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Eigenfunction Concentration for Polygonal Billiards

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In this note, we extend the results on eigenfunction concentration in billiards as proved by the third author in [8]. *There, the methods developed in Burq and Zworski* [3] *to study eigenfunctions for billiards which have rectangular components were applied. Here we take an arbitrary polygonal billiard B and show that eigenfunction mass cannot concentrate away from the vertices; in other words, given any neighborhood U of the vertices, there is a lower bound*

$$\int_U |u|^2 \geq c \int_B |u|^2$$

for some $c = c(U) > 0$ *and any eigenfunction u.*

Keywords Control region; Eigenfunction concentration; Polygonal billiards; Semiclassical measures.

Mathematics Subject Classification 35P20.

1. Introduction

Let B be a plane polygonal domain, not necessarily convex. Let V denote the set of all vertices of B, and let Δ_B denote the Dirichlet or the Neumann Laplacian on $L^2(B)$. In this note, we will prove the following

Theorem 1. *Let B and V be as above and let U be any neighborhood of V. Then there exists $c = c(U) > 0$ such that, for any L^2-normalized eigenfunction u of the Dirichlet*

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(or Neumann) Laplacian Δ_B*, we have*

$$\int_U |u|^2 \geq c. \tag{1.1}$$

That is, U is a control region for B, in the terminology of [1, 2].

To understand the issues involved in this proof, first consider the case that every billiard trajectory meets the set U, or in other words, U geometrically controls B. In this case, propagation estimates along billiard trajectories show that U analytically controls B in the sense of (1.1) [9].

So the main point is to deal with cases when geometric control fails. In this case we exploit the special properties of billiard flow on polygonal domains. It is known that every billiard trajectory which avoids a neighborhood U of the set of vertices is periodic [6]. Clearly, periodic trajectories on a polygonal billiard come in 1-parameter families, which form 'cylinders' in $B\backslash U$. Moreover, there are only finitely many such cylinders [4].

These geometric results, together with propagation results for eigenfunctions, imply that if a sequence of eigenfunctions concentrates away from U, it must concentrate along the families of periodic trajectories that sweep out such cylinders. But this can be ruled out using the argument of Burq and Zworski [2, 3], as developed by the third author in [8], which shows that such concentration is not possible.

We will actually work in the setting of Euclidean surfaces with conical singularities since this is both more general and, we believe, conceptually simpler. A Euclidean surface with conical singularities (ESCS) is a surface X equipped with a metric g such that X may be written $X_0 \cup P$ where the metric g is Euclidean on X_0, and P consists of a finite number of points p_i, such that each p_i has a neighborhood isometric to a Euclidean cone whose tip corresponds to p_i. Any plane polygonal domain B can be doubled across its boundary to produce an ESCS. In this procedure each vertex of B with angle α gives rise to a conic point of X with angle 2α. A billiard trajectory on B gives rise to a geodesic (locally, just a straight line in the plane) on X. We will only consider trajectories that do not meet the vertices (conic points) in this paper.

The Laplacian on X is defined by taking the Friedrichs extension of the operator with domain $C_c^\infty(X_0)$. This is self-adjoint with discrete spectrum tending to infinity. Let u_j be the L^2-normalized eigenfunctions of the Laplacian on X. Our main result is

Theorem 2. *Let X be a compact orientable ESCS and U any neighborhood of the set P of conic points. Then there exists a positive constant* $c = c(U)$ *such that any normalized eigenfunction* u_k *of the Euclidean Laplace operator on X satisfies*

$$\int_U |u_k|^2 \geq c. \tag{1.2}$$

Remark 3. Theorem 2 implies Theorem 1 (see Section 2). However, it also applies to several settings other than polygonal billiards. For example, it applies to polygons with slits, to tori with polygonal holes and slits, and to translation surfaces, i.e., surfaces that can be realized by identifying by translation the sides of a $2n$-gon pairwise (see [10] for a survey of dynamical results on this kind of surface).

Remark 4. In the case that X has at least one conic point p with angle of the form $2\pi/n$, where n is an integer, the result of Theorem 2 can be sharpened, by taking U to be a neighborhood only of those conic points with angle α such that $2\pi/\alpha$ is not an integer. To see that we can exclude p with angle $2\pi/n$ from our control region, we form the n-fold cover $\widetilde{X}$ of X around p. Then the Laplacian on X lifts to the Laplacian on $\widetilde{X}$, acting on functions invariant under rotations of angle $2\pi/n$ about the lift $\tilde{p}$ of p. On $\widetilde{X}$, $\tilde{p}$ is a removable singularity, and the result follows by applying Theorem 2 to the ESCS obtained from $\widetilde{X}$ by excluding $\tilde{p}$ from the set of conic points of $\widetilde{X}$.

The proof of this theorem splits naturally into a geometric/dynamical part and an analytical part. To make this division as transparent as possible, we make the following definition.

Definition 5. Let X be a compact orientable ESCS. A region $U \subset X$ is said to satisfy condition (CC) (the 'cylinder condition') if the following two properties hold.

(1) Any orbit that avoids U is periodic.
(2) There exists a finite collection of cylinders $(\mathscr{C}_i)_{i \leq N}$ such that any orbit that avoids U belongs to some $\mathscr{C}_i$.

Here, by a *cylinder* we mean an isometric immersion of $\mathbb{S}^1_l \times I$ into X_0, where $I \subset \mathbb{R}$ is an interval and $\mathbb{S}^1_l$ is the circle of length l (see Lemma 6 below). Notice that on an orientable ESCS, any periodic orbit is part of a 1-parameter family of parallel periodic orbits, which together form a cylinder. Therefore, the key point in the second condition above is the *finiteness* of the number of cylinders.

We separate the proof of Theorem 2 into Proposition 9, in which we show that any neighborhood of P in X satisfies (CC) (the geometric/dynamical part), and Proposition 12, in which we show that any $U \subset X$ satisfying (CC) is a control region, i.e., satisfies (1.2) (the analytic part). The organization of the paper reflects these two steps of the proof. We first recall in Section 2 some basic facts about ESCSs, semiclassical measures and the doubling procedure that allows one to treat polygonal billiards as ESCSs. In Section 3, we prove Proposition 9. This result is already contained in [4, 6] in the special case of billiards, so our contribution is to extend this to ESCSs. Finally, in Section 4, we will prove Proposition 12, using a straightforward adaptation of the argument in [8] which is based in turn on [2].

2. ESCSs, Polygons and Semiclassical Measures

From now on, we work in the setting of Euclidean surfaces X with conical singularities (ESCSs), which were defined in the Introduction. Let P be the set of conic points and $X_0 = X \backslash P$ as before.

We first show that for any plane polygonal billiard B, possibly with polygonal holes and/or slits, the following doubling procedure gives a ESCS X. Take two copies B and σB of the polygon where σ is a reflection of the plane. The double X is obtained by considering the formal union $B \cup \sigma B$ where two corresponding sides are identified pointwise, see Figure 1. The reflection σ then gives an involution of X that commutes with the Laplace operator. The latter thus decomposes into odd and even functions and the reduced operators are then equivalent to the Laplace operator in P with Dirichlet and Neumann boundary condition respectively. In particular, for

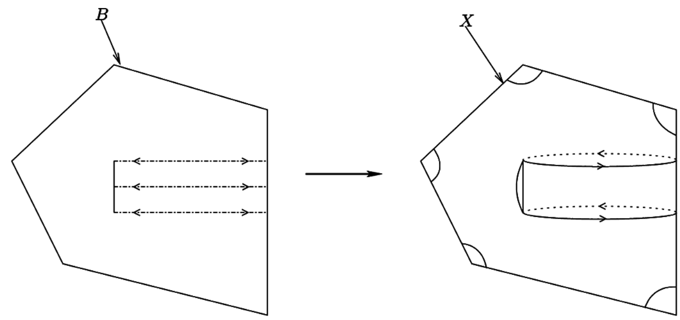

Figure 1. Doubling a billiard B, here with a slit, to form an ESCS X. Each vertex with angle α gives rise to a conic point with angle 2α, and the endpoints of the slit become conic points with angle 4π.

any u_n eigenfunction of the Neumann, resp. Dirichlet Laplace operator in P, we can construct an eigenfunction of the Laplace operator in X by taking u in P and $u \circ \sigma$, resp., $-u \circ \sigma$ in σP. Using this construction we see immediately that Theorem 1 is a consequence of Theorem 2.

On such a surface, we shall consider the geodesic flow induced by the Euclidean metric on X_0. A geodesic that hits a conical point will be called *singular*. A *non-singular* geodesic will thus be a geodesic that can be extended infinitely while staying in X_0. The following lemma shows that a non-singular periodic geodesic on a ESCS is always part of a family.

Lemma 6. *Let X be an orientable ESCS. Let $g : \mathbb{R} \to X$ be a non-singular T-periodic geodesic, then there exists $\delta > 0$ such that g extends to a map h from $\mathbb{R} \times (-\delta, \delta)$ into X_0 such that*

(1) $h(t, 0) = g(t)$,
(2) *h is a local isometry from $\mathbb{R} \times (-\delta, \delta)$ equipped with the flat metric into X_0,*
(3) *h is T-periodic in t.*

Thus h may be viewed as defined on the cylinder $\mathscr{C}_{\delta,T} := \mathbb{S}^1_T \times (-\delta, \delta)$.

Proof. Let T be the smallest period of g. For any $t < T$ there exists δ_t such that the square $(-\delta_t, \delta_t)^2$ is isometric to a neighborhood of $g(t)$. Moreover, this isometry, say h_t, may be chosen so that the horizontal segment $(-\delta_t, \delta_t) \times \{0\}$ maps to $g(t - \delta_t, t + \delta_t)$ with $h_t(t_1, 0) = g(t + t_1)$. Using compactness, $\delta = \inf\{\delta_t, t \in [0, T]\}$ exists and is positive. Gluing the h_t by continuity defines a local isometry $h : \mathbb{R} \times (-\delta, \delta)$ into X_0. By construction, for any s, $h(t, s)$ is at distance $|s|$ of the geodesic g and this distance is realized by $g(t)$. Thus, there are only two possible choices for $h(t + T, s)$. Since X is orientable, necessarily $h(t + T, s) = h(t, s)$. □

Let $\eta \in (-\delta, \delta)$. As $\eta \uparrow \delta$, the periodic geodesics $h(t, \eta)$ converge to a possibly singular periodic geodesic (and similarly for $\eta \downarrow -\delta$). The cylinder $\mathscr{C}_{\delta,T}$ will be called maximal if both these geodesics are singular. In the geometric condition (CC), we may assume that the cylinders are maximal.

We now define the Euclidean Laplace operator on a ESCS. First note that the Euclidean metric on X provides us with a well-defined L^2 norm and that smooth functions compactly supported in X_0 are dense in $L^2(X)$. For any such function, we can also define the quadratic form $q(u) = \int_X |\nabla u|^2 dx$ in which ∇ is taken with respect to the Euclidean metric and dx is the Euclidean area element. The Laplace operator is the self-adjoint operator associated with the closure of this quadratic form. It is also the Friedrichs extension of the usual Euclidean Laplace operator defined on $\mathscr{C}_0^\infty(X_0)$. It is standard that this operator has compact resolvent so that its spectrum is purely discrete and we may consider its eigenvalues and eigenfunctions.

Let u_n be a sequence of eigenfunctions on X associated with a sequence of eigenvalues going to infinity. We want to associate to this sequence a so-called *semiclassical measure*. Since we do not want to look precisely at what is happening at the conical point, our semiclassical measure μ will be a positive distribution acting on $\mathscr{C}_0^\infty(S^*X_0)$, where S^*X_0 denotes the unit cotangent bundle over X_0. Our semiclassical measure is then given by the usual recipe. In particular, for any $a \in \mathscr{C}_0^\infty(S^*X_0)$ and any zeroth-order pseudodifferential operator A on X with principal symbol a we have

$$\lim_{n\to\infty} \langle Au_n, u_n\rangle = \int_{S^*X_0} a\, d\mu.$$

Remark 7. It is considerably simpler to define a pseudodifferential operator on X_0 than on X. In particular we may use local isometries with the Euclidean plane.

Remark 8. It should be noted that, in contrast with the usual semiclassical measure, with this definition, a semiclassical measure need not be a probability measure. In order to be a probability measure one has to prove that, loosely speaking, no mass accumulates at the conical points. In our proof, however, this subtlety is avoided since, by hypothesis, the mass goes to zero in a neighborhood of P (see (4.1)).

The invariance property of this measure by the geodesic flow also has to be taken carefully. The infinitesimal version of this invariance is true at each point in X_0 using the standard commutator argument and Egorov's theorem. One can then integrate this property along any geodesic until it reaches a conic point.

3. Condition (*CC*) for Neighborhoods of the Conic Set *P*

In this section we prove that any neighborhood of the set P of conic points of a compact ESCS X satisfies the cylinder condition (*CC*). Clearly it suffices to consider the ε neighborhood U_ϵ of P, for arbitrary $\epsilon > 0$.

Proposition 9. *Let X be an orientable ESCS with singular set P.*

(i) *For any geodesic γ, either γ is periodic, or the closure of γ meets P.*

(ii) *Let U_ϵ denote the ε neighborhood of P. Then any periodic geodesic avoiding U_ϵ (which is periodic by part* (i)) *belongs to a maximal cylinder and the number of such maximal cylinders is finite.*

That is, U_ϵ satisfies (*CC*).

Before proving Proposition 9, we introduce some notation and definitions. A *strip* is an isometric immersion $h : \mathbb{R} \times I \to X_0$, where I is a nonempty open interval and $\mathbb{R} \times I$ is equipped with the Euclidean metric. We will also sometimes call the image of h a strip. The *width* of the strip is the length of the interval I.

For any strip, the mappings $\gamma_c := h(\cdot, c)$, $c \in I$, are geodesics of X_0. Since h is a local isometry and X is orientable, if one γ_c is periodic of length L then, for any c', $\gamma_{c'}$ is also periodic with length L.

A *maximal strip* is a strip that cannot be extended to $\mathbb{R} \times I'$ for any open I' properly containing I. A strip is maximal if and only if P intersects the closure of $h(\mathbb{R} \times I)$ on its left and on its right.

For any geodesic γ we will denote by $\tilde{\gamma}$ the geodesic lifted to the unit tangent bundle SX_0 and we denote by π the projection of SX_0 in X. We also denote by $d(\cdot, \cdot)$ the distance on X.

Proposition 9 is a straightforward consequence of the following lemma, which is closely related to results of [6].

Lemma 10. *Let $h : \mathbb{R} \times I \to X_0$ be a strip of positive width. Then there exists L such that $h(t + L, s) = h(t, s)$.*

Proof. We follow closely the ideas of [6]. We may assume that h is maximal and $I = (-\delta, \delta)$ and we will prove that γ_0 is periodic. Observe that for any t, $d(\gamma_0(t), P) \geq \delta$. We denote by $Z \subset SX_0$ the forward limit set of the lifted geodesic $\tilde{\gamma}_0$. By continuity, we have that $d(\pi(Z), P) \geq \delta$. This implies first that Z is compact and then that the geodesic flow is continuous on Z. Using Furstenberg's uniform recurrence theorem [5], there exists a point x that is uniformly recurrent in Z. We denote by G the geodesic emanating from x (observe that $\widetilde{G}(\mathbb{R}) \subset Z$). We also denote by $H : \mathbb{R} \times (-\Delta^-, \Delta^+)$ the maximal strip around G. Uniform recurrence means the following: for any neighborhood $\widetilde{W} \subset SM_0$ that intersects $\widetilde{G}(\mathbb{R})$, there exists $L \in \mathbb{R}$ such that

$$\forall t,\ \exists s \in [t, t + L] \quad \text{such that } (G(s), \dot{G}(s)) \in \widetilde{W}.$$

The uniform recurrence and the maximality of the strip imply

$$\forall \varepsilon > 0,\ \exists L \quad \text{such that } \forall t, \quad d(H([t, t + L] \times \{\Delta^+ - \varepsilon\}), P) < 2\varepsilon. \tag{3.1}$$

Indeed, by maximality, for any $\varepsilon > 0$, there exists t_0 such that the geodesic $\bar{\gamma}$ emanating from the point in phase space given by $(G(t_0), \dot{G}(t_0) - \frac{\pi}{2})$ hits a conical point in time less than $\Delta^+ + \frac{\varepsilon}{2}$. In particular

$$d(H(t_0, \Delta^+ - \varepsilon), P) \leq \frac{3\varepsilon}{2}.$$

By continuity, we can find a neighborhood $\widetilde{V}$ of $(G(t_0), \dot{G}(t_0))$ such that, for any (m, θ) in this neighborhood, the geodesic starting from $(m, \theta - \frac{\pi}{2})$ stays in the $\frac{\varepsilon}{2}$ tubular neighborhood of $\bar{\gamma}$ until time $\Delta^+ - \varepsilon$. Using uniform recurrence, there exists L such that, for any t, there exists $s \in [t, t + L]$ so that $(G(s), \dot{G}(s))$ belongs to $\widetilde{V}$. Using the preceding property we have that

$$d(H(s, \Delta^+ - \varepsilon), H(t_0, \Delta^+ - \varepsilon)) < \frac{\varepsilon}{2}.$$

We conclude (3.1) using the triangle inequality.

Observe that (3.1) means that if we represent the strip H by a vertical strip in $\mathbb{R}^2$ then we can find a vertical strip of width 2ε (that contains the right boundary of the strip H) such that any rectangle of height L contained in this strip contains at least one conical point.

We fix local coordinates near x so that $x = ((0,0), \frac{\pi}{2})$. Since x is in the forward limit set, there exists t_n such that $\tilde{\gamma}_0(t_n)$ converges to x. We set $\tilde{\gamma}_0(t_n) = (z_n, \theta_n)$. Represent now in $\mathbb{R}^2$, the strip H around $(0,0)$ and the strip h around z_n. By standard Euclidean geometry, for $\varepsilon \ll \delta$ the intersection of any vertical strip of width 2ε with h contains a vertical rectangle of width 2ε and height that goes to ∞ when θ goes to $\frac{\pi}{2}$ (see Figure 2). Indeed, denoting by $\alpha = \frac{\pi}{2} - \theta$, this height is

$$\frac{2\delta}{\sin|\alpha|} - \frac{2\varepsilon}{\tan|\alpha|}.$$

Using (3.1), since there is no conical point in h, this implies that we have $\theta_n = \frac{\pi}{2}$ for n large enough. The strip h around z_n is thus represented by a vertical strip of width 2δ. Then maximality of both h and H implies that the strips coincide up to a translation in the first variable; in particular, the widths coincide, and z_n is independent of n. But this implies that the geodesic γ_0 is periodic. □

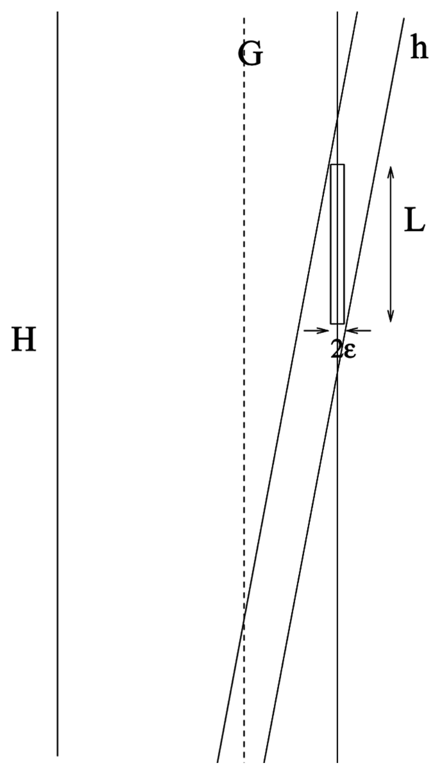

Figure 2. Illustration of the argument in the proof of Lemma 10. For sufficiently large n, if $\theta_n \neq \pi/2$ then the strip h would contain a rectangle of size $2\varepsilon \times L$, as illustrated. This is not possible as any such rectangle intersects P.

Proof of Proposition 9. (i) Let us consider a geodesic g such that $\overline{g(\mathbb{R})}$ contains no conical point. There exists $\varepsilon > 0$ such that $\forall t$, $B(g(t), \epsilon) \cap P = \emptyset$. For, otherwise, we could find sequences ε_n, t_n, and p_n such that $d(g(t_n), p_n) < \varepsilon_n$ contradicting the hypothesis. This implies that the geodesic $g : \mathbb{R} \to X_0$ extends to a strip of positive width and Lemma 10 concludes the proof.

(ii) We only have to prove the finiteness property. Denote by $\mathscr{C}_i$ the maximal cylinders. By definition the *middle geodesic* of $\mathscr{C}_i$ is at distance at least ε of the conical points. So that the $\varepsilon/2$ strip around this geodesic consists in periodic geodesics at distance at least $\varepsilon/2$ of the conical points. Denote by S_i this strip. The proof of Lemma 4.2 of [7] (see also Figure 2 of this reference and [4]) implies that if γ_i and γ_j are two periodic geodesics in strip S_i and S_j respectively then, at any of their intersections, they make an angle θ satisfying

$$\frac{1}{\sin\theta} \leq \frac{\min(L_i, L_j)}{\varepsilon}.$$

We can now adapt the argument of [4]. For any i we consider the following region V_i of SX_0,

$$V_i = \left\{(x, \theta), x \in S_i, |\theta - \theta_i| < \frac{\varepsilon}{2L_i}\right\}.$$

The preceding estimate implies:

(1) V_i is isometric to $(-\varepsilon/2, \varepsilon/2) \times \mathbb{S}_{L_i} \times (-\frac{\varepsilon}{2L_i}, \frac{\varepsilon}{2L_i})$,
(2) for different cylinders, the regions V_i are distinct.

The first point implies that the volume of V_i is bounded away from zero independently of the cylinder, and the second point coupled with the fact that the unit tangent space to X has finite volume yields the result. □

Remark 11. We have seen that the ε neighborhood U_ϵ of the conical points satisfies (CC). Since any geodesic that enters the ϵ-neighborhood also enters the annular region $\epsilon/2 \leq d(x, P) \leq \epsilon$, the union of these annular regions also satisfies (CC). A similar argument also shows that for any $\delta > 0$ and any $\varepsilon > 0$ the ε-neighborhood of the union of the normal co-bundle to the circles $d(x, p) = \delta$ satisfies (CC) (in the cotangent bundle!).

4. Proof that Regions Satisfying (CC) are Control Regions

Let X be a compact orientable ECSC, and let U a domain of X. We will denote by $U_0 = U \backslash P$ and subsequently $U_\epsilon = U \backslash B(P, \epsilon)$, where $B(P, \epsilon)$ is given by the union of balls of radius ϵ about the points $p \in P$.

Proposition 12. *Let X and U be as above, and assume that U satisfies condition* (CC). *Then the conclusion of Theorem* 2 *holds for U.*

Proof. The proof is by contradiction. Suppose that there exists a sequence u_n of normalized eigenfunctions of the Laplacian on X such that

$$\lim_{n\to\infty}\int_U |u_n|^2 = 0. \tag{4.1}$$

Let μ be any semiclassical measure associated to (u_n). Then we have the following (standard) properties of μ:

Lemma 13.

(i) *The support of* μ *is disjoint from* $\pi^{-1}(U_0)$.
(ii) μ *is a probability measure that is invariant under the geodesic flow.*

Proof of Lemma. (i) Suppose that there is a point $q \in \operatorname{supp}\mu$ with $\pi(q) \in U_0$. Choose a nonnegative function $\phi \in C^\infty(X_0)$ supported in U_0, with $\phi \equiv 1$ in a small neighborhood G of $\pi(q)$. Since $\phi \geq 0$ and μ is a positive measure, we have $\langle \mu, \phi\rangle \geq 0$. If $\langle \mu, \phi\rangle = 0$ then $\langle \mu, \chi\rangle = 0$ for every $\chi \in C_0^\infty(S^*X_0)$ supported in $\pi^{-1}(G)$, since we have $\chi = \chi\phi$, and by the positivity of μ and ϕ, $|\langle \mu, \chi\phi\rangle|$ is bounded by $\langle \mu, \phi\rangle \|\chi\|_\infty$. But this would mean that $\pi^{-1}(G)$ is disjoint from the support of μ, which is not the case. Thus we conclude that $\langle \mu, \phi\rangle > 0$. This means that

$$\lim_{n\to\infty}\int_B |u_n|^2\phi > 0,$$

contradicting our assumption about the sequence (u_n). This proves (i).

(ii) Consider a cutoff function ρ that is identically 1 near each conical point and identically 0 outside U. According to statement (i), we have $\int_{X_0}(1-\rho)d\mu = \int_{X_0} 1\, d\mu$. By (4.1), $\int_{X_0}(1-\rho)|u_n|^2 \to 1$, thus proving that μ is a probability measure. The invariance holds since μ is a semiclassical measure. □

Continuation of the Proof of Proposition 12. Let μ be as above, and let $(z, \zeta) \in T^*X_0$ be in the support of μ. According to the preceding lemma and the invariance property of μ, condition (*CC*) implies that z belongs to a cylinder periodic in the direction ζ.

The support of μ is thus included in the union of the maximal cylinders $\mathscr{C}_i$ defined in condition (*CC*).

Let $\mathscr{C}$ be such a cylinder. By definition, there is a local isometry between $\mathbb{S}^1_L \times (0, a)$ and $\mathscr{C}$. Using it, we can pull-back the eigenfunction u_n to $\mathscr{C}$. We now apply the argument of [8] to this function u_n on $\mathscr{C}$. Let us use Cartesian coordinates (x, y) on $\mathscr{C}$, where $x \in [0, L]$, $y \in [0, a]$ with $x = 0$ and $x = L$ identified. Thus $\{y = 0\}$ and $\{y = a\}$ are the two long sides of the cylinder, and the variable y parametrizes periodic geodesics. Choose a cutoff function $\chi \in C_c^\infty[0, a]$ such that $\chi = 1$ on an open set containing all y parametrizing all paths disjoint from U_ϵ (as opposed to $U_{\epsilon/2}$). In other words, we take χ such that $\chi(y) = 1$ for $y \in (\epsilon, a-\epsilon)$ and $\chi(y) = 0$ for $y \in [0, \frac{\epsilon}{2}) \cup (a - \frac{\epsilon}{2}, a]$. Then χu_n vanishes near the long sides of $\mathscr{C}$, and thus may be regarded as a function on a torus T. So we now have a sequence $v_n = \chi u_n$ on T. Consider any semiclassical measure ν associated with the sequence (v_n) on T. By compactness, the v_n are bounded in L^2, so there exists at least one semiclassical measure associated with (v_n) (on the torus). This could be the zero measure; this would be the case if $\|v_n\|_{L^2} \to 0$, for example. Since μ is supported on a finite number of cylinders, there are only a finite number of directions in the support of ν. So we can find a constant-coefficient pseudodifferential operator Φ on T that is microlocally 1 in a neighborhood of directions parallel to dx, i.e., in the direction of the unwrapped periodic paths, but vanishes microlocally in a neighborhood of every other direction in the support of ν. (See [8] for a discussion of constant-coefficient pseudodifferential operators on a torus.)

Consider the sequence of functions (Φv_n) on T. The semiclassical measures ν' associated to this sequence are related to those for the sequence (v_n) by $\nu' = \sigma(\Phi)\nu$, where $\sigma(\Phi)$ is the principal symbol of the operator Φ. Thus, the support of ν' is restricted to directions parallel to dx and to geodesics parametrized by y such that $\chi(y) = 1$ (because of Lemma 13 and the way we chose χ).

Now we apply the proposition on p. 46 of [3] which says:

Proposition 14. *Let $\Delta = -(\partial_x^2 + \partial_y^2)$ be the Laplacian on a rectangle $R = [0, l]_x \times [0, a]_y$. For any open $\omega \subset R^2$ of the form $[0, l]_x \times \omega_y$, there is C independent of λ such that, for any solution of*

$$(\Delta - \lambda^2)w = f + \partial_x g$$

on R, satisfying periodic boundary conditions, we have

$$\|w\|^2_{L^2(R)} \le C\big(\|f\|^2_{L^2(R)} + \|g\|^2_{L^2(R)} + \|w\|^2_{L^2(\omega)}\big).$$

(This proposition is stated in [3] for Dirichlet boundary conditions on a rectangle, but applies equally well to periodic boundary conditions as noted in [8].) We apply this with $w = w_n = \Phi v_n$, $f = f_n = \Phi((\partial_y^2\chi)u_n)$, $g = g_n = -2\Phi((\partial_y\chi)u_n)$, and ω contained in the set $\{\chi = 0\}$. (Note that Φ commutes with Δ and ∂_y.) Since f and g are supported on the support of $\nabla\chi$, their support is disjoint from that of ν', so $\|f_n\|^2_{L^2(R)} + \|g_n\|^2_{L^2(R)} \to 0$ as $n \to \infty$. Also, by our choice of ω, we have $\|w_n\|^2_{L^2(\omega)} = 0$. It follows that $\|w_n\|^2_{L^2(R)} \to 0$. But this means that $\nu' = 0$. This implies that ν has no mass along directions parallel to dx, which means that μ has no mass along the cylinder $\mathscr{C}$. Since $\mathscr{C}$ is arbitrary, and the number of such cylinders is finite, this means that μ has no mass, i.e., it is the zero measure. This contradicts part (ii) of Lemma 13. We conclude that Proposition 12 holds. □

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