i-1}} B_{i-1} \otimes_{R_{i-1}} R_{i-1,i-2} \otimes_{R_{i-2}} B_{i-2} \otimes \cdots \otimes R_{j+1,j} \otimes_{R_j} B_j,$$

with $R_{m,m-1} = \mathcal{O}(V_m \cap V_{m-1})$ for each $m \in [j+1, i]$; if $i = j$ we get $B_i = B_j$. After reduction B is isomorphic to

$$R_i \otimes_i R_{i,i-1} \otimes_{i-1} R_{i-1,i-2} \otimes_{i-2} \cdots \otimes_{j+1} R_{j+1,j} \otimes_j R_j,$$

where $\otimes_m$ means $\otimes_{R_m^{s_m}}$; if $i = j$ we get $R_i \otimes_i R_i$. Thanks to Remark 2.2.1 one then has $R_m^{s_m} \twoheadrightarrow \mathcal{O}(V_m \cap V_{m-1})$ as well as $R_{m-1}^{s_{m-1}} \twoheadrightarrow \mathcal{O}(V_m \cap V_{m-1})$ for each $m \in [j+1, i]$. Hence any tensor

$$a_i \otimes_i a_{i,i-1} \otimes_{i-1} \cdots \otimes_{j+1} a_{j+1,j} \otimes_j a_j \in B$$

is equal to a tensor

$$a \otimes_i 1 \otimes_{i-1} \cdots \otimes_{j+1} 1 \otimes_j a' \in B.$$

As a consequence B is generated as $(\bar{R}, \bar{R})$-bimodule by the degree zero element $1 \otimes_i 1 \otimes_{i-1} \cdots \otimes_{j+1} 1 \otimes_j 1$ which forces indecomposability since the degree zero component of B has dimension 1. ◻

Lemma 2.3.12. *Consider the bimodule B from the proof of Lemma 2.3.11 written in the form*

$$R_i \otimes_i R_{i,i-1} \otimes_{i-1} R_{i-1,i-2} \otimes_{i-2} \cdots \otimes_{j+1} R_{j+1,j} \otimes_j R_j.$$

Any tensor $a \otimes_i 1 \otimes_{i-1} \cdots \otimes_{j+1} 1 \otimes_j a' \in B$ where $a \in R_i$, $a' \in R_j$ can be written in the form

$$(b \otimes_i 1 \otimes_{i-1} \cdots \otimes_{j+1} 1 \otimes_j 1) + (b' \otimes_i 1 \otimes_{i-1} \cdots \otimes_{j+1} 1 \otimes_j f_j),$$

where $b, b' \in R_i$.

Proof. It suffices to decompose $a' = r + r' f_j$ with $r, r' \in R_j^{s_j}$ and move r, r' to the left using the fact that $R_m^{s_m} \twoheadrightarrow \mathcal{O}(V_m \cap V_{m-1})$ as well as $R_{m-1}^{s_{m-1}} \twoheadrightarrow \mathcal{O}(V_m \cap V_{m-1})$ for each $m \in [j+1, i]$. ☐

Notation. Let $i_k \cdots j_k \cdots i_1 \cdots j_1$ be a sequence defining a fully commutative bimodule B. For each $m \in [1, k]$, we write $B(m)$ for the bimodule associated to the subsequence $i_m \cdots j_m$. In particular we have

$$B \cong B(k) \star B(k-1) \star \cdots \star B(1).$$

Proposition 2.3.13. *Let B be an intertwined bimodule associated to the sequence* $\mathrm{seq} = i_k \cdots j_k i_{k-1} \cdots j_{k-1} \cdots i_1 \cdots j_1$.

1. *One has the equality* $\mathrm{supp}(T(\mathrm{seq})) = [i_1 - 2(k-1), i_1 + 1]$. *Moreover, the set* $T(\mathrm{seq})$ *contains the reflection* $(i_1 - 2(k-1), i_1 + 1)$ *(in other words, it has a single block).*

2. *The bimodule B is indecomposable.*

Proof. The first claim is shown by induction on k. If $k = 1$, one has $\mathrm{seq} = i_1 \cdots j_1$ and $T(\mathrm{seq}) = \{s_{i_1}\}$ (see Lemma 2.1.9) whose support is $\{i_1, i_1 + 1\}$.

Now suppose that $k > 1$ and that the result holds for any sequence of rank at most $k - 1$ and consider the case where the sequence has rank k. If $W = W_{i_{k-1} \cdots j_1}$, then by induction $\mathrm{supp}(T_W) = [i_1 - 2(k-2), i_1 + 1]$ and T_W contains the reflection $(i_1 - 2(k-2), i_1 + 1)$. Now consider the subsequence $i_k \cdots j_k$ of seq, which is equal to the concatenation of the decreasing sequences $\mathrm{seq}_1 = i_k \cdots (i_1 - 2(k-1) + 1)$ and $\mathrm{seq}_2 = (i_1 - 2(k-1)) \cdots j_k$ (since the bimodule is intertwined). Any reflection s_j with j in seq_2 commutes with any reflection in T_W hence one gets using Lemma 2.1.10 that $T_{\mathrm{seq}_2 \cdot W} = T_W \cup \{s_{i_1 - 2(k-1)}\}$. We now study the effect of applying seq_1 to $\mathrm{seq}_2 \cdot W$. Using again Lemma 2.1.10, applying the first index on the right of seq_1, that is $(i_1-2(k-1)+1)$, replaces the two reflexions $s_{i_1-2(k-1)}$ and $(i_1-2(k-2),i_1+1)$ in $T_W\cup\{s_{i_1-2(k-1)}\}$ by $s_{i_1-2(k-1)+1}$ and $(i_1-2(k-1),i_1+1)$ and applying the following indices only removes and adds reflexions supported in $[i_1-2(k-1)+1,i_1]$, showing that $T_{\mathrm{seq}\cdot W}$ has support equal to $[i_1-2(k-1),i_1+1]$ and contains the reflection $(i_1-2(k-1),i_1+1)$.

To show indecomposability of B, we first compute the $\star$-products occurring in the bimodules $B(m)$ associated to each decreasing subsequence $\mathrm{seq}_m = i_m\cdots j_m$ of our sequence. These ones occur to be indecomposable thanks to Lemma 2.3.11 and we will write them as in the proof of this lemma in the form

$$R_{i_m}\otimes_{i_m}R_{i_m,i_m-1}\otimes_{i_m-1}R_{i_m-1,i_m-2}\otimes\cdots\otimes R_{j_m+1,j_m}\otimes_{j_m}R_{j_m}.$$

We will abuse notation and write $B(m)$ for the isomorphic bimodule above. It remains to make a choice of brackets for computing the product $B(k)\star B(k-1)\star\cdots\star B(2)\star B(1)$. We will compute it "from the right", i.e., as

$$B\cong B(k)\star(B(k-1)\star(\cdots\star(B(3)\star(B(2)\star B(1)))\cdots)).$$

Thanks to Theorem 2.2.15 together with the first part of the proposition, one has that for $1<\ell\leq k$, the left annihilator of the intertwined bimodule

$$B(\ell-1)\star(B(\ell-2)\star(\cdots\star(B(3)\star(B(2)\star B(1)))\cdots))$$

is equal to the ideal of functions vanishing on $\bigcap_{t\in Q_\ell}V_t$ where $Q_\ell\subset\mathcal{T}$ is a dense set satisfying $\mathrm{supp}(Q_\ell)=[i_1-2(\ell-2),i_1+1]$ and containing the reflection $(i_1-2(\ell-2),i_1+1)$. The right annihilator of $B(\ell)$ is equal to $I(V_{j_\ell})$. Since the bimodule B is intertwined one has that $j_\ell\leq i_1-2(\ell-1)=i_1-2(\ell-2)-2$ and in particular, s_{j_ℓ} commutes with any reflection in Q_ℓ. Set $X_\ell:=\bigcap_{t\in Q_\ell}V_t$, $W_\ell:=V_{s_{j_\ell}}\cap X_\ell$ for $\ell>1$ and $W_1=V_{j_1}$. For any $1\leq\ell\leq k$, one has that W_ℓ is s_{j_ℓ}-invariant and hence we can decompose

$$\mathcal{O}(W_\ell)=\mathcal{O}(W_\ell)^{s_{j_\ell}}\oplus\mathcal{O}(W_\ell)^{s_{j_\ell}}f_{j_\ell}|_{W_\ell}. \tag{2.2}$$

We will abuse notation and write f_i instead of $f_i|_X$ for the image of f_i in $\mathcal{O}(X)$ where $X\subset Z$ is an algebraic set to avoid using too much indices and since this will make no possible confusion in the next computations. Computing recursively our product with the choice of brackets described above we get that B is isomorphic to

$$B(k)\otimes_{R_{j_k}}\mathcal{O}(W_k)\otimes_{\mathcal{O}(X_k)}B(k-1)\otimes\cdots\otimes B(2)\otimes_{R_{j_2}}\mathcal{O}(W_2)\otimes_{R_{i_1}}B(1).$$

Again we abuse notation and write B for this isomorphic bimodule. We have seen in the proof of Lemma 2.3.11 that the bimodule $B(\ell)$ is indecomposable and generated by the element $1_\ell := 1 \otimes_{i_\ell} 1 \otimes_{i_\ell - 1} 1 \otimes \cdots \otimes_{j_\ell} 1 \in B(\ell)$ for each ℓ. Hence using Lemma 2.3.12 any tensor in the above tensor product can be written as a sum of two elements of the form

$$a \cdot 1_k \otimes_{R_{j_k}} a_k \otimes_{\mathcal{O}(X_k)} 1_{k-1} \otimes \cdots \otimes a_3 \otimes_{\mathcal{O}(X_3)} 1_2 \otimes_{R_{j_2}} a_2 \otimes_{R_{i_1}} 1_1 \cdot a'$$

the first one with $a' = 1$, $a \in \bar{R}$, $a_\ell \in \mathcal{O}(W_\ell)$ and the second one having the same properties but with $a' = f_{j_1}$. Our strategy is the same as in Lemma 2.3.11: we will show that our bimodule can be generated by the element

$$1_k \otimes_{R_{j_k}} 1 \otimes_{\mathcal{O}(X_k)} 1_{k-1} \otimes \cdots \otimes 1_2 \otimes_{R_{j_2}} 1 \otimes_{R_{j_1}} 1_1.$$

In that case, because of the s_{j_ℓ}-invariance of the variety W_ℓ, we use relation 2.2 to move the invariant parts of each a_k to the left in the same way as at the end of the proof of Lemma 2.3.11: we begin with a_2, writing $a_2 = r_2 + r'_2 f_{j_2}$ where r_2 and r'_2 are s_{j_2}-invariant. But then one has that in

$$\mathcal{O}(W_3) \otimes_{\mathcal{O}(X_3)} B(2) \otimes_{R_{j_2}} \mathcal{O}(W_2),$$

$a_3 \otimes 1_2 \otimes r_2 = q \otimes 1_2 \otimes 1$ and $a_3 \otimes 1_2 \otimes r'_2 f_{j_2} = q' \otimes 1_2 \otimes f_{j_2}$ with $q, q' \in \mathcal{O}(W_3)$. In other words a tensor in B of the form

$$a \cdot 1_k \otimes_{R_{j_k}} a_k \otimes_{\mathcal{O}(X_k)} 1_{k-1} \otimes \cdots \otimes a_3 \otimes_{\mathcal{O}(X_3)} 1_2 \otimes_{R_{j_2}} a_2 \otimes_{R_{i_1}} 1_1 \cdot a'$$

is equal to a tensor of the form

$$a \cdot 1_k \otimes_{R_{j_k}} a_k \otimes_{\mathcal{O}(X_k)} 1_{k-1} \otimes \cdots \otimes (q \otimes_{\mathcal{O}(X_3)} 1_2 \otimes_{R_{j_2}} 1 + q' \otimes_{\mathcal{O}(X_3)} 1_2 \otimes_{R_{j_2}} f_{j_2}) \otimes_{R_{i_1}} 1_1 \cdot a'.$$

Now one can decompose q, q' and again "move" the s_{j_3}-invariant parts to the left, and so on. At the end of the process we get a sum of elements $\sum_i a_i \cdot t_i$ where t_i are tensors in B with f_ℓ or 1 in the $\mathcal{O}(W_\ell)$-component of B and 1 in any other component. It remains to show that each of these t_i can be written as a sum of elements of the form $b \cdot 1 \otimes 1 \otimes \cdots \otimes 1 \otimes 1 \cdot b'$ with $b, b' \in \bar{R}$ to show that the arbitrary tensor in B we began with can be obtained from the tensor $1 \otimes 1 \otimes \cdots 1 \otimes 1 \in B$. In fact we will show that we can write any of the t_i as a single tensor of the form $b \cdot 1 \otimes 1 \otimes \cdots \otimes 1 \otimes 1 \cdot b'$ with $b = 1$ (in other words, all the remaining f_ℓ in our tensors will be "moved" to the right) and b' being equal to a polynomial in f_i for $i \leq i_1$. For this we need the following technical lemma :

Lemma 2.3.14. *Let $B(i)$, W_i, X_i be as above for each $2 \leq i \leq k$ and set $W_1 = V_{j_1}$.*

Let $\ell \in [2,k]$ and suppose $m \leq i_1 - 2(\ell-1)$. Then the tensor $f_m \otimes 1_{\ell-1} \otimes 1$ in $\mathcal{O}(W_\ell) \otimes_{\mathcal{O}(X_\ell)} B(\ell-1) \otimes_{R_{j_{\ell-1}}} \mathcal{O}(W_{\ell-1})$ is equal to a tensor of the form $1 \otimes 1_{\ell-1} \otimes \sum_j f_j$ in the same tensor product with all indices $j \leq i_1 - 2(\ell-2)$.

Proof. The first case is the case where $m < j_{\ell-1} - 1$. In that case f_m is invariant by any reflection $s_{m'}$ with m' an index occurring in the sequence $i_{\ell-1} \cdots j_{\ell-1}$ and hence $f_m \otimes 1_{\ell-1} \otimes 1 = 1 \otimes 1_{\ell-1} \otimes f_m$ since all the tensor products in $B(\ell-1)$ are over various $R^{s_{m'}}$ for m' occurring in the sequence $i_{\ell-1} \cdots j_{\ell-1}$.

The second case is the case where $m = j_{\ell-1} - 1 < i_1 - 2(\ell-2) - 1$, then m and $i_{\ell-1}$ are distant: if $\ell > 2$ it is clear since $[j_{\ell-1}, i_{\ell-1}]$ contains at least two indices. If $\ell = 2$, the condition $m = j_1 - 1$ forces $i_1 > j_1$ since otherwise one would have $m = i_1 - 1$ contradicting our assumption that $m \leq i_1 - 2(\ell-1)$. Hence our tensor is equal to the tensor

$$1 \otimes_{\mathcal{O}(X_k)} 1 \otimes_{i_{\ell-1}} 1 \otimes \cdots f_m \otimes_{j_{\ell-1}} 1 \otimes_{R_{j_{\ell-1}}} 1$$

with f_m lying in $\mathcal{O}(V_{j_{\ell-1}+1} \cap V_{j_{\ell-1}})$. But in this ring we have $f_{j_{\ell-1}+1} + f_{j_{\ell-1}} = 0$ since $V_{j_{\ell-1}+1} \cap V_{j_{\ell-1}} \subset H$ where H is the reflecting hyperplane of $s_{j_{\ell-1}+1} s_{j_{\ell-1}} s_{j_{\ell-1}+1}$ (lemma 2.1.4), hence in $\mathcal{O}(V_{j_{\ell-1}+1} \cap V_{j_{\ell-1}})$ we get

$$f_m = f_{j_{\ell-1}-1} = f_{j_{\ell-1}+1} + f_{j_{\ell-1}} + f_{j_{\ell-1}-1},$$

which is $s_{j_{\ell-1}}$-invariant, hence the sum in the right hand side can be moved to the last component of the tensor product ; but this is a sum of f_j for $j \leq j_{\ell-1} + 1 = m + 2 \leq i_1 - 2(\ell-2)$.

The last case is the case where $m \geq j_{\ell-1}$. This forces m to occur as an index of the sequence $i_{\ell-1} \cdots j_{\ell-1}$ and $m+1$, $m+2$ also occur since the bimodule is intertwined and $m \leq i_1 - 2(\ell-1)$. In that case our tensor $f_m \otimes 1_{\ell-1} \otimes 1$ is equal to a tensor

$$1 \otimes_{\mathcal{O}(X_\ell)} 1 \otimes_{i_{\ell-1}} 1 \otimes \cdots \otimes_{m+2} f_m \otimes_{m+1} 1 \otimes_m 1 \otimes \cdots \otimes 1,$$

with f_m lying in $\mathcal{O}(V_{m+1} \cap V_{m+2})$. In that ring one has $f_m = f_m + f_{m+1} + f_{m+2}$ which is s_{m+1}-invariant, hence the sum can be moved to the next factor which is $\mathcal{O}(V_m \cap V_{m+1})$. But in that ring, one has $f_m + f_{m+1} = 0$, hence our tensor is equal to the tensor

$$1 \otimes_{\mathcal{O}(X_\ell)} 1 \otimes_{i_{\ell-1}} 1 \otimes \cdots \otimes 1 \otimes_{m+1} f_{m+2} \otimes_m 1 \otimes_{m-1} 1 \otimes \cdots \otimes 1,$$

and the f_{m+2} can be moved to the right since it is invariant under the operation of all s_j for $j \leq m$. Hence the tensor is equal to

$$1 \otimes_{\mathcal{O}(X_\ell)} 1 \otimes_{i_{\ell-1}} 1 \otimes \cdots \otimes 1 \otimes_{m+1} 1 \otimes_m 1 \otimes_{m-1} 1 \otimes \cdots \otimes 1 \otimes f_{m+2},$$

and $m+2 \leq i_1 - 2(\ell-2)$, which concludes. □

End of the proof of the proposition. Using the above lemma we can move our f_ℓ's in the $\mathcal{O}(W_\ell)$ components of our bimodule B to the right inductively, beginning from the left with $\ell = k$ by moving f_ℓ to the right in the $\mathcal{O}(W_{\ell-1})$ component and so on. □

We now consider the indecomposability of a slightly more general family of bimodules.

Definition 2.3.15. *A fully commutative bimodule associated to a sequence*

$$i_k \cdots j_k \cdots i_1 \cdots j_1$$

will be called a generalized intertwined bimodule *if the following condition holds : each set $\{i_\ell, \ldots, j_\ell\}$ contains a nonempty subset S_ℓ of cardinality at most two such that the following inductive condition is satisfied : $S_1 = \{i_1\}$, and if $n(\ell)$ is the lowest index in S_ℓ, then the set $\{i_{\ell+1}, \ldots, j_{\ell+1}\}$ contains the index $n(\ell)-1$ and we put*

$$S_{\ell+1} = \begin{cases} \{n(\ell)-1\} & \text{if } n(\ell)-1 = j_{\ell+1} \\ \{n(\ell)-2, n(\ell)-1\} & \text{otherwise.} \end{cases}$$

The union of the sets S_ℓ for $\ell = 1, \ldots, k$ is called the set of intertwining indices *of the corresponding sequence or bimodule.*

Example 2.3.16 In case $n \geq 9$, the fully commutative bimodules associated to the sequences

$$(\underline{1})(\underline{32})(\underline{4})(7\underline{65})(\underline{87})(\underline{9}),\ (\underline{1})(\underline{2})(\underline{43})(7\underline{65}4)(\underline{87}65)(\underline{9}876),\ (\underline{87})(\underline{9})$$

are generalized intertwined bimodules; the indices belonging to the set $\bigcup_\ell S_\ell$ are underlined. The bimodules associated to the sequences

$$(1)(32)(65)(87)(9),\ (7)(98)$$

are not generalized intertwined bimodules.

The following technical result will allow us to use the same kind of arguments as for intertwined bimodules to show indecomposability; to this end, we order the set $\mathcal{S}$ of simple reflections by setting $s_i < s_j$ if and only if $i < j$, for $i, j \in [1, n]$.

Lemma 2.3.17. *Let $i_k \cdots j_k \cdots i_1 \cdots j_1$ be a sequence defining a generalized intertwined bimodule with corresponding variety $W \in \mathcal{V}_n$. Then*

1. *The smallest index in* $\operatorname{supp}(T_W)$ *is equal to $n(k)$ where $n(k)$ is as in definition 2.3.15,*
2. *The lowest simple reflection occurring in T_W is s_{i_k}.*

Proof. We argue by induction on k; if $k = 1$, the result is trivially true since $T_W = \{s_{i_1}\}$ and $n(1) = i_1$. Now suppose $k > 1$. By induction the smallest index occurring in $T_{W'}$ where W' is associated to the sequence $i_{k-1} \cdots j_{k-1} \cdots i_1 \cdots j_1$ is $n(k-1)$ (in particular there exists $j > n(k-1)$ such that $(n(k-1), j) \in T_{W'}$) and the lowest simple reflection occurring in $T_{W'}$ is $s_{i_{k-1}}$.

First consider the case $|S_k| = 1$, we then have $n(k) = j_k = n(k-1) - 1$. We get $T_{j_k \cdot W'} = (T_{W'} \backslash \{(n(k-1), j)\}) \cup \{s_{n(k)}\}$. If $(n(k-1), j)$ is simple, then $n(k-1) = i_{k-1}$ and $i_k = j_k$ since $j_k = i_{k-1} - 1$ and $j_k \leq i_k < i_{k-1}$; in that case we are done. Otherwise, the first two blocks (from the left) of the set $T_{j_k \cdot W'}$ have the form given by figure 2.6, where all reflections having in their supports an index in

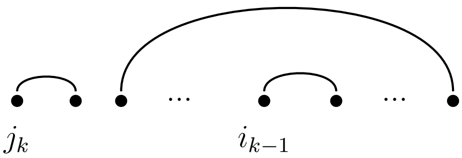


FIG. 2.6: The first two blocks of the set $T_{j_k \cdot W}$.

$[j_k + 2, i_{k-1} - 1]$ must have the other index of their support bigger than or equal to $i_{k-1} + 2$ (otherwise $s_{i_{k-1}}$ would not be the lowest simple reflection in $T_{W'}$). Thanks to this property together with Lemma 2.1.10 and the fact that $i_k < i_{k-1}$, applying $i_k \cdots (j_k + 1)$ to $j_k \cdot W'$ does not change the support of the corresponding dense set and gives a set whose lowest simple reflection is s_{i_k} (see figure 2.7 for an illustration: in that case $n(k) = j_k$).

Now suppose $|S_k| = 2$; applying the sequence $n(k) \cdots (j_k + 1) j_k$ to W' we get a variety W'' with corresponding set equal to $T_{W'} \cup \{s_{n(k)}\}$ since $n(k-1) = n(k) + 2$ is the lowest index in $T_{W'}$. We can then argue exactly as in the first case to get the conclusion (see figure 2.7). □

Proposition 2.3.18. *Let B be a generalized intertwined bimodule with associated sequence $i_k \cdots j_k \cdots i_1 \cdots j_1$. Then B is indecomposable. More precisely, when writing B in the form*

$$B(k) \otimes_{R_{j_k}} \mathcal{O}(W_k) \otimes_{\mathcal{O}(X_k)} B(k-1) \otimes \cdots \otimes B(2) \otimes_{R_{j_2}} \mathcal{O}(W_2) \otimes_{R_{i_1}} B(1)$$

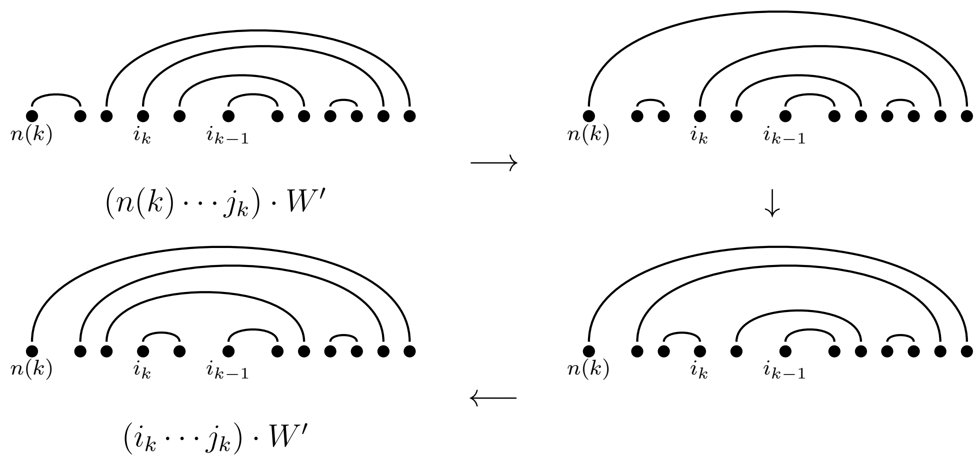


FIG. 2.7: Example of the process of applying the sequence $i_k \cdots (n(k)+1)$ to $(n(k) \cdots j_k) \cdot W'$; in case $|S_k| = 1$ we have $n(k) = j_k$.

where we made the same choice of brackets as in Proposition 2.3.13, with X_ℓ the variety associated to the subsequence $i_{\ell-1} \cdots j_{\ell-1} \cdots i_1 \cdots j_1$ and $W_\ell = X_\ell \cap V_{j_\ell}$, any tensor in B can be written as a sum of elements of the form

$$a \cdot 1 \otimes 1 \otimes \cdots \otimes 1 \otimes 1 \cdot p(f_1, \ldots, f_{i_1})$$

where the $\cdot$ holds for the operation of $\bar{R}$ on both sides and $p(f_1, \ldots, f_{i_1})$ is a polynomial in $f_1, f_2, \ldots, f_{i_1}$.

Moreover if $j+2$ is smaller than or equal to the smallest index in S_k, then there exists a polynomial $p(f_1, \ldots, f_{i_1})$ such that

$$f_j \cdot 1 \otimes 1 \otimes \cdots 1 \otimes 1 = 1 \otimes 1 \otimes \cdots 1 \otimes 1 \cdot p(f_1, \cdots f_{i_1}).$$

Proof. We first consider in which case the variety X_ℓ is s_{j_ℓ}-invariant ; if $|S_\ell| = 2$, we have that $j_\ell \leq n(\ell-1) - 2$ by definition 2.3.15 hence X_ℓ is s_{j_ℓ}-invariant by the first assertion of Lemma 2.3.17 together with Proposition 2.1.11. If $|S_\ell| = 1$, then $j_\ell = n(\ell-1) - 1$ by definition 2.3.15 hence X_ℓ is not s_{j_ℓ}-invariant by the first assertion of Lemma 2.3.17 together with Proposition 2.1.11. Therefore in case $|S_\ell| = 2$ one can decompose

$$\mathcal{O}(W_\ell) = \mathcal{O}(W_\ell)^{s_{j_\ell}} \oplus \mathcal{O}(W_\ell)^{s_{j_\ell}} f_{j_\ell}|_{W_\ell},$$

hence for each ℓ such that $|S_\ell| = 2$ we can decompose the $\mathcal{O}(W_\ell)$-component of any tensor in B and move the invariant parts to the left in $B(\ell)$ and then in $\mathcal{O}(W_{\ell+1})$ as we did in 2.3.13 for the intertwined case. In the case where $|S_\ell| = 1$, we have seen that X_ℓ is not s_{j_ℓ}-invariant. Thanks to Corollary 2.1.12 together with Remark 2.2.1, $R_{j_\ell}^{s_{j_\ell}} \twoheadrightarrow \mathcal{O}(W_\ell)$, hence the $\mathcal{O}(W_\ell)$ component of any tensor in B can be moved to the left in $B(\ell)$ and then in the $\mathcal{O}(W_{\ell+1})$-component. As a consequence, any tensor $b \in B$ can be written as a sum $\sum_i a_i \cdot t_i$, where $a_i \in \bar{R}$ and t_i are tensors in B with f_{j_ℓ} or 1 in the $\mathcal{O}(W_\ell)$-component for ℓ such that $|S_\ell| = 2$, with 1 in the $\mathcal{O}(W_\ell)$-component for ℓ such that $|S_\ell| = 1$ and with 1 in the components coming from the bimodules $B(\ell)$. It remains to show that if $|S_\ell| = 2$, the f_{j_ℓ} in the $\mathcal{O}(W_\ell)$-components can be "moved to the right".

Now we consider an element f_j in the $\mathcal{O}(W_\ell)$-component of one of the t_i, with $j \leq n(\ell)$ as we did at the end of the proof of 2.3.14. If $|S_{\ell-1}| = 1$, then the only index in $S_{\ell-1}$ is $j_{\ell-1}$ and one has $j_{\ell-1} \geq j+2$ since $|S_\ell| = 2$. In that case, any index occurring in the sequence $i_{\ell-1} \cdots j_{\ell-1} \cdots i_1 \cdots j_1$ is distant from j and hence f_j can be moved in the very first component on the right of our tensor product (that is $\mathcal{O}(W_1) = R_{j_1}$). The other case is the case where $|S_{\ell-1}| = 2$. Since $i_{\ell-1} > n(\ell)+1$, f_j is $s_{i_{\ell-1}}$-invariant and hence can be moved to the right in $B(\ell-1)$. We then argue exactly as in Lemma 2.3.14, distinguishing the three cases: $j < j_{\ell-1} - 1$, $j = j_{\ell-1} - 1$ and $j \geq j_{\ell-1}$, to conclude that we can "move" our f_j to the right in the $\mathcal{O}(W_{\ell-1})$-component where we obtain a sum of $f_{j'}$ for $j' \leq j+2$. But since $|S_\ell| = 2$, $j' \leq n(\ell-1)$. Hence we can inductively "move" the f_j's to the $\mathcal{O}(W_m)$-component with $m < \ell$ as far as $|S_i| = 2$ for each $i \in [m, \ell-1]$ obtaining in that component a polynomial in $p(f_1, \cdots, f_{n(m)})$ and if then $|S(m-1)| = 1$, we apply the first case to move our polynomial in the very first component on the right of the tensor product (that is $\mathcal{O}(W_1) = R_{j_1}$). Hence we can inductively move any f_j to the right and one obtains in that component polynomials in the f_i's for i smaller than or equal to $n(1) = i_1$. This also shows the last statement since if $j+2$ is less than or equal to $n(k)$, then arguing as above our f_j lying in the very first component on the left of the tensor product can be moved in the $\mathcal{O}(W_k)$-component and one obtains a sum of $f_{j'}$ for j' less than or equal to $j+2 \leq n(k)$. □

We have all the required tools to prove :

Theorem 2.3.19. *Let B be a fully commutative bimodule. Then B is indecomposable in $\bar{R} - \mathrm{mod}_{\mathbb{Z}} - \bar{R}$.*

Proof. We consider the sequence $i_k \cdots j_k \cdots i_1 \cdots j_1$ our bimodule is associated to. We consider the biggest index ℓ such that the bimodule associated to the subsequence $\mathrm{seq}_1 = i_\ell \cdots j_\ell \cdots i_1 \cdots j_1$ is a generalized intertwined bimodule and write $G(1)$ for the corresponding bimodule. Then one can do the same with the subsequence $i_k \cdots j_k \cdots i_{\ell+1} \cdots j_{\ell+1}$ to obtain a generalized intertwined bimodule $G(2)$ associated to a subsequence seq_2. At the end of the process we obtain a sequence

$G(1), \dots, G(m)$ of generalized intertwined bimodules associated to subsequences $\mathrm{seq}_1, \dots, \mathrm{seq}_m$ such that

$$B \cong G(m) \star G(m-1) \star \dots \star G(2) \star G(1)$$

and $\mathrm{seq} = \mathrm{seq}_m \cdots \mathrm{seq}_2 \mathrm{seq}_1$. We compute the various $\star$ products occurring in each of the bimodule $G(i)$ with the same choice of brackets as in Propositions 2.3.13 and 2.3.18; we then compute the above product "from the right", i.e. with the following choice of brackets:

$$G(m) \star (G(m-1) \star (\cdots (G(3) \star (G(2) \star G(1))) \cdots)).$$

By maximality of the rank of the subsequence $i_\ell \cdots j_\ell \cdots i_1 \cdots j_1$ defining $G(1)$, $j_{\ell+1} \leq i_{\ell+1} < n(\ell) - 1$. But we know from Lemma 2.3.17 that the lowest index in the support of T_{U_2} where U_2 is the variety associated to seq_1 is precisely $n(\ell)$. The variety Z_2 occurring when computing the $\star$ product between $G(1)$ and $G(2)$, which is equal to $U_2 \cap V_{j_{\ell+1}}$, is then $s_{j_{\ell+1}}$-invariant. Moreover, since $i_{\ell+1}$ is the biggest index occurring in seq_2, one has that

$$T_{W_{\mathrm{seq}_2 \mathrm{seq}_1}} = T_{W_{\mathrm{seq}_1}} \cup T_{W_{\mathrm{seq}_2}}$$

and the same holds using induction when replacing 1 by i for $1 < i < m$. Hence our bimodule is isomorphic to

$$G(m) \otimes_{R_{k_m}} \mathcal{O}(Z_m) \otimes_{\mathcal{O}(U_m)} G(m-1) \otimes \cdots \otimes_{R_{k_2}} \mathcal{O}(Z_2) \otimes_{\mathcal{O}(U_2)} G(1)$$

where U_j is the variety associated to the sequence $\mathrm{seq}_{j-1} \cdots \mathrm{seq}_2 \mathrm{seq}_1$, k_j is the last index of the sequence seq_j and $Z_j = U_j \cap V_{k_j}$ is s_{k_j}-invariant. Now consider any tensor

$$a_m \otimes_{R_{k_m}} b_m \otimes_{\mathcal{O}(U_m)} a_{m-1} \otimes \cdots \otimes_{R_{k_2}} b_2 \otimes_{\mathcal{O}(U_2)} a_1$$

in the above tensor product with $a_j \in G(j)$, $b_j \in \mathcal{O}(Z_j)$. Since $R_{k_j} \twoheadrightarrow \mathcal{O}(Z_j)$ we can suppose that each b_i equals 1. Now using Proposition 2.3.18 inductively, beginning with a_1, we can rewrite our tensor as a sum of tensors of the form

$$a \cdot 1 \otimes_{R_{k_m}} p(f_1, \dots, f_{n_m}) \otimes_{\mathcal{O}(U_m)} 1 \otimes \cdots \otimes_{R_{k_2}} p(f_1, \dots, f_{n_2}) \otimes_{\mathcal{O}(U_2)} 1 \cdot p(f_1, \cdots, f_{n_1}),$$

where n_j is the biggest index in the sequence seq_j (in particular $n_1 = i_1$ and $n_2 = i_{\ell+1}$. Now each $n_j + 2$ is less than or equal to the smallest index in the set of intertwining indices of seq_{j-1} because this sequence was chosen to be maximal such that the corresponding bimodule is a generalized intertwined bimodule. Hence we can apply the last statement of Proposition 2.3.18 inductively, beginning on the left. This concludes. □

2.3.5 Categorification of the diagram basis

Notice that the category of finitely generated graded $\bar{R}$-bimodules has the Krull-Schmidt property (see Remark 1.3.1). We denote by $\mathcal{B}_{\mathrm{TL}_n}$ the full subcategory of $\bar{R} - \mathrm{mod}_{\mathbb{Z}} - \bar{R}$ whose objects are all the bimodules isomorphic to direct sums of (possibly shifted) fully commutative bimodules.

Theorem 2.3.19 allows us to extend the $\star$-product to direct sums of fully commutative bimodules and their shifts by bilinearity; since in general Krull-Schmidt decompositions are not unique but essentially unique this is well defined only up to isomorphism (in particular the $\star$-product is not necessarily a functor) but it descends to a well-defined operation on the level on the split Grothendieck group of the category $\mathcal{B}_{\mathrm{TL}_n}$ which we denote by $\langle \mathcal{B}_{\mathrm{TL}_n} \rangle$. Indeed, the isomorphism classes of indecomposable objects form a basis of it and $\langle M \rangle = \langle N \rangle$ if and only if $M \cong N$. In particular $\langle \mathcal{B}_{\mathrm{TL}_n} \rangle$ comes equipped with a ring structure and even a $\mathbb{Z}[\tau, \tau^{-1}]$-algebra structure via $\tau \cdot \langle M \rangle = \langle M[1] \rangle$.

Recall that for $w \in \mathcal{W}_f$ a fully commutative element, we write b_w for the corresponding element of the Temperley-Lieb algebra and B_w for a corresponding fully commutative bimodule. Combining our efforts from the previous sections it is now easy to prove:

Theorem 2.3.20 (Categorification of the diagram basis of the Temperley-Lieb algebra). *The category $\mathcal{B}_{\mathrm{TL}_n}$ categorifies the Kazhdan-Lusztig basis of the Temperley-Lieb algebra TL_n. More precisely, there is a unique isomorphism of $\mathbb{Z}[\tau, \tau^{-1}]$-algebras*

$$\mathcal{E} : \mathrm{TL}_n \xrightarrow{\sim} \langle \mathcal{B}_{\mathrm{TL}_n}, \oplus, \star \rangle ,$$

such that $\mathcal{E}(b_w) = \langle B_w \rangle$ for each $w \in \mathcal{W}_f$, $\mathcal{E}[\tau] = \left\langle \bar{R}[1] \right\rangle$.

Proof. We know from Theorem 2.3.3 that the bimodules B_w satisfy the Temperley-Lieb relations. This shows that we have a surjective morphism of $\mathbb{Z}[\tau, \tau^{-1}]$-algebras with the claimed properties. In order to see that this morphism is injective, it suffices to show that if $w \neq w'$ are two fully commutative elements in $\mathcal{W}$, then the corresponding bimodules B_w and $B_{w'}$ are non-isomorphic. This has already been proven in 2.3.6. □

2.4 Fully commutative bimodules as rings of regular functions

In the previous section, we showed that any fully commutative bimodule B_w is generated as bimodule by a nonzero element in its one-dimensional degree zero component, which implies indecomposability. Since there is an equivalence of categories between quasi-coherent sheaves on $Z \times Z$ and $\bar{R} \otimes_k \bar{R}$-modules, cyclicity is an indication that we may realize our bimodules as rings of regular functions on subvarieties of $Z \times Z$. The two projections on Z then give the bimodule structure. The aim of this section is to realize fully commutative bimodules B_w where $w \in \mathcal{W}_f$ is associated to a sequence of rank one as rings of regular functions on a subvariety of $Z \times Z$.

A proof of this fact for any fully commutative bimodule would provide a new proof of their indecomposability, since the ring of regular functions on a closed subscheme of $Z \times Z$ is obviously generated as bimodule by the constant function 1.

We also use some geometric properties of the bimodules to compute the spaces of homorphisms between B_i and B_j for $i, j \in [1, n]$ at the end of the section.

2.4.1 Regular functions on twisted diagonals

Recall that s_i acts on V_i, hence on $R_i = \mathcal{O}(V_i)$. Write $(R_i)_{s_i}$ for the graded $(\bar{R}, \bar{R})$-bimodule which is equal to R_i as left $\bar{R}$-module but with right operation of $\bar{R}$ twisted by s_i that is, if $b \in (R_i)_{s_i}$, $r \in \bar{R}$, $b \cdot r = bs_i(r|_{V_i})$. One checks that there are short exact sequences

$$0 \longrightarrow R_i[-2] \xrightarrow{\varphi_i^+} B_i \xrightarrow{\mu_i^-} (R_i)_{s_i} \longrightarrow 0,$$

where $\varphi_i^+(r) = r \otimes f_i + rf_i \otimes 1$ and $\mu_i^-(a \otimes b) = as_i(b)$, and

$$0 \longrightarrow (R_i)_{s_i}[-2] \xrightarrow{\varphi_i^-} B_i \xrightarrow{\mu_i^+} (R_i) \longrightarrow 0,$$

where $\varphi_i^-(r) = r \otimes f_i - rf_i \otimes 1$ and $\mu_i^+(a \otimes b) = ab$ (this is easily shown using the decomposition $R_i \cong R_i^{s_i} \oplus R_i^{s_i}[-2]$ in $R_i^{s_i} - \mathrm{mod}_{\mathbb{Z}}$).

Notation. Let $Q \subset \mathcal{T}$, $A \subset \mathcal{W}$. We set

$$\mathrm{Gr}_Q^R(A) = \bigcup_{x \in A} \{(xv, v) \mid v \in \bigcap_{t \in Q} V_t\},$$

$$\mathrm{Gr}_Q^L(A) = \bigcup_{x \in A} \{(v, xv) \mid v \in \bigcap_{t \in Q} V_t\}.$$

If Q consists of one or two simple reflections and $A = \{x_1, \dots, x_k\}$, for example if $Q = \{s_i, s_j\}$, we will write $\mathrm{Gr}^{L/R}_{i,j}(x_1, \dots, x_k)$ instead of $\mathrm{Gr}^{L/R}_Q(A)$ for notational compactness.

Lemma 2.4.1. *In $\bar{R} - \mathrm{mod}_{\mathbb{Z}} - \bar{R}$ one has isomorphisms*

$$B_i \cong \mathcal{O}(\mathrm{Gr}^R_i(e, s_i)) \cong \mathcal{O}(\mathrm{Gr}^L_i(e, s_i)).$$

Proof. Notice that the last isomorphism is trivial and is even an equality since $\mathrm{Gr}^R_i(e, s_i) = \mathrm{Gr}^L_i(e, s_i)$. The map $\bar{R} \otimes_k \bar{R} \twoheadrightarrow \mathcal{O}(\mathrm{Gr}^R_i(e, s_i))$ factors through B_i, yielding a surjective map $B_i \twoheadrightarrow \mathcal{O}(\mathrm{Gr}^R_i(e, s_i))$. To show that this map is an isomorphism, one first shows that there exists a short exact sequence

$$R_i[-2] \hookrightarrow \mathcal{O}(\mathrm{Gr}^R_i(e, s_i)) \twoheadrightarrow (R_i)_{s_i}$$

with suitable maps. To show this, notice that $R_i \cong \mathcal{O}(\mathrm{Gr}^R_i(e))$, $(R_i)_{s_i} \cong \mathcal{O}(\mathrm{Gr}^R_i(s_i))$. One defines the surjective map in the sequence above by restriction and the injective map by multiplication by a linear functional $f \in V^* \times V^*$ which vanishes on $\mathrm{Gr}^R_i(s_i)$ but not on $\mathrm{Gr}^R_i(e)$, that is, $f_i \otimes \mathrm{id} + \mathrm{id} \otimes f_i$ (up to a scalar). One has injectivity since $f_i(v) \neq 0$ for $v \in V_i - \{0\}$. Now suppose that f is a regular function on $\mathrm{Gr}^R_i(e, s_i)$ which vanishes on $\mathrm{Gr}^R_i(s_i)$. Since $t := s_i \times \mathrm{id}$ is a reflection in $V \times V$ one can then decompose f in the form $r + r'(f_i \otimes \mathrm{id})$ where r, r' are $s_i \times \mathrm{id}$-invariant. But since f vanishes on $\mathrm{Gr}^R_i(s_i)$ one has $r(v, v) = r'(v, v) f_i(v)$ for each $v \in V_i$. Hence $f = r'(f_i \otimes \mathrm{id} + \mathrm{id} \otimes f_i)$, and the sequence is exact. We get a commutative diagram

$$\begin{array}{ccccc}
\mathcal{O}(\mathrm{Gr}^R_i(e)) & \hookrightarrow & \mathcal{O}(\mathrm{Gr}^R_i(e, s_i)) & \twoheadrightarrow & \mathcal{O}(\mathrm{Gr}^R_i(s_i)) \\
\cong \uparrow & & \uparrow & & \cong \uparrow \\
R_i & \hookrightarrow & B_i & \twoheadrightarrow & (R_i)_{s_i}
\end{array},$$

which forces the map in the middle to be an isomorphism. □

We would like to generalize Lemma 2.4.1, as least in the case where $w \in \mathcal{W}_f$ has the form $s_i s_{i-1} \cdots s_{j+1} s_j$ where $i > j$. That is, we would like to find an isomorphism between B_w and the ring of regular functions on some closed subscheme $X_w \subset Z \times Z$. To this end, we need some technical results; first recall that any choice of brackets for computing the product

$$B_i \star B_{i-1} \star \cdots \star B_j$$

gives as a consequence of Lemma 2.1.9 the following tensor product of commutative rings:

$$R_i \otimes_{R_i^{s_i}} R_{i,i-1} \otimes_{R_{i-1}^{s_{i-1}}} R_{i-1,i-2} \otimes_{R_{i-2}^{s_{i-2}}} \cdots \otimes_{R_{j+1}^{s_{j+1}}} R_{j+1,j} \otimes_{R_j^{s_j}} R_j,$$

where $R_{k+1,k} = \mathcal{O}(V_k \cap V_{k-1})$. We will write the tensor product above simply as

$$R_i \otimes_i R_{i,i-1} \otimes_{i-1} R_{i-1,i-2} \otimes_{i-2} \cdots \otimes_{j+1} R_{j+1,j} \otimes_j R_j.$$

Now recall from Lemma 2.2.7 that for $m \in \{k+1, k-1\}$, one has an isomorphism of $R_k^{s_k}$-modules

$$R_k^{s_k} \xrightarrow{\sim} R_{k,m}$$

given by the following composition of inclusion and restriction

$$R_k^{s_k} \hookrightarrow R_k \twoheadrightarrow R_{k,m}.$$

As a consequence, given an element $a_j \in R_j^{s_j}$, one can associate to it a unique $a_i \in R_i^{s_i}$ since all the arrows in the diagram below are isomorphisms (but isomorphisms in the category of graded k-modules)

$$\begin{array}{ccccccccccc} R_i^{s_i} & & & & R_{i-1}^{s_{i-1}} & & & & \cdots & & R_{j+1}^{s_{j+1}} & & & & R_j^{s_j} \\ & \searrow^{\sim} & & \swarrow_{\sim} & & \searrow^{\sim} & & \swarrow_{\sim} & & \swarrow_{\sim} & & \searrow^{\sim} & & \swarrow_{\sim} & \\ & & R_{i,i-1} & & & & R_{i-1,i-2} & & \cdots & & & & R_{j+1,j} & & \end{array}$$

We want to understand the element a_i in terms of a_j; this will be done by induction on $i-j$. To this end, write a_k, $k \in \{j, j+1, \ldots, i\}$ for the unique element of $R_k^{s_k}$ corresponding to a_j via the chain of isomorphisms given in the diagram above. For $u, v \in \mathcal{W}$, we write ${}^u v$ for uvu^{-1}. Let $j \leq k$. Set

$$w(j,k) = ({}^{s_{j+1}}s_j)({}^{s_{j+2}}s_{j+1}) \cdots ({}^{s_k}s_{k-1})({}^{s_{k+1}}s_k)$$

for $j < k$ and $w(j,k) = e$ if $j = k$. Now if $v \in V_k$, associate an integer $n_v^k \in \mathbb{Z}_{\geq 0}$ by

$$n_v^k = |\{m \in \{j, j+1, \ldots, k-1\} \mid v \in H_{s_m}\}|.$$

Lemma 2.4.2. *Let $v \in V_k$. We have*

$$a_k(v) = \begin{cases} a_j(w(j,k-1)v) & \text{if } n_v^k \text{ is even,} \\ a_j(w(j,k-1)s_k v) & \text{if } n_v^k \text{ is odd.} \end{cases}$$

Proof. As mentioned above, the proof is by induction on $k-j$. If $k-j=0$, then $v \in V_k$ implies $n_v^k = 0$ and $a_k(v) = a_j(v) = a_j(w(k,k)v)$. Now suppose that $k-j > 0$. By definition, we have that $a_k|_{V_k \cap V_{k-1}} = a_{k-1}|_{V_k \cap V_{k-1}}$. Let $v \in V_k$. If

$v \in V_{k-1}$, then $a_k(v) = a_{k-1}(v)$. If $v \in H_{s_{k-1}}$, then $v \in V_{s_k s_{k-1} s_k}$ otherwise one would have $v \in H_{s_k}$, a contradiction. It implies that $s_k v \in V_{k-1}$. But $a_k \in R_k^{s_k}$, implying $a_k(v) = a_k(s_k v) = a_{k-1}(s_k v)$. By induction we get

$$a_k(v) = \begin{cases} a_{k-1}(v) = a_j(w(j,k-2)v) & \text{if } v \in V_{k-1} \text{ and } n_v^{k-1} \text{ is even,} \\ a_{k-1}(v) = a_j(w(j,k-2)s_{k-1}v) & \text{if } v \in V_{k-1} \text{ and } n_v^{k-1} \text{ is odd,} \\ a_{k-1}(s_k v) = a_j(w(j,k-2)s_k v) & \text{if } v \in H_{s_{k-1}} \text{ and } n_{s_k v}^{k-1} \text{ is even,} \\ a_{k-1}(s_k v) = a_j(w(j,k-2)s_{k-1}s_k v) & \text{if } v \in H_{s_{k-1}} \text{ and } n_{s_k v}^{k-1} \text{ is odd.} \end{cases}$$

Now in the first two cases, one has $v \in V_k \cap V_{k-1}$ hence $v \in H_{s_k s_{k-1} s_k}$ by Lemma 2.1.4. In the first case we get $a_k(v) = a_j(w(j,k-2)v)$ but since $v = s_k s_{k-1} s_k v$ this is equal to $a_j(w(j,k-2)s_k s_{k-1} s_k v) = a_j(w(j,k-1)v)$ and $n_v^k = n_v^{k-1}$ hence even. In the second case we get $a_k(v) = a_j(w(j,k-2)s_{k-1}s_k s_{k-1} s_k v) = a_j(w(j,k-1)s_k v)$ and $n_v^k = n_v^{k-1}$ hence odd. In the third case we have $a_k(v) = a_{k-1}(s_k v) = a_j(w(j,k-2)s_k v)$ but since $v \in H_{s_{k-1}}$ this is equal to $a_j(w(j,k-2)s_k s_{k-1} v) = a_j(w(j,k-1)s_k v)$; moreover $n_v^k = n_{s_k v}^{k-1} + 1$ since $s_k v \in V_m$ for $m < k-1$ if and only if $v \in V_m$ since m and k are distant, and $v \in H_{s_{k-1}}$. Hence n_v^k is odd in that case. In the last case we have $a_k(v) = a_{k-1}(s_k v) = a_j(w(j,k-2)s_{k-1}s_k v)$ but since $v \in H_{s_{k-1}}$ this is equal to $a_{k-1}(s_k v) = a_j(w(j,k-2)s_{k-1}s_k s_{k-1} v) = a_k(w(j,k-1)v)$. In that case by the same argument as for the third case we have $n_v^k = n_{s_k v}^{k-1} + 1$ hence $n_k(v)$ is even. To summarize we have

$$a_k(v) = \begin{cases} a_j(w(j,k-1)v) & \text{if } n_v^k \text{ is even,} \\ a_j(w(j,k-1)s_k v) & \text{if } n_v^k \text{ is odd.} \end{cases}$$

This is exactly what we wanted to establish. □

Lemma 2.4.3. *Let $i, j \in \{1, \dots, n\}$, $i > j$. Let $w = s_i s_{i-1} \dots s_j$. There is an operation of* $\mathrm{id} \times s_j$ *on the closed subscheme*

$$X(i,j) := \mathrm{Gr}_j^R(w(j,i-1)^{-1}, s_i w(j,i-1)^{-1}) \hookrightarrow V_i \times V_j \hookrightarrow Z \times Z.$$

As a consequence, there is a decomposition

$$\mathcal{O}(X(i,j)) = \mathcal{O}(X(i,j))^{\mathrm{id} \times s_j} \oplus \mathcal{O}(X(i,j))^{\mathrm{id} \times s_j}(1 \times f_j)$$

and $\mathcal{O}(X(i,j))^{\mathrm{id} \times s_j}$ is isomorphic to R_i.

Proof. Set

$$\mathrm{Gr}_1 := \mathrm{Gr}_j^R(w(j,i-1)^{-1}),$$

$$\mathrm{Gr}_2 := \mathrm{Gr}_j^R(s_i w(j,i-1)^{-1}),$$

so that $X(i,j) = \mathrm{Gr}_1 \cup \mathrm{Gr}_2$.

The fact that there is an operation of $\mathrm{id} \times s_j$ on $X(i,j)$ is a consequence of the fact that $s_j w(j, i-1) s_i w(j, i-1)^{-1} = e$, which is a straightforward computation.

Now $\mathrm{id} \times s_j$ is a reflection in $V \times V$ with reflecting hyperplane equal to $V \times H_{s_j}$. As a consequence there is no irreducible component of $X(i,j)$ included in the reflecting hyperplane of $\mathrm{id} \times s_j$ whence the claimed decomposition. Now $w(j,i-1)^{-1}(V_j) = V_i = (s_i w(j,i-1)^{-1})(V_j)$. Hence the projection

$$\mathrm{pr} : X(i,j) \to V_i$$

yields a morphism of k-algebras

$$\varphi : R_i = \mathcal{O}(V_i) \to \mathcal{O}(X(i,j))$$

whose image lies in the subring of $\mathrm{id} \times s_j$ invariant functions, given by $a \mapsto a \otimes \mathrm{id}$. It is clear that this map is injective. For surjectivity, one considers $f \in \mathcal{O}(X(i,j))^{\mathrm{id} \times s_j}$. Since $X(i,j) = \mathrm{Gr}_1 \cup \mathrm{Gr}_2$, we can restrict f to a single graph, for example to Gr_1. The inclusion $V_i \hookrightarrow \mathrm{Gr}_1$ and projection $\mathrm{Gr}_1 \twoheadrightarrow V_i$ are morphisms of varieties inverse to each other, giving an isomorphism of rings $\mathcal{O}(\mathrm{Gr}_1) \cong \mathcal{O}(V_i)$. Hence there is a single element in R_i corresponding to $f|_{\mathrm{Gr}_1}$. One checks that this element is a preimage of f under φ using $\mathrm{id} \times s_j$-invariance. $\square$

Proposition 2.4.4. *Let $i, j \in \{1, \dots, n\}$, $i > j$. Let $w = s_i s_{i-1} \cdots s_j$. Then*

$$B_w \cong \mathcal{O}(X(i,j)).$$

Proof. Recall that B_w is free of rank two as left R_i-module, with basis given by $\{1 \otimes 1 \otimes \cdots \otimes 1 \otimes 1, 1 \otimes 1 \otimes \cdots \otimes 1 \otimes f_j\}$. We define a map

$$\varphi : B_w \to \mathcal{O}(X(i,j))$$

by the assignments $1 \otimes 1 \otimes \cdots \otimes 1 \otimes 1 \mapsto 1 \otimes 1$, $1 \otimes 1 \otimes \cdots \otimes 1 \otimes f_j \mapsto 1 \otimes f_j$. This defines an isomorphism of left R_i-modules since $\mathcal{O}(X(i,j))$ is a free left $\mathcal{O}(X(i,j))^{\mathrm{id} \times f_j}$-module of rank two with basis given by $\{1 \otimes 1, 1 \otimes f_j\}$ and $\mathcal{O}(X(i,j))^{\mathrm{id} \times f_j}$ is canonically isomorphic to R_i (see Lemma 2.4.3). It remains to show that such a map is also a morphism of right R_j-modules. Let $a \in R_j$ act on $1 \otimes 1 \otimes \cdots \otimes 1 \otimes 1$ on the right. Write $a = a_1 + a_2 f_j$ with $a_m \in R_j^{s_j}$, $m = 1, 2$. Write $(a_1)_i$ and $(a_2)_i$ for the corresponding elements of $R_i^{s_i}$ given by Lemma 2.4.2. We then have that the image of $1 \otimes 1 \otimes \cdots \otimes 1 \otimes a_1$ under our map is given by $(a_1)_i \otimes 1$, that is, $((a_1)_i \otimes 1)(v', v) =$

$(a_1)_i(v')$. If $n^i_{v'}$ is even and $v' = w(j, i-1)^{-1}v$ we have

$$(a_1)_i(v') = a_1(w(j, i-1)v') = a_1(v).$$

If $n^i_{v'}$ is odd and $v' = w(j, i-1)^{-1}v$ we have

$$(a_1)_i(v') = a_1(w(j, i-1)s_i v') = a_1(w(j, i-1)s_i w(j, i-1)^{-1}v).$$

But $a_1 \in R_j^{s_j}$ and $s_j w(j, i-1)s_i w(j, i-1)^{-1} = e$, hence we get that $(a_1)_i(v') = a_1(v)$. Similar arguments show that $(a_1)_i(v') = a_i(v)$ in case $v' = s_i w(j, i-1)^{-1}v$. Hence we have proven that

$$\varphi((1 \otimes \cdots \otimes 1) \cdot a_1) = \varphi(1 \otimes \cdots \otimes 1) \cdot a_1.$$

We can give exactly the same arguments for a_2 applied on the right of $1 \otimes 1 \otimes \cdots \otimes 1 \otimes f_j$ to prove that

$$\varphi((1 \otimes \cdots \otimes f_j) \cdot a_2) = \varphi(1 \otimes \cdots \otimes f_j) \cdot a_2.$$

Putting the two equalities together one obtains

$$\varphi((1 \otimes \cdots \otimes 1) \cdot a) = (1 \otimes 1) \cdot a = \varphi(1 \otimes \cdots \otimes 1) \cdot a.$$

Now using the fact that φ is a morphism of left-modules this equality implies that

$$\varphi((f_i \otimes \cdots \otimes 1) \cdot a) = (f_i \otimes 1) \cdot a = \varphi(f_i \otimes \cdots \otimes 1) \cdot a.$$

But since B_w is a free right R_j-module of rank two with basis given by $\{1 \otimes 1 \otimes \cdots 1 \otimes 1\}$ and $\{f_i \otimes 1 \otimes \cdots 1 \otimes 1\}$ we have just shown that φ is also a morphism of right R_j-modules. □

2.4.2 Morphisms between elementary bimodules

Notation. If $i, j \in [1, n]$ are distinct, we write $[f_i, f_j]$ for the element of $\bar{R}$ equal to $f_i + 2f_{i+1} + \cdots + 2f_{j-1} + f_j$ if $j > i$ and $f_i + 2f_{i-1} + \cdots + 2f_{j+1} + f_j$ if $j < i$.

Remark 2.4.5. The element $[f_i, f_j]$ lies in $I(V_i \cap V_j)$ as noticed in [16], 3.13. This can also be proved directly using the elementary properties of Weyl lines from Lemma 2.1.4: if $|i-j| = 1$, then we know that $V_i \cap V_j \subset H_{s_i s_j s_i}$ and in that case $[f_i, f_j] = f_i + f_j$ is an equation of $H_{s_i s_j s_i}$. In case $|i-j| > 1$, consider $v \in V_i \cap V_j$ and assume that $v \in H_t$ where $t = (i+1, j+1)$. This forces $v \in V_r$ for $r = (i, j+1)$ and hence $v \in V_j \cap V_r \subset H_q$ with $q = (i, j)$. Hence $v \in H_t \cap H_q$ which implies that

$\sum_{k=i+1}^{j} f_i(v) = 0 = \sum_{k=i}^{j-1} f_k(v)$ and so $[f_i, f_j](v) = 0$. Now if $v \notin H_t$ one has $v \in V_t$. By similar arguments it implies that $v \in H_r \cap H_{r'}$ with $r' = (i+1, j)$ also giving $[f_i, f_j](v) = 0$.

Proposition 2.4.6. *Let $L \subset Z$ be a Weyl line. The element $f_L \in \bar{R}$ defined by*

$$f_L := \prod_{t, L \subset V_t} f_t$$

is nonzero on any $v \in (L - \{0\})$ but zero everywhere else.

Proof. Let $L' \subset Z$ be a Weyl line different from L. Since L and L' are lines which are intersections of reflecting hyperplanes there must exist a reflection $t \in \mathcal{T}$ such that $L' \subset H_t$ but $L \not\subset H_t$. Hence $L \subset V_t$ and f_t vanishes on L' but not on L. □

Corollary 2.4.7. *Let $\mathcal{A}_L$ be a set of Weyl lines in Z. Let $W := \bigcup_{L \in \mathcal{A}_L} L$. The element $f_W \in \bar{R}$ defined by*

$$f_W := \sum_{L \in \mathcal{A}_L} f_L$$

is nonzero on any $v \in (W - \{0\})$ but zero everywhere else.

Proof. This is an immediate consequence of 2.4.6. □

Example 2.4.8 In type A_2 the Weyl lines are exactly the reflecting hyperplanes, hence there are three Weyl lines given by $H_{s_1}, H_{s_2}, H_{s_1 s_2 s_1}$. Therefore $V_1 = H_{s_2} \cup H_{s_1 s_2 s_1}$. One has

$$f_{V_1} = f_1 f_{s_1 s_2 s_1} + f_1 f_2 = f_1(2f_2 + f_1).$$

Proposition 2.4.9. *Let M, N be two graded $\bar{R}$-bimodules such that M is cyclic. Let V_M, V_N be unions of Weyl lines and assume that M is a free left $\mathcal{O}(V_M)$-module and N is a free left $\mathcal{O}(V_N)$-module. If $\varphi : M \to N$ is a homogeneous morphism of bimodules and if $r \in \mathcal{O}(V_N)$ is such that $r|_{V_N \cap V_M} = 0$, then $r \cdot \varphi = 0$.*

Proof. By assumption $M = \langle g \rangle$ for some $g \in M$. Consider the variety $W := \bigcup_{L, L \not\subset V_M} L$. Then f_W vanishes on V_M by the Corollary 2.4.7. Hence $f_W \cdot \varphi(g) = \varphi(f_W \cdot g) = 0$. Let $r \in \mathcal{O}(V_N)$ be such that $r|_{V_M \cap V_N} = 0$. Let $\mathcal{B}$ be a basis of N as a left $\mathcal{O}(V_N)$-module and write $\varphi(g) = \sum_{b \in \mathcal{B}} \alpha_b \cdot b$ where $\alpha_b \in \mathcal{O}(V_N)$. Now $r \cdot (f_W \cdot \varphi(g)) = 0$ which we rewrite

$$\sum_{b \in \mathcal{B}} (r f_W|_{V_N} \alpha_b) \cdot b = 0.$$

We know that N is free as a left $\mathcal{O}(V_N)$-module. Therefore we have that for each $b \in \mathcal{B}$,

$$r f_W|_{V_N} \alpha_b = 0.$$

By Corollary 2.4.7, f_W does not vanish on a Weyl line L such that $L \subset V_N$ but $L \not\subset V_M$. This forces $r\alpha_b$ to vanish on such lines for each $b \in \mathcal{B}$. But since $r|_{V_N \cap V_M} = 0$, we get that $r\alpha_b = 0$ for each $b \in \mathcal{B}$, hence $r \cdot \varphi(g) = 0$. Since M is cyclic generated by g, it implies that $r \cdot \varphi = 0$. □

Remark 2.4.10. In particular, Proposition 2.4.9 applies to fully commutative bimodules since we know that they are free as left modules over the ring of regular functions on an inductively defined union of Weyl lines (Theorem 2.2.15) and cyclic: to establish Theorem 2.3.19 we showed that a fully commutative bimodule is generated by the unit of its commutative ring structure.

Remark 2.4.11. Let $\mathcal{A} \subset \{1, \ldots, n\}$. Let $W = \bigcap_{i \in \mathcal{A}} V_i$. In ([16], §6, claim 4), a k-basis of $\bar{R}/I(W)$ is given. In particular, a k-basis of R_i is given for each $i \in \{1, \ldots, n\}$. It consists of monomials in the f_j and is indexed by pairs (A, k) where $A \subset \{1, \ldots, n\} \backslash i$ and $k \in \mathbb{Z}_{\geq 0}$. The monomial corresponding to the pair (A, k) is given by

$$M(A, k) = f_i^k \prod_{m \in A} f_m.$$

The idea of Elias is the following: by Remark 2.4.5, $[f_i, f_k] \in I(V_i \cap V_k)$ implying that $f_k[f_i, f_k] \in I(V_i)$. Assume $k > i$, the other case being symmetric. Then

$$f_k[f_i, f_k] = f_k(f_i + 2f_{i+1} + \cdots + 2f_{k-1} + f_k),$$

which implies that a monomial having an f_k^2 as factor can be replaced by a linear combination of monomials with no f_k^2 as factors. Iterating this process one gets rid of all the f_j^2, $j \neq i$. Elias then uses Bergman's Diamond Lemma to conclude.

Proposition 2.4.12. *Let i, j be distant. Then*

$$\mathrm{Hom}(B_j, B_i) \cong \mathcal{O}(V_i \cap V_j)[-4]$$

as graded $\bar{R}$-bimodules, where the generating map is given by

$$1 \otimes 1 \mapsto f_j f_i \otimes 1 + f_j \otimes f_i.$$

In particular, left and right operation of $\bar{R}$ are the same on $\mathrm{Hom}(B_j, B_i)$.

Proof. The three following claims will prove the proposition:

1. Any homogeneous map $\varphi : B_j \to B_i$ sends 1 to $r_\varphi f_i \otimes 1 + r_\varphi \otimes f_i$ where $r_\varphi \in R_i$ is of the form $r'_\varphi f_j$ with $r'_\varphi \in R_i$.

2. The assignment $\varphi(1 \otimes 1) = f_j f_i \otimes 1 + f_j \otimes f_i$ defines a morphism of graded bimodules $B_j \to B_i[4]$.

3. A homogeneous morphism $\varphi : B_j \to B_i$ is zero if and only if $r'_\varphi|_{V_i \cap V_j} = 0$.

Before proving the three claims, let us make a few remarks; since the B_i, $i \in [1, n]$ are cyclic generated by $1 \otimes 1$, any morphism between B_j and B_i is determined by its value at $1 \otimes 1$, whence the formulation of the two first claims. One may wonder why the left and right operations of $\bar{R}$ should coincide on $\mathrm{Hom}(B_j, B_i)$ if the three claims hold; this is true since $f_i \otimes 1 + 1 \otimes f_i$ is the image of $1 \otimes 1$ under the injective map φ_i^+ of the short exact sequence

$$0 \longrightarrow R_i[-2] \xrightarrow{\varphi_i^+} B_i \xrightarrow{\mu_i^-} (R_i)_{s_i} \longrightarrow 0,$$

given in subsection 2.4.1; on $R_i[-2]$, the left and right operations of $\bar{R}$ are obviously the same.

Proof of the first claim. Let $\varphi : B_j \to B_i$ a morphism of bimodules. The set $\{1 \otimes 1, 1 \otimes f_i\}$ is a basis of B_i as left R_i-module; write $\varphi(1 \otimes 1) = q_1 \otimes 1 + q_2 \otimes f_i$ with $q_k \in R_i$, $k = 1, 2$. Since i and j are distant, the equation f_i lies in $R_j^{s_j}$. It implies that $f_i \cdot \varphi(1 \otimes 1) = \varphi(1 \otimes 1) \cdot f_i$ on B_j, hence

$$f_i q_1 \otimes 1 + f_i q_2 \otimes f_i = q_1 \otimes f_i + q_2 f_i^2 \otimes 1,$$

which implies that $q_1 = f_i q_2$ since $\{1 \otimes 1, 1 \otimes f_i\}$ is a basis of B_i as left R_i-module. It remains to show that $q_2 = r' f_j$ for some $r' \in R_i$. Using Proposition 2.4.9 and Remark 2.4.5, $[f_i, f_j] \cdot \varphi(1 \otimes 1) = 0$, which implies that $[f_i, f_j] q_2 = 0$ in R_i. Now using Remark 2.4.11, one can express q_2 as a k-linear combination of monomials M_ℓ of the form

$$M_\ell = f_i^{k_\ell} \prod_{m \in A_\ell} f_m$$

for ℓ in some finite indexing set L, where $A_\ell \subset [1, n]$ are subsets which do not contain f_i. Write $q_2 = \sum_\ell \alpha(\ell) M_\ell$ with $\alpha(\ell)$ a scalar. Since in R_i one has $f_j[f_i, f_j] = 0$ (see Remark 2.4.5), we get

$$0 = q_2[f_i, f_j] = \sum_{\{\ell \in L \ | \ j \notin A_\ell\}} \alpha(\ell) M_\ell [f_i, f_j]$$

We now suppose $i < j$, the argument is similar in the other case. The idea is to rewrite the element above in the basis given by Elias; multiplying a monomial M_ℓ by f_i just gives another basis monomial. Multiplying M_ℓ by a f_k for $i < k < j$ gives another basis monomial in case A_ℓ does not contain the index k. In case $k \in A_\ell$, one gets

$$M'_\ell = f_i^{k_\ell} f_k^2 \prod_{m \in A_\ell, m \neq k} f_m,$$

and in that case one replaces f_k^2 by $-f_k(f_i + 2f_{i+1} + \cdots + 2f_{k-1})$ (see Remark 2.4.11) hence f_k^2 is replaced by $-f_k$ times a sum of f_m for $m < k$ and hence by iterating the process one expresses M'_ℓ as a linear combination of monomials of the basis which do not have f_j as a factor. As a consequence the only monomials occurring when writing $q_2[f_i, f_j]$ in the basis which do have f_j as a factor are the $M_\ell f_j$ with $j \notin A_\ell$, and therefore one has $\alpha(\ell) = 0$ if ℓ is such that $j \notin A_\ell$, which proves that $q_2 = f_j r'$ for some $r' \in R_i$. □

Proof of the second claim. Recall the identification of B_i with the algebra of regular functions on $\mathrm{Gr}_i^R(e, s_i)$ from Lemma 2.4.1. We have to show that if $a, b \in R_j$, $r \in R_j^{s_j}$ and a', b', r' are preimages of a, b, r in $\bar{R}$, then

$$(a'r')|_{V_i}\varphi(1 \otimes 1)b'|_{V_i} = a'|_{V_i}\varphi(1 \otimes 1)(r'b')|_{V_i}$$

and that it is independent of the chosen preimages a', b', r' of a, b, r. The fact that it is independent of the chosen preimages is a consequence of the fact that

$$\varphi(1 \otimes 1) = f_j f_i \otimes 1 + f_j \otimes f_i = f_i \otimes f_j + 1 \otimes f_i f_j$$

together with Lemma 2.4.1: indeed, if we identify B_i with the ring of regular functions on $\mathrm{Gr}_i^R(e, s_i)$, we have that $\varphi(1 \otimes 1)(w, v) = 0$ whenever w or $v \notin V_j$. Moreover, using the fact that the left and right operations on $\varphi(1 \otimes 1)$ are the same, we get the equality above. This proves that φ is a well-defined morphism of graded bimodules. □

Proof of the third claim. Suppose that $\varphi : B_j \to B_i$ is zero. Then $\varphi(1 \otimes 1) = r'_\varphi(f_i f_j \otimes 1 + f_j \otimes f_i) = 0$ which forces $r'_\varphi f_j = 0$. This forces $r'_\varphi|_{V_i \cap V_j} = 0$. The converse is given by Proposition 2.4.9. □

□

Proposition 2.4.13. *Let i, j be adjacent. Then*

$$\mathrm{Hom}(B_j, B_i) \cong \mathcal{O}(V_i \cap V_j)[-4]$$

as graded $\bar{R}$-bimodules, where the generating map is given by

$$1 \otimes 1 \mapsto f_j f_i \otimes 1 + f_j \otimes f_i.$$

In particular, left and right operation of $\bar{R}$ are the same on $\mathrm{Hom}(B_j, B_i)$.

Proof. The proof is almost the same as for distant i, j (Proposition 2.4.12) since all the argument work for adjacent indices except one: the only step which needs a different proof in the adjacent case is the fact used at the beginning of the proof of claim 1 that if $\varphi : B_j \to B_i$ is a morphism of bimodules with $\varphi(1\otimes 1) = q_1 \otimes 1 + q_2 \otimes f_i$, $q_i \in R_i$, then $q_1 = f_i q_2$. Since $f_i = f_i + \frac{f_j}{2} - \frac{f_j}{2}$ and $f_i + \frac{f_j}{2}$ is s_j-invariant one has in B_j that

$$f_i \cdot 1 \otimes 1 = 1 \otimes 1 \cdot \left(f_i + \frac{f_j}{2}\right) - \frac{f_j}{2} \cdot 1 \otimes 1.$$

Applying φ and expressing the right hand side in the basis $\{1 \otimes 1, 1 \otimes f_i\}$ of B_i as left R_i-module one gets using the fact that B_i is free of rank 2 and comparing the coefficients that $q_1 = f_i q_2$. □

Proposition 2.4.14. *One has an isomorphism*

$$\mathrm{Hom}(B_i, B_i) \cong B_i$$

as graded $\bar{R}$-bimodules.

Proof. The assignment $\varphi(1 \otimes 1) = a$ for any $a \in B_i$ clearly defines a morphism of bimodules $B_i \to B_i$. Conversely, any morphism $B_i \to B_i$ is given by its value at $1 \otimes 1$. □

Putting 2.4.12, 2.4.13 and 2.4.14 together we get:

Theorem 2.4.15. *Let $i, j \in [1, n]$. One has*

$$\mathrm{Hom}(B_j, B_i) \cong \begin{cases} \mathcal{O}(V_i \cap V_j)[-4] & \text{if } i \neq j \\ B_i & \text{if } i = j. \end{cases}$$

Remark 2.4.16. In Soergel category, morphisms between B_{s_i} and B_{s_j} for $s_i \neq s_j$ are generated in degree 2 by the composition

$$R \otimes_{R^{s_i}} R \twoheadrightarrow R \hookrightarrow R \otimes_{R^{s_j}} R[2],$$

where the surjective map is given by multiplication and the injective map by $r \mapsto r f_j \otimes 1 + r \otimes f_j$. In our case, B_i surjects to R_i, hence another degree two map must occur in the composition to obtain the generating map decribed in Propositions 2.2.12 and 2.4.13:

$$R_i \otimes_{R^{s_i}} R_i \overset{\mu_i^+}{\twoheadrightarrow} R_i \longrightarrow \bar{R}[2] \twoheadrightarrow R_j[2] \overset{\varphi_j^+}{\hookrightarrow} R_j \otimes_{R_j^{s_j}} R_j[4],$$

where the map $R_i \longrightarrow \bar{R}[2]$ is given by multiplication by f_i and the map $\bar{R}[2] \twoheadrightarrow R_j[2]$ is the natural quotient map. Notice that R_i and R_j do not lie in $\mathcal{B}_{TL_n}$.

As a consequence the degrees of the morphisms are sometimes different from the degrees of the analogous morphisms in Soergel category and as a consequence, they are sometimes different from the degrees of the corresponding morphisms in Elias category (see [16]) which is a quotient of Soergel category.

Chapter 3

Combinatorics of Zinno basis

During the whole chapter, $(\mathcal{W}, \mathcal{S})$ will denote a Coxeter system of type A_n where $\mathcal{W}$ is identified with the symmetric group $\mathfrak{S}_{n+1}$ on $n+1$ letters and $\mathcal{S}$ is identified with $\{s_1, \ldots, s_n\}$, where s_i is the transposition $(i, i+1)$. We write $\mathcal{T}$ for the set of reflections or transpositions of $\mathcal{W}$. We denote by c the Coxeter element $c = s_1 s_2 \cdots s_n$ and by c' any Coxeter element, that is, any product of all the elements of $\mathcal{S}$.

We will write B_{n+1} or simply B for the braid group of type A_n, that is, the braid group on $n+1$ strands. Recall that it has the following presentation

$$B = \left\langle \mathbf{s}_1, \ldots, \mathbf{s}_n \;\middle|\; \begin{array}{c} \mathbf{s}_i \mathbf{s}_{i+1} \mathbf{s}_i = \mathbf{s}_{i+1} \mathbf{s}_i \mathbf{s}_{i+1}, \quad \forall i \in \{1, \ldots, n-1\} \\ \mathbf{s}_i \mathbf{s}_j = \mathbf{s}_j \mathbf{s}_i, \quad \text{if } |i-j| > 1. \end{array} \right\rangle.$$

3.1 Geometry of noncrossing partitions for arbitrary Coxeter elements

Recall the geometrical representation of a noncrossing partition $x \in \mathcal{P}_c$ from subsection 1.4.3. If one considers an arbitrary Coxeter element c', there is still a geometrical representation of elements of $\mathcal{P}_{c'}$ as disjoint unions of polygons having as vertices marked points on a circle; what changes is the order of the integers labelling the points. To get the ordering, one needs to express the Coxeter element as an $(n+1)$-cycle beginning by 1, that is, in the form $c' = (i_1, i_2, \dots, i_{n+1})$ with $i_1 = 1$. On then labels the points on the circle in clockwise order by the integers $1 = i_1, i_2, \dots, i_{n+1}$. Since we are working with the fixed simple system $\mathcal{S} = \{s_i\}_{i=1}^n$, any $(n+1)$-cycle does not yield a Coxeter element; the Coxeter elements obtained are given by the following two lemmas. One can find an example of a labelling of the points on the circle in figure 3.1.

Lemma 3.1.1. *If c' is a Coxeter element and $(i_1, \dots, i_{n+1})$ is the corresponding cycle with $i_1 = 1$, $i_k = n+1$, then $1 = i_1 < i_2 < \cdots < i_k$ and $1 = i_1 < i_{n+1} < i_n < \cdots < i_{k+1} < n+1$.*

Proof. We write $\sigma_{c'}$ for the permutation corresponding to c'. Since $\sigma_{c'}(1) = i_2$, s_j must occur before s_{j-1} in any $\mathcal{S}$-reduced expression of c' for all $j \in \{2, 3, \dots, i_2 - 1\}$. Moreover, if $k \neq 2$, s_{i_2} cannot occur before s_{i_2-1} because it would contradict $\sigma_{c'}(1) = i_2$. Hence $k \neq 2$ implies that s_{i_2} is after s_{i_2-1}; but these two reflections are the only elements of $\mathcal{S}$ which do not fix i_2. It implies that $i_3 = \sigma_{c'}(i_2) > i_2$. Iterating this process gives $1 = i_1 < i_2 < \cdots < i_k$. A similar argument gives the second sequence of inequalities. ☐

Notation. Let c' be a Coxeter element. We set $R_{c'} := \{i_1, i_2, \dots, i_k\}$ and $L_{c'} := \{i_k, i_{k+1}, \dots, i_n, i_{n+1}, i_1\}$, where the i_j's are given by Lemma 3.1.1. In particular, $L_{c'} \cup R_{c'} = \{1, 2, \dots, n+1\}$ and $L_{c'} \cap R_{c'} = \{1, n+1\}$.

Lemma 3.1.2. *Let $(i_1, \dots, i_{n+1})$ be an $(n+1)$-cycle with $i_k = n+1$ such that $1 = i_1 < i_2 < \cdots < i_k = n+1$ and $1 = i_1 < i_{n+1} < i_n < \cdots < i_{k+1} < i_k = n+1$. Then $(i_1, \dots, i_{n+1})$ is a Coxeter element.*

Proof. We argue by induction on n. If $n = 2$, then such a cycle is either equal to $(1, 2, 3)$ or $(1, 3, 2)$. The first one is $s_1 s_2$ and the second one $s_2 s_1$. Now consider the cycle $(i_1, \dots, i_{n+1})$. If the inequalities of the lemma are true, then either $i_2 = 2$ or $i_{n+1} = 2$. If $i_2 = 2$ then $s_1(i_1, \dots, i_{n+1}) = (i_2, \dots, i_{n+1})$ and the result follows by applying the induction hypothesis to the n-cycle $(i_2, \dots, i_{n+1})$ in the parabolic subgroup W_I where $I = \{s_2, \ldots, s_{n+1}\}$. If $i_{n+1} = 2$ then $(i_1, \ldots, i_{n+1})s_1 = (i_2, \ldots, i_{n+1}) = (i_{n+1}, i_2, \ldots, i_n)$ and we can apply the induction hypothesis to the n-cycle $(i_{n+1}, i_2, \ldots, i_n)$ in the parabolic subgroup W_I. □

Remark 3.1.3. As a consequence, it follows that for any $k \in \{1, \ldots, n+1\}$ there exists a line containing the point labeled with k and cutting the circle with marked points corresponding to the Coxeter element c' such that the set of points on the circle labeled with $E_{\leq k} := \{i \in \{1, \ldots, n+1\} \mid i \leq k\}$ lies in one of the half-plane defined by that line and the set of points labeled with $E_{\geq k} := \{i \in \{1, \ldots, n+1\} \mid i \geq k\}$ lies in the other half-plane. For example, the points labeled with the elements of $E_{\leq 4}$ are drawn in white in figure 3.1. In that case the line contains the point labeled with 4 and a point on the circle lying between the point labeled with 2 and the point labeled with 5.

Lemmas 3.1.1 and 3.1.2 tell us that an $(n+1)$-cycle is a Coxeter element c' if and only if when writing the cycle $(i_1 \ \ldots \ i_{n+1})$ with $i_1 = 1$, all the i_j are increasing between $i_1 = 1$ and $i_k = n+1$ and decreasing between $i_k = n+1$ and i_{n+1}. The consequence for the geometry of noncrossing partitions and simple elements of the dual braid monoid associated to c' is that the indices which label the points on the circle are increasing when reading them from the point labeled with 1 to the one labeled with $n+1$ in clockwise order and decreasing when reading them from the point labeled with $n+1$ to the one labeled with 1 in clockwise order. Conversely any such choice of indices labelling the points on the circle corresponds to some Coxeter element. As a consequence, if one draws a path from 1 to $n+1$ by drawing a line segment between any two points with indices j, $j+1$, one obtains a path without crossings (that is, a zigzag) between the point with index 1 and the one with index $n+1$. The segment between the point indexed by j and the one indexed by k corresponds to the transposition (j, k). One can look at the picture on the left in Figure 3.1 for such a labelling. It turns out that it will be more convenient for many proofs to slighlty change the geometric representation (except in case the Coxeter element is $s_1 s_2 \cdots s_n$) as follows: instead of drawing the points on the circle such that the length of any arc between two successive points is the same, we will draw the point labeled by 1 at the top of the circle, the points labeled with $n+1$ at the bottom, the points with index in $L_{c'}$ on the left and the points with index in $R_{c'}$ on the right, each point having a specific heigth depending on its index. If P_1, P_2 are two points with index i_1, i_2 such that $i_1 < i_2$, then the height of P_1 will be bigger than the height of P_2. We represented the new way the points are drawn on the picture on the right in Figure 3.1. In case the Coxeter element is $s_1 s_2 \cdots s_n$, since all the points would be on the right, we keep the traditional geometric representation that we used before, except in case we want to compare a situation involving a noncrossing partition in $\mathcal{P}_c$ with another situation involving a noncrossing partition in $\mathcal{P}_{c'}$. Since points may be very close to each other in the new geometric representation, in case we represent a noncrossing partition we may use curvilinear polygons instead or regular polygons for a more comfortable reading.

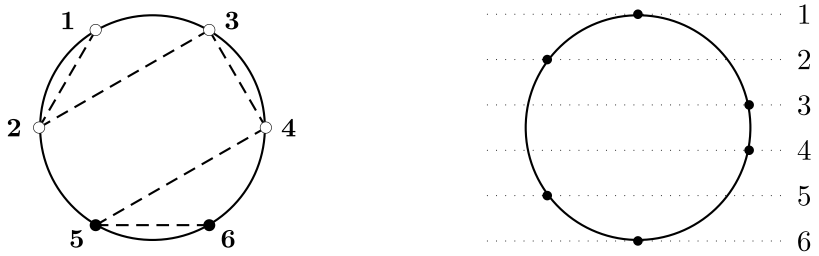


FIG. 3.1: Example of a labelling of the vertices given by a Coxeter element c'; here $c' = s_2s_1s_3s_5s_4 = (1,3,4,6,5,2)$. When going from the point labeled with 1 to the one labeled with 6 one obtains a noncrossing zigzag.

To summarize, there are bijections

$$\left\{ \begin{array}{c} \text{Coxeter} \\ \text{elements} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{Orientations} \\ \text{of the} \\ \text{Dynkin diagram} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{Ordered pairs } (C_1, C_2) \text{ of} \\ \text{disjoint subsets of } \{2, \ldots, n\} \\ \text{s.t. } C_1 \dot{\cup} C_2 = \{2, \ldots, n\} \end{array} \right\}$$

where we use the convention that the set C_1 is the set of integers that label points on the circle lying strictly between $n+1$ and 1 (in clockwise order), and C_2 the is the set of integers that label points lying strictly between 1 and $n+1$. Notice that the sets C_1 and C_2 corresponding to a Coxeter element c' are given by $L_{c'} = C_1 \dot{\cup} \{1, n+1\}$, $R_{c'} = C_2 \dot{\cup} \{1, n+1\}$.

3.2 Bijections between noncrossing partitions and fully commutative elements

3.2.1 Noncrossing partitions and fully commutative elements

The set $\mathcal{W}_f$ of fully commutative elements of $\mathcal{W}$ which indexes the diagram (or Kazhdan-Lusztig) basis of the corresponding Temperley-Lieb algebra from chapter 2 has Catalan enumeration. Recall that any fully commutative element $w \in \mathcal{W}_f$ can be written uniquely in the form

$$(s_{i_\ell}s_{i_\ell-1}\cdots s_{j_\ell})(s_{i_{\ell-1}}s_{i_{\ell-1}-1}\cdots s_{j_{\ell-1}})\cdots(s_{i_1}s_{i_1-1}\cdots s_{j_1})$$

with all indices in $\{1,\ldots,n\}$ and $i_\ell < i_{\ell-1} < \cdots < i_1$, $j_\ell < j_{\ell-1} < \cdots < j_1$ and $j_m \leq i_m$ for each $m = 1,\ldots,\ell$. Conversely, any such expression is an $\mathcal{S}$-reduced expression of an element of $\mathcal{W}_f$ (see Proposition 1.1.2).

Notation. We write I_w for the set $\{i_1, i_2, \ldots, i_\ell\}$ and J_w for the set $\{j_1, j_2, \ldots, j_\ell\}$.

Remark 3.2.1. Notice that $i \in I_w$ if and only if in any $\mathcal{S}$-reduced expression of w, there is no occurrence of s_{i+1} before the first occurrence of s_i. Similarly, one has that $i \in J_w$ if and only if in any $\mathcal{S}$-reduced expression of w, there is no occurrence of s_{i-1} after the last occurrence of s_i.

The set of noncrossing partitions $\mathcal{P}_c$ which also has Catalan enumeration turns out to index another basis of that algebra which we call the *Zinno basis*, obtained by considering the images in the Temperley-Lieb algebra of the simple elements of the dual braid monoid, which are lifts of elements of $\mathcal{P}_c$ (see section 1.4). We will introduce the basis in the next sections and more details can be found in [42] and [31]. We will use extensively the geometric representation of elements of $\mathcal{P}_c$ (and of their lifts in the braid group) given in section 1.4.3. It turns out that if one takes an arbitrary Coxeter element c', one gets a basis given by the images of the simple elements of the dual braid monoid associated to c' in the Temperley-Lieb algebra as shown by Vincenti ([41]), generalizing Zinno's Theorem. At the very end of chapter 3, we will provide a new proof of this fact. We will sometimes refer to the *generalized Zinno basis* in case we are working with an arbitrary Coxeter element.

Zinno's strategy is as follows: he gives in [42] a bijection $a : \mathcal{P}_c \to \mathcal{W}_f$ as well as a partial ordering on $\mathcal{P}_c$ such that there exists an upper triangular matrix (with respect to any linear extension of his partial order on $\mathcal{P}_c$ and the order induced on $\mathcal{W}_f$ by a) allowing one to pass from the basis $\{b_w\}_{w\in\mathcal{W}_f}$ to the set of images of the simple elements. However, his approach does not allow a direct generalization to arbitrary Coxeter elements as well as an explicit description of the inverse bijection. This last aspect should be a first step towards a better understanding of the change of basis matrix between the diagram basis and the Zinno basis.

The aim of the next subsections is therefore to introduce for any Coxeter element c' a bijection

$$\varphi_{c'} : \mathcal{P}_{c'} \to \mathcal{W}_f$$

with an explicit description of the inverse bijection

$$\psi_{c'} : \mathcal{W}_f \to \mathcal{P}_{c'}$$

such that in case $c' = c$, one s Zinno's bijection, that is, one has $\varphi_c = a$.

3.2.2 Bijections generalizing Zinno's bijection

Notation. Let $k \in \mathbb{Z}_{\geq 0}$. We denote by $\mathcal{I}_k$ the set of pairs (D, U) where $D = \{d_1, d_2, \ldots, d_k\}$, $U = \{e_1, e_2, \ldots, e_k\}$, $e_i, d_i \in \{1, \ldots, n+1\}$, $d_i < d_{i+1}$, $e_i < e_{i+1}$ for each $1 \leq i < k$, $d_i < e_i$ for each $1 \leq i \leq k$. Set $\mathcal{I} := \coprod_{k=0}^{n} \mathcal{I}_k$.

Remark 3.2.2. There is a bijection $\mathcal{W}_f \to \mathcal{I}$ given by $w \mapsto (J_w, I_w + 1)$. This is just a reformulation of the fact that we recalled at the beginning of subsection 3.2.1.

Notation. Let $x \in \mathcal{P}_{c'}$. We denote by $\mathrm{Pol}(x)$ the set of polygons appearing in the geometrical representation of x and by $\mathrm{Vert}(x)$ the set of integers indexing the vertices of the elements of $\mathrm{Pol}(x)$. We ofter abuse notation and write $k \in P$ to mean that k indexes a vertex of $P \in \mathrm{Pol}(x)$.

Definition 3.2.3. *Given any $P \in \mathrm{Pol}(x)$ with set of integers indexing its vertices given by $\{d_1, \ldots, d_k\}$ where $d_i < d_{i+1}$, we say that d_1 is the* initial *index of P or an initial index of x. We say that d_k is the* terminal *index of P or a terminal index of x. If we do not want to write down the set of indexing vertices we simply write* $\min P$ *for the initial index of P and* $\max P$ *for the terminal index of P. We say that an integer $\ell \in \{2, \ldots, n\}$ is* nested *in $P \in \mathrm{Pol}(x)$ if $\ell \notin P$ but $\min P < k < \max P$. Notice that it does not imply that $\ell \notin \mathrm{Vert}(x)$ since one may have $\ell \in Q$ for $Q \in \mathrm{Pol}(x)$, $Q \neq P$.*

Example 3.2.4 In the example of figure 3.2, the integer 4 is nested in P_1 and P_2. The integer 3 is nested only in P_2.

Let $x \in \mathcal{P}_{c'}$. Write $D_x^{c'}$ for the set of indices in $\mathrm{Vert}(x)$ that are not terminal and $U_x^{c'}$ for the set of indices in $\mathrm{Vert}(x)$ that are not initial. In particular we have the equality $|D_x^{c'}| = |U_x^{c'}|$.

Lemma 3.2.5. *Let $x \in \mathcal{P}_{c'}$. Let $I_x^{c'} := D_x^{c'} \backslash (D_x^{c'} \cap U_x^{c'})$ be the set of initial indices and $T_x^{c'} := U_x^{c'} \backslash (D_x^{c'} \cap U_x^{c'})$ be the set of terminal indices. Then*

$$(I_x^{c'}, T_x^{c'}) \in \mathcal{I}.$$

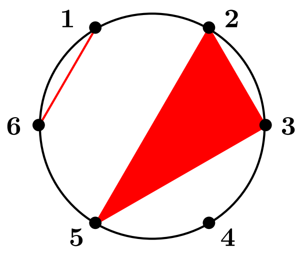


FIG. 3.2: Geometric representation of $x = (1,6)(2,3,5)$. We have two polygons $P_1 = [235]$ and $P_2 = [16]$.

Proof. Indeed, one can argue by induction on the number of polygons of x. If x has no polygon then $x = e$ and the claim is trivially true since $I_x^{c'} = T_x^{c'} = \emptyset$. Now assume that x has at least one polygon. Consider the polygon P of x having the biggest minimal index and remove P from x. It gives a noncrossing partition y for which $(I_y^{c'}, T_y^{c'}) \in \mathcal{I}$ by induction. We have that $(I_x^{c'}, T_x^{c'}) = (I_y^{c'} \cup \{\min P\}, T_y^{c'} \cup \{\max P\})$. Since $\max P > \min P$ and $\min P$ is the biggest index in $I_x^{c'}$, if we add $\max P$ to the set $T_y^{c'}$ we still get that after ordering, the i^{th} index in $I_x^{c'}$ is smaller than the i^{th} index in $T_x^{c'} = T_y^{c'} \cup \{\max P\}$ which is exactly saying that $(I_x^{c'}, T_x^{c'}) \in \mathcal{I}$. □

Lemma 3.2.6. *Let c' a Coxeter element and consider the circle with marked points and labelling of the points given by c'. Let $(D, U) \in \mathcal{I}$ with the additionnal property that $D \cap U = \emptyset$. There is a unique bijection $f : D \to U$ such that any two segments in the collection of segments joining the point with index $i \in D$ to the point with index $f(i) \in U$ are noncrossing.*

Proof. By definition of $\mathcal{I}$, one has $|D| = |U|$ and if we order the sets as $D = \{d_1, \ldots, d_k\}$, $U = \{u_1, \ldots, u_k\}$, such that $d_j < d_{j+1}$, $u_j < u_{j+1}$ for each $1 \leq j < k$, we have that $d_j < u_j$ for any $1 \leq j \leq k$ since $(D, U) \in \mathcal{I}$.

We argue by induction on k. If $k = 0$, there is nothing to prove. Now assume that $k \geq 1$. The point labeled with d_k must be joined to a point labeled by an element $u_j \in U$ such that $d_k < u_j$. But there is only one possible such u_j if we want at the end to obtain a noncrossing family: if $d_k \in L_{c'}$, then u_j is the element of U labelling the first point with index in U which is met when going along the circle from d_k in counterclockwise order. If $d_k \in R_{c'}$, then u_j if the element of U labelling the first point with index in U which is met when going along the circle from d_k in clockwise order. Indeed, if the point labeled with d_k joins the point labeled with another element $u_m \in U$, then the segment joining these two points defines two half-planes, one containing at least one point u_j labeled with an element of U but no point labeled with an element of D: this is a consequence of the fact that d_k is the biggest index of D. One half-plane contains exclusively points labeled with indices bigger than $\min(d_k, u_m) = d_k$ and since u_m is not the point of U bigger than d_k and as close as possible to it, there must be a point labeled with an element u_j of U in that halplane. But such a point must be joined to a point labeled with an index in $D\backslash\{d_k\}$ which contradicts the noncrossing property since the points labeled by numbers in $D\backslash\{d_k\}$ do not lie in the same half-plane as u_j. Hence d_k must be joined to u_j and in one of the halplanes defined by the segment (d_k, u_j), there is no point having as an element of $D \cup U$. Now by induction if we remove d_k from D and u_j from U, we get two sets D' and U' and if we order them, the element in the i^{th} position in D' is still smaller than the element in the i^{th} position in U' since we removed the biggest index of D. Hence $(D', U') \in \mathcal{I}$.

Hence by induction there exists a unique bijection $f' : D' \to U'$ with the required property and since all the points labeled with $D' \cup U'$ lie in the same halplane defined by the segment (d_k, u_j), this segment does not cross the family of segments obtained by induction. Hence $f : D \to U$ is defined by $f(i) = f'(i)$ if $i \in D'$, $f(d_k) = u_j$. $\square$

Definition 3.2.7. *Consider the circle with marked points labeled with integers corresponding to the Coxeter element c'. Let (m_i, n_i) be a collection of pairwise noncrossing segments between points labeled with n_i and m_i on that circle such that if $i \neq j$, $\{n_i, m_i\} \cap \{n_j, m_j\} = 0$. If $k \in \{1, \ldots, n+1\}$, we say that k is* exposed *to the segment (m_j, n_j) if the segment joining the point labeled with k to the point labeled with m_j (or equivalently n_j) does not cross any segment of our family.*

Lemma 3.2.8. *If $x, y \in \mathcal{P}_{c'}$ and $x \neq y$, then $(D_x^{c'}, U_x^{c'}) \neq (D_y^{c'}, U_y^{c'})$.*

Proof. Given $(D := D_x^{c'}, U := U_x^{c'})$, one recovers the set I_x of initial (resp. the set T_x of terminal) indices by $D\backslash(D \cap U)$ (resp. $U\backslash(D \cap U)$). One has $(I_x, T_x) \in \mathcal{I}$ by Lemma 3.2.5 and $I_x \cap T_x = \emptyset$. Using Lemma 3.2.6, there is a unique bijection $f : I_x \to T_x$ such that any two segments in the collection $\{(a, f(a))\}_{a \in I_x}$ are noncrossing (when drawn on the circle with the ordering corresponding to c'). Hence if x, y were distinct but such that $(D_x^{c'}, U_x^{c'}) = (D_y^{c'}, U_y^{c'})$, they would have the same family of longest segments of polygons (we call *segment* of a polygon an edge or diagonal of the polygon). It remains to show that for any $j \in D \cap U$, there is a unique possible polygon (given here by its longest segment) which can have the point indexed by j as vertex. If this property fails for an index $j \in D \cap U$, then j should be exposed to (at least) two noncrossing segments $(d_1, u_1) \neq (d_2, u_2)$ of our family such that $d_k < j < u_k$, $k = 1, 2$. But the set $E_{\leq j-1} = \{m \in \{1, \ldots, n+1\} \mid m < j\}$ consists of points labelling vertices that are successive on the circle thanks to Remark 3.1.3. The same holds for the set $E_{\geq j+1} = \{m \in \{1, \ldots, n+1\} \mid m > j\}$. Notice that $d_k \in E_{\leq j-1}$ and $u_k \in E_{\geq j+1}$. Both segments (d_1, u_1) and (d_2, u_2) join a point labeled with an index in $E_{\leq j-1}$ a point labeled with an index in $E_{\geq j+1}$. Therefore any index

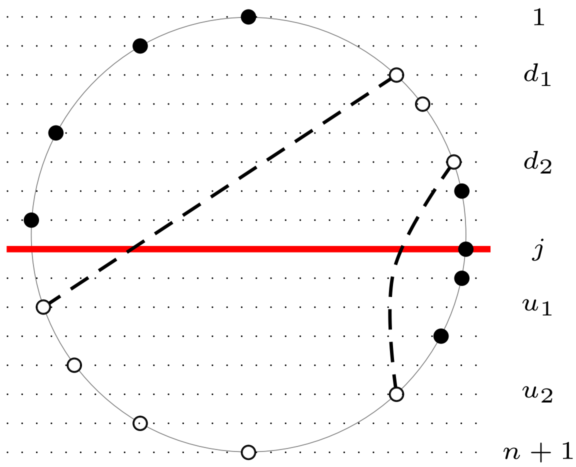


FIG. 3.3: Illustration of the proof of 3.2.8. When going from d_2 to d_1 with the orientation such that the points u_1 and u_2 are not met (here in counterclockwise order), all the points must have their index in $E_{\leq j-1}$. Similarly all the points that are met when going from u_2 to u_1 (with the good orientation) must have their index in $E_{\geq j+1}$. But j is neither in $E_{\leq j-1}$ nor in $E_{\geq j+1}$, hence it must label one of the remaining black points (notice that some of the black points may also lie in $E_{\leq j-1}$ or in $E_{\geq j+1}$). As a consequence, it cannot be exposed to both (d_1, u_1) and (d_2, u_2).

exposed to the two segments either has its index in $E_{\leq j-1}$ or in $E_{\geq j+1}$ (see figure 3.3). This is a contradiction since j is assumed to be exposed to both segments but lies neither in $E_{\leq j-1}$ nor in $E_{\geq j+1}$. □

Set $Y_x^{c'} =: D_x^{c'} \cap U_x^{c'} \cap L_{c'}$ and write $N_x^{c'}$ for the set of indices which lie in $L_{c'}$ but not in $\mathrm{Vert}(x)$ and are nested in at least one polygon of x. In particular $Y_x^{c'} \cap N_x^{c'} = \emptyset$. Consider the two modified sets

$$((D_x^{c'} \backslash Y_x^{c'}) \cup N_x^{c'}, (U_x^{c'} \backslash Y_x^{c'}) \cup N_x^{c'}).$$

For an example the reader can look at 3.2.16.

We define an involution $^- : \mathcal{P}_{c'} \to \mathcal{P}_{c'}$ as follows: given $x \in \mathcal{P}_{c'}$, the sets $Y_x^{c'}$ and $N_x^{c'}$ are disjoint. Both are subsets of $L_{c'}$; the first one contains those which are non terminal indices of polygons while the second one contains those which are nested indices and which moreover do not label a vertex of a polygon. Given the family of longest segments of polygons of x, we have that any index in $N_x^{c'}$ is nested in at least one segment and as seen in the proof of Lemma 3.2.8, it cannot be exposed to more than one segment in which it is nested; therefore it is exposed

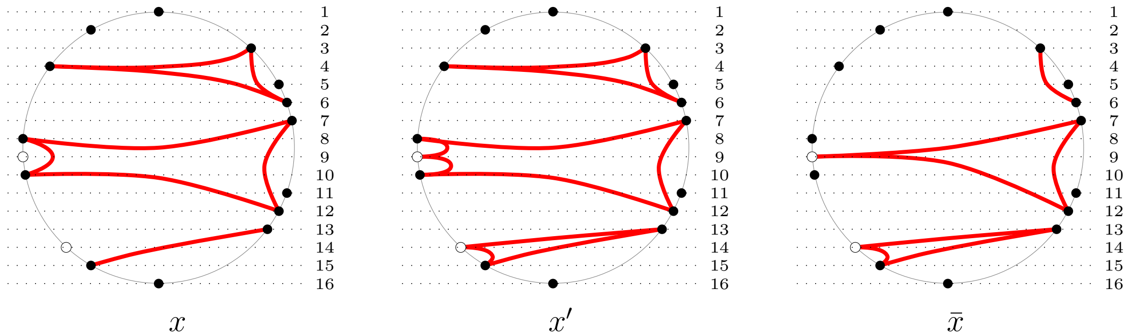


FIG. 3.4: An example of the involution $x \mapsto \bar{x}$. The points with index in $N_x^{c'}$ are drawn in white; it is the points indexed by 9 and 14. The points in $Y_x^{c'}$ are those indexed by 4, 8 and 10.

to exactly one segment in which it is nested. It implies that for any $k \in N_x^{c'}$, there is a unique polygon $P \in \mathrm{Pol}(x)$ which can be enlarged in a polygon P' with one more vertex, namely the vertex indexed by k, and such that the noncrossing property stays satisfied. By adding any vertex indexed by a number in $N_x^{c'}$ to the unique possible polygon, we obtain a noncrossing partition x' such that $N_{x'}^{c'} = \emptyset$. We then consider the noncrossing partition $\bar{x}$ obtained from x' by removing from any polygon $P \in \mathrm{Pol}(x')$ the vertices with index in $Y_x^{c'}$, that is, the non extremal indices of polygons of x which moreover lie in $L_{c'}$. These indices become nested in polygons of $\bar{x}$ and they do not lie in $\mathrm{Vert}(x)$. In fact, we have $N_{\bar{x}}^{c'} = Y_x^{c'}$. We also have that $Y_{\bar{x}}^{c'} = N_x^{c'}$. Nothing changes for the indices in $R_{c'}$. Therefore we get an involution $x \mapsto \bar{x}$ on $\mathcal{P}_{c'}$ such that morevoer

$$(D_{\bar{x}}^{c'}, U_{\bar{x}}^{c'}) = ((D_x^{c'} \backslash Y_x^{c'}) \cup N_x^{c'}, (U_x^{c'} \backslash Y_x^{c'}) \cup N_x^{c'}).$$

As a consequence we have that $(D_x^{c'}, U_x^{c'}) \in \mathcal{I}$ for any $x \in \mathcal{P}_{c'}$ if and only if

$$((D_x^{c'} \backslash Y_x^{c'}) \cup N_x^{c'}, (U_x^{c'} \backslash Y_x^{c'}) \cup N_x^{c'}) \in \mathcal{I}$$

for any $x \in \mathcal{P}_{c'}$.

An example of the involution $x \mapsto \bar{x}$ is given in figure 3.4.

Lemma 3.2.9. *The pair of sets $(D_x^{c'}, U_x^{c'})$ lies in $\mathcal{I}$.*

Proof. We write $D_x^{c'} = \{d_1, \ldots, d_k\}$ where $d_i < d_{i+1}$ and $U_x^{c'} = \{u_1, \ldots, u_k\}$, where $u_i < u_{i+1}$ for any $1 \leq i < k$. By definition, $N_x^{c'}$ is disjoint from both $D_x^{c'}$ and $U_x^{c'}$. The set $D_x^{c'} \backslash (D_x^{c'} \cap U_x^{c'})$ is the set of initial indices of x while the set $U_x^{c'} \backslash (D_x^{c'} \cap U_x^{c'})$ is the set of terminal indices. Set $(I_x^{c'}, T_x^{c'}) := (D_x^{c'} \backslash (D_x^{c'} \cap U_x^{c'}), U_x^{c'} \backslash (D_x^{c'} \cap U_x^{c'}))$. We have that $(I_x^{c'}, T_x^{c'}) \in \mathcal{I}$ thanks to Lemma 3.2.5.

Let $D_x^{c'} \cap U_x^{c'} := \{n_1, n_2, \dots, n_\ell\}$, where $n_i < n_{i+1}$ if $1 \leq i < \ell$. We show by induction on j that $(I_x^{c'} \cup \{n_1, \dots, n_j\}, T_x^{c'} \cup \{n_1, \dots, n_j\}) \in \mathcal{I}$. If $j = 0$ we already know that $(I_x^{c'}, T_x^{c'}) \in \mathcal{I}$ by Lemma 3.2.5.

Assume that $(I_x^{c'} \cup \{n_1, \dots, n_{j-1}\}, T_x^{c'} \cup \{n_1, \dots, n_{j-1}\}) \in \mathcal{I}$. Write x_i for the elements of the first set after ordering (that is, $x_i < x_{i+1}$ for all i) and x'_i for the elements of the second after ordering, in particular $x_i < x'_i$ for any i since the pair of sets lies in $\mathcal{I}$. Since n_j is nested in at least one polygon of x, there exists at least one pair $(x_p, x'_m) \in (I_x^{c'} \cup \{n_1, \dots, n_{j-1}\}) \times (T_x^{c'} \cup \{n_1, \dots, n_{j-1}\})$ such that $x_p < n_j < x'_m$ (recall that $I_x^{c'}$ is the set of initial indices and $T_x^{c'}$ the set of terminal indices). Consider the pair satisfying such a property with p maximal and m minimal. We therefore have that $|\{r \mid x_r < n_j\}| = p$. Since $(I_x^{c'} \cup \{n_1, \dots, n_{j-1}\}, T_x^{c'} \cup \{n_1, \dots, n_{j-1}\}) \in \mathcal{I}$ we have that $|\{r \mid x'_r < n_j\}| \leq p$. Indeed, otherwise we would have an index q for which $x'_q < x_q$, contradicting our assumption. If it is an equality, it implies that

$$|\{d \in I_x^{c'} \mid d < n_j\}| = |\{d \in T_x^{c'} \mid d < n_j\}|,$$

which means that the number of initial indices smaller that n_j is equal to the number of terminal indices smaller than n_j. We claim that it is a contradiction with the fact that n_j is by definition nested in a polygon of x. Indeed, if we draw the line segment separating the points with index smaller than or equal to n_j from the points with index bigger than or equal to n_j (see Remark 3.1.3), then such a line should meet any polygon in which n_j is nested and there is by assumption at least one. Such a polygon has its initial index smaller than n_j while its terminal index is bigger than n_j. Any polygon in which n_j is not nested has its initial and terminal indices both either bigger than n_j or smaller than n_j. This proves the claim. Hence we have that $m - 1 = |\{r \mid x'_r < n_j\}| < p$, implying $p + 1 > m$. Therefore since $(I_x^{c'} \cup \{n_1, \dots, n_{j-1}\}, T_x^{c'} \cup \{n_1, \dots, n_{j-1}\}) \in \mathcal{I}$ we claim that it implies that $(I_x^{c'} \cup \{n_1, \dots, n_j\}, T_x^{c'} \cup \{n_1, \dots, n_j\}) \in \mathcal{I}$. Indeed, write $\tilde{x}_i$ for the elements of the first set after ordering and $\tilde{x}'_i$ for the elements of the second after ordering. Notice that $\tilde{x}_{p+1} = n_j$ while $\tilde{x}'_m = n_j$. Since $m < p + 1$ we have that $\tilde{x}_i = x_i < x'_i = \tilde{x}'_i$ whenever $i < m$. In case $i = m$ we have that $\tilde{x}_m = x_m < n_j = \tilde{x}'_m$. For $m < i \leq p+1$ we have that $\tilde{x}_i \leq n_j = \tilde{x}'_m < \tilde{x}'_i$. For $i > p + 1$ we have $\tilde{x}_i = x_{i-1} < x'_{i-1} = \tilde{x}_i$. □

Proposition 3.2.10. *The map $\mathcal{P}_{c'} \to \mathcal{I}$ defined by*

$$x \mapsto ((D_x^{c'} \backslash Y_x^{c'}) \cup N_x^{c'}, (U_x^{c'} \backslash Y_x^{c'}) \cup N_x^{c'})$$

is a bijection.

Proof. Notice that thanks to Lemma 3.2.9 together with the remarks above Lemma

3.2.5, the map is well-defined and we only need to show that the map $x \mapsto (D_x^{c'}, U_x^{c'})$ is a bijection. The fact that such a map is injective in given by Lemma 3.2.8. Since as noticed at the beginning of the subsection there is a bijection $\mathcal{W}_f \to \mathcal{I}$ and that we know that both $\mathcal{W}_f$ and $\mathcal{P}_{c'}$ have Catalan enumeration, Lemma 3.2.8 allows us to conclude. However, since we claimed to have an "explicit" description of the inverse $\psi_{c'}$ of the bijection $\varphi_{c'}$ we are trying to build, we show surjectivity, which will explain how to recover a noncrossing partition $x \in \mathcal{P}_{c'}$ from the data given by an element of $\mathcal{I}$. The surjectivity is given by Lemma 3.2.11. □

Lemma 3.2.11. *For any $(D,U) \in \mathcal{I}$, there exists an element $x \in \mathcal{P}_{c'}$ such that $(D,U) = (D_x^{c'}, U_x^{c'})$.*

Proof. If (D,U) was equal to $(D_x^{c'}, U_x^{c'})$ for some $x \in \mathcal{P}_{c'}$, then the sets $(I,T) := (D\backslash(D\cap U), U\backslash(D\cap U))$ would necessarily give the initial and terminal indices of the polygons of x. By Lemmas 3.2.5 and 3.2.6, there is a unique bijection $f : I \to T$ such that the corresponding family of segments is noncrossing. We saw in the proof of Lemma 3.2.8 that an index k which lies neither in I nor in T cannot be exposed to two segments (d_1, u_1) and (d_2, u_2) of our family with $d_j < k < u_j$, $j = 1, 2$. It remains to show that any element of $D \cap U$ is exposed to at least one such segment. Let $k \in D \cap U$. Consider the line L containing the point labeled with k and defining two half-planes, one containing all the points labeled with indices smaller than k and the other one containing the remaining points labeled with indices bigger than or equal to $k+1$ (see Remark 3.1.3). Assume that k is exposed to no segment (d,u) with $d < k < u$. It implies that L crosses no segment of our family. Hence any segment of our family joins two points which are labeled with indices either both bigger than k, either both smaller than k. It implies that $|\{m \in D \;|m < k\}| = |\{m \in U \;|m < k\}|$, implying that when ordering D and U, k appears at the same position in D and U in contradiction with our assumption that $(D,U) \in \mathcal{I}$. □

Remark 3.2.12. With the proof of Lemma 3.2.11, we have an algorithm to recover x from the sets (D,U): firstly we obtain the unique possible family of longest segments of polygons (Lemma 3.2.6). Then we add any index from $D \cap U$ to the only possible longest segment of the obtained family. Then a segment together with the points which we attached to it give us the vertices of a polygon of x. An example of the algorithm is given below in 3.2.13.

Example 3.2.13 Consider the pair of sets

$$(D,U) = (\{2,3,4,5,8,10,12,13\}, \{3,4,7,8,9,11,13,15\}) \in \mathcal{I}$$

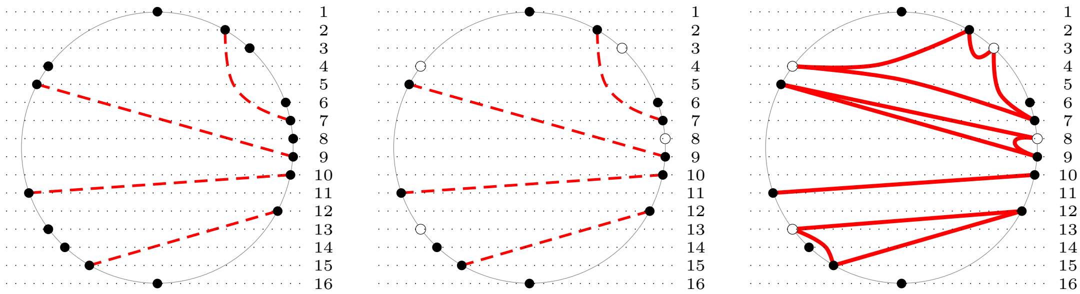


FIG. 3.5: Illustration of the algorithm allowing one to recover $x \in \mathcal{P}_{c'}$ from the corresponding element in $\mathcal{I}$.

and the Coxeter element $c' = (1, 2, 3, 6, 7, 8, 9, 10, 12, 16, 15, 14, 13, 11, 5, 4)$. We want to find the unique noncrossing partition $x \in \mathcal{P}_{c'}$ such that $(D_x^{c'}, U_x^{c'}) = (D, U)$. The set T of terminal indices is obtained by removing $D \cap U = \{3, 4, 8, 13\}$ from U giving $T = \{7, 9, 11, 15\}$ while the set I of initial indices is obtained by removing $D \cap U$ from D giving $I = \{2, 5, 10, 12\}$. The picture on the left in figure 3.5 shows the unique associated family of noncrossing arcs. The figure in the middle shows in white the points labeled with numbers in $D \cap U$; one sees that for any white point with index k, there is a single segment (i, j) of our family to which k is exposed and such that $i < k < j$. The last picture shows the noncrossing partition where a polygon has as vertices the two vertices of a segment of the family and the added white point(s) (if there are any).

Putting Proposition 3.2.10 together with Remark 3.2.2 we have proven:

Theorem 3.2.14. *Let c' be a Coxeter element. There is a bijection $\varphi_{c'} : \mathcal{P}_{c'} \to \mathcal{W}_f$ obtained by composing the bijection $\mathcal{P}_{c'} \to \mathcal{I}$ from Proposition 3.2.10 with the bijection $\mathcal{I} \to \mathcal{W}_f$ from Remark 3.2.2. The inverse bijection $\psi_{c'}$ is obtained by the composition of the bijection $\mathcal{W}_f \to \mathcal{I}$ from Remark 3.2.2 with the bijection $\mathcal{I} \to \mathcal{P}_{c'}$ which is the inverse of the bijection given in 3.2.10.*

Remark 3.2.15. Let $x \in \mathcal{P}_c$, $w = \varphi_c(x)$. Notice that we have

$$k \in J_w \Leftrightarrow \begin{cases} k \in R_{c'} \cap D_x^{c'}, \text{ or} \\ k \in L_{c'} \text{ and } k \text{ is an initial index of a polygon of } x, \text{ or} \\ k \in L_{c'} \text{ and } k \notin \mathrm{Vert}(x) \text{ but } k \text{ is nested in a polygon of } x. \end{cases}$$

We also have that

$$k - 1 \in I_w \Leftrightarrow \begin{cases} k \in R_{c'} \cap U_x^{c'}, \text{ or} \\ k \in L_{c'} \text{ and } k \text{ is a terminal index of a polygon of } x, \text{ or} \\ k \in L_{c'} \text{ and } k \notin \mathrm{Vert}(x) \text{ but } k \text{ is nested in a polygon of } x. \end{cases}$$

Example 3.2.16 Consider the Coxeter element

$$c' = (1, 2, 3, 6, 7, 8, 9, 10, 11, 12, 16, 15, 14, 13, 11, 5, 4).$$

Let $x \in \mathcal{P}_{c'}$ be the noncrossing partition represented in figure 3.6. Write $w := \psi_{c'}(x)$. One has $D_x^{c'} = \{3, 4, 5, 8, 10, 11, 12\}$ and $U_x^{c'} = \{4, 7, 8, 9, 11, 13, 15\}$. We have $Y_x^{c'} =$

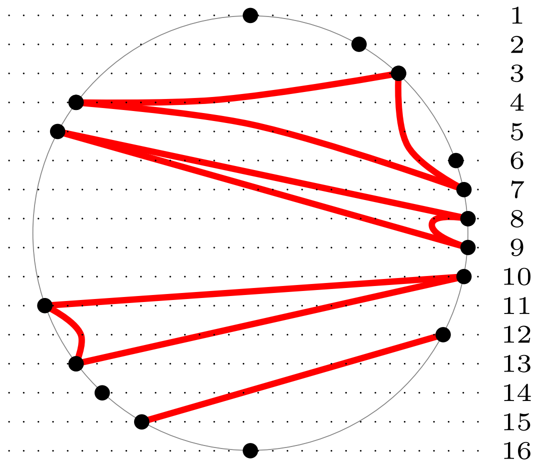


FIG. 3.6

$D_x^{c'} \cap U_x^{c'} \cap L_{c'} = \{4, 11\}$ and $N_x^{c'} = \{14\}$. We then consider the two modified sets

$$((D_x^{c'} \backslash Y_x^{c'}) \cup N_x^{c'}, (U_x^{c'} \backslash Y_x^{c'}) \cup N_x^{c'}) = (\{3, 5, 8, 10, 12, 14\}, \{7, 8, 9, 13, 14, 15\}).$$

The first obtained set is then equal to J_w while the second one is nothing but $I_w + 1$. Hence we get

$$(J_w, I_w) = (\{3, 5, 8, 10, 12, 14\}, \{6, 7, 8, 12, 13, 14\})$$

giving

$$w = (s_6 s_5 s_4 s_3)(s_7 s_6 s_5)(s_8)(s_{12} s_{11} s_{10})(s_{13} s_{12}) s_{14} \in \mathcal{W}_f.$$

Remark 3.2.17. In case $c' = c$, the involution $^{-}: \mathcal{P}_c \to \mathcal{P}_c$ is the identity since the equality $L_c = \{1, n+1\}$ implies that for any $x \in \mathcal{P}_c$, we have $N_x^c = \emptyset = Y_x^c$. We therefore have that

$$(D_x^c, U_x^c - 1) = (J_{\varphi_c(x)}, I_{\varphi_c(x)}).$$

The bijection is hence much easier to compute in that case.

Let us now explain the relationship with results by Zinno. We need to introduce some results and vocabulary of [42]. The simple elements $\{i_c(x) \mid x \in \mathcal{P}_c\}$ of the dual braid monoid B_c^* which we can see as lifts of elements of $\mathcal{P}_c$ in the braid group correspond to the so called *canonical factors* (shortly *canfacs*) from [42], but the way the lifts of reflections (called *band generators* in [42]) are written by Zinno corresponds to the choice of Coxeter element $s_n s_{n-1} \cdots s_1$. Since we are working rather with the Coxeter element $s_1 s_2 \cdots s_n$ we will adapt Zinno's results to our setting. Let us assume that $c = s_1 s_2 \cdots s_n$. Given a reflection $t = (i, k+1)$, $i \leq k$, consider the braid word $\mathbf{s}_{i,k+1} := \mathbf{s_k}^{-1}\mathbf{s_{k-1}}^{-1} \cdots \mathbf{s_{i+1}}^{-1}\mathbf{s_i}\mathbf{s_{i+1}} \cdots \mathbf{s_k}$; it represents the image of the element $i_c(t)$ in the braid group. In fact, the embedding $B_c^* \hookrightarrow B$ sends $i_c(s)$ to $\mathbf{s}$ for any simple reflection $s \in \mathcal{S}$ and one finds an expression of any $i_c(t)$ for any $t \in \mathcal{T}$ in the braid group B by using the dual braid relations; for example, one has that $s_1 s_2 <_{\mathcal{T}} c$ whence

$$i_c(s_1) i_c(s_2) = i_c(s_2) i_c(s_2 s_1 s_2)$$

and since $i_c(s_i)$ is equal to $\mathbf{s}_i$ if viewed in B, one has that $i_c(s_2 s_1 s_2)$ is equal to $\mathbf{s}_2^{-1}\mathbf{s}_1\mathbf{s}_2 = \mathbf{s}_{1,3}$ if viewed in the braid group. By induction on $k - i$ one shows that $i_c((i, k+1))$ is represented by the braid word $\mathbf{s}_{i,k+1}$ in B.

We will often abuse notation and also write $i_c(x)$ for the image of $i_c(x)$ in B. A braid word such as $\mathbf{s}_{i,k+1}$ is called a *syllable* by Zinno. The braid group generator $\mathbf{s_i}$ is the *center* of the syllable, splitting the syllable into a *left* part $\mathbf{s_k}^{-1}\mathbf{s_{k-1}}^{-1} \cdots \mathbf{s_{i+1}}^{-1}$ and a *right* part $\mathbf{s_{i+1}} \cdots \mathbf{s_k}$. A noncrossing partition $x \in \mathcal{P}_c$ which is a cycle, that is, such that $|\mathrm{Pol}(x)| = 1$ is still called a *cycle* in [42] after lifting in the braid group. Zinno uses the following braid word to represent $i_c(x)$: firstly he writes $x = (i_1, i_2, \ldots, i_k)$, where $i_1 < i_2 < \cdots < i_k$. Then $i_c(x)$ is represented by the braid word $\mathbf{s}_{i_1,i_2}\mathbf{s}_{i_2,i_3} \cdots \mathbf{s}_{i_{k-1},i_k}$. We will represent a simple element $i_c(x)$ of the dual braid monoid by the braid word obtained by concatenating the cycles, ordered by the maximal index in each cycle (that is, the terminal index of the associated polygon), in ascending order, and refer to such a word as to the *standard form* of a simple element of the dual braid monoid. We denote the obtained braid word by $m_{\mathbf{x}}$.

Remark 3.2.18. Notice that a braid group generator can be the center of at most one syllable, hence it occurs twice in any other syllable in which it occurs, once in the left part with negative exponent and once in the right part with positive exponent. The way the polygons (equivalently the cycles) are ordered implies that if $\mathbf{s}_i^{\pm 1}$ is the center of a syllable (which is equivalent as saying that i is a non terminal index of a polygon of x), then the first occurrence of $\mathbf{s}_i^{\pm 1}$ in $m_{\mathbf{x}}$ when reading the word from the left to the right is at the center of that syllable.

It turns out that if we replace each $\mathbf{s_i}^{\pm 1}$ by s_i in $m_{\mathbf{x}}$, we obtain an $\mathcal{S}$-reduced decomposition m_x of $x \in \mathcal{P}_c$ which we also call the *standard form* of x (and we will also call m_t for $t \in \mathcal{T}$ a *syllable* with a *center*, *left part*, etc.); we will give a more general definition of this form in subsection 3.7.2 where we will work with arbitrary Coxeter elements. It turns out that the Coxeter word m_x plays an important role in the study of the orders making the change of bases matrices between the Zinno basis (which we will introduce in a few lines) and the diagram basis upper triangular.

The image of the braid group generator $\mathbf{s_i}$ in the Temperley-Lieb algebra will be written Z_i. It is equal to $v^{-1} - b_i$ (for the conventions on the quotient map from the braid group to the Temperley-Lieb algebra, see subsection 3.3.1). If $x \in \mathcal{P}_c$, we write Z_x for the image of $i_c(x)$ in the Temperley Lieb algebra. Notice that $Z_i = Z_{s_i}$ for any $s_i \in \mathcal{S}$.

Zinno's bijection $a : \mathcal{P}_c \to \mathcal{W}_f$ mentioned in the introduction and at the end of subsection 3.2.1 is built as follows: given $x \in \mathcal{P}_c$, Zinno considers the braid word $m_\mathbf{x}$ representing $i_c(x)$. He then extracts a subword $w_\mathbf{x}$ of $m_\mathbf{x}$ in the following way: let us call *letter* any $\mathbf{s}_i^{\pm 1}$ occurring in a syllable. If a syllable has at least one letter indexed by i (the letters indexed by i are $\mathbf{s}_i$ and $\mathbf{s}_i^{-1}$), then that syllable must contribute to the subword exactly one of its letters indexed by i. In particular each center contributes since it is the only letter with its index in a syllable. The contributions are as follows: if $\mathbf{s}_i$ is the center of a syllable and occurs in another syllable, then such a syllable contributes the $\mathbf{s}_i$ which has positive exponent. If $\mathbf{s}_i$ is not the center of a syllable but there are syllables containing letters indexed by i, then these syllables must contribute their $\mathbf{s}_i^{-1}$ to the subword. In this way we extract a subword $w_\mathbf{x}$. By replacing in that word the $\mathbf{s}_i^{\pm 1}$ by s_i we get an element w_x of the Coxeter group. These rules are equivalent to the ones given by the following algorithm: read the word $m_\mathbf{x}$ from the left to the right. If the first letter $\mathbf{s_i}^{\pm 1}$ occurring in $m_\mathbf{x}$ has positive (resp. negative) exponent, then all the occurrences of $\mathbf{s_i}$ (resp. of $\mathbf{s_i}^{-1}$) in $m_\mathbf{x}$ and only those must contribute to the subword $w_\mathbf{x}$. Apply the same process to the next generator $\mathbf{s_j}^{\pm 1}$, $j \neq i$ occurring right to the first $\mathbf{s_i}^{\pm 1}$ in $m_\mathbf{x}$, until you have considered all the indices k such that $\mathbf{s_k}^{\pm 1}$ occurs in $m_\mathbf{x}$.

Zinno then shows that w_x is fully commutative and that the map $a : \mathcal{P}_c \to \mathcal{W}_f$ defined by $x \mapsto w_x$ is surjective. Since $|\mathcal{P}_c| = |\mathcal{W}_f|$ the map is bijective. An example of Zinno's algorithm where we apply the rules given above to extract the fully commutative element w_x as a subword of a standard form $m_\mathbf{x}$ of a simple element $i_c(x)$, $x \in \mathcal{P}_c$ is given in example 3.2.19 below.

Example 3.2.19 (Zinno's algorithm from [42] for extracting $w_x = a(x)$ from $m_\mathbf{x}$)
Let $x = (2, 3, 5)(1, 6) \in \mathcal{P}_c$

$$m_\mathbf{x} = \mathbf{s_2}(\mathbf{s_4}^{-1}\mathbf{s_3}\mathbf{s_4})(\mathbf{s_5}^{-1}\mathbf{s_4}^{-1}\mathbf{s_3}^{-1}\mathbf{s_2}^{-1}\mathbf{s_1}\mathbf{s_2}\mathbf{s_3}\mathbf{s_4}\mathbf{s_5})$$
$$m_\mathbf{x} = \underline{\mathbf{s_2}}(\mathbf{s_4}^{-1}\mathbf{s_3}\mathbf{s_4})(\mathbf{s_5}^{-1}\mathbf{s_4}^{-1}\mathbf{s_3}^{-1}\mathbf{s_2}^{-1}\mathbf{s_1}\underline{\mathbf{s_2}}\mathbf{s_3}\mathbf{s_4}\mathbf{s_5})$$

$$
\begin{aligned}
m_{\mathbf{x}} &= \underline{\mathbf{s_2}}(\underline{\mathbf{s_4}}^{-1}\mathbf{s_3}\mathbf{s_4})(\mathbf{s_5}^{-1}\underline{\mathbf{s_4}}^{-1}\mathbf{s_3}^{-1}\mathbf{s_2}^{-1}\mathbf{s_1}\underline{\mathbf{s_2}}\mathbf{s_3}\mathbf{s_4}\mathbf{s_5})\\
m_{\mathbf{x}} &= \underline{\mathbf{s_2}}(\underline{\mathbf{s_4}}^{-1}\underline{\mathbf{s_3}}\mathbf{s_4})(\mathbf{s_5}^{-1}\underline{\mathbf{s_4}}^{-1}\mathbf{s_3}^{-1}\mathbf{s_2}^{-1}\mathbf{s_1}\underline{\mathbf{s_2}\mathbf{s_3}}\mathbf{s_4}\mathbf{s_5})\\
m_{\mathbf{x}} &= \underline{\mathbf{s_2}}(\underline{\mathbf{s_4}}^{-1}\underline{\mathbf{s_3}}\mathbf{s_4})(\underline{\mathbf{s_5}}^{-1}\underline{\mathbf{s_4}}^{-1}\mathbf{s_3}^{-1}\mathbf{s_2}^{-1}\mathbf{s_1}\underline{\mathbf{s_2}\mathbf{s_3}}\mathbf{s_4}\mathbf{s_5})\\
m_{\mathbf{x}} &= \underline{\mathbf{s_2}}(\underline{\mathbf{s_4}}^{-1}\underline{\mathbf{s_3}}\mathbf{s_4})(\underline{\mathbf{s_5}}^{-1}\underline{\mathbf{s_4}}^{-1}\mathbf{s_3}^{-1}\mathbf{s_2}^{-1}\underline{\mathbf{s_1}\mathbf{s_2}\mathbf{s_3}}\mathbf{s_4}\mathbf{s_5})
\end{aligned}
$$

$$\rightsquigarrow w_{\mathbf{x}} = \mathbf{s_2}\mathbf{s_4}^{-1}\mathbf{s_3}\mathbf{s_5}^{-1}\mathbf{s_4}^{-1}\mathbf{s_1}\mathbf{s_2}\mathbf{s_3}$$

$$\rightsquigarrow w_x = s_2s_4s_3s_5s_4s_1s_2s_3 = (s_2s_1)(s_4s_3s_2)(s_5s_4s_3) \in \mathcal{W}_f.$$

Proposition 3.2.20. *The bijection φ_c and the bijection described in ([42], Theorems* 3 *and* 6*) which we denoted by a are the same.*

Proof. For $w \in \mathcal{W}_f$, we will use the characterization of the sets I_w and J_w given in Remark 3.2.1.

Let $x \in \mathcal{P}_c$ and write $w_x = a(x)$ for the corresponding fully commutative element as described above and given by Theorem 6 of [42]. By definition of the bijection φ_c (see also Remark 3.2.17), we have to show that $i \in I_{w_x}$ if and only if $i+1 \in U_x^c$ and $i \in J_{w_x}$ if and only if $i \in D_x^c$. Let $i \in I_{w_x}$. It implies that the first occurrence of s_i in w_x must come from a $\mathbf{s}_i^{\pm 1}$ which is the first letter of its syllable w: otherwise $\mathbf{s}_{i+1}^{-1}$ would occur in w on the left of the $\mathbf{s}_i^{\pm 1}$ contributed and that $\mathbf{s}_{i+1}^{-1}$ would be contributed in case $\mathbf{s}_{i+1}$ is not a center; in case $\mathbf{s}_{i+1}$ is a center, the occurrence of $\mathbf{s}_{i+1}$ at the center must be the first in the word (see Remark 3.2.18), hence before the syllable w and must be contributed. Hence the first occurrence of s_i in w_x must come from a $\mathbf{s}_i^{\pm 1}$ which is the first letter of its syllable w. But $\mathbf{s}_i^{\pm 1}$ is the first letter of a syllable if and only if $\mathbf{s}_i^{\pm 1}$ is at the top of a syllable if and only if $i+1 \in U_x^c$. Hence we have that $i+1 \in U_x^c$. Conversely, consider an index $i+1$ which labels a vertex of a polygon $P \in \mathrm{Pol}(x)$ and which is not initial (that is, $i+1 \in U_x^c$). Write $(i_1, \ldots, i_m)$, $i_1 < i_2 < \cdots < i_m$ for the cycle corresponding to P. If $i+1 = i_m$, $m \neq 1$, then $\mathbf{s}_i^{\pm 1}$ is the first letter of the syllable $\mathbf{s}_{i_{m-1},i_m}$ occurring in the cycle corresponding to P. We will show that this letter $\mathbf{s}_i^{\pm 1}$ contributes, that it is the first occurrence of $\mathbf{s}_i^{\pm 1}$ in $m_{\mathbf{x}}$ and that there is no occurrence of $\mathbf{s}_{i+1}^{\pm 1}$ in $m_{\mathbf{x}}$ at its left. All these properties together imply that $i \in I_{w_x}$. If there is another letter $\mathbf{s}_i^{\pm 1}$ before ours, then it must be in a cycle corresponding to a polygon $Q \neq P$. Suppose that it occurs as a center of the syllable corresponding to Q. It means that x has a polygon Q with a non terminal vertex indexed by i and another polygon P with a non initial vertex indexed by $i+1$, contradicting the noncrossing property. If it is not as a center, it cannot be at a top since we already have a syllable with $\mathbf{s}_i^{\pm 1}$ at its top and there can be at most one. But if $\mathbf{s}_i$ is not at the top, it has to be in a syllable $\mathbf{s}_{k,k'}$ where $k < \min P$, $k' > \max P$ otherwise there would be a contradiction with the noncrossing property. But the terminal index of the polygon Q containing $\mathbf{s}_{k,k'}$ would then be bigger than the terminal index of P, hence $\mathbf{s}_{k,k'}$ cannot occur before $\mathbf{s}_{i_{m-1},i_m}$ in $m_{\mathbf{x}}$. Hence our $\mathbf{s}_i^{\pm 1}$ at the top of its syllable is the first occurrence of $\mathbf{s}_i^{\pm 1}$ in the word $m_{\mathbf{x}}$. Now if $\mathbf{s}_{i+1}$ occurs in a syllable, if it is at the center then it is in P and the syllable appears just after $\mathbf{s}_{i_{m-1},i_m}$. If it is not at the center, then to respect the noncrossing property one must again have that the syllable containing it appears after $\mathbf{s}_{i_{m-1},i_m}$. Therefore we have $i \in I_{w_x}$.

Similarly one shows without difficulty that $i \in J_{w_x}$ if and only if $i \in D_x^c$. As a consequence one gets that $a(x) = w_x = \varphi_c(x)$. □

Remark 3.2.21. At the very end of the chapter, we will give a more general proof of this fact. We will introduce a process generalizing Zinno's algorithm to extract a fully commutative subword from a standard form in the case of dual braid monoids associated to arbitrary Coxeter elements and will show that it gives us the bijections given by Theorem 3.2.14.

Notation. Let $i, j \in \{1, \dots, n\}$, $j \leq i$. We write $\mathcal{W}(i,j)$ for the set of fully commutative elements $w \in \mathcal{W}_f$ such that in the notation of subsection 3.2.1, one has $i = i_1, j = j_\ell$. These are exactly the fully commutative elements whose $\mathcal{S}$-reduced expressions contain the reflections s_i and s_j exactly once and any other simple reflection s_k occurring in such an $\mathcal{S}$-reduced expression satisfies $j < k < i$.

The next proposition is not used further but can help to compute images of some elements under the bijection $\psi_c : \mathcal{W}_f \to \mathcal{P}_c$, which is the inverse of φ_c, hence of Zinno's bijection by Proposition 3.2.20.

Proposition 3.2.22. *Let $c = s_1 s_2 \cdots s_n$. Let $j \leq i < m \leq k$. Let $w_1 \in \mathcal{W}(i,j)$, $w_2 \in \mathcal{W}(k,m)$. Then $w_1 w_2$ lies in $\mathcal{W}(k,j)$ and*

$$\psi_c(w) = \psi_c(w_1)\psi_c(w_2).$$

Proof. The fact that the product is fully commutative is clear since the smallest index of a reflection occurring in any reduced expression of w_2 is larger than the largest index occurring in a reduced expression for w_1; the largest index in the product is then k and the smallest one is j proving the first claim.

Our condition on i, j, k and m implies that an $\mathcal{S}$-reduced expression for $w_1 w_2$ is given by concatenating $\mathcal{S}$-reduced expressions for w_1 and w_2 and that the obtained $\mathcal{S}$-reduced expression for $w_1 w_2$ does have the usual form with successive sequences of decreasing indices if the ones for w_1 and w_2 do. It also implies that the sequences associated to w_1 and w_2 cannot break in w inside a decreasing subsequence for w

but only between two of them. Hence

$$U^c_{\psi_c(w_1w_2)} = U^c_{\psi_c(w_1)} \dot{\cup} U^c_{\psi_c(w_2)},$$

$$D^c_{\psi_c(w_1w_2)} = D^c_{\psi_c(w_1)} \dot{\cup} D^c_{\psi_c(w_2)}.$$

First, suppose that $U^c_{\psi_c(w_1)}$ and $D^c_{\psi_c(w_2)}$ are disjoint. It implies that any polygon of $\psi_c(w_1)$ has all its indexing numbers bigger than those of any polygon of $\psi_c(w_2)$. As a consequence, the union of all the polygons of both of them gives the geometrical representation of a noncrossing partition $x = \psi_c(w_1)\psi_c(w_2) = \psi_c(w_2)\psi_c(w_1)$ such that obviously $U^c_x = U^c_{\psi_c(w_1w_2)}$ and $D^c_x = D^c_{\psi_c(w_1w_2)}$. Thanks to Proposition 3.2.10 this forces $x = \psi_c(w_1w_2)$.

Now consider the case where $U^c_{\psi_c(w_1)} \cap D^c_{\psi_c(w_2)} \neq \emptyset$, forcing that intersection to contain the single element m (equal to $i+1$ in that case). Then $\psi_c(w_1)$ has a polygon with maximal index equal to m and $\psi_c(w_2)$ has a polygon with minimal index equal to m. This implies that the product $x = \psi_c(w_1)\psi_c(w_2)$ has a representation given by the disjoint union of the polygons of w_1 and w_2 different from the two mentioned aboved together with the polygon obtained by taking the polygon given by the convex hull of these two polygons. Again, we have that $U^c_x = U^c_{\psi_c(w_1w_2)}$ and $D^c_x = D^c_{\psi_c(w_1w_2)}$. □

3.3 Zinno basis and diagram basis

3.3.1 Zinno basis

Let us recall some facts from subsection 1.2.2. Recall that the Temperley-Lieb algebra $\mathrm{TL}_n = \mathrm{TL}_n(v+v^{-1})$ is the associative unital $\mathbb{Z}[v, v^{-1}]$-algebra with n generators $b_1, \ldots, b_n$ and relations

$$b_j b_i b_j = b_j \text{ if } |i-j| = 1,$$

$$b_i b_j = b_j b_i \text{ if } |i-j| > 1,$$

$$b_i^2 = (v+v^{-1}) b_i.$$

The algebra TL_n has a basis $\{b_w\}_{w \in \mathcal{W}_f}$ indexed by fully commutative elements where if $s_{i_1} \cdots s_{i_k}$ is an $\mathcal{S}$-reduced expression of $w \in \mathcal{W}_f$, then $b_w := b_{i_1} \cdots b_{i_k} \in \mathrm{TL}_n(v+v^{-1})$ is independent of the choice of the $\mathcal{S}$-reduced expression we made for w. For any $s_i \in \mathcal{S}$, $b_{s_i} := b_i$. The basis $\{b_w\}_{w \in \mathcal{W}_f}$ has an interpretation by planar diagrams and is the projection of the Kazhdan-Lusztig basis C'_w of the Hecke algebra via a quotient map θ. It is also the projection (up to signature) of the basis C_w through an alternative quotient map θ'. The map $\theta : \mathcal{H} \to \mathrm{TL}_n(v+v^{-1})$ has as kernel the ideal generated by all the $C'_{s_i s_{i+1} s_i}$, $i = 1, \ldots, n$, while the second map $\theta' : \mathcal{H} \to \mathrm{TL}_n(v+v^{-1})$ has as kernel the ideal generated by all the $C_{s_i s_{i+1} s_i}$, $i = 1, \ldots, n$. One has $\theta(C'_w) = b_w$ while $\theta'(C_w) = (-1)^{\ell_{\mathcal{S}}(w)} b_w$, where $\ell_{\mathcal{S}}$ is the Coxeter length and $w \in \mathcal{W}_f$; if $w \notin \mathcal{W}_f$ the images are zero. In the previous subsection, we mentioned that the braid group generator $\mathbf{s}_i$ maps to $v^{-1} - b_i$; this is given by the composition

$$\mathbb{Z}[v, v^{-1}]B_{n+1} \twoheadrightarrow \mathcal{H} \overset{\theta'}{\twoheadrightarrow} \mathrm{TL}_n(v+v^{-1}),$$

$$\mathbf{s}_i \mapsto vT_{s_i} \mapsto v^{-1} - b_i,$$

where B_{n+1} is the braid group on $n+1$ strands. The existence of the first quotient map is clear since the T_{s_i} satisfy the braid relations which are homogeneous, hence the same holds for the vT_{s_i}. In the following we will always consider the Temperley-Lieb algebra as the quotient of the group algebra of the braid group via the composition above. If one prefers to work with the quotient map θ instead of θ', our results can of course easily be adapted. We will work with elements of the dual braid monoid viewed as elements of the braid group. Given a braid word in the $\mathbf{s}_i^{\pm 1}$, we will consider the image of such a word in the Temperley-Lieb by replacing $\mathbf{s}_i$ by $v^{-1} - b_i$ and $\mathbf{s}_i^{-1}$ by $v - b_i$ since $(v^{-1} - b_i)(v - b_i) = 1$.

Zinno shows in [42] that the set $\{Z_x\}_{x \in \mathcal{P}_c}$ is a basis of $\mathrm{TL}_n(v+v^{-1})$, where Z_x is the image of the simple element $i_c(x)$ as introduced in the previous subsection.

Remark 3.3.1. Notice that the geometric representation of an element of $\mathcal{P}_c$ also works to represent its lift in the braid group, hence the proof of Proposition 3.2.22 gives us more precisely that under the assumptions on w_1, w_2 one has

$$i_c(\psi_c(w_1 w_2)) = i_c(\psi_c(w_1)) i_c(\psi_c(w_2)),$$

where the relation holds in the dual braid monoid or in the braid group: it implies that

$$Z_{\psi_c(w_1 w_2)} = Z_{\psi_c(w_1)} Z_{\psi_c(w_2)},$$

where the relation holds in the Temperley-Lieb algebra.

Recall that a triple of transpositions (s, t, u) is *admissible* if $s \neq t$, $st \in \mathcal{P}_c$ and $u = sts$. In particular we have a dual relation $st = tu = us$. We recall from [31] the so-called *dual presentation* of Temperley-Lieb algebras, given in terms of the images of the atoms of the dual braid monoid in the Temperley-Lieb algebra, that is, the Z_t for $t \in \mathcal{T}$:

Theorem 3.3.2 (**Dual presentation of Temperley-Lieb algebras**, Lee, Lee, [31]). *The Temperley-Lieb algebra is generated by Z_t, $t \in \mathcal{T}$ with relations*

$$Z_s Z_t = Z_t Z_s \text{ if } st = ts \text{ and } st \in \mathcal{P}_c,$$

$$Z_s Z_t = Z_t Z_u = Z_u Z_s \text{ if } (s,t,u) \text{ is admissible },$$
$$-vZ_s Z_t - v^{-1} Z_t Z_s + Z_u + v^{-2}(Z_s + Z_t) - v^{-3} = 0 \text{ if } (s,t,u) \text{ is admissible },$$
$$Z_t^2 = (v^{-1} - v) Z_t + 1 \text{ if } t \in \mathcal{T}.$$

Vincenti gave a dual presentation of the Temperley-Lieb algebra in type B (see [40]).

3.3.2 A new basis of the Temperley-Lieb algebra

We recall the definition and various characterizations of the Bruhat order on a Coxeter system $(\mathcal{W}, \mathcal{S})$. For $w, w' \in \mathcal{W}$, we define a relation by $w \to w'$ if there exists $t \in \mathcal{T}$ such that $w' = tw$ and $\ell_{\mathcal{S}}(w') > \ell_{\mathcal{S}}(w)$. We then extend this relation to a partial order $<_{\mathcal{S}}$ by setting $w <_{\mathcal{S}} w'$ if there exists $w_1, \dots, w_k \in \mathcal{W}$ such that $w \to w_1 \to w_2 \to \cdots \to w_k \to w'$. It is the *Bruhat order* of the Coxeter system $(\mathcal{W}, \mathcal{S})$. The following characterization is classical (see for example [5], Corollary 2.2.3):

Proposition 3.3.3. *For $w, w' \in \mathcal{W}$, the following are equivalent:*

1. *One has $w <_{\mathcal{S}} w'$,*
2. *Any $\mathcal{S}$-reduced expression for w' has a subword that is an $\mathcal{S}$-reduced expression for w,*
3. *There exists an $\mathcal{S}$-reduced expression for w' which has a subword that is an $\mathcal{S}$-reduced expression for w.*

Again, $(\mathcal{W}, \mathcal{S})$ will be of type A_n, with the same identifications and notations as before. In this subsection we are working exclusively with the Coxeter element $c = s_1 s_2 \cdots s_n$. For $w \in \mathcal{W}_f$ we set

$$L(w) = \{s \in \mathcal{S} \mid sw <_{\mathcal{S}} w\},$$

$$R(w) = \{s \in \mathcal{S} \mid ws <_{\mathcal{S}} w\}.$$

Remark 3.3.4. Notice that if $s, t \in L(w)$, then $ts = st$. The same holds if both s, t lie in $R(w)$. Moreover, if $s \in L(w)$ (resp. $R(w)$), then sw (resp. ws) lies again in $\mathcal{W}_f$.

Recall from subsection 3.3.1 that $b_i = v^{-1} - Z_i$. For the next results one may keep in mind the properties of the bijections φ_c and ψ_c given in Remark 3.2.17. Given a transposition (i, j), assume that a polygon P occurring in the geometrical representation of $x \in \mathcal{P}_c$ has an edge joining the point with index i to the point with index j. We will also denote this edge by (i, j) (notice that if (i, j) is an edge or even a diagonal of a polygon of x, it implies that $(i, j) <_{\mathcal{T}} x$; this is an easy consequence of the remark made in the last paragraph of subsection 1.4.3).

Proposition 3.3.5. *Let $w \in \mathcal{W}_f$, $s = s_i \in \mathcal{S}$. Then*

1. *$s = s_i \in L(w)$ if and only if $\{(i, i+1)$ is an edge of a polygon $P \in \mathrm{Pol}(\psi_c(w))$ with i initial$\}$ or $\{$the point with index i is not a vertex of a polygon of $\psi_c(w)$ but there exists a polygon $P \in \mathrm{Pol}(\psi_c(w))$ having an edge $(k, i+1)$ for some $k < i\}$.*

2. *$s \in R(w)$ if and only if $\{(i, i+1)$ is an edge of a polygon $P \in \mathrm{Pol}(\psi_c(w))$ with $i+1$ terminal$\}$ or $\{$the point with index i is not a vertex of a polygon of $\psi_c(w)$ but there exists a polygon $P \in \mathrm{Pol}(\psi_c(w))$ having an edge $(i, i+k)$ with $k > 1\}$.*

Proof. One has that $s \in L(w)$ if and only if $i \in I_w$, $i-1 \notin I_w$ if and only if $i+1 \in U^c_{\psi_c(w)}$, $i \notin U^c_{\psi_c(w)}$ if and only if $(i, i+1)$ is an edge at the bottom of a polygon of $\psi_c(w)$ or i is not a vertex of a polygon of $\psi_c(w)$ but there exists an edge $(k, i+1)$ of a polygon with $k < i$. One argues similarly for $s \in R(w)$. □

Corollary 3.3.6. *Let $w \in \mathcal{W}_f$.*

1. *If $s \in L(w)$, then $s\psi_c(w) \in \mathcal{P}_c$ and $\ell_{\mathcal{S}}(s\psi_c(w)) = \ell_{\mathcal{S}}(\psi_c(w)) - 1$.*

2. *If $s \in R(w)$, then $\psi_c(w)s \in \mathcal{P}_c$ and $\ell_{\mathcal{S}}(\psi_c(w)s) = \ell_{\mathcal{S}}(\psi_c(w)) - 1$.*

Proof. Thanks to the previous proposition we know what the assumption $s \in L(w)$ means in terms of the geometrical representation of $\psi_c(w)$ by disjoint unions of polygons. In case $(i, i+1)$ is an edge of a polygon P of $\psi_c(w)$ with i initial, it means that the cycle $y \in \mathcal{P}_c$ corresponding to P is equal to $s_i y'$ where $y' \in \mathcal{P}_c$ is the cycle corresponding to the polygon P' obtained from P by removing the vertex with index i. Since i is the miminal index of P one then has that $\ell_{\mathcal{S}}(y') = \ell_{\mathcal{S}}(y) - 1$. But if a noncrossing partition $x \in \mathcal{P}_c$ has decomposition into disjoint cycles $y_1 y_2 \cdots y_k$, one has (see Remark 3.4.1) that

$$\ell_{\mathcal{S}}(y) = \sum_{j=1}^{k} \ell_{\mathcal{S}}(y_j),$$

which concludes. In case i is not an index of a vertex of a polygon of $\psi_c(w)$ but there is a polygon P having an edge $(k, i+1)$ for $k < i$, consider again the cycle $y \in \mathcal{P}_c$ corresponding to P. The product $s_i y$ is again a noncrossing partitions corresponding to the polygon P' obtained from P by adding the vertex labeled by i. If the set of indices of vertices of P is given by $d_1, \dots, d_k$, $d_j < d_{j+1}$ with $d_m = k, d_{m+1} = i+1$, an $\mathcal{S}$-reduced expression of y is given by the concatenation

$$[d_1, d_2][d_2, d_3] \cdots [d_{k-1}, d_k],$$

where $[j, \ell] = s_{\ell-1} s_{\ell-2} \cdots s_{j+1} s_j s_{j+1} \cdots s_{\ell-2} s_{\ell-1}$ (see Remark 3.4.1). Adding the vertex i replaces in the product above the subword $[d_m, d_{m+1}]$ by $[d_m, i][i, d_{m+1}]$ and this just removes one occurrence of s_i. Hence we again have $\ell_{\mathcal{S}}(s_i y) = \ell_{\mathcal{S}}(y) - 1$ and the same argument as for the first case gives the conclusion. The proof of the case where $s \in R(w)$ is similar. ☐

Corollary 3.3.7. *Let $w \in \mathcal{W}_f$. Then*

$$s = s_i \in L(w) \cap R(w) \;\Leftrightarrow\; s <_{\mathcal{T}} \psi_c(w) \text{ and } s\psi_c(w) = \psi_c(w)s$$

$$\Leftrightarrow \text{There exists a polygon of } \psi_c(w) \text{ which is reduced to the edge } (i, i+1).$$

Proof. It is a consequence of Proposition 3.3.5 which is proven with the same kind of arguments as the corollary above. ☐

Corollary 3.3.8. *Let $w \in \mathcal{W}_f$. Let $s \in L(w)$ and $t \in R(w)$, with $s \neq t$*

$$s\psi_c(w) = \psi_c(w)t \;\Leftrightarrow\; s = s_j,\; t = s_{j-1} \text{ for some index } j.$$

Proof. Let $s = s_j$, $t = s_k$ and suppose $s\psi_c(w) = \psi_c(w)t$. Thanks to Proposition 3.3.5, applying s_j on the left of $\psi_c(w)$ either adds or removes the vertex with index j (and possibly the vertex with index $j+1$ but in that case, one would have $s \in R(w)$; since $t \neq s$ the reflection t would then remove a vertex with index k distant from j since any two reflections in $R(w)$ commute with each other, a contradiction to $s\psi_c(w) = \psi_c(w)t$ since the operation of s in the left hand side does not change the vertex with index k). So we can suppose that s removes or adds the vertex with index j, leaving all other vertices of the polygons unchanged. This means that t also has to remove or add the vertex with index j. This is possible only if $t = s_{j-1}$ or $t = s_j$ but the last case is excluded. Conversely, the assumption implies by the above proposition that $\psi_c(w)$ has a polygon P having an edge $(j-1, j+1)$. We then have that $s_j\psi_c(w) = \psi_c(w)s_{j-1}$ and in the geometrical representation, it corresponds to adding the vertex with index j to the polygon P. □

Notation. Let $w \in \mathcal{W}_f$, $L \subset L(w)$ and $R \subset R(w)$. We build new sets L', R' from L and R by doing the following: if $s \in L \cap R$, we either remove s from L or remove it from R. If $s_j \in L$ and $s_{j-1} \in R$, then we either remove s_j from L or remove s_{j-1} from R. At the end of the process we get two (non canonically defined) sets $L' \subset L$ and $R' \subset R$. It is clear that if (L', R') and $(\tilde{L}', \tilde{R}')$ are two distinct sets with these properties, one has $|L' \cup R'| = |\tilde{L}' \cup \tilde{R}'|$.

Example 3.3.9 Let $w = s_2s_1s_3$. Then $L(w) = \{s_2\}$, $R(w) = \{s_1, s_3\}$. Let $L = L(w)$, $R = R(w)$. One can choose $L' = \{s_2\}$, $R' = \{s_3\}$. Another possible choice is $L' = \emptyset$, $R' = \{s_1, s_3\}$.

The following proposition is a generalization of Corollary 3.3.6.

Proposition 3.3.10. *Let $w \in \mathcal{W}_f$, $L \subset L(w)$, $R \subset R(w)$. Then*

$$x_{L',R'} := (\prod_{s\in L'} s)\psi_c(w)(\prod_{s\in R'} s)$$

is independent of the choice of L' and R' and will therefore be denoted by $x_{L,R}$. Moreover, $x_{L,R}$ lies in $\mathcal{P}_c$, $x_{L,R} <_{\mathcal{S}} \psi_c(w)$ and $\ell_{\mathcal{S}}(x_{L,R}) = \ell_{\mathcal{S}}(\psi_c(w)) - |L' \cup R'|$.

Proof. One can argue by induction on $|L' \cup R'|$. If it is equal to zero, it means that $L = \emptyset = R$, in which case the claim is trivially true. If $L' \cup R'$ is a singleton, the claim is true by corollaries 3.3.6, 3.3.7 and 3.3.8. Now suppose that $|L' \cup R'| > 1$ and remove an arbitrary reflection s_j from $L' \cup R'$, say from L', the other case being similar. Write $L'' = L' \backslash \{s_j\}$. One can choose $(L'')' = L''$, $(R')' = R'$. Since $s \in L(w)$, it means by Proposition 3.3.5 that in the representation of $\psi_c(w)$ by disjoint unions of polygons, we has one of the two following configurations: either $(j, j+1)$ is an edge of a polygon of $\psi_c(w)$ with j initial, or j does not index any vertex of a polygon of $\psi_c(w)$ but one has a polygon of $\psi_c(w)$ with an edge $(k, j+1)$ where $k < j$. Now any reflection in L'' is distant from s_j and R' contains neither s_j nor s_{j-1}. Using Proposition 3.3.5 again this implies that any of the two possible configurations are preserved when reducing from $\psi_c(w)$ to $y := (\prod_{s\in L''} s)\psi_c(w)(\prod_{s\in R'} s)$ (the configuration with an edge $(j, j+1)$ is preserved and since $s_{j-1} \notin R'$ the only thing that can change the edge $(k, j+1)$ of the second configuration is in case we have an edge $(k, j+1)$ with $k < j-1$ and $s_k \in R'$; in that case the edge $(k, j+1)$ is replaced by an edge $(k+1, j+1)$ in y and y still has the second configuration since $k+1 < j$). In particular, using the same proposition, we get $s_j \in L(\varphi_c(y))$. Induction together with Corollary 3.3.6 conclude. □

Definition 3.3.11. *To each fully commutative element $w \in \mathcal{W}_f$, we will associate an element X_w of the Temperley-Lieb algebra called the* simplex *of w. Set*

$$Q_w := \{ \; x_{L,R} \mid L \subset L(w), R \subset R(w) \}.$$

We then define X_w by its coefficients when expressed in Zinno's basis:

$$X_w := \sum_{x \in \mathcal{P}_c} p_x^w Z_x,$$

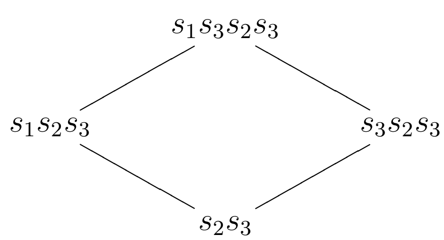
where $p_x^w = 0$ unless $x \in Q_w$. If $x \in Q_w$ then set

$$p_x^w := (-1)^{\ell_{\mathcal{S}}(w) + \ell_{\mathcal{S}}(\psi_c(w)) - \ell_{\mathcal{S}}(x)} v^{\ell_{\mathcal{T}}(x) - \ell_{\mathcal{T}}(\psi_c(w))}.$$

Remark 3.3.12. As a consequence of Proposition 3.3.10, one has $sQ_w = Q_w$ for any $s \in L(w)$ and $Q_w s = Q_w$ for any $s \in R(w)$. In particular, $|Q_w|$ is always a power of two and is at least two if $w \neq e$ since for any $w \in \mathcal{W}_f \backslash e$, $L(w) \cup R(w) \neq \emptyset$.

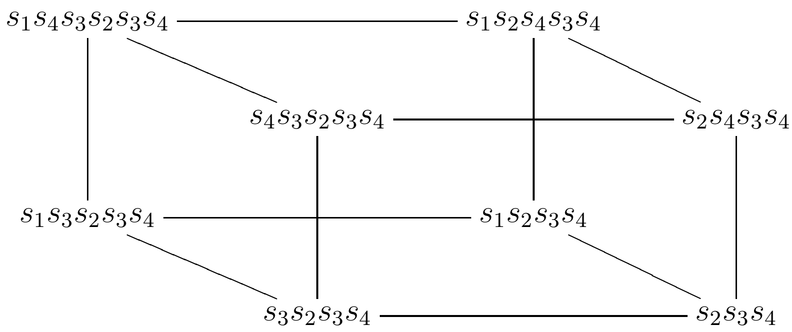
Example 3.3.13 $w = s_1 s_3 s_2$, $\psi_c(w) = s_1 s_3 s_2 s_3$

$Q_w =$



Example 3.3.14 $w = s_1 s_4 s_3 s_2$, $\psi_c(w) = s_1 s_4 s_3 s_2 s_3 s_4$

$Q_w =$



Remark 3.3.15. Notice that for $s \in \mathcal{S}$,

$$X_s = p_s^s Z_s + p_e^s = -Z_s + v^{-1} = b_s.$$

In general $X_w \neq b_w$.

Proposition 3.3.16. *The set $\{X_w\}_{w \in \mathcal{W}_f}$ is a basis of the Temperley-Lieb algebra.*

Proof. It suffices to order Zinno basis by the order on $\mathcal{P}_c$ given by any linear extension of the Bruhat order. One then orders the set $\{X_w\}_{\mathcal{W}_f}$ by the order on $\mathcal{W}_f$ obtained as the image of the order we put on $\mathcal{P}_c$ under the bijection φ_c. Thanks to Proposition 3.3.10, one then gets an upper triangular matrix with the invertible coefficients $\{p^w_{\psi_c(w)}\}_{w \in \mathcal{W}_f}$ on the diagonal, passing from the basis $\{Z_x\}_{x \in \mathcal{P}_c}$ to the set $\{X_w\}_{w \in \mathcal{W}_f}$. □

Remark 3.3.17. The order giving triangularity of the change of basis matrix between the bases X_w and Z_x is any linear extension of the Bruhat order on $\mathcal{P}_c$, together with the order induced on $\mathcal{W}_f$ by the bijection φ_c. It is the same order giving triangularity of the change of basis matrix between the diagram basis b_w and Z_x. As a consequence, these orders also give triangularity of the change of basis matrix between X_w and b_w, with invertible coefficient on the diagonal.

3.3.3 Application: change of basis matrix between the diagram and Zinno bases

Remark 3.3.18. Let $x \in \mathcal{P}_c$. Recall that $m_{\mathbf{x}}$ is a braid word representing the simple element $i_c(x)$. As we previously noticed, of one replaces any $\mathbf{s}_i^{\pm}$ by s_i in the braid word $m_{\mathbf{x}}$, then one obtains a Coxeter word m_x that is an $\mathcal{S}$-reduced expression of x. After being mapped to the Temperley-Lieb algebra, any letter $\mathbf{s}_i$ is replaced by $v^{-1} - b_i$ while each letter $\mathbf{s}_i^{-1}$ is replaced by $v - b_i$. As a consequence, if we expand the image of $m_{\mathbf{x}}$ in TL_n, we obtain a linear combination of elements of the form $b_{i_1}b_{i_2}\cdots b_{i_k}$ where $s_{i_1}s_{i_2}\cdots s_{i_k}$ is a subword of m_x. If $s_{i_1}s_{i_2}\cdots s_{i_k}$ is not an $\mathcal{S}$-reduced expression of a fully commutative element, then $b_{i_1}b_{i_2}\cdots b_{i_k}$ is not a reduced word, but it is equal to $(v+v^{-1})^m b_w$ for a unique pair $(m,w)\in\mathbb{Z}_{>0}\times\mathcal{W}_f$ and it is not difficult to show that w has an $\mathcal{S}$-reduced expression which is a subword of $s_{i_1}s_{i_2}\cdots s_{i_k}$. But $s_{i_1}s_{i_2}\cdots s_{i_k}$ was itself a subword of m_x. Since m_x is an $\mathcal{S}$-reduced expression of x, it follows that $w<_{\mathcal{S}} x$. Hence in the linear combination of Z_x in the diagram basis, the $w\in\mathcal{W}_f$ indexing the b_w which occur must satisfy $w<_{\mathcal{S}} x$.

Zinno orders the set of noncrossing partitions by the length of the braid word $m_{\mathbf{x}}$, which thanks to our observation is nothing but the Coxeter length $\ell_{\mathcal{S}}(x)$ of x. He then proves the following theorem, which is rewritten here using our notations and the observation above:

Theorem 3.3.19 (Zinno, [42], Theorem 5)**.** *Let $x\in\mathcal{P}_c$ and assume $w<_{\mathcal{S}} x$. If $w\neq\varphi_c(x)$, there exists an element $y\in\mathcal{P}_c$ such that $w<_{\mathcal{S}} y$ and $\ell_{\mathcal{S}}(y)<\ell_{\mathcal{S}}(x)$.*

Remark 3.3.20. It turns out that if one looks carefully at Zinno's proof, one sees that we can refine his conclusion by $y<_{\mathcal{S}} x$, $y\neq x$, which will be useful for a study of the coefficients of the change of basis matrix between the Zinno and diagram bases. In the following we will use this refinement. Zinno then uses this Theorem to prove that with the same assumptions as in the Theorem, one has then $\ell_{\mathcal{S}}(\psi_c(w))<\ell_{\mathcal{S}}(x)$. Again, it is not difficult to see from Zinno's proof that one can refine the conclusion by $\psi_c(w)<_{\mathcal{S}} x$, $\psi_c(w)\neq x$. However we will give a new approach in the following sections which will allow us to prove this result directly and in a much more general setting at the very end of the chapter. For the meanwhile we will just assume it. In fact, the surprising consequence of this result is that an order making the mentioned change of basis matrix upper triangular is any linear extension of the Bruhat order on $\mathcal{P}_c$, which is a non-natural order on $\mathcal{P}_c$ which we will study extensively in the next sections.

Lemma 3.3.21. *Let $w\in\mathcal{W}_f$, $s\in L(w)$. Then*

$$b_s X_w=(v+v^{-1})X_w.$$

Proof. Let $x\in Q_w$ such that $sx<_{\mathcal{S}} x$. Since $s\in L(w)$ one has that $sx\in Q_w\subset\mathcal{P}_c$ thanks to Remark 3.3.12. One has either $sx<_{\mathcal{T}} x$, in which case $Z_x=Z_sZ_{sx}$, or $x<_{\mathcal{T}} sx$, in which case $Z_x=Z_s^{-1}Z_{sx}$. Assume that $sx<_{\mathcal{T}} x$. One has $\ell_{\mathcal{T}}(sx)=$

$\ell_{\mathcal{T}}(x) - 1$ hence $p_x^w = -vp_{sx}^w$ so we get

$$\begin{aligned} b_s(p_{sx}^w Z_{sx} + p_x^w Z_x) &= (v^{-1} - Z_s)(p_{sx}^w Z_{sx} + p_x^w Z_x) \\ &= v^{-1}p_{sx}^w Z_{sx} - p_{sx}^w Z_s Z_{sx} + v^{-1}p_x^w Z_x - p_x^w Z_s^2 Z_{sx} \\ &= v^{-1}p_{sx}^w Z_{sx} + 2v^{-1}p_x^w Z_x - p_x^w(Z_s(v^{-1} - v) + 1)Z_{sx} \\ &= (v + v^{-1})(p_{sx}^w Z_{sx} + p_x^w Z_x). \end{aligned}$$

Now assume that $x <_{\mathcal{T}} sx$. One has $\ell_{\mathcal{T}}(sx) = \ell_{\mathcal{T}}(x) + 1$ hence $p_x^w = -v^{-1}p_{sx}^w$ so we get

$$\begin{aligned} b_s(p_{sx}^w Z_{sx} + p_x^w Z_x) &= (v - Z_s^{-1})(p_{sx}^w Z_{sx} + p_x^w Z_x) \\ &= vp_{sx}^w Z_{sx} - p_{sx}^w Z_x + vp_x^w Z_x - p_x^w Z_s^{-1} Z_x \\ &= vp_{sx}^w Z_{sx} + 2vp_x^w Z_x - p_x^w((v - v^{-1}) + Z_s)Z_x \\ &= (v + v^{-1})(p_{sx}^w Z_{sx} + p_x^w Z_x). \end{aligned}$$

Summing these equalities on all the couples (sx, x) one gets the result. ☐

Remark 3.3.22. One has of course a similar statement for $s \in R(w)$.

We now consider the linear expansion of an element b_w in the basis $X_{w'}$

$$b_w = \sum_{w' \in \mathcal{W}_f} q_{w'}^w X_{w'}$$

and we would like to understand for which w' one can have $q_w^{w'} \neq 0$. To this end, we write the element X_w in the Kazhdan-Lusztig basis as

$$X_w = \sum_{y \in \mathcal{W}_f} r_y^w b_y.$$

Notation. To each fully commutative element $w \in \mathcal{W}_f$ we associate a subset $F_w \subset \mathcal{W}_f$ defined by

$$F_w = \{y \in \mathcal{W}_f \mid L(y) \supset L(w), R(y) \supset R(w) \text{ and } \psi_c(y) <_{\mathcal{S}} \psi_c(w)\}.$$

Remark 3.3.23. Obviously one has $w \in F_w$ and if $y \in F_w$, then $F_y \subset F_w$.

Proposition 3.3.24. *If $r_y^w \neq 0$, then $y \in F_w$.*

Proof. Let $s \in L(w)$. Thanks to Lemma 3.3.21 one has that

$$\underbrace{b_s \left(\sum_{y \in \mathcal{W}_f} r_y^w b_y \right)}_{X_w} = (v + v^{-1}) \left(\sum_{y \in \mathcal{W}_f} r_y^w b_y \right).$$

Among all the y for which r_y^w is nonzero, choose an element y such that $\ell_{\mathcal{S}}(y)$ is maximal. It follows from this equality and the maximality of $\ell_{\mathcal{S}}(y)$ that in case $\ell_{\mathcal{S}}(sy) > \ell_{\mathcal{S}}(y)$, then sy cannot be a fully commutative element. In other words, when reducing $b_s b_y$, one has to apply the relation $b_s^2 = (v+v^{-1})b_s$ (in case $sy <_{\mathcal{S}} y$) or the relation $b_{s_i} b_{s_{i\pm1}} b_{s_i} = b_{s_i}$ where $s = s_i$ (in case $sy >_{\mathcal{S}} y$). In the first case y has an $\mathcal{S}$-reduced expression beginning with s, hence $s \in L(y)$ implying $b_s b_y = (v + v^{-1})b_y$. In the second case since b_y also appears in the right hand side of the equality above it means that there exists a fully commutative element y' such that $r_{y'}^w \neq 0$ having an $\mathcal{S}$-reduced expression beginning with $b_{s_{i\pm1}} b_{s_i}$. But such an element also occurs in the right hand side and cannot obviously come from an element $b_s b_{y''}$ with $y'' \in \mathcal{W}_f$, a contradiction. Hence it means that our element y has an $\mathcal{S}$-reduced expression beginning with s, that it, $s \in L(y)$ and that we can remove $b_s b_y = (v + v^{-1})b_y$ from both sides of the equality above obtaining

$$b_s \left(\sum_{z \in \mathcal{W}_f, z \neq y} r_z^w b_z \right) = (v + v^{-1}) \left(\sum_{z \in \mathcal{W}_f, z \neq y} r_z^w b_z \right).$$

One can then choose another element z with maximal Coxeter length among the remaining ones with nonzero coefficient and give the same argument to obtain that $s \in L(z)$ and so on until we run out of all the elements with nonzero coefficient. This proves that for any $s \in L(w)$, $s \in L(y)$ for any y such that $r_y^w \neq 0$. Doing the same for any $s \in R(w)$ one gets that for any y such that $r_y^w \neq 0$, $L(y) \supset L(w)$ and $R(y) \supset R(w)$.

Now if y is such that $r_y^w \neq 0$, one must have $y <_{\mathcal{S}} x$ for at least one $x \in Q_w$ by Remark 3.3.18. Thanks to Remark 3.3.20 we have $\psi_c(y) <_{\mathcal{S}} x$ and thanks to Proposition 3.3.10 one also has that $x <_{\mathcal{S}} \psi_c(w)$ giving $\psi_c(y) <_{\mathcal{S}} \psi_c(w)$. Therefore we have that $y \in F_w$. ☐

Proposition 3.3.25. *If $q_{w'}^w \neq 0$, then $w' \in F_w$.*

Proof. One argues by induction of $\ell_{\mathcal{S}}(\psi_c(w))$. If $\ell_{\mathcal{S}}(\psi_c(w)) = 1$ then w is a simple reflection. In that case by Remark 3.3.15 one has $b_w = X_w$ and the claim is trivially true since $F_w = \{w\}$. Now suppose that $\ell_{\mathcal{S}}(\psi_c(w)) > 1$. Thanks to the previous proposition we have that

$$X_w = \sum_{y \in F_w} r^w_y b_y,$$

in particular, $\psi_c(y) <_{\mathcal{S}} \psi_c(w)$, hence $\ell_{\mathcal{S}}(\psi_c(y)) < \ell_{\mathcal{S}}(\psi_c(w))$ in case $w \neq y$. Hence by induction one has that

$$b_y = \sum_{z \in F_y} q^y_z X_z$$

which we replace in the previous equality:

$$X_w = r^w_w b_w + \sum_{y \in F_w, y \neq w} r^w_y \left(\sum_{z \in F_y} q^y_z X_z \right).$$

But since $y \in F_w$, one has that $F_y \subset F_w$ (see Remark 3.3.23), hence the equality can be rewritten as

$$X_w = r^w_w b_w + \sum_{y \in F_w, y \neq w} \widetilde{q}^w_y X_y$$

for suitable polynomials $\widetilde{q}^w_y$, which concludes since r^w_w is invertible (3.3.17). □

Now write the expansion of an element b_w in Zinno basis as

$$b_w = \sum_{x \in \mathcal{P}_c} h^w_x Z_x.$$

As an immediate consequence of the proposition above we get:

Corollary 3.3.26. *If $x \notin \bigcup_{y \in F_w} Q_y$, then $h^w_x = 0$.*

Lemma 3.3.27 (Zinno, [42])**.** *Let $w \in \mathcal{W}_f$, $x = \psi_c(w)$. The coefficient of b_w in the expansion of Z_x in the diagram basis is equal to*

$$(-1)^{\ell_{\mathcal{S}}(w)} v^{-2k_w + \ell_{\mathcal{S}}(w) - \ell_{\mathcal{T}}(x)},$$

where k_w is the number of letters of $m_{\mathbf{x}}$ which have negative exponent and contribute to $w_{\mathbf{x}}$.

Proof. The coefficient on the diagonal is explicitely computed by Zinno in [42] at the end of section 6. Since we have different notations and conventions we sketch a proof. Let $x = \psi_c(w)$. Recall that Z_x is the image of the element of the braid group represented by the word $m_{\mathbf{x}}$ in the Temperley-Lieb algebra. It is obtained by replacing each letter $\mathbf{s}_i$ in $m_{\mathbf{x}}$ by $v^{-1} - b_i$ and each letter $\mathbf{s}_i^{-1}$ by $v - b_i$. Hence if we expand without reducing, we obtain $2^{\ell_{\mathcal{S}}(x)}$ different terms: for each $\mathbf{s}_i^{\pm 1}$ occurring in

$m_{\mathbf{x}}$ we can either choose the $-b_i$ or the $v^{\pm 1}$. Recall that there is a rule to read w_x which is a reduced expression for w as a subword of $m_{\mathbf{x}}$ that we recalled in example 3.2.19 and in the paragraphs above it. Zinno proves that among the $2^{\ell_{\mathcal{S}}(x)}$ terms which are (possibly non reduced) words in the b_i multiplied by a power of v, the term obtained by taking the b_i from any $\mathbf{s}_i^{\pm 1}$ contributing to w_x and taking the $v^{\pm 1}$ from any other $\mathbf{s}_i^{\pm 1}$ is the only term among the $2^{\ell_{\mathcal{S}}(x)}$ which is proportional to b_w[1]. But its coefficient is easily computed: each b_i which is contributed is multiplied by -1, and since a b_i is contributed exactly from the $\mathbf{s}_i^{\pm 1}$ contributing to w_x and since moreover w_x is an $\mathcal{S}$-reduced expression of w, this gives rise to a sign $(-1)^{\ell_{\mathcal{S}}(w)}$. Now each $\mathbf{s}_i^{\pm 1}$ not contributing to $w_{\mathbf{x}}$ must contribute its $v^{\pm 1}$. For any $\mathbf{s}_i^{-1}$ contributing to w_x, there is an $\mathbf{s}_i \mapsto v^{-1} - b_i$ at its right which does not contribute, giving a coefficient v^{-k_w}. Now if a $\mathbf{s}_i$ contributes to w_x, it means that $\mathbf{s}_i$ is the center of a syllable. As a consequence all the $\mathbf{s}_i^{-1}$ do not contribute to w_x. We need to count them. The number of occurrences of all the various $\mathbf{s}_i^{\pm 1}$ with s_i occurring at a center is given by $\ell_{\mathcal{S}}(x) - 2k_w$. We then need to subtract the centers and there are $\ell_{\mathcal{T}}(x)$ many of them. We then need to divide the result by two since we have here all the $\mathbf{s}_i^{\pm 1}$ such that the instance with positive exponent contribute with the centers removed, but any instance $\mathbf{s}_i$ of one of these comes with an instance of $\mathbf{s}_i^{\pm 1}$ in the same syllable since we removed the centers. Hence the power of v we obtain from the $\mathbf{s}_i$ not contributing to w_x is equal to

$$\frac{\ell_{\mathcal{S}}(x) - 2k_w - \ell_{\mathcal{T}}(x)}{2}$$

so the power of v we obtain before our b_w in the expansion is

$$-k_w + \frac{\ell_{\mathcal{S}}(x) - 2k_w - \ell_{\mathcal{T}}(x)}{2}.$$

One gets the claim using the equality

$$\frac{\ell_{\mathcal{S}}(x) - \ell_{\mathcal{T}}(x)}{2} = \ell_{\mathcal{S}}(w) - \ell_{\mathcal{T}}(x)$$

which holds since all the centers contribute to w_x: hence the left hand side is equal to all the contribution to w different from the centers (recall that any syllable contributes any of its reflections exactly once to w and that if s_i is not at the center, it occurs twice in the syllable). □

[1] This fact will be needed later to prove the triangularity of the change of basis matrices between Zinno bases (for arbitrary Coxeter elements) and the diagram basis at the end of chapter 3, where we will provide a proof of it in that more general setting.

Theorem 3.3.28. *Let $w \in \mathcal{W}_f$. Let $x \in Q_w$. Then*

$$h_x^w = (-1)^{\ell_{\mathcal{S}}(w)+\ell_{\mathcal{S}}(\psi_c(w))-\ell_{\mathcal{S}}(x)} v^{2k_w+\ell_{\mathcal{T}}(x)-\ell_{\mathcal{S}}(w)},$$

where k_w is the number of letters of $m_{\psi_{\mathbf{c}}(\mathbf{w})}$ which have negative exponent and contribute to $w_{\psi_{\mathbf{c}}(\mathbf{w})}$.

Proof. This is a consequence of the fact that if $y \in F_w, y \neq w$, then $Q_w \cap Q_y = \emptyset$. Indeed, assume that $x \in Q_w \cap Q_y$. Then there exists two sets $L' \subset L(w)$, $R' \subset R(w)$ such that

$$(\prod_{s \in L'} s) x (\prod_{s \in R'} s) = \psi_c(w).$$

Since $L(y) \supset L(w)$ and $R(y) \supset R(w)$ and $x \in Q_y$, one also has using Remark 3.3.12 that

$$(\prod_{s \in L'} s) x (\prod_{s \in R'} s) \in Q_y.$$

But by definition of F_w, $\psi_c(y) <_{\mathcal{S}} \psi_c(w) = x$, $\psi_c(w) \neq \psi_c(y)$ and any element $z \in Q_y$ satisfies $z <_{\mathcal{S}} \psi_c(y)$. Hence $x <_{\mathcal{S}} \psi_c(y) <_{\mathcal{S}} \psi_c(w) = x$, a contradiction.

As a consequence of this observation together with Corollary 3.3.26, if one knows the coefficient of $Z_{\psi_c(w)}$ in the expansion of b_w, one knows the coefficient of any Z_x for $x \in Q_w$ since the only element of the simplex-basis which can contribute elements Z_x for $x \in Q_w$ is X_w. Using Lemma 3.3.27 we have that the inverse coefficient of b_w in the expansion of $Z_{\psi_c(w)}$ is equal to $(-1)^{\ell_{\mathcal{S}}(w)} v^{-2k_w+\ell_{\mathcal{S}}(w)-\ell_{\mathcal{T}}(\psi_c(w))}$. Therefore since the change of basis matrix is upper triangular with invertible coefficient on the diagonal one has that the coefficient of $Z_{\psi_c(w)}$ in the expansion of b_w is given by

$$(-1)^{\ell_{\mathcal{S}}(w)} v^{2k_w-\ell_{\mathcal{S}}(w)+\ell_{\mathcal{T}}(\psi_c(w))}.$$

Using the fact that

$$b_w = \sum_{w' \in F_w} q_{w'}^w X_{w'}$$

and that any element Z_x with $x \in Q_w$ is contributed exclusively by X_w, one has that

$$q_w^w p_{\psi_c(w)}^w = (-1)^{\ell_{\mathcal{S}}(w)} v^{2k_w-\ell_{\mathcal{S}}(w)+\ell_{\mathcal{T}}(\psi_c(w))},$$

hence $q_w^w = v^{2k_w-\ell_{\mathcal{S}}(w)+\ell_{\mathcal{T}}(\psi_c(w))}$ since $p_{\psi_c(w)}^w = (-1)^{\ell_{\mathcal{S}}(w)}$. Hence for any $x \in Q_w$ we obtain

$$h_x^w = q_w^w p_x^w = (-1)^{\ell_{\mathcal{S}}(w)+\ell_{\mathcal{S}}(\psi_c(w))-\ell_{\mathcal{S}}(x)} v^{2k_w+\ell_{\mathcal{T}}(x)-\ell_{\mathcal{S}}(w)},$$

as claimed. ∎

Remark 3.3.29. In the next section we will give a combinatorial definition of the coefficient k_w that will avoid using the braid word $m_{\psi_{\mathbf{c}}(\mathbf{w})}$ lifting $\psi_c(w)$ in the braid group (see Remark 3.4.2).

3.4 Noncrossing partitions and vectors with parity conditions

3.4.1 Vectors with parity conditions

Notation. We will often distinguish between elements of the Coxeter group and Coxeter words representing them; in case $w, w' \in \mathfrak{S}_{n+1}$, we will denote their product by ww'. In case w, w' are Coxeter words, we will denote by $w \star w'$ the word obtained by concatenation.

We order the set of polygons $P_1, \dots P_r$ occurring in the geometric representation of $x \in \mathcal{P}_c$ by ascending order of the terminal index of each polygon. To each $x \in \mathcal{P}_c$ and each $m \in \{1, \dots, n\}$ one defines an integer $x_m \geq 0$:

$$x_m := 2|\{i \mid m \text{ is nested in } P_i\}| + 1_{m \in D_x^c},$$

where $1_{m \in D_x^c} = 1$ if $m \in D_x^c$ and $1_{m \in D_x^c} = 0$ if $m \notin D_x^c$. In particular x_m is odd if and only if $m \in D_x^c$. For convenience we will write $N(m) := |\{i \mid m \text{ is nested in } P_i\}|$ and omit the dependance on x. We write v_x for the element of $(\mathbb{Z}_{\geq 0})^n$ having x_k as k^{th} component.

Remark 3.4.1. Any polygon $P_i = [i_1 i_2 \cdots i_k]$ occurring in the geometric representation of $x \in \mathcal{P}_c$ represents an element $y_i \in \mathcal{P}_c$. As element of the symmetric group y_i is the cycle $(i_1, i_2, \dots, i_k)$. Let $j < k$. In the framework of Coxeter theory, a reduced expression for a transposition (j, k) is given by the word $[j, k] := s_{k-1}s_{k-2} \cdots s_{j+1}s_j s_{j+1} \cdots s_{k-2}s_{k-1}$. The word m_i obtained by the concatenation of such words

$$m_i := [i_1, i_2] \star [i_2, i_3] \star \cdots \star [i_{k-1}, i_k]$$

yields a reduced expression for y_i. With $P_1, \dots P_r$ ordered as above one has $x = y_1 y_2 \cdots y_r$ and the concatenation $m_1 \star m_2 \star \cdots \star m_r$ of the words yields a Coxeter word which we shall write m_x. The Coxeter word m_x is an $\mathcal{S}$-reduced expression for $x \in \mathcal{P}_c$ (this is easy to see by induction on the number of polygons of x if one keeps in mind that the Coxeter length $\ell_{\mathcal{S}}(\sigma)$ of a permutation $\sigma \in \mathfrak{S}_{n+1}$ is equal to the number of $i < j$ such that $\sigma(i) > \sigma(j)$). In particular, one has that $\sum_{i=1}^{n} x_i = \ell_{\mathcal{S}}(x)$. This is exactly the word m_x we already defined in subsection 3.2.2.

Remark 3.4.2. The integer k_w defined in Theorem 3.3.28 can easily be defined using the vector $v_{\psi_c(w)}$: recall that it was the number of generators with negative exponent in the word $m_{\mathbf{x}}$ which contribute to the subword $w_{\mathbf{x}}$ (here $x = \psi_c(w)$). In other words, these are all the contributions to the subword w_x which come from the left parts of the various syllables of m_x. But an element from the left of a syllable contributes if and only if it is not a center. As a consequence, if one sums all the even components of $v_{\psi_c(w)}$, one gets twice the contributions from generators with negative exponent in $m_{\mathbf{x}}$ since for any occurrence of a simple reflection in a left part of a syllable, there is an occurrence of the same reflection in the right part of the same syllable. Therefore we have that

$$k_w = \frac{1}{2} \sum_{\psi_c(w)_i \text{ even}} \psi_c(w)_i.$$

Definition 3.4.3. *Let $x \in \mathcal{P}_c$. We will say that x is in* standard form *if it is represented by the Coxeter word m_x or that m_x is the* standard form *of x. One then has*

$$x_m = \textit{number of occurrences of } s_m \textit{ in } m_x.$$

Notice that we made a specific choice of $\mathcal{S}$-reduced expression of x and that such a definition of x_m depends on that choice. The subwords $[i_{j-1}, i_j]$ of y_i or m_x will be called the syllables *of y_i or m_x, the reflection $s_{i_{j-1}}$ will be the* center *of that syllable and the reflection s_{i_j-1} will be said to occur at the* top *of the syllable (it occurs twice, on the very left and the very right of the syllable).*

As a consequence, note that the sum of the x_i with i running between 1 and n is just the Coxeter length $\ell_S(x)$ of x.

Lemma 3.4.4. *The vector v_x where $x \in \mathcal{P}_c$ has the following properties:*

1. *If x_m is even and $x_{m+1} > x_m$, then $x_{m+1} = x_m + 1$.*
2. *If x_m is odd and $x_{m+1} > x_m$, then $x_{m+1} = x_m + 1$ or $x_m + 2$.*
3. *If x_m is even and $x_{m+1} < x_m$, then ($x_{m+1} = x_m - 1$ or $x_{m+1} = x_m - 2$) and ($m + 1 \in U_x^c$).*
4. *If x_m is odd and $x_{m+1} < x_m$, then $x_{m+1} = x_m - 1$ and $m + 1 \in U_x^c$ ($m + 1$ is even maximal in its polygon).*

5. *The integer $m \neq n+1$ lies in U_x^c in exactly two situations:*

   (a) *if x_m is odd and ($x_{m-1} = x_m$ or $x_{m-1} = x_m + 1$)*

   (b) *if x_m is even and $x_m < x_{m-1}$.*

   The integer $m = n+1$ lies in U_x^c if and only if $x_{m-1} > 0$.

Proof. 1 and 3. If x_m is even, then $N(m) = N(m+1)$ except in case $m+1 \in U_x^c$ where one has $N(m+1) = N(m) - 1$. In that last case, one has $x_{m+1} = x_m - 2$ if $m+1 \notin D_x^c$ and $x_{m+1} = x_m - 1$ if $m+1 \in D_x^c$. If $N(m) = N(m+1)$ one has $x_m = x_{m+1}$ if $m+1 \notin D_x^c$ and $x_{m+1} = x_m + 1$ if $m+1 \in D_x^c$.

2 and 4. One has $m \in D_x^c$. In particular, there exists a polygon P of x such that $m \in P$ but m is not terminal in P. If $m+1 \in P$ and $m+1$ is terminal then $x_{m+1} = x_m - 1$ since $m \in D_x^c$, $m+1 \notin D_x^c$ and $N(m) = N(m+1)$. If $m+1 \in P$ with $m+1$ not terminal then $x_m = x_{m-1}$. If $m+1 \notin P$ then $m+1$ is nested in P (since $m \in D_x^c \cap P$) implying $N(m+1) = N(m)+1$. We then have $x_{m+1} = x_m + 1$ if $m+1 \notin D_x^c$ and $x_{m+1} = x_m + 2$ if $m+1 \in D_x^c$ (which is possible if $m+1$ is initial).

5. First suppose $m \in U_x^c$ and write P for the polygon such that $m \in P$. If $m \in D_x^c$, then x_m is odd and one gets $x_{m-1} = x_m$ if $m-1 \in P$ and $x_{m-1} = x_m + 1$ if $m-1 \notin P$ (in which case $m-1$ has to be nested in P since $m \in U_x^c$ but cannot lie in D_x^c). If $m \notin D_x^c$ then x_m is even. One then has $x_{m-1} = x_m + 1$ if $m-1 \in P$ and $x_{m-1} = x_m + 2$ if $m-1 \notin P$ since in that case $m-1$ must be nested in P (because m is maximal in P). For the converse, all the situations where $x_m \neq x_{m-1}$ are given by points 3 and 4. It remains to show that if $x_m = x_{m-1}$ and both are odd, then $m \in U_x^c$. If m and $m-1$ are not lying in the same polygon P, the assumption $m-1 \in D_x^c$ implies that m is nested in P which would contradict $x_m = x_{m-1}$. Hence they lie in the same polygon P and $m \in U_x^c$. Now consider the case of $n+1$; if $n+1 \in U_x^c$ then there is a polygon P with $n+1 \in P$. If $n \in P$, then $x_n = 1$; if $n \notin P$, then n is nested in P and $x_n = 2$. Conversely if $x_n > 0$, then either $n \in D_x^c$ forcing $n+1 \in U_x^c$ or n is nested in a polygon P which therefore needs to have $n+1$ as vertex. □

Corollary 3.4.5. *Let $x, y \in \mathcal{P}_c$. Then $x = y$ if and only if $v_x = v_y$.*

Proof. If $x \neq y$, then $(D_x^c, U_x^c) \neq (D_y^c, U_y^c)$ by Lemma 3.2.8. Since $m \in D_x^c$ if and only if x_m is odd and by point 5 of the lemma above, one then has $v_x \neq v_y$. □

Proposition 3.4.6. *The map $x \mapsto v_x$, $x \in \mathcal{P}_c$ defines a bijection from $\mathcal{P}_c$ to the set of vectors $w \in (\mathbb{Z}_{\geq 0})^n$ with the following properties, where w_k is the k-th component of w:*

- *If k is the smallest k with $w_k \neq 0$, then $w_k = 1$.*
- *If k is the largest k with $w_k \neq 0$, then $w_k = 1$ or $w_k = 2$.*
- *If w_m is even and $w_{m+1} > w_m$ then $w_{m+1} = w_m + 1$.*
- *If w_m is odd and $w_{m+1} > w_m$ then $w_{m+1} = w_m + 1$ or $w_{m+1} = w_m + 2$.*
- *If w_m is even and $w_{m+1} < w_m$, then $w_{m+1} = w_m - 1$ or $w_{m+1} = w_m - 2$.*
- *If w_m is odd and $w_{m+1} < w_m$, then $w_{m+1} = w_m - 1$.*

It is convenient to represent the last four conditions as follows:

$w_{m+1} - w_m$	w_{m+1} even	w_{m+1} odd
w_m even	-2 or 0	1 or -1
w_m odd	1 or -1	2 or 0

Examples of vectors satisfying these conditions are given in example 3.4.7.

Proof. The fact that $x \mapsto v_x$ is an injection is a consequence of the previous lemma and corollary. Conversely, we show by induction on the number of k such that $w_k \neq 0$ that any w with the above properties is equal to v_x for some $x \in \mathcal{P}_c$. If all components are zero then $w = v_e$. Now suppose that w has $p > 0$ nonzero components. Consider k largest with $w_k \neq 0$. If $w_k = 1$, then the above conditions imply that $w_{k-1} = 0, 1$ or 2. The vector w' obtained from w by replacing w_k by 0 still satisfies the above six conditions: this is obvious if $w_{k-1} = 1$ or 2 and if $w_{k-1} = 0$, one has to show that $w_{k-\ell}$ where ℓ is the smallest interger $\ell > 1$ such that $w_{k-\ell} > 0$ is either 1 or 2 in order to satisfy the second condition; but this is a consequence of the last two conditions on w. By induction, $w' = v_x$ for some $x \in \mathcal{P}_c$ and since $(w')_m = 0$ for $m \geq k$, xs_k also lies in $\mathcal{P}_c$ and $v_{xs_k} = w$.

If $w_k = 2$, the conditions imply that $w_{k-1} = 1, 2, 3$ or 4. If $w_{k-1} = 1$, one has $w_{k-2} = 0, 1$ or 2 and if one replaces w by the vector w' obtained from w by replacing w_k and w_{k-1} by zero one still gets a vector satisfying the conditions, hence by induction there must exist $x \in \mathcal{P}_c$ with $v_x = w'$. Moreover the largest integer indexing a vertex of a polygon of x must be smaller than or equal to $k - 1$. As a consequence $y := xs_k s_{k-1} s_k \in \mathcal{P}_c$ and $v_y = w$. If $w_{k-1} \geq 2$, consider the smallest $\ell > 1$ such that $w_{k-\ell} < 2$. The conditions imply that $w_{k-\ell}$ must be equal to 1. We then subtract from w the vector z such that $z_{k-\ell} = 1$, $z_{k-\ell+i} = 2$ for all $1 \leq i \leq \ell$ and with all other components equal to zero. Write w' for $w - z$. The conditions imply that the last nonzero component of w' is equal to $0, 1$ or 2 and subtracting 2 from any component between $w_{k-\ell+1}$ and w_k does not change the parity of these integers, hence for m and $m+1$ both lying in $\{k-\ell+1, \ldots, k\}$, the conditions on w'_m and w'_{m+1} still hold. It remains to check that the conditions hold for $w'_{k-\ell-1}$, $w'_{k-\ell} = 0$ and $w'_{k-\ell+1}$. Since $w_{k-\ell-1}$ is followed by $w_{k-\ell} = 1$, it has to be equal to $0, 1$ or 2. So we have $w'_{k-\ell-1} = 0, 1$ or 2 followed by $w'_{k-\ell} = 0$ still satisfying our conditions. Now $w_{k-\ell+1} = 2$ or 3 (recall that it is ≥ 2) hence $w'_{k-\ell+1} = 0, 1$ with $w'_{k-\ell} = 0$ which still satisfy our conditions. Again, it is easy to see that the largest nonzero component of w' is either equal to 1 or 2 hence by induction $w' = v_x$ for $x \in \mathcal{P}_c$ with $w'_{k-\ell} = 0 = w'_k$; it means that in the geometrical representation of x, $k-\ell$ is either terminal in a polygon P, or alone, and $k+1$ is alone. One can add the edge $(k-\ell, k+1)$ to P or simply add it if $k-\ell$ is alone to get a noncrossing partition y with $v_y = w$. □

We will denote by $\mathcal{V}$ the set of vectors in $(\mathbb{Z}_{\geq 0})^n$ satisfying the properties of Proposition 3.4.6. Since noncrossing partitions have Catalan enumeration, the same holds for $\mathcal{V}$ by the proposition above.

Example 3.4.7 If $n=2$ the five elements of $\mathcal{V}$ are given by $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$, $(1,2)$. If $n=3$ the fourteen elements of $\mathcal{V}$ are given by $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$, $(1,1,1)$, $(1,2,0)$, $(0,1,2)$, $(1,2,1)$, $(1,1,2)$, $(1,2,2)$, $(1,3,2)$.

3.5 A criterion for Bruhat order on noncrossing partitions

3.5.1 The criterion

We denote by $<_{\mathcal{S}}$ the Bruhat order on $\mathfrak{S}_{n+1}$. We will use the characterization given by Theorem 2.1.5 of [6] which we recall below. We use the same notation as in [6], that is, for $x \in \mathfrak{S}_{n+1}$, $i, j \in \{1, \ldots, n+1\}$,

$$x[i,j] := |\{a \in \{1, \ldots, i\} \mid x(a) \geq j\}|.$$

Theorem 3.5.1 ([6], Theorem 2.1.5)**.** *Let $x, y \in \mathfrak{S}_{n+1}$. The following are equivalent:*

1. $x <_{\mathcal{S}} y$
2. $x[i,j] \leq y[i,j]$, *for all* $i, j \in \{1, \ldots, n+1\}$.

According to [5], this criterion is due to Ehresmann (see [15]).

Proposition 3.5.2. *For $x, y \in \mathcal{P}_c$, one has*

$$x <_{\mathcal{S}} y \Rightarrow \forall k, x_k \leq y_k.$$

Proof. Suppose that there exists k with $x_k > y_k$. If suffices to see that for a non-crossing partition x, one has $x[k-1, k+1] = j$ if $x_k = 2j$ or if $x_k = 2j+1$, and $x[k, k+1] = j$ if $x_k = 2j$ and $j+1$ if $x_k = 2j+1$. If x_k and y_k have the same parity we then have $x[k, k+1] > y[k, k+1]$. If x_k is even and y_k is odd we have $x[k-1, k+1] > y[k-1, k+1]$. If x_k is odd and y_k is even we have $x[k, k+1] > y[k, k+1]$. Hence by Theorem 3.5.1 we have $x \not<_{\mathcal{S}} y$. □

The aim now is to prove the converse of Proposition 3.5.2. Set

$$\Sigma := \sum_k y_k - x_k.$$

Lemma 3.5.3. *Suppose that $x_k \leq y_k$ for all k. If $\Sigma = 1$, then $x <_{\mathcal{S}} y$.*

Proof. Since $\Sigma = 1$, there exists a unique k such that $x_k \neq y_k$ and $y_k = x_k + 1$. If x_k is odd and $k \notin U_y^c$ consider the reflection (i, j), $j > i$, with smallest $j - i$ and such that (i, j) is an edge of a polygon P of y in which k is nested (it always exists since y_k is even and $y_k > 0$). It suffices to add the vertex k to the polygon P to obtain an element $x' \in \mathcal{P}_c$ satisfying $v_{x'} = v_x$ hence $x = x'$ by Corollary 3.4.5; this is possible since k lies neither in U_y^c nor in D_y^c. Indeed, we replaced the subword $[i, j]$ in m_y by the product $[i, k] \star [k, j]$ of the words representing the reflections (i, k) and (k, j), removing one occurrence of s_k. Hence the word m_x which is a reduced expression for x is equal (up to commuting syllables) to a subword of m_y implying $x <_{\mathcal{S}} y$. If x_k is odd and $k \in U_y^c$, we prove that $k - 1 \notin D_x^c$: if $k - 1 \in D_x^c$ one has $k - 1 \in D_y^c$ hence $k - 1$ and k lie in the same polygon P of y with k terminal (because y_k is even) hence

$$x_{k-1} = y_{k-1} = y_k + 1 = x_k + 2$$

which by Lemma 3.4.4 is impossible since both x_{k-1} and x_k are odd. But $k-1 \notin D_x^c$ if and only if $k - 1 \notin D_y^c$, which implies that y_{k-1} is even with $y_{k-1} > y_k$ since $k - 1$ is nested in the polygon of y having k as vertex (because $k \in U_y^c$). But y_k and y_{k-1} are both even, hence we have a contradiction with Lemma 3.4.4 again since

$$x_{k-1} = y_{k-1} = y_k + 2 = x_k + 3.$$

If x_k is even, then y_k is odd. In that case, there exists a polygon P of y such that k is a non terminal vertex of P. Let ℓ be the vertex of P following k. If $\ell \neq k + 1$, one has a contradiction with 3.4.4 since

$$x_{k+1} = y_{k+1} \geq y_k + 1 = x_k + 2.$$

If $\ell = k + 1$, splitting the polygon P into two polygons by removing the edge $(k, k+1)$ yields an element $x' \in \mathcal{P}_c$;in the reduced expression m_y of y we just removed an occurrence of s_k to obtain (up to commuting syllables) a word which is still a standard form $m_{x'}$ of an element $x' \in \mathcal{P}_c$, implying $v_{x'} = v_x$ and hence $x' = x$ by 3.4.5. Since $m_{x'}$ is a subword of m_y we have $x <_{\mathcal{S}} y$. □

Before proving the case $\Sigma > 1$ we give some properties of standard forms:

Lemma 3.5.4. *Let $x \in \mathcal{P}_c$ with corresponding standard form m_x.*

1. *A simple reflection s_k is the center of at most one syllable of m_x. If it is the center of a syllable, then its first occurence in m_x when reading m_x from the left to the right is at the center of that syllable.*

2. *If s_k and s_{k+1} occur in the same syllable w_i, then they have the same number of occurrences in m_x on the right of w_i and that number is even.*

3. *Suppose that s_k is at the center of a syllable w_i of m_x and that s_{k+1} occurs on the left of w_i. Then s_{k+1} is at the center of a syllable and occurs in w_i.*

4. *Write $m_x = w_1 \star w_2 \star \cdots \star w_m$ as the concatenation of its syllables. If s_k and s_{k-1} both occur in w_i and do not occur in w_j for $j < i$, then we can replace $w_i = [q, p+1]$ in the word m_x by the product of the two syllables $w_i' = [q, k] \star [k, p]$ to obtain a word which is still a standard form of an element $y \in \mathcal{P}_c$.*

Proof. 1. The fact that s_k occurs at most once as a center is obvious. An occurrence at the center (equivalent to $k \in D_x^c$, in particular $k \in P$ for some polygon P of x) must be a first one since other occurrences of s_k have to come from polygons in which k is nested, hence the subwords corresponding to such polygons must occur in m_x after the subword corresponding to P since their maximal index is bigger than the maximal index of P.

2. Using point 1, s_k and s_{k+1} cannot be the centers of syllables on the right of w_i. Since they occur both in w_i, they have to occur together in any syllable on the right of w_i in which they occur, otherwise the noncrossing condition would not be satisfied (recall that the polygons are ordered by ascending order of their maximal indices). Since they are not at the center, they occur twice in any such syllable.

3. If s_{k+1} is not the center of any syllable, then the noncrossing condition is not satisfied. If $w_i = s_k$, the syllable on the left of w_i containing s_{k+1} corresponds to an

edge of a polygon having biggest index bigger than $k+1$, which is a contradiction, since the subword corresponding to such a polygon should appear after w_i in m_x. Hence $w_i \neq s_k$. Since s_k is by assumption at the center, s_{k+1} has to occur in w_i.

4. This means exactly that k is nested in the polygon P corresponding to w_i, that there cannot be any other polygon located between the point labeled with k and P in which k is nested, and that k lies in no polygon. Therefore we can enlarge P by adding the vertex labeled with k. From the point of view of standard forms this operation corresponds exactly to what is claimed (see the first situation of figure 3.7). □

We now deal with the case $\Sigma > 1$ giving a converse to Proposition 3.5.2. Recall that the word $m_y = y_1 y_2 \cdots y_r$ is the concatenation of the words y_i associated to the polygons of y, each of which is the concatenation of the syllables (that is, the chosen reduced expressions for reflections; see Remark 3.4.1).

Proposition 3.5.5. *Suppose that $x_k \leq y_k$ for all k. If $\Sigma > 1$, then $x <_{\mathcal{S}} y$.*

Proof. Using induction on Σ (the case $\Sigma = 1$ is given by Lemma 3.5.3) it suffices to find an element $x' \in \mathcal{P}_c$ with $x' \neq x, y$ and such that for all k, $x_k \leq x'_k \leq y_k$. Consider the smallest k with $x_k < y_k$. Write $m_y = w_1 \star w_2 \star \cdots \star w_m$ as the concatenation of its syllables. Consider the smallest index i such that s_k occurs in w_i.

Let $w_i = [q, p+1]$ and assume $q < k < p$. In particular, s_{k-1} must occur in w_i. If s_{k-1} does not appear in w_j for $j < i$ then thanks to point 4 of 3.5.4 we can replace w_i in m_y by the concatenation of the syllables $[q,k] \star [k, p+1]$ giving a word m' still representing an element $x' \in \mathcal{P}_c$ in standard form, with $x'_k = y_k - 1 \geq x_k$, $x'_r = y_r \geq x_k$ for $r \neq k$ (see the first situation of figure 3.7 for an illustration). If s_{k-1} occurs in w_j for $j < i$ then by point 1 of 3.5.4, s_{k-1} cannot be at the center of w_i; hence using point 2 of Lemma 3.5.4 we see that $y_{k-1} > y_k$. But $x_{k-1} = y_{k-1}$ by definition of k. Hence $x_{k-1} > y_k > x_k$. By Lemma 3.4.4 it implies that x_{k-1} and x_k are even. But y_k is even since s_k is not at the center of w_i and does not occur in w_j for $j < i$, hence cannot be a center by point 1 of Lemma 3.5.4. We obtain that $x_{k-1} \geq x_k + 4$ which contradicts Lemma 3.4.4.

Now suppose that s_k is the center of w_i. If $w_i = s_k$, then we can remove $w_i = s_k$ from m_y to obtain a required word which is (up to commuting syllables) a standard form of an element $x' \in \mathcal{P}_c$. So we can suppose that w_i is not reduced to s_k, that is, $w_i = [k, p+1]$ with $p > k$. If s_{k+1} does not occur in w_j for $j < i$ and $x_{k+1} < y_{k+1}$ then using point 4 of 3.5.4 we can replace w_i by $[k, k+1] \star [k+1, p+1]$ which concludes since we remove one occurrence of s_{k+1} (see the second situation of figure 3.7). If $x_{k+1} = y_{k+1}$, we get that y_{k+1} is even (since in that case s_{k+1} is not the center of a syllable) and $y_{k+1} > y_k$ by point 2 of Lemma 3.5.4, giving that $x_{k+1} \geq x_k + 2$ with

$x_{k+1} = y_{k+1}$ even, contradicting Lemma 3.4.4. If s_{k+1} occurs in w_j for $j < i$ then it has to be at the center of its syllable by point 3 of Lemma 3.5.4. If $x_{k+1} = y_{k+1}$ one has

$$x_{k+1} = y_{k+1} = y_k + 2 > x_k + 2$$

which contradicts Lemma 3.4.4 again. If $x_{k+1} + 1 = y_{k+1}$ one has

$$x_{k+1} + 1 = y_{k+1} = y_k + 2 > x_k + 2$$

but x_{k+1} is even (since y_{k+1} is odd and $y_{k+1} = x_{k+1} + 1$) which is impossible. Hence we can suppose that $x_{k+1} + 2 \leq y_{k+1}$. Now consider the smallest $\ell \geq 1$ such that $s_{k+\ell+1}$ is not at the center of any w_j for $j < i$. Using point 3 of Lemma 3.5.4 inductively we see that for any $1 < i \leq \ell$,

$$y_{k+i} = y_{k+i-1} + 2.$$

Lemma 3.4.4 together with the inequality above give $x_{k+i} + 2 \leq y_{k+i}$ which holds in particular for $i = \ell$. If $s_{k+\ell}$ is alone in the syllable $[k+\ell, p+1]$ of which it is the center (that it, $k + \ell = p$), then we can remove it from m_y to obtain a word in standard form which is (up to commuting syllables) a standard form of a required element $x' \in \mathcal{P}_c$. Otherwise $y_{k+\ell+1} = y_{k+\ell}+1$ implying by Lemma 3.4.4 that $x_{k+\ell+1} < y_{k+\ell+1}$. By definition of ℓ we have that $s_{k+\ell+1}$ is not at the center of any w_j for $j < i$; hence by point 3 of Lemma 3.5.4 we have that $s_{k+\ell+1}$ does not occur in m_y on the left of $[k+\ell, p+1]$. So we can replace $[k+\ell, p+1]$ in m_y by the product $s_{k+\ell} \star [k+\ell+1, p+1]$ to obtain a word which is still a standard form of a required element $x' \in \mathcal{P}_c$; this is also illustrated by the second situation of figure 3.7, but the k on the picture is our $k + \ell$.

Now suppose that s_k is at the top of w_i (with $w_i \neq s_k$). If s_{k-1} is not at the top of w_j for $j < i$ then one can replace $w_i = [q, k+1]$ in m_y by the product of two syllables $[q, k] \star [k, k+1]$ giving again a word in standard form (see the third situation of figure 3.7). So we can suppose that s_{k-1} is at the top of w_j for $j < i$. By definition of k we have $x_{k-1} = y_{k-1}$. Since $w_i \neq s_k$, s_{k-1} has to occur in w_i hence we get

$$x_{k-1} = y_{k-1} = y_k + 2 > x_k + 2$$

which contradicts Lemma 3.4.4 again. □

Putting 3.5.2, 3.5.3 and 3.5.5 together we have proven

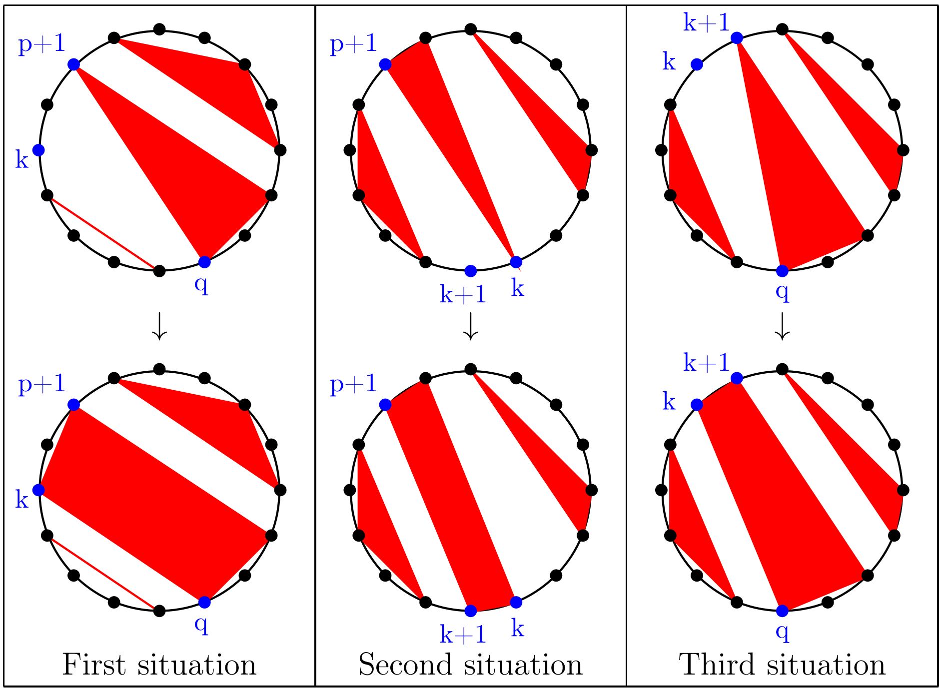


FIG. 3.7: Illustration of various local operations on noncrossing partitions:we associate to any noncrossing partition x at the top a noncrossing partition y at the bottom such that the sum of the components of the corresponding vectors decreases by one.

Theorem 3.5.6. *let $x, y \in \mathcal{P}_c$. Then,*

$$x <_{\mathcal{S}} y \text{ if and only if } \forall k, x_k \leq y_k.$$

3.5.2 Covering relations

In Proposition 3.5.5, we showed that if $\Sigma > 1$, there always exists a noncrossing partition x' such that for any k, $x_k \leq x'_k \leq y_k$. But the sum of the components of the vector is the Coxeter length of the noncrossing partition. This implies in particular that x is covered by y if and only if $x < y$, $\ell_{\mathcal{S}}(x) = \ell_{\mathcal{S}}(y) - 1$. In that case we are exactly in the setting of Lemma 3.5.3; the proof of that lemma shows what the covering relations are: suppose that $x_k = y_k - 1$ and all other components agree; then either x is obtained by y by splitting a polygon P of y having both k and $k+1$ labelling its vertices into two (possibly empty) polygons P_1 and P_2 by removing the edge joining k and $k+1$; or x is obtained from y by adding the vertex k to the polygon of P which is the closest to y among the polygons in which k is nested.

As a consequence, the poset of noncrossing partitions with the Bruhat order is a graded poset, the rank function being simply the Coxeter length.

3.6 New lattice structure on $\mathcal{P}_c$ and related combinatorial considerations

3.6.1 Lattice property

We now associate to any two elements $x, y \in \mathcal{P}_c$ two vectors $z(x, y)$ and $w(x, y)$ in $(\mathbb{Z}_{\geq 0})^n$ by setting

$$z(x, y)_k = \min(x_k, y_k),\ w(x, y)_k = \max(x_k, y_k),$$

with the notations from the previous section. We shall use them to prove:

Theorem 3.6.1. *The poset $(\mathcal{P}_c, \leq)$ is a lattice.*

Proof. Thanks to Proposition 3.4.6 and Theorem 3.5.6, it suffices to show that $z := z(x, y)$ and $w := w(x, y)$ lie in $\mathcal{V}$, hence to prove that both satisfy the conditions of Proposition 3.4.6. The first two conditions are clearly true for both z and w. We prove the last four conditions first for z. One can without loss of generality suppose

$z_m = x_m \neq y_m$ and $z_{m+1} = y_{m+1} \neq x_{m+1}$, otherwise the conditions are inherited from the conditions on x and y. If $z_{m+1} > z_m$ one has

$$z_m = x_m < z_{m+1} = y_{m+1} < x_{m+1}$$

which by Lemma 3.4.4 is possible only if x_m is odd and $x_{m+1} = x_m + 2$, hence z_m is odd and $z_{m+1} = z_m + 1$. If $z_{m+1} < z_m$ then

$$z_{m+1} = y_{m+1} < z_m = x_m < y_m$$

which is possible only if $y_m = y_{m+1}+2$, y_m is even hence z_m is odd and $z_{m+1} = z_m-1$. Therefore the vector z satisfies all the conditions.

We now prove the last four conditions for w. We can suppose without loss of generality that $w_m = x_m \neq y_m$ and $w_{m+1} = y_{m+1} \neq x_{m+1}$ otherwise the conditions are inherited from the conditions on x and y. If $w_m < w_{m+1}$ one hence has

$$y_m < w_m = x_m < w_{m+1} = y_{m+1}$$

which by Lemma 3.4.4 is possible only if w_m is odd, hence w_m even with $w_{m+1} = w_m + 1$. If $w_{m+1} < w_m$ one then has

$$x_{m+1} < w_{m+1} = y_{m+1} < w_m = x_m$$

which by Lemma 3.4.4 is possible only if x_{m+1} is even, hence w_{m+1} is odd and $w_m = w_{m+1} + 1$. □

Hasse diagrams of the lattice of noncrossing partitions with Bruhat order are given for type A_2 and A_3 on the right of figure 3.11 and on the left of figure 3.12, respectively.

3.6.2 Bijection with Dyck paths

Recall that a Dyck path is a path from $(0, 0)$ to $(2n, 0)$ with steps of the form $+(1, 1)$ or $+(1, -1)$, which stays above the x-axis. Denote by $\mathcal{D}_n$ the set of Dyck paths with $2n$ steps. We represent any noncrossing partition $x \in \mathcal{P}_c$ by arcs joining points on a line labeled with integers from 1 to $n + 1$ as in figure 3.8. This is very close to the geometrical representation on a circle, but the longest edge of each polygon is not represented. We associate to each point labeled with an integer between 1 and $n + 1$ a part of a Dyck path depending on arcs beginning or ending at the point and then collapse these various parts from the left to the right to obtain a Dyck path p_x

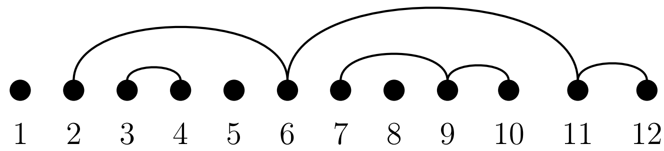


FIG. 3.8: A diagram for $n = 11$ representing the noncrossing partition $x = (2, 6, 11, 12)(3, 4)(7, 9, 10)$.

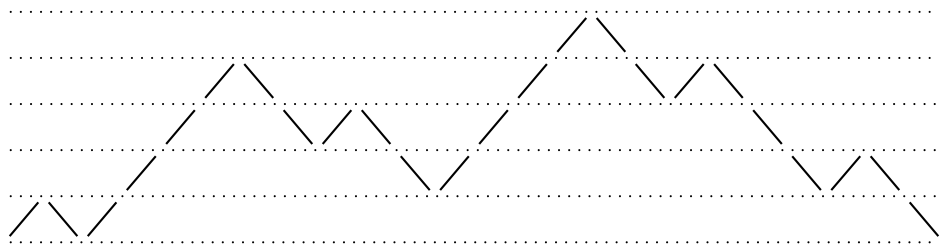


FIG. 3.9: Dyck path for $x = (2, 6, 11, 12)(3, 4)(7, 9, 10)$.

associated to x; the four possible steps are the following:

- (Step I) If there is no arc starting or ending at a point, this corresponds to an upward step followed by a downward step,
- (Step II) If there is an arc ending at a point and another one starting, this corresponds to a downward step followed by an upward step,
- (Step III) If there is a single arc starting at a point, this corresponds to two upward steps,
- (Step IV) If there is a single arc ending at a point, this corresponds to two downward steps.

As an example the path associated with the noncrossing partition from figure 3.8 above is given in figure 3.9

Proposition 3.6.2. *The assignment* $x \mapsto p_x$ *is a well-defined bijection* $\mathcal{P}_c \to \mathcal{D}_{2n+2}$.

Proof. We refer to [11] §3.2, where this bijection is also considered. □

Proposition 3.6.3. *Under the bijection above, the Bruhat order on noncrossing partitions corresponds to the order on Dyck paths by inclusion.*

Proof. The covering relations for the inclusion order on Dyck paths are easily described: it corresponds to replacing and upward move followed by a downward move into a downward move followed by an upward move. In case the first move begins at a point with even coordinates, it just corresponds to replacing a step I with a step II which corresponds for the noncrossing partition to breaking a long reflection into two smaller ones (see the first picture of figure 3.10). This is exactly the second of the two operations describing the covering relations as given in subsection 3.5.2. If the starting point of the first move has odd coordinates, one has to look at the various possible cases to see that this corresponds to doing on the corresponding noncrossing partition exactly the first operation given in 3.5.2: all the possible cases are detailed in figure 3.10 below. □

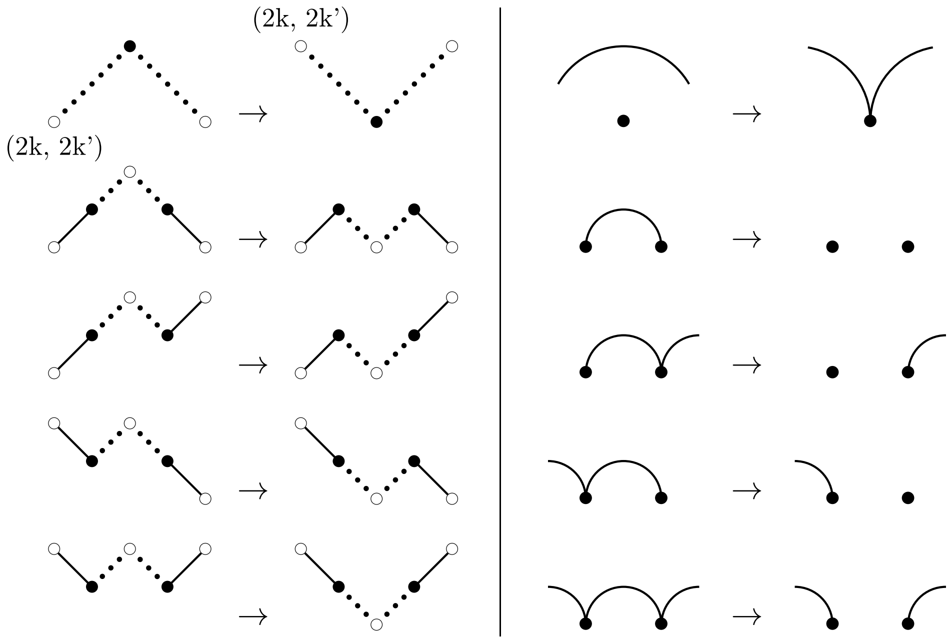


Fig. 3.10: Covering relations in the lattice of Dyck paths and the corresponding relations on noncrossing partitions. The parts which are in dotted style are the parts which are changed. The white points are the points with even coordinates, that is, corresponding to the beginning or end of a step.

3.6.3 Alternative direct proof of the lattice property

Using the bijection with Dyck paths from the previous section, one might look for a direct proof of Theorem 3.6.1 without using Theorem 3.5.6. In Proposition 3.6.3,

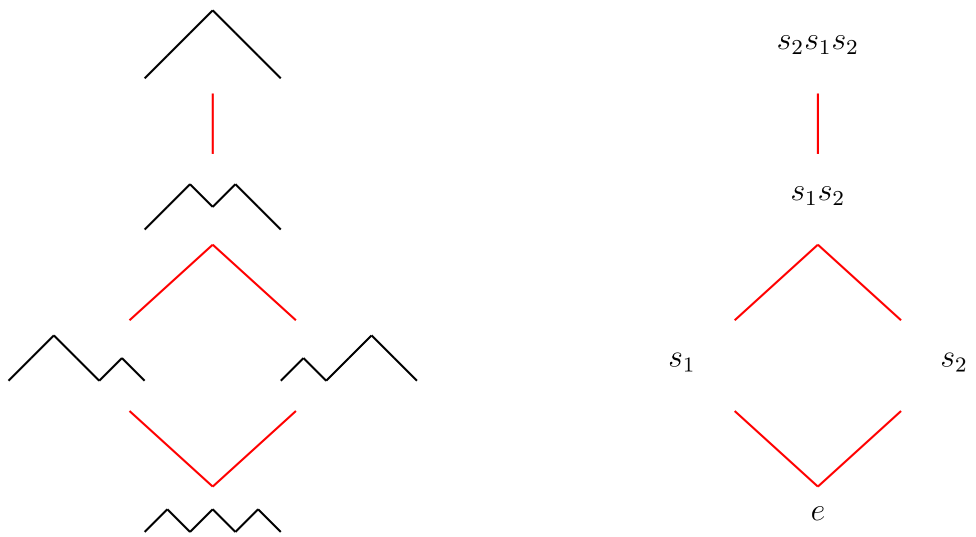


FIG. 3.11: Lattice of Dyck paths for inclusion and the corresponding lattice of noncrossing partitions with Bruhat order in type A_2.

the fact that $p_x < p_y \Rightarrow x <_{\mathcal{S}} y$ does not use the criterion. Hence the purpose here is to prove the converse.

Proposition 3.6.4. *Let $x, y \in \mathcal{P}_c$, and assume that $x <_{\mathcal{S}} y$. Then $p_x < p_y$.*

Proof. Thanks to Theorem 3.5.1, we have $x[i,j] \leq y[i,j]$ for any i, j. The height of the path p_x after i steps as defined at the beginning of subsection 3.6.2 is equal to $2k$ where k is the number of arcs of the noncrossing partition x above the point labeled with i (including any arc starting at i, but excluding the arcs ending at it). But $k = x[i, i+1]$. As a consequence, the points $(2i, j)$ of p_x and the point $(2i, j')$ of p_y are such that $j \leq j'$, in other words, any point of p_x with even coordinates has height smaller than or equal to the height of the point of p_y with the same first coordinate. Hence p_x is always below p_y at points with even coordinates. It remains to show that it cannot be above at points with odd coordinates. Using the fact that it is below at points with even coordinates, we only need to check that if the points $(2i, 2k)$ and $(2(i+1), 2k)$ belong to both p_x and p_y and if the i^{th} step of p_x is step I, then the i^{th} step of p_y is also step I. Notice that we have the equalities $k = x[i, i+1] = y[i, i+1]$. But if the i^{th} step of p_x is step I, it means that i is a fixed point of the permutation x and $x[i-1, i+1] = x[i, i+1] = k$. If the i^{th} step of p_y is not step I, then it must be step II, in which case i is the endpoint of an arc and the startpoint of another arc. It means that $y[i-1, i+1] + 1 = y[i, i+1] = x[i, i+1] = x[i-1, i+1]$ hence $y[i-1, i+1] < x[i-1, i+1]$, a contradiction. ∎

3.7 Changing the Coxeter element

3.7.1 Failure of the lattice property

The lattice property of $\mathcal{P}_{c'}$ with Bruhat order fails in type A_3 for $c' = s_1 s_3 s_2$; there are two maximal elements $x_1 = s_3 s_2 s_1 s_2 s_3$ and $x_2 = s_2 s_1 s_3 s_2$ in the poset (see figure 3.12). One might wonder why such a property fails. The Coxeter element from $c = s_1 s_2 \cdots s_n$ has a single reduced expression, which fails for $s_2 s_1 s_3 = s_2 s_3 s_1$. But by looking for example in type B_2, one sees that the noncrossing partitions associated to the Coxeter element $c = st$ (which has a single reduced expression as well) does not form a lattice either (see figure 3.13). As explained in the previous sections, Bruhat

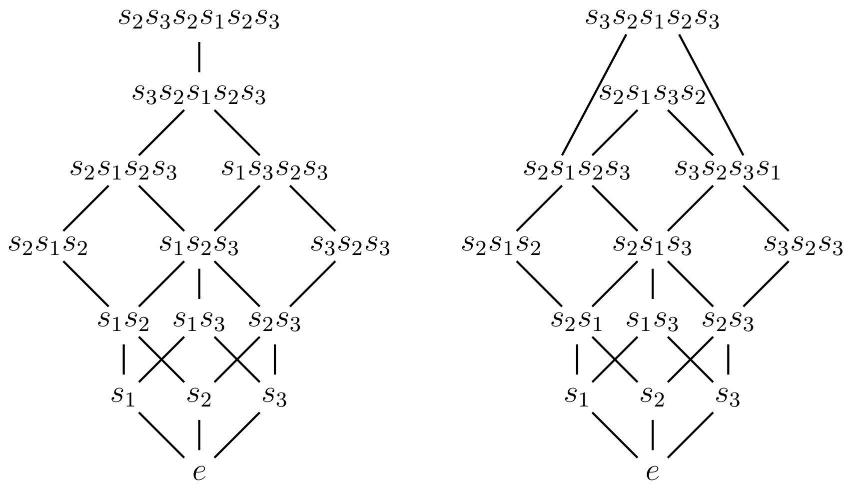


FIG. 3.12: Two diagrams for type A_3; on the left is the Hasse diagram of the lattice of noncrossing partitions for $c = s_1 s_2 s_3$ with Bruhat order; on the right is the Hasse diagram of the poset of noncrossing partitions for $c' = s_2 s_1 s_3$ with Bruhat order, which is not a lattice!

order on noncrossing partitions gives triangularity of a change of basis matrix in the Temperley-Lieb algebra, between the diagram basis and the Zinno basis. It turns out that the Zinno basis can be defined for an arbitrary Coxeter element and one can still order the noncrossing partitions to get triangularity, but it will not be Bruhat order any more. This will be investigated in the next sections. For defining the order, we need to define exactly the same vectors as for the case $c = s_1 s_2 \cdots s_n$ and these also correspond to counting reflections in expressions of noncrossing partitions, but these expressions need not be reduced; as an example for the A_3 case with $c' = s_2 s_1 s_3$, if one just replaces the word $s_2 s_1 s_3 s_2$ by the equivalent word $s_2 s_1 s_2 s_3 s_2 s_3$ in figure 3.12 and orders the set $\mathcal{P}_c$ using the corresponding vectors with parity conditions, then one gets exactly the same Hasse diagram as for the case $c = s_1 s_2 s_3$. The purpose

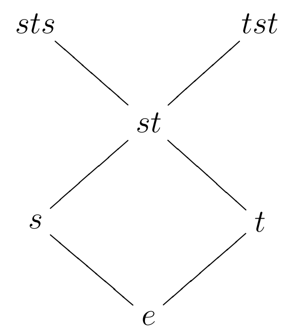


FIG. 3.13: The Hasse diagram of noncrossing partitions in type B_2 with Bruhat order.

of this section is to explain how to associate a standard form, hence a vector with parity conditions, to a noncrossing partition for an arbitrary Coxeter element.

3.7.2 Standard forms

Recall that for $x \in \mathcal{P}_c$, we denote by x_i the i^{th} component of the tuple v_x, that is, the number of occurrences of s_i in m_x.

Lemma 3.7.1 (Standard forms for cycles). *Let $c = s_1 s_2 \dots s_n$ and c' be a Coxeter element. Let $x' \in \mathcal{P}_{c'}$ having a single polygon P'. Consider the element $x \in \mathcal{P}_c$ having a single polygon P and such that $k \in P' \Leftrightarrow k \in P$ (that is, both x and x' are cycles with the same support). There exists a word $m_{x'}^{c'}$ representing x' in the Coxeter group and having the following properties:*

- *The number of occurrences of s_i in $m_{x'}^{c'}$ is equal to x_i for each $1 \leq i \leq n$,*
- *The word $m_{x'}^{c'}$ is an $\mathcal{S}$-reduced expression of x',*
- *If we write set of numbers indexing the vertices of P' (equivalently P) as $\{d_1, d_2, \dots, d_k\}$ where $d_i < d_{i+1}$ for $1 \leq i < k$, then $m_{x'}^{c'}$ is a product of all the words $[d_i, d_{i+1}]$ in some order.*

Proof. We argue by induction on $\ell_{\mathcal{T}}(x')$, that is, by induction on the number of edges of P'; if $\ell_{\mathcal{T}}(x') = 1$, then $x' \in \mathcal{T}$, $x' = x$ and since $\mathcal{T} \subset \mathcal{P}_c \cap \mathcal{P}_{c'}$ we can set $m_{x'}^{c'} = m_x$; we know that such a word is an $\mathcal{S}$-reduced decomposition of x and the other properties obviously hold (notice that the word has a single syllable).

Now assume that P' has more than one edge. As a permutation, x' is a cycle $(i_1, i_2, \dots, i_k)$ with $k > 2$. Using the description of the configuration of indices on the marked points of our circle from subsection 3.1, we can without loss of generality assume that i_1 and i_2 are the two smallest indices in the set $\{i_1, i_2, \ldots, i_k\}$. We can rewrite our cycle x' as the product $(i_1, i_2)(i_2, i_3, \ldots, i_k)$ in case $i_2 > i_1$, resp. $(i_1, i_3, \ldots, i_k)(i_1, i_2)$ in case $i_2 < i_1$. The cycle $y' = (i_2, i_3, \ldots, i_k)$ (resp. $(i_1, i_3, \ldots, i_k)$) is again a noncrossing partition with respect to c' and by induction there exists an $\mathcal{S}$-reduced expression $m_{y'}^{c'}$ of y' in which the number of occurrences of simple reflections is the same as in m_y, where $y \in \mathcal{P}_c$ is the cycle with same support as y'; now the set of simple reflections occurring in a reduced expression of (i_1, i_2) is disjoint from the set of reflections occurring in $m_{y'}^{c'}$ and indexed by smaller indices. As a consequence, the Coxeter word $[i_1, i_2] \star m_y$ if $i_1 < i_2$ or $[i_2, i_1] \star m_y$ if $i_2 < i_1$ is in standard form and corresponds to an element $x \in \mathcal{P}_c$. One then chooses the word $[i_1, i_2] \star m_{y'}^{c'}$ if $i_1 < i_2$ or $m_{y'}^{c'} \star [i_2, i_1]$ if $i_2 < i_1$ for representing x' with the required properties. □

For an example of an element x' and the corresponding element x together with the words $m_{x'}^{c'}$ and m_x, the reader can look at example 3.7.2 below and at example 3.8.3.

Example 3.7.2 In type A_4 let c' be the (bipartite) Coxeter element $s_4 s_2 s_1 s_3 = (1, 3, 5, 4, 2)$ and x' the 4-cycle $(1, 3, 5, 2) \in \mathcal{P}_{c'}$. The reflections in standard form that are used to built $m_{x'}^{c'}$ are $(1, 2)$, $(2, 3)$ and $(3, 5)$. We have $c' = (2, 3, 5)(1, 2) = (2, 3)(3, 5)(1, 2)$. The word $m_{x'}^{c'}$ is given by $s_2(s_4 s_3 s_4)s_1$. The corresponding element $x \in \mathcal{P}_c$ is (in standard form) $s_1 s_2(s_4 s_3 s_4) = (1, 2, 3, 5)$. We see that x and x' are built with the same reflections in standard form but concatenated in a different order.

Definition 3.7.3. *A word $m_{x'}^{c'}$ as in Lemma 3.7.1 will be called a* standard form *of x'. The various subwords $[d_i, d_{i+1}]$ as described in the lemma are again called the* syllables *of $m_{x'}^{c'}$.*

Remark 3.7.4. The word $m_{x'}^{c'}$ with the listed properties is not unique in general since the construction given in the proof may give at the end adjacent syllables $[d_i, d_{i+1}] \star [d_j, d_{j+1}]$ with $j > i + 1$ (or $j + 1 < i$) in the word which commute as elements of the Coxeter group, hence permuting them still yield a word representing x' with the same number of occurrences of simple reflections; however the position of a syllable $[d_i, d_{i+1}]$ relatively to the position of $[d_{i+1}, d_{i+2}]$ is always unique (that is, $[d_i, d_{i+1}]$ cannot be before $[d_{i+1}, d_{i+2}]$ in one word with the required properties and after it in another word) and that is the only thing which will matter further.

We would like to generalize such a process to arbitrary noncrossing partitions $x' \in \mathcal{P}_{c'}$, that is, associate to any $x' \in \mathcal{P}_{c'}$ a standard form $m_{x'}^{c'}$ and a noncrossing partition $x \in \mathcal{P}_c$ such that the number of occurrences of simple reflections in $m_{x'}^{c'}$ is the same as in m_x. The situation becomes slightly more complicated when x' has more than one polygon; as an example, consider the case given at the beginning of the section of the fully commutative element $s_2s_1s_3s_2$, which is a noncrossing partition for the Coxeter element $c' = s_2s_1s_3$. The word one needs to consider to represent it as a product of reflections in standard form is $(s_2s_1s_2)(s_3s_2s_3)$ (or $(s_3s_2s_3)(s_2s_1s_2)$). Since the two reflections commute with each other, it is geometrically represented by two disjoint edges and one cannot argue polygon by polygon to associate to x' (a standard form of) an element x of $\mathcal{P}_c$ since the reflections are fixed by the process as shown in Lemma 3.7.1. The corresponding element of $\mathcal{P}_c$ (that is, having the same vector) is $s_2(s_3s_2s_1s_2s_3)$.

Moreover, $(s_2s_1s_2)(s_3s_2s_3)$ is not an $\mathcal{S}$-reduced expression of $s_2s_1s_3s_2$. Hence if x' has more than one polygon, we will not expect $m_{x'}^{c'}$ to be an $\mathcal{S}$-reduced expression of x' in general.

We define a process allowing one to pass from x' to x as follows: we represent a noncrossing partition x' on a line with marked points from 1 to $n+1$ (even for $c' \neq c$) in the following way: to each polygon P associated to x', we order its set $\{d_1, \dots, d_k\}$ of indexing numbers of the vertices such that $d_i < d_{i+1}$ for $i = 1, \dots, k-1$. We represent P by successive arcs joining the point on the line labeled with d_i to the point labeled with d_{i+1}, for $i = 1, \dots, k-1$. Thanks to Lemma 3.7.1 the noncrossing partition corresponding to P is given by a product of the reflections (d_i, d_{i+1}) in some order depending on c'. We do the same for each polygon. Notice that since the points on the line are labeled from 1 to $n+1$, in case $c' \neq c$ the resulting diagram may have crossings. We want to associate to any such diagram a diagram of the same kind but with no crossings, which will therefore represent a noncrossing partition x with respect to $c = s_1s_2 \dots s_n$. We will then prove that our process defines a bijection $\mathcal{P}_{c'} \to \mathcal{P}_c$ and use it after having defined standard forms for noncrossing partitions in $\mathcal{P}_{c'}$ to build a bijection $\mathcal{P}_{c'} \to \mathcal{V}$.

The idea of the process is the following: if the diagram associated to x' has no crossings, then it is the diagram of a noncrossing partition x for c, so we are done. If there are at least two arcs (i,k), (j,ℓ) which cross each other, say with $i < j < k < \ell$, we replace them by the two arcs (i,ℓ), (k,j) which do not cross. Since at each step the number of crossing decreases by one we obtain eventually a diagram with no crossings which represents an element of $\mathcal{P}_c$. Moreover, since the operations can be seen as changing local configurations of the diagram, the order in which we remove the crossings does not affect the final diagram. After resolving each crossing we obtain a diagram of an element $x \in \mathcal{P}_c$. Hence it defines a map $\phi_{c',c} : \mathcal{P}_{c'} \to \mathcal{P}_c$, $x' \mapsto x$; figure 3.14 gives a concrete example of the process. It is clear that the set of vertices as well as the sets of initial, terminal, non initial and non terminal vertices are all preserved by our process.

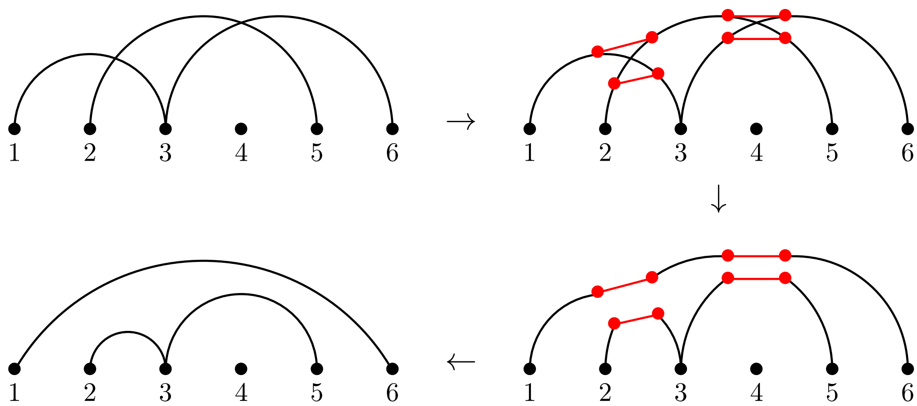


FIG. 3.14: Process associating to a noncrossing partition $x' \in \mathcal{P}_{c'}$ a noncrossing partition $x \in \mathcal{P}_c$. Here $c' = s_4s_3s_1s_2s_5 = (1,2,5,6,4,3)$ and $x' = (2,5)(1,6,3)$.

We now associate to any $x' \in \mathcal{P}_{c'}$ a Coxeter word $m_{x'}^{c'}$. We consider the various noncrossing partitions $y_1, \dots, y_k$ occurring in the decomposition of x' into a product of disjoint cycles, that is, corresponding to the various polygons $P_1, \dots, P_k$ of x'. We consider the word $m_{x'}^{c'}$ obtained as the contatenation of the various words $m_{y_i}^{c'}$ from Lemma 3.7.1, that is,

$$m_{x'}^{c'} = m_{y_1}^{c'} \star m_{y_2}^{c'} \star \cdots \star m_{y_k}^{c'}.$$

Recall that this is a product of reflections in standard form, namely, the reflections obtained in the following way: if P is a polygon of x' with set of vertices indexed by $\{d_1, \dots, d_m\}$ where $d_i < d_{i+1}$, the reflections using to build the various subwords $m_{y_i}^{c'}$ are by construction the reflections (d_i, d_{i+1}), $i = 1, \dots, m-1$ and we will call them *special* (with respect to x'). We write $x_i'^{c'}$ for the number of occurrences of s_i in $m_{x'}^{c'}$. Thanks to the order we chose on polygons, notice that we have $m_x^c = m_x$ and hence $x_i^c = x_i$ for any $x \in \mathcal{P}_c$. Notice that $m_{x'}^{c'}$ is not necessarily a reduced expression for x'; for example, in case $c' = s_2s_1s_3$ and $x' = (1,3)(2,4) = s_2s_1s_3s_2$, one has $m_{x'}^{c'} = s_2s_1s_2s_3s_2s_3$.

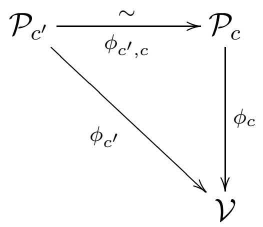
Definition 3.7.5. *A word $m_{x'}^{c'}$ obtained as described above is called a* standard form *of $x' \in \mathcal{P}_{c'}$*

Remark 3.7.6. Again, notice that since the decomposition into disjoint cycles is unique only up to permutation of the cycles and since moreover the standard forms $m_{y_i}^{c'}$ are not unique in general as pointed out in Remark 3.7.4, it follows that $m_{x'}^{c'}$ is not unique in general. But various such words are built with the same syllables but occurring in a possibly different order, hence the number of occurrences of simple reflections is the same. In case $c' = c$, we had chosen a specific order on the polygons because it was useful for the proof of the criterion but we can as well define a standard form as any concatenation of the standard forms of polygons (which in case $c' = c$ are unique).

3.7.3 Bijections between $\mathcal{P}_{c'}$ and $\mathcal{P}_c$ and lattice structure on $\mathcal{P}_{c'}$

Proposition 3.7.7. *The map $\phi_{c',c} : \mathcal{P}_{c'} \to \mathcal{P}_c$, $x' \mapsto x$ is a bijection. Moreover, one has ${x'}_i^{c'} = x_i$.*

Remark 3.7.8. One can summarize the situation as follows: let d be any Coxeter element, let $\phi_d : \mathcal{P}_d \to \mathcal{V}$, $x \mapsto (x_i^d)_{i=1}^n$. Then the following diagram of bijections commutes



Notice that ϕ_c is an isomorphism of posets while $\phi_{c',c}$ and $\phi_{c'}$ are bijections; one can therefore order $\mathcal{P}_{c'}$ by the componentwise order on $\mathcal{V}$ which corresponds to the Bruhat order on $\mathcal{P}_c$; such an order is not in general the Bruhat order on $\mathcal{P}_{c'}$!

Proof. Since the set of vertices as well as the sets of initial, terminal, non initial and non terminal vertices are all preserved by the geometrical process, one has $U_{x'}^{c'} = U_x^c$ and $D_{x'}^{c'} = D_x^c$. But $(D_{x'}^{c'}, U_{x'}^{c'}) = (D_x^c, U_x^c)$ lies in $\mathcal{I}$ (Lemma 3.2.9) and there is a bijection $\mathcal{P}_d \to \mathcal{I}$, $y \mapsto (D_y^d, U_y^d)$ for any Coxeter element d, in particular for $d = c, c'$ (obtained by composing the bijection from Proposition 3.2.10 with the involution $(D_{\bar{x}}^d, U_{\bar{x}}^d) \mapsto (D_x^d, U_x^d)$. It implies that our map $x' \mapsto x$ is bijective.

Now it suffices to modify our word $m_{x'}^{c'}$ at each elementary step of our geometrical process such that the number of occurrences of simple reflections does not change and such that the word obtained at the end of the process has the same number of occurrences of simple reflections as m_x. This is done in the following way: the arcs in the initial diagram correspond exactly to the special reflections, which have standard forms occurring as disjoint subwords building the word $m_{x'}^{c'}$. The geometrical process consists at each step of replacing two crossing reflections $(i,k)(j,\ell)$ where $i < j < k < \ell$ and represented by arcs by the two noncrossing reflections (i,ℓ) and (j,k). It suffices to replace the standard form of (i,k) in $m_{x'}^{c'}$ by the standard form of (i,ℓ)

and that of (j, ℓ) by that of (j, k) (or vice-versa; the order in which the standard forms of (i, ℓ) and (j, k) occur in the modified word do not matter since we are just interested by the number of simple reflections occurring in the word). This replacement does not change the number of simple reflections occurring in the new word. Iterating the process, we get at the end a word which is a product of the special reflections with respect to x in standard form (but they are not necessarily in the right order, that is, such a word does not necessarily represent x!). As a consequence such a word has a same number of occurrences of simple reflections as m_x, since at each step the total number of occurrences of a given simple reflection in the word does not change, which concludes. Notice that the order in which we choose to replace crossings does not affect the result since it is geometrically clear that the special reflections obtained at the end are the same for any order; what can change is the order in which the special reflections appear in the final word, but we are only interested by the number of occurrences of simple reflections in the final word which is not affected by permuting the special reflections. □

Example 3.7.9 Figure 3.15 gives the Hasse diagram of this new ordering in type A_3 in case $c' = s_2s_1s_3$ compared with the Bruhat order which did not yield a lattice structure in that case (see figure 3.12).

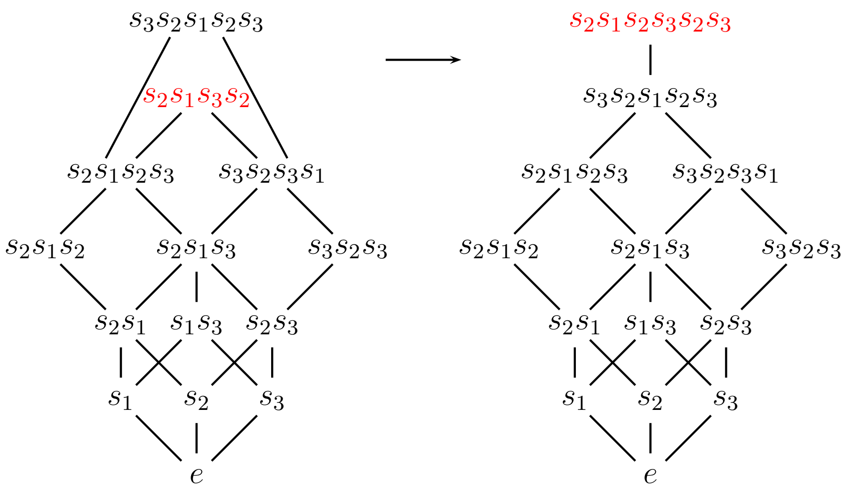


Fig. 3.15: With the new ordering, the Hasse diagram (which was not a lattice for the Bruhat order, represented here on the left) in type A_3 for $c' = s_2s_1s_3$ is given here on the right. The element $s_2s_1s_3s_2$ from the left is replaced on the right by an equivalent word which is a product of reflections in standard forms and moves to the top of the diagram, yielding a lattice.

Remark 3.7.10. Notice that the bijections $\phi_{c',c}$ we obtain are distinct from those we can obtain by conjugations in the Coxeter group (any two Coxeter elements are conjugate and this defines bijections $\mathcal{P}_{c'} \to \mathcal{P}_c$). For example, it follows from the construction that the set $\mathcal{T}$ of reflections which lies in both $\mathcal{P}_{c'}$ and $\mathcal{P}_c$ is pointwise fixed by $\phi_{c',c}$.

3.8 Triangularity

The aim of this section is to show that if we order $\mathcal{P}_{c'}$ by any linear extension of the order $<_{\mathcal{V}}$ from the previous subsection (see Remark 3.7.8) and if we consider the total order which is induced on $\mathcal{W}_f$ by this order $<_{\mathcal{V}}$ under the bijection $\varphi_{c'} : \mathcal{P}_{c'} \to \mathcal{W}_f$, then the change of basis matrix between the diagram basis and the generalized Zinno basis corresponding to the Coxeter element c' is upper triangular with invertible coefficient on the diagonal.

3.8.1 Order of the polygons

Definition 3.8.1. *The various words*

$$[d_i, d_{i+1}] = s_{d_{i+1}-1}s_{d_{i+1}-2}\cdots s_{d_i+1}s_{d_i}s_{d_i+1}\cdots s_{d_{i+1}-1}$$

that are used in Lemma 3.7.1 to build the standard form $m_x^{c'}$ of $x \in \mathcal{P}_{c'}$ with $|\mathrm{Pol}(x)| = 1$ are called the syllables *of $m_x^{c'}$. Notice that a simple reflection cannot occur as letter of two distinct syllables of $m_x^{c'}$ since $d_1 < d_2 < \cdots < d_k$. We say that s_{d_i} is the* center *of the syllable (or a center of $m_x^{c'}$) and that $s_{d_{i+1}-1}$ is at the* top *of the syllable. The center of the syllable splits the syllable into a* left part *$s_{d_{i+1}-1}s_{d_{i+1}-2}\cdots s_{d_i+1}$ and a* right part *$s_{d_i+1}\cdots s_{d_{i+1}-1}$. An integer k satisfying $d_i < k < d_{i+1}$ is said to be* nested *in the syllable $[d_i, d_{i+1}]$; it implies that k is nested (as defined in 3.2.3) in the unique polygon P of x. We will often say that the polygon P* contains *the syllable $[d_i, d_{i+1}]$ or any of its letters. Notice that k is nested in P if and only if s_k appears in $m_x^{c'}$ but not as a center, which is equivalent to say that s_k appears exactly twice in $m_x^{c'}$ since the various syllables have disjoint support and any letter which is not at the center of a syllable occurs twice in the syllable while the centers occur only once. More generally if x has more than one polygon, we say that a polygon Q is* nested *in a polygon P if $\min P < \min Q$ and $\max Q < \max P$. It implies that if s_k is any letter contained in Q, then k is nested in P.*

Lemma 3.8.2. *Let $x \in \mathcal{P}_{c'}$ and assume $|\mathrm{Pol}(x)| = 1$. Assume that s_k is the center of a syllable w of $m_x^{c'}$, that is, k indexes a non terminal vertex of the unique polygon P of x. Assume that s_{k-1} occurs in $m_x^{c'}$, that is, k is also non initial (if s_{k-1} occurs it must be at the top of its syllable, otherwise s_k would occur in two different syllables* *of the same polygon). Then if $k \in R_{c'}$ (resp. $k \in L_{c'}$), the unique syllable of $m_x^{c'}$ containing s_{k-1} occurs before (resp. after) w in $m_x^{c'}$.*

Proof. Recall that if $\{d_1, d_2, \ldots, d_m\}$ are the integers labelling the vertices of P where $d_i < d_j$ if $i < j$, then the standard form is a concatenation of the syllables $[d_i, d_{i+1}]$, $1 \leq i < k$ in some order, the order depending on the Coxeter element. Under our assumptions we have that $k = d_i$ for some $i \neq 1, m$. We can without loss of generality assume that $m = 3$ with $d_2 = k$. Since $k \in R_{c'}$ (resp. $k \in L_{c'}$), if we go along the circle in clockwise order from d_1 to d_3 (resp. from d_3 to d_1), we must meet d_2; therefore the standard form corresponding to such a polygon is $[d_1, d_2 = k] \star [d_2 = k, d_3]$ (resp. $[d_2 = k, d_3] \star [d_1, d_2 = k]$) and s_{k-1} occurs in $[d_1, d_2]$, hence before (resp. after) $w = [d_2, d_3]$. $\square$

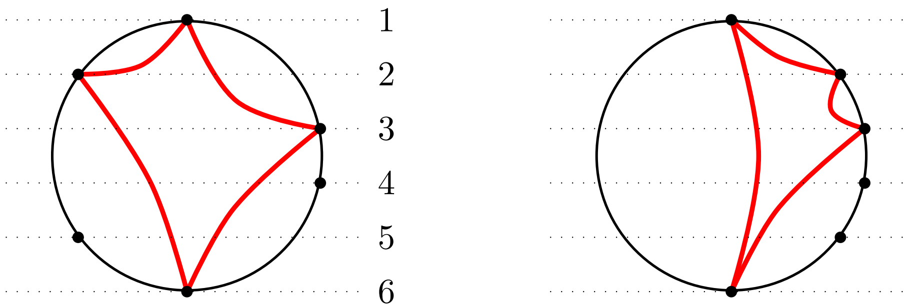


FIG. 3.16

Example 3.8.3 Consider the Coxeter element $c' = s_2 s_1 s_3 s_5 s_4 = (1, 3, 4, 6, 5, 2)$. The noncrossing partition on the left in figure 3.16 is the cycle $x' = (1, 3, 6, 2) \in \mathcal{P}_{c'}$. The $x \in \mathcal{P}_c$ which is the image of x' under the bijection $\phi_{c',c}$ (also given by Lemma 3.7.1) is equal to the cycle $(1, 2, 3, 6)$ as drawn on the right in figure 3.16. A standard form of x' as built in the proof of 3.7.1 is given by $m_{x'}^{c'} = s_2(s_5 s_4 s_3 s_4 s_5)s_1$. The standard form m_x of x is given by $m_x = s_1 s_2(s_5 s_4 s_3 s_4 s_5)$. The number of occurrences of simple reflections in m_x and $m_{x'}^{c'}$ are the same. Moreover, $2 \in L_{c'}$ while $2 \in R_c$. As an illustration of Lemma 3.8.2, one sees that s_1 occurs after s_2 in $m_{x'}^{c'}$ while it occurs before in m_x.

We now introduce a non obvious order on $\mathrm{Pol}(x)$. Recall that $L_{c'} \cap R_{c'} = \{1, n+1\}$. Among all the polygons of x, first consider the polygons such that at least one vertex is indexed by an integer in $L_{c'}$ and at least one vertex is indexed by an integer in $R_{c'}$. We call such a polygon an *alternating* polygon. We order the alternating polygons inductively on the number of such polygons in the following way: when going along the diagonal joining the point labeled by 1 to the point labeled by $n+1$, every alternating polygon is crossed exactly once. Write $P \prec Q$ if P occurs before Q when going from 1 to $n+1$ and notice that it defines a total ordering on the set of alternating polygons. We define a new order inductively by explaining where the last polygon P met by the segment going from the point with index 1 to the point with index $n+1$ is put relatively to the polygons met previously by the segment. It is done as follows: if the smallest index of P is in $L_{c'}$, then P will be put after all the previously met polygons. If it is in $R_{c'}$, the P will be put before all the previously met polygons. This defines inductively a total order on the set of alternating polygons which we denote by $<$. We want to extend this order to $\mathrm{Pol}(x)$. To this end, consider the polygons having none of their vertices indexed by integers in $L_{c'}$ (resp. $R_{c'}$) and call them *right* (resp. *left*) polygons. We then order the right (resp. left) polygons by ascending order (resp. decreasing order) of their maximal indices and decide that any right (resp. left) polygon is smaller than (resp. greater than) any alternating polygon. This gives a total ordering of the polygons of x. We denote this order by $<$. Notice that if we write the ordered set of polygons of a noncrossing partition x as

$$P_1 < P_2 < \cdots < P_m,$$

the left polygons occur on the right while the right polygons occur on the left!

Remark 3.8.4. Notice that if we invert the Coxeter element c', from the point of view of geometry the labelling of the indices on the circle is the mirror image of the labelling corresponding to c', in other words, one has $L_{c'} = R_{c'^{-1}}$ and $R_{c'} = L_{c'^{-1}}$. For $x \in \mathcal{P}_{c'}$ one has $x^{-1} \in \mathcal{P}_{c'^{-1}}$ and the geometrical representation of x^{-1} is the mirror image of the geometric representation of x. In particular they have the same number of polygons with same sets of numbers indexing the vertices (this is also clear if one keeps in mind that these are just the supports of the various cycles occurring in the decomposition into a product of disjoint cycles). The order on $\mathrm{Pol}(x)$ is the reversed order of the order on $\mathrm{Pol}(x^{-1})$. A reflection s_k is a center of $m_x^{c'}$ if and only if it is a center of $m_{x^{-1}}^{c'^{-1}}$. An index k is nested in a polygon P of x if and only if it is nested in a polygon P' of x^{-1}.

Lemma 3.8.5. *Let $x \in \mathcal{P}_{c'}$, $P \in \mathrm{Pol}(x)$. Let $k \in P$ be a non terminal index of P, or equivalently, let s_k be a center of a syllable of $m_x^{c'}$.*

1. *If $k \in L_{c'}$ and k is nested in $Q \in \mathrm{Pol}(x)$, then $Q < P$.*
2. *If $k \in R_{c'}$ and k is nested in $Q \in \mathrm{Pol}(x)$, then $Q > P$.*

Proof. Using Remark 3.8.4, it suffices to prove the first claim. Notice that P is either left or alternating since $k \in L_{c'}$.

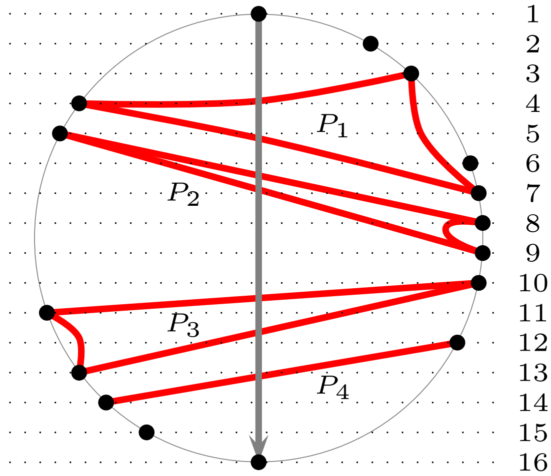


FIG. 3.17

Example of the order on alternating polygons. We have four alternating polygons; one has $P_1 \prec P_2 \prec P_3 \prec P_4$ and

$$P_4 < P_3 < P_1 < P_2.$$

If P is left, the only case to consider is the case where Q is also left since otherwise one has by definition of the order that $Q < P$. In that case, since k is nested in Q, one must have $\max P < \max Q$, hence $Q < P$.

Now assume that P is alternating. Notice that k cannot be nested in a left polygon since there is at least one vertex of P indexed by an integer k' lying in $R_{c'}$: the segment joining k and k' (which is either an edge or a diagonal of P) would then cross a segment joining two vertices of Q with indices lying in $L_{c'}$ in which k would be nested, contradicting the noncrossing property (since this segment is either an edge or a diagonal of Q). Hence Q is either alternating or right. If it is right, then by definition $Q < P$. If Q is alternating, first assume that $Q \prec P$. We make two cases: if $\min P \in R_{c'}$, since $k \in P$ and k lies in $L_{c'}$, then $\max Q < k$ since Q is met before P, contradicting the fact that k is nested in Q. Now assume $\min P \in L_{c'}$; P must therefore be the last polygon in the order among the alternating polygons P' with $P' \prec P$, in particular, $Q < P$. Now assume that $Q \succ P$. If $\min Q \in L_{c'}$, since $k \in L_{c'}$ and Q occurs after P, then $k < \min Q$, hence k cannot be nested in Q. Therefore $\min Q$ needs to be right, forcing $Q < P'$ for any alternating polygon P' satisfying $P' \prec Q$, in particular, $Q < P$. □

Example 3.8.6 In the example of figure 3.17, we have that $5 \in P_2 \cap L_{c'}$ is non terminal and nested in P_1 and we have seen that $P_1 < P_2$. We have that $12 \in P_4 \cap R_{c'}$ is non terminal and nested in P_3 and we have already seen that $P_3 > P_4$.

3.8.2 Canonical forms

Definition 3.8.7. *Given any $x \in \mathcal{P}_{c'}$, write* $\mathrm{Pol}(x) = \{P_1, \ldots, P_k\}$ *where $d < d'$ if and only if $P_d < P_{d'}$. For each $1 \leq i \leq k$ let $y_i \in \mathcal{P}_c$ correspond to P_i and let $m_{y_i}^{c'}$ be* *the standard form of y_i. We define the Coxeter word q_x to be the concatenation*

$$q_x = m^{c'}_{y_1} \star m^{c'}_{y_2} \star \cdots \star m^{c'}_{y_k}.$$

This is the canonical form *of x. It is not necessarily an $\mathcal{S}$-reduced expression. The syllables of the various standard forms $m^{c'}_{y_i}$ are also called the* syllables *of q_x. A center of a syllable of q_x will be a* center *of q_x. Notice that s_k is a center of q_x if and only if $k \in D^{c'}_x$.*

Remark 3.8.8. Notice that the terminology is a bit abusive since the various $m^{c'}_{y_i}$ are unique up to commutation of some syllables corresponding to reflections with commute with each other (see Remark 3.7.4). We want to insist here on the fact that the order of the subwords which are standard forms of polygons is unique. The canonical form is a standard form where we also fix the order of the subwords corresponding to the polygons.

Example 3.8.9 Let $c' = s_2s_1s_3$. One has $L_{c'} = \{1, 2, 4\}$, $R_{c'} = \{1, 3, 4\}$. Consider the noncrossing partition $x = (1, 3)(2, 4) \in \mathcal{P}_{c'}$. It has two polygons P_1, P_2 that are both alternating: P_1 corresponding to the cycle $y_1 = (1, 3)$ and P_2 corresponding to the cycle $y_2 = (2, 4)$ with $P_1 < P_2$. We have

$$q_x = m^{c'}_{y_1} \star m^{c'}_{y_2} = (s_2s_1s_2) \star (s_3s_2s_3) = s_2s_1s_2s_3s_2s_3.$$

Notice that q_x is not an $\mathcal{S}$-reduced expression of x. A reduced expression is given by $s_2s_1s_3s_2$.

We can reformulate Lemma 3.8.5 using the canonical form and remarks we made about nested indices in definition 3.8.1, which will turn out to be more convenient for the next proofs:

Lemma 3.8.10. *Let $x \in \mathcal{P}_{c'}$, $P \in \mathrm{Pol}(x)$. Write*

$$q_x = w_1 \star w_2 \star \cdots \star w_p$$

where w_i are the various syllables. Let s_k be the center of w_i for some i.

1. *If $k \in L_{c'}$ and if s_k occurs in w_j for $j \neq i$, then $j < i$. In other words, the occurrence of s_k at the center of w_i is the last occurrence of s_k in q_x.*
2. *If $k \in R_{c'}$ and if s_k occurs in w_j for $j \neq i$, then $j > i$. In other words, the occurrence of s_k at the center of w_i is the first occurrence of s_k in q_x.*

We should think of the order we put on the set of polygons as an order giving the property above, that is, an order such that a center s_k with $k \in L_{c'}$ is the last occurrence of s_k in the word q_x and a center s_k with $k \in R_{c'}$ is the first occurrence of s_k in q_x. This order also has consequences on the locations of the tops of the syllables in q_x, which we state in the following lemma:

Lemma 3.8.11. *Let $x \in \mathcal{P}_{c'}$. Write $q_x = w_1 \star w_2 \star \cdots \star w_p$, where w_i are the syllables of q_x. Assume that s_k occurs at the top of a syllable w_i of q_x. Then (at least) one of the following is true*

- *For any $j < i$, w_j does not contain s_k,*
- *For any $j > i$, w_j does not contain s_k.*

In other words, w_i is either the first or the last syllable containing s_k. More precisely, one has

1. *If s_k is a center of q_x and $k \in R_{c'}$ (resp. $L_{c'}$), then*
    - *if $w_i \neq s_k$, then w_i is the last (resp. first) syllable containing s_k,*
    - *if $w_i = s_k$, then w_i is the first (resp. last) syllable containing s_k.*
2. *If s_k is not a center of q_x, then*
    - *if $k+1 \in R_{c'}$, then w_i is the first syllable containing s_k,*
    - *if $k+1 \in L_{c'}$, then w_i is the last syllable containing s_k.*

Proof. Using Remark 3.8.4, we can assume that $k \in R_{c'}$. If s_k is a center, then $k \in D_x^{c'}$. We also have that $k+1 \in U_x^{c'}$ since s_k is at the top of w_i. If $w_i = s_k$, then s_k is the center of w_i, hence w_i is the first syllable containing s_k by Lemma 3.8.10. Now assume that $w_i \neq s_k$. If $k+1 \in R_{c'}$, then the only way to have $k \in D_x^{c'}$, $k+1 \in U_x^{c'}$ and both k and $k+1$ in $R_{c'}$ is in case w_i (which has s_k at its top) also has s_k at its center, that is, $w_i = s_k$, a contradiction. So we can assume $k+1 \in L_{c'}$.

We now prove that if $k+1 \in L_{c'}$, then w_i is the last syllable containing s_k (without assuming that s_k is a center or not, which therefore proves simultaneously the first point of 1 and the second point of 2). Write P for the polygon having w_i as syllable. If P is left, then $P > Q$ for each Q right or alternating by definition of $<$; hence we must show that $P > Q$ for any left polygon Q containing s_k; but since s_k is at the top of w_i, if another left polygon Q contains s_k, then P must be nested in Q implying $\max P < \max Q$, whence $Q < P$. Hence w_i is the last syllable containing s_k. If P is alternating, then no left polygon Q can contain s_k since P

has an edge or a diagonal from $k+1 \in L_{c'}$ to an index in $R_{c'}$ which would cross Q. Hence a polygon Q containing s_k is either right or alternating. If Q is right, then $Q < P$. If Q is alternating, first assume that $Q \succ P$. It implies that $\min Q \in R_{c'}$ (since Q contains s_k any of its indices lying in $L_{c'}$ is bigger than $k+1 \in P$ because $Q \succ P$), whence $Q < P$ by definition of $<$. Now assume that $Q \prec P$. We make two cases. If $\min P \in L_{c'}$, we have $P > Q$ by definition of $<$. If $\min P \in R_{c'}$, then no polygon P' with $P' \prec P$ can contain s_k, contradicting $Q \prec P$. Hence P is in all the cases the last polygon containing s_k.

Now assume that s_k is not a center. The case $k+1 \in L_{c'}$ has already been proven in the previous paragraph, hence assume that $k+1 \in R_{c'}$. If P is right, then $P < Q$ for each Q right and containing s_k (since $\max Q > \max P$), hence $P < Q$ for any Q containing s_k since right polygons are the smallest polygons by definition of $<$. If P is alternating and $Q \neq P$ contains s_k, then Q is not right since otherwise it would cross the diagonal or vertex of P joining the point with index $k+1$ to any point with index in $L_{c'} \cap P \neq \emptyset$. If Q is left, then $Q > P$ by definition of $<$. If Q is alternating, first assume $\min P \in R_{c'}$, implying $P < Q$ whenever $Q \prec P$. If $Q \succ P$, then $\min Q \in L_{c'}$ (since $k+1 \in P \cap R_{c'}$) implying $Q > P$. Now assume that $\min P \in L_{c'}$. Then no P' with $P' \prec P$ can contain s_k. Hence $Q \succ P$. Since Q contains s_k we have $\min Q \in L_{c'}$, whence $Q > P$. Hence $P < Q$ in all the cases, proving that P is always the first polygon containing s_k.

□

3.8.3 Fully commutative subword

Let $x \in \mathcal{P}_c$. We consider a subword w_x of q_x defined by the following rules:

- Each syllable of q_x contributes to w_x any simple reflection occurring in it exactly once. In particular each center of syllable must contribute.
- If s_i is a center of a syllable and $i \in L_{c'}$, then the s_i are contributed from the left part of the other syllables in which they occur,
- If s_i is a center of a syllable and $i \in R_{c'}$, then the s_i are contributed from the right part of the other syllables in which they occur,
- If s_i is not a center and $i \in L_{c'}$, then the s_i are contributed from the right part of the syllables in which they occur,
- If s_i is not a center and $i \in R_{c'}$, then the s_i are contributed from the left part of the syllables in which they occur.

Example 3.8.12 Let us consider again the noncrossing partition $x \in \mathcal{P}_{c'}$ from example 3.8.9. Recall that we have

$$q_x = (s_2 s_1 s_2) \star (s_3 s_2 s_3) = s_2 s_1 s_2 s_3 s_2 s_3.$$

We have that s_1 is the center of the first syllable and s_2 is the center of the second syllable. If we apply the rules given above we get the subword whose letters are underlined

$$\underline{s_2 s_1} s_2 \underline{s_3 s_2} s_3$$

hence $w_x = s_2 s_1 s_3 s_2$. In that case as element of the Coxeter group we have $w_x = x$ but this fails in general; for example if $x' = (1, 3) \in \mathcal{P}_{c'}$, the canonical form is $q_{x'} = s_2 s_1 s_2$ while $w_{x'} = s_1 s_2$. In fact we will show at the end of the subsection that the subword we defined is always fully commutative.

Remark 3.8.13. These rules generalize Zinno's rules from [42] recalled in subsection 3.2.2 for the case $c' = c$.

Before proving that the subword is fully commutative and that we recover in that way our bijection $\varphi_{c'} : \mathcal{P}_{c'} \to \mathcal{W}_f$ from subsection 3.2.2, we prove the following technical lemma which will be useful in the next (also technical) proofs:

Lemma 3.8.14. *Let $x \in \mathcal{P}_{c'}$. Write $q_x = w_1 \star w_2 \star \cdots \star w_p$ where the w_i are the syllables.*

1. *Assume that $k, k+1 \in R_{c'}$ (or $k, k+1 \in L_{c'}$). If s_k is the center of w_i and the top of w_j, then $i = j$ and $w_i = s_k$.*
2. *Assume that $k \in L_{c'}$, $k+1 \in R_{c'}$ (or $k \in R_{c'}$, $k+1 \in L_{c'}$). Assume that $w_i = s_k$. Then s_k does not occur in w_j for $j \neq i$. Moreover, there is at most one occurrence of s_{k+1} in q_x (which must be as a center of a syllable of P, where P is the polygon containing s_k).*

Proof. 1. Thanks to Remark 3.8.4, we can assume that $k, k+1 \in R_{c'}$. If a syllable has s_k at its center (resp. at its top), it means that such a syllable is in a polygon P (resp. P') which has k as non terminal (resp. $k+1$ as non initial) index; in particular, if $P \neq P'$, then P (resp. P') must have an edge or a diagonal joining the point indexed by k (resp. $k+1$) to a point indexed by $m > k$ and $m \neq k+1$ whence $m > k+1$ (resp. $m' < k+1$ and $m' \neq k$, whence $m' < k$). If follows that the two segments (k, m) and $(m', k+1)$ are crossing (see figure 3.18), which is a contradiction. Indeed, since the segment (k, m) defines two half planes, one of which

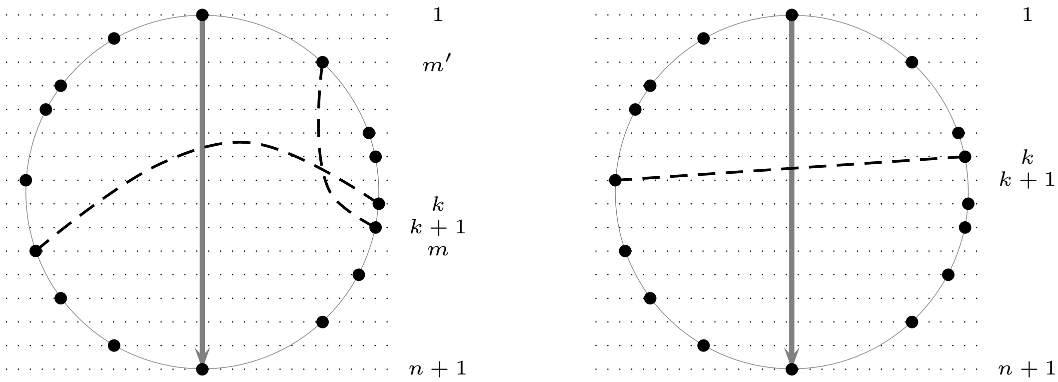


FIG. 3.18: A figure (on the left) for the proof of point 1. of Lemma 3.8.14 and one (on the right) for the proof of point 2.

contains the point with index $k+1$ and any other point contained in it has index bigger than k, implying that (k, m) and $(m', k+1)$ are crossing. Therefore we must have $P = P'$ and $w_i = w_j = s_k$.

2. Under these assumptions, there is a polygon $P \in \mathrm{Pol}(x)$ having both k and $k+1$ as vertices, hence alternating. Another polygon Q containing s_k would then cross the segment joining the point with index k to the point with index $k+1$ since this segment cuts the plane in two half planes, one containing the points labeled by indices smaller than k and another containing the points labeled by indices bigger than $k+1$. Hence P is the unique polygon containing s_k and its standard form has a unique occurrence of s_k since it is a center. The letter s_{k+1} can then only appear as a center since otherwise we would again have a polygon Q crossing the segment $(k, k+1)$ as one can see on figure 3.18. ☐

Proposition 3.8.15. *Let $x \in \mathcal{P}_{c'}$. Between any two successive occurrences of s_k in w_x, there is exactly one occurrence of s_{k-1}.*

Proof. Thanks to Remark 3.8.4, we can assume that $k \in R_{c'}$.

First assume that s_k is the center of a syllable w of q_x. Therefore w is the first syllable containing s_k in q_x (Lemma 3.8.10), hence the first s_k in w_x comes from w. Write P for the polygon containing w. Since a letter can be the center of at most one syllable, any other syllable containing s_k must also contain s_{k-1}. There is therefore at least one occurrence of s_{k-1} between any two successive occurrences of s_k in w_x since the s_k is taken from the right part of these syllables; one just has to make sure that there cannot be another occurrence of s_{k-1} at some place between two successive occurrences of s_k. But if it would be the case, such an s_{k-1} would come from a syllable w' where it is at the top since otherwise s_k has to occur; but w' would then be in P since P has a syllable with s_k at its center. Hence by Lemma

3.8.2 w' would occur before w. This is absurd, since the s_k coming from w is the first one in q_x.

Now assume that s_k is not a center. It implies that s_{k-1} occurs in any syllable in which s_k occurs. Since $k \in R_{c'}$, the syllables contribute the s_k from their left part. The only possible way to have a contribution of s_{k-1} from a syllable w' not involving s_k is with s_{k-1} at the top of w'. Such a syllable would either be the first or the last one containing s_{k-1} (Lemma 3.8.11). We claim that in all the possible cases, such a syllable is the first one, which concludes since the s_k are always contributed from the left parts. We now prove the claim: if s_{k-1} is not a center, then point 2 of 3.8.11 tells us that the syllable is the first (since we have $k \in R_{c'}$); if s_{k-1} is a center and $k-1 \in R_{c'}$, since $k \in R_{c'}$ the only possible way to have a syllable w with s_{k-1} at its center and a syllable w' with s_{k-1} at its top is that $w = w' = s_{k-1}$ (point 1 of Lemma 3.8.14) in which this syllable is again the first in which s_{k-1} occurs (Lemma 3.8.10). If $k-1 \in L_{c'}$ then the syllable is the first one if it is not reduced to s_{k-1} (point 1 of Lemma 3.8.11). If it is reduced to s_{k-1}, then since $k \in R_{c'}$ and $k-1 \in L_{c'}$ we have a single occurrence of s_{k-1} in w_x by point 2 of 3.8.14, hence w' is the only syllable containing s_{k-1}, in particular the first one. □

Proposition 3.8.16. *Let $x \in \mathcal{P}_{c'}$. Between any two successive occurrences of s_k in w_x, there is exactly one occurrence of s_{k+1}.*

Proof. Thanks to Remark 3.8.4, we can assume that $k \in R_{c'}$.

First suppose that s_k is the center of a syllable w of a polygon $P \in \mathrm{Pol}(x)$. Then w is the first syllable containing s_k by Lemma 3.8.10. Assume $k+1 \in R_{c'}$. If s_{k+1} is a center then it may be in P (which is equivalent to $w = s_k$) or not, in which case it is in a polygon Q with $Q < P$ as one sees easily. In the first case, the s_{k+1} coming from the center is just after the first s_k since $k+1 \in R_{c'}$ (Lemma 3.8.2); any other s_{k+1} must come from all the other syllables containing s_k except possibly the last one (which may have s_k at its top, see Lemma 3.8.11) and they must come from the right part of their syllable. Therefore the claim holds since in any syllable containing both s_k and s_{k+1}, the s_{k+1} is contributed from the right part, hence the s_k is contributed before it. In the second case, since $Q < P$ and $w \neq s_k$, any syllable containing s_k (including w) except possibly the last one contributes an s_{k+1} from its right part; these syllables together with the syllable of Q having s_{k+1} as center are the only syllables containing s_{k+1}, giving the claim again. If s_{k+1} is not a center, then it appears exactly in all the syllables containing s_k except possibly in a syllable w' having s_k at its top; if there is no such syllable, the claim holds. If there is such a syllable w' then thanks to point 1 of Lemma 3.8.14 one has $w = w' = s_k$. In that case the claim holds again since the s_{k+1} are contributed from the left of the syllables.

Now assume $k+1 \in L_{c'}$. The letter s_{k+1} has to occur in any syllable in which s_k occurs except possibly w and the last one w' (with s_k at its top). If it appears in some syllable w'' where s_k does not occur, then it is the center of w'' and there is only one such w''. In that case, if s_k is at the top of w', then w'' appears in the same polygon as w' and before it (Lemma 3.8.2) implying the claim since the s_{k+1} are contributed from the left of the syllables different from w'' where they occur: the order of the syllables (containing either s_k or s_{k+1}) is given by

$$w \cdots w'' w'$$

and w contributes (possibly an s_{k+1} from the left part and) an s_k from the center, any syllable in the $\cdots$ part contributes both s_{k+1} and s_k in this order, w'' contributes a single s_{k+1} and w' a single s_k. If s_k is not at the top of w', then w'' occurs after w' since it must be the last occurrence of s_{k+1} by Lemma 3.8.10 giving the claim again. Now if s_{k+1} is not a center, it can only appear in syllables in which s_k appears and is contributed from the right of the syllables in which it occurs, in which case the claim holds except possibly between w and the second syllable in which s_k appears in case w is reduced to s_k; but in that case there is a single occurrence of s_k in w_x thanks to point 2 of Lemma 3.8.14.

Now assume that s_k is not a center. Then s_{k+1} occurs in any syllable in which s_k occurs except possibly in a syllable with s_k at its top (which must be either the first or the last syllable containing s_k); any other contribution of s_{k+1} comes from a center. If there is no syllable with s_k at its top, then using the fact that s_{k+1} occurs in any syllable in which s_k occurs together with Lemma 3.8.10 one sees that the claim holds. Hence assume that there is a syllable w with s_k at its top; if $k+1 \in R_{c'}$ then by Lemma 3.8.11 w is the first syllable containing s_k. In that case if s_{k+1} is a center of w' then w and w' come from the same polygon and w' appears after w (Lemma 3.8.2); moreover, any other s_{k+1} is contributed from the right implying the claim. If s_{k+1} is not a center then it appears in any syllable containing s_k but distinct from w and is contributed from the left, also giving the claim. Now if $k+1 \in L_{c'}$, a syllable w with s_k at its top is the last syllable containing s_k. All the s_{k+1} are contributed from the right if s_{k+1} is not a center giving the claim; if s_{k+1} is a center of w', then w and w' come from the same polygon and w' comes before w. Since any other contribution from s_{k+1} is from the left, we get the result again. ☐

Corollary 3.8.17. *Let $x \in \mathcal{P}_{c'}$. Then w_x is an $\mathcal{S}$-reduced expression of a fully commutative element.*

Proof. Put 3.8.15 and 3.8.16 together. By point 4 of Proposition 1.1.2, the obtained statement is equivalent to the full commutativity of w_x. ☐

Before reading the two next proofs, the reader may have a look at Remark 3.2.15 to keep in mind how the bijections $\varphi_{c'}$ and $\psi_{c'}$ are built. We also recall that a reflection s_k is a center of q_x if and only if $k \in D_x^{c'}$ and that k is a non initial index of a polygon if and only if $k \in U_x^{c'}$. In the following two lemmas, we abuse notation and use w_x to denote the defined subword of q_x as well as the corresponding (fully commutative) element of the Coxeter group $\mathcal{W}$.

Lemma 3.8.18. *Let $x \in \mathcal{P}_{c'}$. Let $k \in \{1, \ldots, n\}$. Then if $k = 1$, $k \in J_{w_x}$ if and only if s_k appears in q_x. If $k \neq 1$, then*

$$k \in J_{w_x} \Leftrightarrow \begin{cases} k \in R_{c'} \text{ and } s_k \text{ is a center of } q_x, \text{ or} \\ k \in L_{c'} \text{ and } k \text{ is an initial index of a polygon of } x, \text{ or} \\ k \in L_{c'} \text{ and } k \notin \mathrm{Vert}(x) \text{ but } k \text{ is nested in a polygon of } x. \end{cases}$$

Proof. We recall that $k \in J_{w_x}$ if and only if in any $\mathcal{S}$-reduced expression of w_x, there is no occurrence of s_{k-1} after the last occurrence of s_k in w_x. In case $k = 1$, notice that there is at most one occurrence of s_1 in q_x since s_1 can only occur as a center; by definition of w_x, each center has to contribute, hence the claim for $k = 1$ is clear. In case $k \neq 1$, we first show that if any of the three conditions given on the right hand side holds, then $k \in J_{w_x}$. We then prove that if they fail, then k cannot lie in J_{w_x}. Notice that in case $k \neq 1$, one has that $k \notin L_{c'}$ if and only if $k \in R_{c'}$.

If $k \in R_{c'}$ and s_k is a center, then the first occurrence of s_k in q_x is as a center (Lemma 3.8.10). If there is a syllable w with s_{k-1} at its top, it must be in the polygon P having s_k as a center and w occurs before s_k in q_x (Lemma 3.8.2). Therefore there is no contribution of s_{k-1} after the last s_k since if there are other syllables containing either s_k or s_{k-1}, they must contain both s_k and s_{k-1} and s_k is always contributed from the right of these syllables by definition of w_x.

If $k \in L_{c'}$ and k is a minimal index of a polygon, then s_k is a center, hence the last occurrence of s_k in q_x is as a center (Lemma 3.8.10) and by assumption there is no syllable with s_{k-1} at its top since k is the minimal index of its polygon. Hence there is no s_{k-1} after the last occurrence of s_k in w_x implying that $k \in J_{w_x}$.

If $k \in L_{c'}$, $k \notin \mathrm{Vert}(x)$ but k is nested in a polygon of x, then by assumption there is no syllable with $k-1$ at its top (otherwise k would be terminal, in particular $k \in \mathrm{Vert}(x)$) and s_k is contributed from the right of any syllable in which it occurs, implying again that there is no s_{k-1} after the last occurrence of s_k, hence $k \in J_{w_x}$.

Now assume that $k \in R_{c'}$ but that s_k is not a center of q_x. Therefore if s_k appears in a syllable of q_x, s_{k-1} also appears; since s_k is contributed from the left in such syllables, there is always an s_{k-1} on its right implying $k \notin J_{w_x}$.

If $k \in L_{c'}$ and s_k appears in q_x, assume that k is not initial and that either

$k \in \mathrm{Vert}(x)$ or k is not nested in a polygon of x. In particular, s_k is either a center or a terminal index. If it is a center, then the last occurrence of s_k in q_x is as a center, but since k is not initial, there is a syllable w with s_{k-1} at its top and w appears after the s_k which is at the center (Lemma 3.8.2). If k is terminal, it means exactly that there is a syllable w with s_{k-1} at its top. By Lemma 3.8.11, w is the last syllable containing s_{k-1}. Moreover, since k is terminal, if s_k appears in q_x, then s_k appears in any syllable containing s_{k-1} and different from w and it occurs only in these syllables. Again, the last contribution of s_k is therefore followed by an s_{k-1} contributed from w implying $k \notin J_{w_x}$. □

Lemma 3.8.19. *Let $x \in \mathcal{P}_{c'}$. Let $k \in \{2, \dots, n+1\}$. If $k \neq n+1$, then*

$$k-1 \in I_{w_x} \Leftrightarrow \begin{cases} k \in R_{c'} \text{ and } k \text{ is a non initial index of a polygon of } x, \text{ or} \\ k \in L_{c'} \text{ and } k \text{ is a terminal index of a polygon of } x, \text{ or} \\ k \in L_{c'} \text{ and } k \notin \mathrm{Vert}(x) \text{ but } k \text{ is nested in a polygon of } x. \end{cases}$$

while $n \in I_{w_x}$ if and only if s_n appears in q_x.

Proof. We recall that $j \in I_{w_x}$ if and only if in any $\mathcal{S}$-reduced expression of w_x, there is no occurrence of s_{j+1} before the first occurrence of s_j in w_x. If s_n appears in q_x, then there is an s_n contributed to w_x, which forces $n \in I_{w_x}$ since the biggest index for a simple reflection is n. In case $k \neq n+1$, we first show that if any of the three conditions given on the right hand side holds, then $k-1 \in I_{w_x}$. We then prove that if they fail, then $k-1$ cannot lie in I_{w_x}. Notice that in case $k \neq n+1$, one has that $k \notin L_{c'}$ if and only if $k \in R_{c'}$.

First assume that $k \in R_{c'}$ and that k indexes a non initial vertex of a polygon. It means that s_{k-1} is at the top of a syllable w of q_x. Write P for the polygon containing w. If s_k is contained in P then s_k is at the center of a syllable w' of P which appears after w (Lemma 3.8.2). We need to show that any other polygon containing s_k satisfies $Q > P$. If P is right it is clear since any Q right and containing s_k will satisfy $\max Q > \max P$. Assume that P is alternating. Then a right polygon cannot contain s_k. Any $Q \succ P$ with Q containing s_k must have its miminal index in $L_{c'}$ implying $Q > P$. If Q contains s_k with $Q \prec P$, then $\min P \in R_{c'}$ whence $Q > P$. Hence there is no occurrence of s_k before the first occurrence of s_{k-1} in w_x since any polygon $Q \neq P$ containing s_k satisfies $Q > P$, hence the standard form of the cycle corresponding to Q occurs after that of the cycle corresponding to P in q_x. But in P, we have since that the contributions of s_k if there are any come from a syllable w' appearing after w. Hence we have that $k-1 \in I_{w_x}$.

Now assume that $k \in L_{c'}$ and k is a terminal index of a polygon of x. It means that s_{k-1} is at the top of a syllable w contained in a polygon P which does not contain s_k. In particular, s_{k-1} occurs in any syllable in which s_k occurs and the only other occurrences of s_{k-1} are in w. If s_{k-1} is a center and $k-1 \in R_{c'}$, then there is no occurrence of s_k (there is at most one and if there is one it would be as a center of a syllable of P, see Lemma 3.8.14). If s_{k-1} is a center and $k-1 \in L_{c'}$, then w is the first syllable containing s_{k-1} in case $w \neq s_{k-1}$ by Lemma 3.8.11 and the last one in case $w = s_{k-1}$; in the first case, we have the claim since s_k is not a center, hence any syllable containing s_k must contain s_{k-1}. In the second case, since s_k is not a center and $k \in L_{c'}$ the s_k are all contributed from the right, hence there is always an s_{k-1} contributed at their left since s_{k-1} appears in any syllable in which s_k appears, whence $k-1 \in I_{w_x}$. If s_{k-1} is not a center then since $k \in L_{c'}$ we obtain by Lemma 3.8.11 again that w is the last syllable containing s_{k-1} but since $k \in L_{c'}$ and k is not a center, all the s_k are contributed from the right, hence there is always an s_{k-1} on their left.

Now assume that $k \in L_{c'}$ and $k \notin \mathrm{Vert}(x)$ but k is nested in a polygon of x. This means exactly that both s_k and s_{k-1} appear in q_x and that they appear in the same syllables; since $k \in L_{c'}$ and k is not a center, all the s_k are contributed from the right of the syllables implying again $k-1 \in I_{w_x}$ since there is always an s_{k-1} contributed from the same syllable on their left.

Now assume $k \in R_{c'}$ and either k indexes an initial vertex of a polygon or $k \notin \mathrm{Vert}(x)$. In the first case s_k is a center and the first occurrence of s_k in q_x is at the center; therefore there is no occurrence of s_{k-1} before it since otherwise s_{k-1} would be at the top of a syllable, hence in the same polygon as the first s_k contradicting the assumption that k is initial. In the latter case this means exactly that s_k and s_{k-1} occur in the same syllables of q_x if they occur, but since $k \in R_{c'}$ and k is not a center, it is contributed from the left, and there is no s_{k-1} before the first s_k. In both cases we have $k-1 \notin I_{w_x}$.

If $k \in L_{c'}$ and the two corresponding conditions in the right hand side fail, it means that k is not a terminal index of a polygon and that either $k \in \mathrm{Vert}(x)$ or k is not nested in a polygon. If k is not a terminal index of a polygon and $k \in \mathrm{Vert}(x)$, it means exactly that k is a non terminal index of a polygon or equivalently that $k \in D_x^{c'}$. If k is not a terminal index of a polygon and k is not nested in a polygon, it implies that either $k \in D_x^{c'}$ or that $k \notin \mathrm{Vert}(x)$ and k is not nested in a polygon. In that last case, there is no syllable with $k-1$ at its top; if there is a syllable with s_{k-1} appearing not at the top, then k would be nested in such a syllable, a contradiction. Hence if k is not in $\mathrm{Vert}(x)$ and k is not nested, we have that s_{k-1} does not occur in q_x, implying that it cannot occur in w_x hence $k-1 \notin I_{w_x}$. We have to show that if $k \in D_x^{c'}$. then $k-1 \notin I_{w_x}$. Hence assume that s_k is a center of q_x. If there is a syllable with s_{k-1} at its top then it is a syllable of P which occurs after s_k (Lemma

3.8.2). In any other syllable in which s_{k-1} occurs, s_k also occurs and since s_k is a center and $k \in L_{c'}$, the s_k are contributed from the left, hence there is an s_k before the first s_{k-1}, implying $k-1 \notin I_{w_x}$. ☐

Corollary 3.8.20. *Let $x \in \mathcal{P}_{c'}$. Then $w_x = \varphi_{c'}(x)$.*

Proof. Thanks to Remark 3.2.15, this is an immediate consequence of 3.8.18 and 3.8.19. ☐

3.8.4 Triangularity

We recall from the introduction that an element $w \in \mathcal{W}$ with $\mathcal{S}$-reduced expression given by $s_{i_1}s_{i_2}\cdots s_{i_k}$ is fully commutative if and only if for any $1 \le i \le n$,

$$n_i(w) := |\{j \mid i_j = i\}|$$

depends only on w and not on the choice of the $\mathcal{S}$-reduced expression.

We also recall from the previous sections that for $x \in \mathcal{P}_{c'}$, the notation $x_i^{c'}$ is for the number of occurrences of s_i in any standard form of x (for example in q_x).

Lemma 3.8.21. *Let $w \in \mathcal{W}_f$. Then*

- *If s_i is a center of q_x, then $2n_i(w) - 1 = \psi_{c'}(w)_i^{c'}$,*
- *If s_i is not a center of q_x then $2n_i(w) = \psi_{c'}(w)_i^{c'}$.*

Proof. Thanks to Corollary 3.8.20, we have that $w = w_{\psi_{c'}(w)}$. The claims are then clear by the first point of the definition of the subword w_x of q_x where $x \in \mathcal{P}_{c'}$, given at the beginning of subsection 3.8.3. ☐

Notation. Given $x \in \mathcal{P}_{c'}$, we write $\mathrm{Sub}_f(x)$ for the set of fully commutative elements having an $\mathcal{S}$-reduced expression which is a subword of q_x. In particular, thanks to subsection 3.8.3, we have that $\varphi_{c'}(x) \in \mathrm{Sub}_f(x)$.

Remark 3.8.22. In case $c = c'$, $\mathrm{Sub}_f(x)$ consists exactly of the fully commutative elements w such that $w <_{\mathcal{S}} x$, where $<_{\mathcal{S}}$ denotes the Bruhat order; more generally this holds if q_x is an $\mathcal{S}$-reduced expression for x (which is always true in case $c = c'$).

Notation. We denote by $<_{\mathcal{V}}$ the order on $\mathcal{P}_{c'}$ giving the lattice structure considered in the previous section, that is, for $x, y \in \mathcal{P}_{c'}$, we have $x <_{\mathcal{V}} y$ if and only if for all $i \in \{1, \dots, n\}$, $x_i^{c'} \le y_i^{c'}$. In case $c = c'$ recall that this is just the restriction of the Bruhat order on $\mathcal{P}_c$.

Lemma 3.8.23. *There is a unique subword of q_x that is an $\mathcal{S}$-reduced expression of $\varphi_{c'}(x)$, namely w_x.*

Proof. Assume that there is a subword w of q_x which is an $\mathcal{S}$-reduced expression of $\varphi_{c'}(x)$. Then any syllable of q_x should contribute to that subword each reflection in its support exactly once:if not, it means that there is a syllable somewhere contributing two instances of a reflection s_i in its support (or no instances of s_i, but in that case there must be another syllable which contributes two s_i to compensate), which ends in a non fully commutative word since there is no occurrence of s_{i+1} between these two s_i.

Now assume $w \neq w_x$. It means that there exists at least one syllable w' of q_x where the contributions are different, that is, one reflection s_i which is contributed from the left of the syllable in one case and from the right in the other case. Hence in one case, the contribution of s_i from w' is before the contribution of s_{i-1} from w' and after in the other case. Since there must be the same number of contributions of s_i and s_{i-1} coming from the left of w' in the two subwords, it implies that one of the two subwords has no occurrence of s_i before the first s_{i-1} while the second one has, hence that they cannot represent the same fully commutative element. ☐

Proposition 3.8.24. *Let $s_{i_1} \cdots s_{i_k}$ be a subword of q_x and consider the corresponding Temperley-Lieb element*

$$b_{i_1} b_{i_2} \cdots b_{i_k},$$

which is equal to $(v+v^{-1})^m b_w$ for a unique $m \in \mathbb{Z}_{\geq 0}$ and a unique $w \in \mathcal{W}_f$. There exists at least one subword of $s_{i_1} \cdots s_{i_k}$ which is an $\mathcal{S}$-reduced decomposition of w and if $m > 0$, there is more than one subword which is a reduced expression for w.

Proof. This is a consequence of the fact that if $s_{i_1} \cdots s_{i_k}$ is not fully commutative (which is equivalent to saying that $m > 0$), then one can apply either the relation $b_i^2 = (v+v^{-1})b_i$ or the relation $b_i b_{i\pm1} b_i = b_i$ in the corresponding Temperley-Lieb element (possibly after having applied commutation relations); but since in any of these two relations, there are two b_i which reduce to a single b_i, after applying successive such relations and possibly commutation relations, the resulting w must have an $\mathcal{S}$-reduced expression which is a subword of the original word. But then it will be a subword in at least two different ways, since if one of the two relations was applied for at least one index i, then there is more than one subword equivalent representing w since one can choose in $s_{i_1} \cdots s_{i_k}$ the s_i located at the same place as the first b_i to contribute to the subword w or the s_i located at the same place as the second b_i. ☐

Lemma 3.8.25. *Let $x \in \mathcal{P}_{c'}$, $w \in \mathrm{Sub}_f(x)$. Then*

- *If s_i is a center of q_x, one has $2n_i(w) - 1 \leq x_i^{c'}$.*
- *If s_i is not a center of x, one has $2n_i(w) \leq x_i^{c'}$.*

Proof. When reading an $\mathcal{S}$-reduced expression of w as a subword of q_x, s_i can occur at most once in any syllable of x (otherwise by applying commutation relations in our $\mathcal{S}$-reduced expression we would get an expression having as substring either $s_i s_i$ or $s_i s_{i-1} s_i$, hence our initial expression could not be an $\mathcal{S}$-reduced expression of a fully commutative element). Depending on whether s_i is the center of a syllable of q_x or not, one gets the claim. □

Theorem 3.8.26 (Generalization of [42], Theorem 5). *Let $x \in \mathcal{P}_{c'}$, $w \in \mathrm{Sub}_f(x)$. Then*

$$\psi_{c'}(w) <_{\mathcal{V}} x.$$

Proof. We have to show that for any $i = 1, \ldots, n$,

$$\psi_{c'}(w)_i^{c'} \leq x_i^{c'}.$$

Set $y := \psi_{c'}(w)$. If s_i is a center of q_y, then we have that $2n_i(w) - 1 = y_i^{c'}$ thanks to Lemma 3.8.21, in which case Lemma 3.8.25 implies that $y_i^{c'} \leq x_i^{c'}$. If s_i is not a center of q_y, there would be a problem if s_i is a center of q_x and $x_i^{c'} = 2n_i(w) - 1$ since we would have $y_i^{c'} = 2n_i(w) > x_i^{c'}$. But we will show that if s_i is a center of q_x and $2n_i(w) - 1 = x_i^{c'}$, then s_i is also a center of q_y.

Hence assume that s_i is a center of q_x and $2n_i(w) - 1 = x_i^{c'}$. We abuse notation and will not distinguish between w and a subword of q_x which is an $\mathcal{S}$-reduced expression for w. Notice that $x_i^{c'} = 2n_i(\varphi_{c'}(x)) - 1$ thanks to Lemma 3.8.21, hence there are as much contributions of s_i from q_x to w as to the subword w_x; as a consequence, any syllable of q_x which contains s_i must contribute a single s_i to w. If $i \in R_{c'}$, it implies that s_i first occurs in q_x as a center (3.8.10) of a syllable of a polygon P. If there is a syllable in q_x with s_{i-1} at its top, then that syllable must be in P and hence it appears before the first s_i thanks to Lemma 3.8.2. Any other syllable w' containing s_{i-1} must also contain s_i (and hence appears after the syllable with s_i at its center) and these are the only other syllables in which s_i appears. In particular, the s_i appearing at the right of any such syllable must contribute to w, otherwise there would be at some place in w two successive occurrences of s_i with no occurrence of s_{i-1} between them. In particular, the last s_i is contributed from the right of a syllable and there is no s_{i-1} on its right, implying $i \in J_w$. Since $i \in R_{c'}$, one then has that s_i is a center of q_y (see Remark 3.2.15). Now if $i \in L_{c'}$ thanks to Remark 3.8.4 one gets that there is no contribution of s_{i-1} before the first occurrence of s_i in q_x, implying that $i-1 \notin I_w$. Since $i \in L_{c'}$, by Remark 3.2.15 this means that either $i \in \mathrm{Vert}(x)$ with i not terminal or i is not nested in any polygon and also not terminal. Since we in addition know that s_i appears in q_x, s_i must be contained in at least one polygon, hence must either be a center or nested in some polygon. Hence the two conditions force s_i to be a center of q_x: the first condition says exactly that s_i is a center. If the second condition is satisfied this forces s_i to be a center since it has to be either a center or nested. □

Lemma 3.8.27. *Let $t \in \mathcal{T}$, $t = (i, k+1)$ with $i \leq k$. The image of $i_{c'}(t)$ under the injection $B^*_{c'} \hookrightarrow B$ is represented by the braid word*

$$\mathbf{s}_k^{\epsilon_k}\mathbf{s}_{k-1}^{\epsilon_{k-1}} \cdots \mathbf{s}_{i+1}^{\epsilon_{i+1}}\mathbf{s}_i\mathbf{s}_{i+1}^{-\epsilon_{i+1}} \cdots \mathbf{s}_{k-1}^{-\epsilon_{k-1}}\mathbf{s}_k^{-\epsilon_k},$$

where for all $i+1 \leq j \leq k$, $\epsilon_j = 1$ if $j \in L_{c'}$, $\epsilon_j = -1$ if $j \in R_{c'}$.

Proof. We argue by induction on $k-i$. If $k-i=0$, then $\mathbf{s}_i$ represents the simple element $i_{c'}(s_i)$ in the braid group. Assume that $k-i>0$. If $k \in R_{c'}$, then the triple $((i,k), s_k, (i,k+1))$ is admissible hence we have in $B^*_{c'}$ the relation

$$i_{c'}((i,k))i_{c'}(s_k) = i_{c'}(s_k)i_{c'}((i,k+1)).$$

If we embed it into the braid group and use induction we have that $i_{c'}((i,k+1))$ is represented by the braid word

$$\mathbf{s}_k^{-1}(\mathbf{s}_{k-1}^{\epsilon_{k-1}} \cdots \mathbf{s}_{i+1}^{\epsilon_{i+1}}\mathbf{s}_i\mathbf{s}_{i+1}^{-\epsilon_{i+1}} \cdots \mathbf{s}_{k-1}^{-\epsilon_{k-1}})\mathbf{s}_k$$

where for all $i+1 \leq j \leq k-1$, $\epsilon_j = 1$ if $j \in L_{c'}$, $\epsilon_j = -1$ if $j \in R_{c'}$.

If $k \in L_{c'}$, then the triple $((i,k+1), s_k, (i,k))$ is admissible hence we have in B^*_c the relation

$$i_{c'}(s_k)i_{c'}((i,k)) = i_{c'}((i,k+1))i_{c'}(s_k)$$

from which we also derive the claimed formula. □

Theorem 3.8.28. *Let $x \in \mathcal{P}_{c'}$ and write $Z_x^{c'} \in \mathrm{TL}_n(v+v^{-1})$ for the image of the simple element $i_{c'}(x)$ in the Temperley-Lieb algebra. Then for $w \in \mathcal{W}_f$, there exist coefficients $c_w^x \in \mathbb{Z}[v,v^{-1}]$, with $c^x_{\varphi_{c'}(x)}$ invertible, such that*

$$Z_x^{c'} = \sum_{w \in \mathcal{W}_f, \psi_{c'}(w) <_{\mathcal{V}} x} c_w^x b_w.$$

In other words, if one considers any linear extension of the order $\mathcal{V}$ on $\mathcal{P}_{c'}$ together with the total order induced on $\mathcal{W}_f$ by $\varphi_{c'}$, then there is an upper triangular matrix *with invertible coefficient on the diagonal which allows one to pass from the diagram basis to the set* $\{Z_x^{c'}\}_{x\in\mathcal{P}_{c'}}$.

Remark 3.8.29. In Zinno's ([42]) and Lee and Lee's ([31]) proofs, the Coxeter element is not arbitrary. We already know from Vincenti's work (see [41]) that the set $\{Z_x^{c'}\}_{x\in\mathcal{P}_{c'}}$ is a basis of the Temperley-Lieb algebra (it is proven using the same method as Lee's). The theorem above gives a new proof of this fact. In addition it shows that there are orders making the change of basis matrix upper triangular. Hence we get a generalization of both Zinno's and Vincenti's results.

Proof. Recall that the standard form q_x is a Coxeter word representing the element x and that it is obtained by concatenating various words $m_{y_j}^{c'}$ which are standard forms of the cycles y_j corresponding to the polygons of x. The words $m_{y_j}^{c'}$ are obtained from specific $\mathcal{T}$-reduced decompositions where we replaced each reflection by an $\mathcal{S}$-reduced decomposition of it called a syllable. As a consequence, if $q_x = w_1 \star w_2 \star \cdots w_m$ where $w_1, \ldots, w_m$ are the syllables, then the corresponding simple element $i_c(x)$ of the dual braid monoid is equal to the product of the atoms $i_{c'}(t_1)\cdots i_{c'}(t_m)$ where $t_1, \ldots, t_m$ are the reflections for which $w_1, \ldots, w_m$ are $\mathcal{S}$-reduced expressions. But recall that the syllable w_ℓ corresponding to the reflection $t_\ell = (i, k+1)$ is equal to the word $w_\ell = s_k s_{k-1}\cdots s_i \cdots s_{k-1}s_k$. Thanks to Lemma 3.8.27, the atom $i_{c'}(t_\ell)$ embedded in the braid group has an expression in the dual braid monoid of the form $\mathbf{s_k}^{\epsilon_k}\mathbf{s_{k-1}}^{\epsilon_{k-1}}\cdots\mathbf{s_i}\cdots\mathbf{s_{k-1}}^{-\epsilon_{k-1}}\mathbf{s_k}^{-\epsilon_k}$ where $\epsilon_j = \pm 1$ for each $i+1 \leq j \leq k$. It implies that $i_{c'}(x)$ is represented by a word obtained from q_x by replacing each of the s_i by $\mathbf{s_i}$ or $\mathbf{s_i}^{-1}$. Now in the Temperley-Lieb algebra, $\mathbf{s_i}$ is mapped to $b_i - v$ while $\mathbf{s_i}^{-1}$ is mapped to $b_i - v^{-1}$. Hence putting together Lemma 3.8.23, Proposition 3.8.24 and Theorem 3.8.26, one gets the result. ∎

Appendix A

Weyl lines and reflection length.

In this appendix, we give a reformulation of the partial order on $\mathcal{P}_c$ in terms of parabolic subgroups.

Let V be the geometric representation of a finite Coxeter system $(\mathcal{W}, S)$. Let $\mathcal{T} \subset \mathcal{W}$ be the set of reflections and $<_{\mathcal{T}}$ the absolute order. Set

$$V^x = \{v \in V \mid xv = v\}.$$

For $x \in \mathcal{W}$, we have (see [10], Lemma 2)

$$\ell_{\mathcal{T}}(x) = \dim V - \dim V^x.$$

Let $c \in \mathcal{W}$ be a Coxeter element. In [7], the following is proven:

Proposition A.0.30. *Let $x, y \in \mathcal{W}$ such that $x, y <_{\mathcal{T}} c$. Then*

$$x <_{\mathcal{T}} y \Leftrightarrow V^y \subset V^x.$$

Recall that a Weyl line is a one dimensional subspace of V which is an intersection of reflecting hyperplanes. Weyl lines are in bijection with maximal parabolic subgroups of $\mathcal{W}$. We write $Z \subset V$ for the union of all the Weyl lines. It is clear that $\mathcal{W}$ acts on Z.

Lemma A.0.31. *Let $x \in \mathcal{W}$. Suppose that $\ell_{\mathcal{T}}(x) < \dim V$. Then there exists $z \in \mathcal{W}$ such that*

$$\ell_{\mathcal{T}}(xz) = \ell_{\mathcal{T}}(x) + \ell_{\mathcal{T}}(z) = \dim V.$$

Proof. Let $x = t_1 \cdots t_k$ be a reduced expression, with $t_i \in \mathcal{T}$. Since $\ell_{\mathcal{T}}(x) < \dim V$ one has that $V^x \neq 0$. It suffices to show that there exists a reflection $t \in \mathcal{T}$ such that $\ell_{\mathcal{T}}(xt) > \ell_{\mathcal{T}}(x)$, in other words that $V^t \cap V^x \neq V^x$, and then iterate. If this fails, then $V^x \subset \bigcap_{t \in \mathcal{T}} V^t = 0$, a contradiction. □

We now prove the following

Lemma A.0.32. *Let $x \in \mathcal{W}$. The vector space generated by Z^x is exactly V^x.*

Proof. Set $n = \dim V$. We use induction on $\dim V^x = n - \ell_{\mathcal{T}}(x)$. If $\ell_{\mathcal{T}}(x) = n-1$, let $x = t_1 \cdots t_{n-1}$ a reduced $\mathcal{T}$-decomposition of x. Then $V^x = H_{t_1} \cap H_{t_2} \cap \cdots \cap H_{t_{n-1}}$, which is a Weyl line. Since any Weyl line fixed by x is in V^x we conclude that $V^x = Z^x$. Now suppose $\ell_{\mathcal{T}}(x) < n-1$. Let $t \in \mathcal{T}$ such that $\ell_{\mathcal{T}}(xt) > \ell_{\mathcal{T}}(x)$. By induction V^{xt} is equal to the vector space generated by Z^{xt}. We have that $V^{xt} \subset V^x$, hence $Z^{xt} \subset Z^x$. Suppose that the vector space U generated by Z^x is different from V^x. Since $\dim(V^x/V^{xt}) = 1$ this forces $U = V^{xt}$. But $V^{xt} \subset H_t$ since $t <_{\mathcal{T}} xt$, hence $U \subset H_t$. Using the lemma above, let $z \in \mathcal{W}$ such that $\ell_{\mathcal{T}}(xtz) = \ell_{\mathcal{T}}(xt) + \ell_{\mathcal{T}}(z) = n$. We have $xtz = x(tzt)t$. We have $\ell_{\mathcal{T}}(xtzt) = \dim V - 1$. Write $xtzt = q_1 \cdots q_{n-1}$, $q_i \in \mathcal{T}$ a reduced decomposition. Then $H_{q_1} \cap \cdots \cap H_{q_{n-1}}$ is a Weyl line which is x-fixed but cannot be t-fixed since $x(tzt)t$ fixes no line. This is a contradiction with $U \subset H_t$. ☐

Putting A.0.30 and A.0.32 together we get

Proposition A.0.33. *Let $x, y \in \mathcal{W}$ such that $x, y <_{\mathcal{T}} c$. Then*

$$x <_{\mathcal{T}} y \Leftrightarrow Z^y \subset Z^x.$$

Let $x \in \mathcal{W}$. Write $\mathcal{P}(x)$ for the set of maximal parabolic subgroups of $\mathcal{W}$ containing x.

Corollary A.0.34. *Let $x, y <_{\mathcal{T}} c$. Then*

$$x <_{\mathcal{T}} y \Leftrightarrow \mathcal{P}(y) \subset \mathcal{P}(x).$$

Proof. Use 2.1.3. ☐

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