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Spectral Simplicity and Asymptotic Separation of Variables

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Abstract: We describe a method for comparing the spectra of two real-analytic families, (a_t) and (q_t), of quadratic forms that both degenerate as a positive parameter t tends to zero. We suppose that the family (a_t) is amenable to 'separation of variables' and that each eigenspace of a_t is 1-dimensional for some t. We show that if (q_t) is asymptotic to (a_t) at first order as $t \to 0$, then the eigenspaces of (q_t) are also 1-dimensional for all but countably many t. As an application, we prove that for the generic triangle (simplex) in Euclidean space (constant curvature space form) each eigenspace of the Laplacian acting on Dirichlet functions is 1-dimensional.

1. Introduction

In this paper we continue a study of generic spectral simplicity that began with [HlrJdg09] and [HlrJdg10]. In particular, we develop a method that allows us to prove the following.

Theorem 1.1. *For almost every Euclidean triangle $T \subset \mathbf{R}^2$, each eigenspace of the Dirichlet Laplacian associated to T is one-dimensional.*

Although we establish the existence of triangles with simple Laplace spectrum, we do not know the exact geometry of a single triangle that has simple spectrum. Up to homothety and isometry, there are only two Euclidean triangles whose Laplace spectrum has been explicitly computed, the equilateral triangle and the right isoceles triangle, and in both of these cases the Laplace spectrum has multiplicities [Lame, Pinsky80, Berard79, Harmer08]. Numerical results indicate that other triangles might have spectra with multiplicities [BryWlk84]. Non-isometric triangles have different spectra [Durso88, Hillairet05].

More generally, we prove that almost every simplex in Euclidean space has simple Laplace spectrum. Our method applies to other settings as well. For example, we have the following.

Theorem 1.2. *For all but countably many α, each eigenspace of the Dirichlet Laplacian associated to the geodesic triangle T_α in the hyperbolic plane with angles 0, α, and α, is one-dimensional.*

If $\alpha = \pi/3$, then T_α is isometric to a fundamental domain for the group $SL_2(\mathbf{Z})$ acting on the upper half-plane as linear fractional transformations. P. Cartier [Cartier71] conjectured that $T_{\pi/3}$ has simple spectrum. This conjecture remains open (see [Sarnak03]).

Until now, the only extant methods for proving that a domain has simple Laplace spectrum consisted of either explicit computation of the spectrum, a perturbation of a sufficiently well-understood domain, or a perturbation within an infinite dimensional space of domains. As an example of the first approach, using separation of variables one can compute the Laplace spectrum of each rectangle exactly and find that this spectrum is simple iff the ratio of the squares of the sidelengths is not a rational number. In [HlrJdg09] we used this fact and an analytic perturbation to show that almost every polygon with at least four sides has simple spectrum. The method for proving spectral simplicity by making perturbations in an infinite dimensional space originates with J. Albert [Albert78] and K. Uhlenbeck [Uhlenbeck72]. In particular, it is shown in [Uhlenbeck72] that the generic compact domain with smooth boundary has simple spectrum.

In the case of Euclidean triangles, the last method does not apply since the space of triangles is finite dimensional. We also do not know how to compute the Laplace spectrum of a triangle other than the right-isoceles and equilateral ones. One does know the eigenfunctions of these two triangles sufficiently well to apply the perturbation method, but unfortunately the eigenvalues do not split at first order and it is not clear to us what happens at second order.

As a first step towards describing our approach, we consider the following example. Let T_t be the family of Euclidean right triangles with vertices (0,0), (1,0), and $(1, t)$ and let q_t denote the associated Dirichlet energy form

$$q_t(u) = \int_{T_t} \|\nabla u\|^2 \, dx \, dy.$$

For each $u, v \in C_0^\infty(T_t)$, we have $q_t(u, v) = \langle \Delta_t u, v \rangle$, where Δ_t is the Laplacian, and hence the spectrum of Δ_t equals the spectrum of q_t on the domain $H_0^1(T_t)$ with respect to the L^2-inner product on T_t.

As t tends to zero, the triangle T_t degenerates to the segment that joins (0,0) and (1,0). The spectrum of an interval is simple and hence one can hope to use this to show that T_t has simple spectrum for some small $t > 0$ (Fig. 1).

Indeed, the spectral study of domains that degenerate to a one-dimensional object is quite well developed. In particular, the asymptotic behaviour of the spectrum of ordered

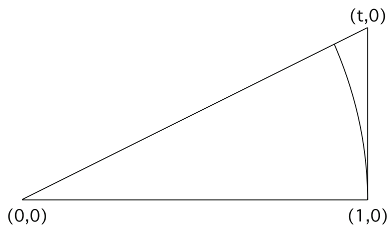

Fig. 1. The triangle T_t and the sector S_t.

eigenvalues involves a limiting one-dimensional Schrödinger operator (see, for example, [ExnPst05,FrdSlm09 and Grieser]). Using these kinds of results it can be proved that for each $n \in \mathbf{N}$, there exists $t_n > 0$ so that the first n eigenvalues of T_{t_n} are simple (as in [LuRowl]).

Unfortunately, this does not imply the existence of a triangle *all* of whose eigenvalues are simple. This subtle point is perhaps best illustrated by a different example whose spectrum can be explicitly calculated: Let C_t be the cylinder $[0, 1] \times \mathbf{R}/t\mathbf{Z}$. The spectrum of the Dirichlet Laplacian on C_t is

$$\left\{\pi^2 \cdot \left(k^2 + 4 \cdot \ell^2/t^2\right) \mid (k, \ell) \in \mathbf{N} \times (\mathbf{N} \cup \{0\})\right\}.$$

Moreover, for each $t > 0$ and $(k, \ell) \in \mathbf{N} \times \mathbf{N}$, each eigenspace is 2-dimensional. On the other hand, the first n eigenvalues of the cylinder C_t are simple iff $t < 2(n^2 - 1)^{-\frac{1}{2}}$.

The example indicates that the degeneration approach to proving spectral simplicity does not work at the 'zeroth order' approximation. The method that we describe here is at the next order. In the case of the degenerating triangles T_t, there is a second quadratic form a_t to which q_t is asymptotic in the sense that $a_t - q_t$ is controlled by $t \cdot a_t$. Geometrically, the quadratic form a_t corresponds to the Dirichlet energy form on the sector, S_t, of the unit disc with angle $\arctan(t)$ and it is quite a standard idea to analyse the spectra of thin right triangles using thin sectors (see for example [BryWlk84]).

The spectrum of the sectorial form a_t can be analyzed using polar coordinates and separation of variables. In particular, we obtain the Dirichlet quadratic form b associated to the interval of angles $[0, \arctan(t)]$, and, asssociated to each eigenvalue $(\ell \cdot \pi / \arctan(t))^2$ of b, we have a quadratic form a_t^ℓ on the radial interval $[0, 1]$. Each eigenfunction of a_t^ℓ is of the form $r \mapsto J_\nu(\sqrt{\lambda} \cdot r)$, where J_ν is a Bessel function of order $\nu = \ell\pi / \arctan(t)$ and where the eigenvalue, λ, is determined by the condition that this function vanish at $r = 1$. The spectrum of a_t is the union of the spectra of a_t^ℓ over $\ell \in \mathbf{N}$.

Figure 2 presents the main qualitative features of the spectrum of a_t after renormalization by multiplying by t^2. For each $\ell \in \mathbf{N}$, the (renormalized) real-analytic eigenvalue

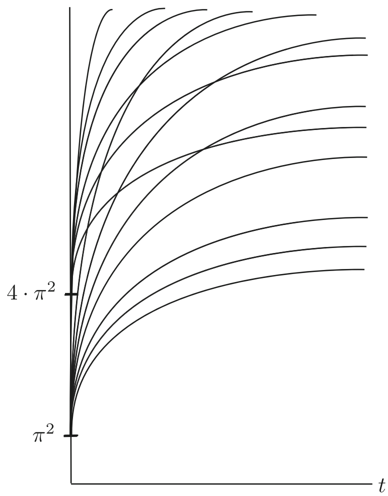

Fig. 2. The spectrum of the family a_t

branches of a coming from a_t^ℓ converge to the threshold $(\ell \cdot \pi)^2$. The eigenvalues of a_t^ℓ are simple for all t, and for all but countably many t, the spectrum of a_t is simple.

From the asymptotics of the zeroes of the Bessel function, one can show that the distance between any two (renormalized) real-analytic eigenbranches of a_t^ℓ is of order at least $t^{\frac{2}{3}}$. This 'super-separation' of eigenvalues is central to our method. Indeed, simplicity would then follow if one were to prove that each real-analytic eigenvalue branch of q_t lies in an $O(t)$ neighborhood of a real-analytic eigenvalue branch of a_t and that at most one eigenfunction branch of q_t has its eigenvalue branch lying in this neighborhood.

In fact, as sets, the distance between the spectrum of a_t and the spectrum of q_t is $O(t)$, and, with some work, one can prove that each (renormalized) real-analytic eigenvalue branch of q_t converges to a threshold in $\{(\ell \cdot \pi)^2 \mid \ell \in \mathbf{N}\}$ (Theorem 13.1). Nonetheless, infinitely many real-analytic eigenbranches of a_t converge to each threshold and the crossing pattern of these branches and the branches of q_t can be quite complicated. Semiclassical analysis predicts that the eigenvalues of $a_t^{\ell'}$ become separated at order t away from the threshold $(\ell \cdot \pi)^2$ (see Remark 10.5). On the other hand, two real-analytic eigenbranches that converge to the same threshold stay separated at order $t^{\frac{2}{3}}$. In order to use the super-separation of eigenvalues, we will need to show that each eigenvector branch of q_t whose eigenvalue branch converges to a particular threshold does not interact with eigenvector branches of a_t that converge to another threshold (see Lemmas 12.3 and 12.4). In this sense, we will asymptotically separate variables.

One somewhat novel feature of this work is the melding of techniques from semiclassical analysis and techniques from analytic perturbation theory. We apply quasimode and concentration estimates to make comparative estimates of the eigenvalues and eigenfunctions of a_t and q_t. We then feed these estimates into the variational formulae of analytic perturbation theory in order to track the real-analytic branches.

So far, our description of the method has been limited to the special case of degenerating right triangles. In §15 we make a change of variables that places the problem for right triangles into the following more general context. We suppose that there exists a positive abstract quadratic from b with simple discrete spectrum, and define

$$a_t(u \otimes \varphi) = t^2 \cdot (\varphi, \varphi) \int_0^\infty |u'(x)|^2\, dx + b(\varphi) \int_0^\infty |u(x)|^2\, dx. \tag{1}$$

We consider this family of quadratic forms relative to the weighted L^2-inner product defined by

$$\langle u \otimes \varphi, v \otimes \psi \rangle \;= (\varphi, \psi) \int_0^\infty u \cdot v\, \sigma\, dx,$$

where σ is a smooth positive function with $\sigma' < 0$ and $\lim_{x\to\infty} \sigma(x) = 0$. See §11. The spectrum of a_t decomposes into the joint spectra of

$$a_t^\mu(u) = t^2 \int_0^\infty |u'(x)|^2\, dx + \mu \int_0^\infty |u(x)|^2\, dx, \tag{2}$$

where μ is an eigenvalue of b (and hence is positive). Because σ is a decreasing function, an eigenfunction of a_t^μ with eigenvalue E oscillates for $x << x_E = (E - \mu \cdot \sigma)^{-1}(0)$ and decays rapidly for $x >> x_E$. Since $\sigma' < 0$, one can approximate the eigenfunction (or a quasimode at energy E) with Airy functions in a neighborhood of x_E. A good deal of the present work is based on this approximation by Airy functions. For example, the asymptotics of the zeroes of Airy functions underlies the super-separation of eigenvalues.

The following is the general result.

Theorem 1.3 (Theorem 14.1). *If q_t is a real-analytic family of positive quadratic forms that is asymptotic to a_t at first order* (*see Definition* 3.1), *then for all but countably many t, the spectrum of q_t is simple.*

Using an induction argument that begins with the triangle, we obtain the following:

Corollary 1.4. *For almost every simplex in Euclidean space, each eigenspace of the associated Dirichlet Laplacian is one-dimensional.*

Dirichlet boundary conditions can be replaced by any boundary condition that corresponds to a positive quadratic form b. In particular, one can choose any mixed Dirichlet-Neumann condition on the faces of the simplex except for all Neumann.

Using the 'pulling a vertex' technique of [HlrJdg09], we can extend generic simplicity to certain classes of polyhedra. For example, a d-dimensional polytope P is called k-*stacked* if P can be triangulated by introducing only faces of dimension $d-1$ [Grünbaum].

Corollary 1.5. *Almost every $d-1$-stacked convex polytope $P \subset \mathbf{R}^d$ with n vertices has simple Dirichlet spectrum.*

Finally, we note that by perturbing the curvature of Euclidean space as in §4 of [HlrJdg09] we obtain the following:

Corollary 1.6. *Almost every simplex in a constant curvature space form has simple Dirichlet Laplace spectrum.*

Organization of the paper. In §2 we use standard resolvent estimates to quantify the assertion that if two quadratic forms are close, then their spectra are close. In particular, we consider the projection, $P_a^I(u)$, of an eigenfunction q with eigenvalue E onto the eigenspaces of a whose eigenvalues lie in an interval $I \ni E$. We show that this projection is essentially a quasimode at energy E for a.

In §3 we specialize these estimates to the case of two real analytic families of quadratic forms a_t and q_t. We define what it means for q_t to be asymptotic to a_t at first order. We show that if the first order variation, $\dot{a}_t$, of a_t is nonnegative, then each real-analytic eigenbranch of a_t converges as t tends to zero, and if q_t is asymptotic to a_t at first order, then the eigenbranches of q_t also converge.

In Sect. 4, we use the variational formula along a real-analytic eigenfunction branch u_t to derive an estimate on the projection $P_{a_t}^I(u_t)$. This results in the assertion that the function

$$t \mapsto \frac{\|\dot{a}_t(P_{a_t}^I(u_t))\|}{\|P_{a_t}^I(u_t)\|^2}$$

is integrable (Theorem 4.2). The integrability will be used several times in the sequel to control the projection $P_{a_t}^I(u_t)$, and in particular, it will be used to prove that the eigenspaces essentially become one-dimensional in the limit. This result depends on both analytic perturbation theory and resolvent estimates.

Sections 5 through 10 are devoted to the study of the one dimensional quadratic forms a_t^{μ} in (2). Most of the material in these sections is based on asymptotics of solutions to second order ordinary differential equations (see, for example, [Olver]). In §6 we provide uniform estimates on the L^2-norm of quasimodes and on the exponential decay of eigenfunctions for large x. In §7 we make a well-known change of variables to transform the second order ordinary differential equation associated to a_t^{μ} into the inhomogeneous Airy equation. In §8 we use elementary estimates of the Airy kernel to estimate both quasimodes and eigenfunctions near the turning point x_E.

In §9 we use the preceding estimates to prove Proposition 9.1 which essentially says that the L^2-mass of both eigenfunctions and quasimodes of a_t^{μ} does not concentrate at x_E as t tends to zero. This proposition is an essential ingredient in proving the projection estimates of §12. But first we use it in §10 to prove that each real-analytic eigenvalue branch of a_t^{μ} converges to a threshold $\mu/\sigma(0)$.

In §10 we also establish the 'super-separation' of eigenvalue branches for a_t^{μ}. In the case of degenerating right-triangles, we may use the uniform asymptotics of the Bessel function (see [Olver]) to obtain the 'super-separation' near the threshold. We prove it directly in Proposition 10.4 for general σ.

In §11 we establish some basic properties of the quadratic form defined in (1). In §12 we combine results of §2, §4, and §9 to derive estimates on $P_{a_t}^{I}(u_t)$, where u_t is a real-analytic eigenfunction branch of q_t with eigenvalue branch E_t converging to a point E_0 belonging to the interior of an interval I.

In §13 we show that each eigenvalue branch of q_t converges to some threshold $\mu/\sigma(0)$ (Theorem 13.1). This leads to the following natural question: which thresholds $\mu/\sigma(0)$ are limits of some real-analytic eigenvalue branch of q_t? Strangely enough, we do not answer it here.

In §14 we prove the generic simplicity of q_t. In §15, we show how simplices and other domains in Euclidean space fit into the general framework presented here. Finally, in §16 we prove a generalization of Theorem 1.2.

2. Quasimode Estimates for Quadratic Forms

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Let a be a real-valued, densely defined, closed quadratic form on $\mathcal{H}$. Let $\mathrm{dom}(a) \subset \mathcal{H}$ denote the domain of a.

In the sequel, we will assume that the spectrum $\mathrm{spec}(a)$ of a with respect to $\langle\cdot,\cdot\rangle$ is discrete. Moreover, we will assume that for each $\lambda \in \mathrm{spec}(a)$, the associated eigenspace V_λ is finite dimensional, and we will assume that there exists an orthonormal collection, $\{\psi_\ell\}_{\ell\in\mathbf{N}}$, of eigenfunctions such that the span of $\{\psi_\ell\}$ is dense in $\mathcal{H}$.

The following estimate is standard:

Lemma 2.1 (Resolvent estimate). *Suppose that the distance, δ, from E to the spectrum of a is positive. If*

$$|a(w,v) - E\cdot\langle w,v\rangle| \leq \epsilon\cdot\|v\|$$

then

$$\|w\| \leq \frac{\epsilon}{\delta}.$$

Given a closed interval $I \subset [0, \infty)$, define P_a^I to be the orthogonal projection onto $\bigoplus_{\lambda \in I} V_\lambda$.

Definition 2.2. *Let q be a real-valued, closed quadratic form defined on* $\mathrm{dom}(a)$. *We will say that q is* **ε-close** *to a if and only if for each $v, w \in$* $\mathrm{dom}(a)$*, we have*

$$|q(v,w) - a(v,w)| \leq \varepsilon \cdot a(v)^{\frac{1}{2}} \cdot a(w)^{\frac{1}{2}}. \tag{3}$$

For each quadratic form q defined on $\mathrm{dom}(a)$, define

$$n_q(u) = \left(\|u\|^2 + q(u)\right)^{\frac{1}{2}}.$$

If q is ε-close to a, then the norms n_q and n_a are equivalent on $\mathrm{dom}(a)$. Thus, the form domains of q and a with respect to $\|\cdot\|$ coincide. We will denote this common form domain by $\mathcal{D}$.

Lemma 2.3. *Let q and a be quadratic forms such that q is ε-close to a. If u is an eigenfunction of q with eigenvalue E contained in the open interval $I \subset \mathbf{R}$, then*

$$a\left(u - P_a^I(u)\right) \leq \varepsilon^2 \cdot a(u) \cdot \left(1 + \frac{E}{\delta}\right)^2, \tag{4}$$

where δ is the distance from E to the complement $\mathbf{R}\backslash I$.

Proof. Let $v \in \mathcal{D}$. Since $q(u,v) = E \cdot \langle u, v\rangle$, from (3) we have

$$|E \cdot \langle u, v\rangle - a(u,v)| \leq \varepsilon \cdot a(u)^{\frac{1}{2}} \cdot a(v)^{\frac{1}{2}}. \tag{5}$$

There exists a linear functional f such that for all $v \in \mathcal{D}$ we have

$$E \cdot \langle u, v\rangle - a(u,v) = \langle f, v\rangle. \tag{6}$$

Write $f = \sum f_\ell \cdot \psi_\ell$ and define $v_{\text{test}} = \sum \lambda_\ell^{-1} f_\ell \cdot \psi_\ell$. Observe that

$$E \cdot \langle u, v_{\text{test}}\rangle - a(u, v_{\text{test}}) = \langle f, v_{\text{test}}\rangle = \sum_\ell \frac{|f_l|^2}{\lambda_l} = a(v_{\text{test}}).$$

By substituting $v = v_{\text{test}}$ into (5), we find that

$$\sum_\ell \frac{|f_l|^2}{\lambda_l} \leq \varepsilon^2 \cdot a(u). \tag{7}$$

Let $u = \sum_\ell u_\ell \cdot \psi_\ell$. From (6) we find that $u_\ell = (E - \lambda_l)^{-1} \cdot f_\ell$ for each $\ell \in \mathbf{N}$. Therefore,

$$a(u - P_a^I(u)) = \sum_{\lambda_l \notin I} \lambda_l \cdot \frac{|f_l|^2}{|E - \lambda_l|^2} \leq \varepsilon^2 \cdot a(u) \cdot \sup_{\lambda_l \notin I} \frac{\lambda_\ell^2}{|E - \lambda_\ell|^2},$$

where the inequality follows from (7). We have

$$\sup_{\lambda_l \notin I} \frac{\lambda_\ell^2}{|E - \lambda_\ell|^2} \leq \sup_{|1-x| > \delta/E} \frac{x^2}{|1-x|^2} = \left(\frac{E}{\delta} + 1\right)^2.$$

The desired bound follows. □

The preceding lemma provides control of the norm of $P_a^I(u)$. In particular, we have the following:

Corollary 2.4. *Let q and a be quadratic forms such that q is ε-close to a. If u is an eigenfunction of q with eigenvalue E contained in the open interval $I \subset \mathbf{R}$, then*

$$\|P_a^I(u)\|^2 \geq \left[1 - \varepsilon^2 \cdot \left(1 + \frac{E}{\delta}\right)^2\right] \cdot \frac{a(u)}{\sup(I)}, \tag{8}$$

where δ is the distance from E to the complement $\mathbf{R}\backslash I$.

Proof. Since $a(u - P_a^I(u), P_a^I(u)) = 0$, we have

$$a(P_a^I(u)) = a(u) - a\left(u - P_a^I(u)\right).$$

Thus, it follows from Lemma 2.3 that

$$a(P_a^I(u)) \geq \left(1 - \varepsilon^2 \left(1 + \frac{E}{\delta}\right)^2\right) \cdot a(u).$$

Since, on the other hand,

$$a(P_a^I(u)) \leq \sup(I) \cdot \|P_a^I(u)\|^2,$$

the claim follows. □

We use the preceding to prove the following.

Lemma 2.5. *Let I be an interval, let $E \in I$, and let δ denote the distance from E to the complement $\mathbf{R}\backslash I$. Let u be an eigenfunction of q with eigenvalue E. If $\varepsilon < (1 + E/\delta)^{-1}$ and q is ε-close to a, then for each $v \in \mathcal{D}$, we have*

$$\left|a\left(P_a^I(u), v\right) - E \cdot \langle P_a^I(u), v\rangle\right| \leq \frac{\varepsilon \cdot \sup(I)}{\left(1 - \varepsilon^2\left(1 + \frac{E}{\delta}\right)^2\right)^{\frac{1}{2}}} \cdot \|P_a^I(u)\| \cdot \|v\|. \tag{9}$$

Proof. Let $\tilde{u} = P_a^I(u)$, and $\tilde{v} = P_a^I(v)$. Since P_a^I is an orthogonal projection that commutes with a, we have $a(\tilde{u}, v) = a(\tilde{u}, \tilde{v}) = a(u, \tilde{v})$ and $\langle \tilde{u}, v\rangle = \langle \tilde{u}, \tilde{v}\rangle = \langle u, \tilde{v}\rangle$. Therefore, by replacing v with $\tilde{v}$ in (5) we obtain

$$|a(\tilde{u}, v) - E \cdot \langle \tilde{u}, v\rangle| \leq \varepsilon \cdot a(u)^{\frac{1}{2}} \cdot a(\tilde{v})^{\frac{1}{2}}.$$

Since $\tilde{v} \in P_a^I(\mathcal{H})$, we have

$$a(\tilde{v}) \leq \sup(I) \cdot \|\tilde{v}\|^2 \leq \sup(I) \cdot \|v\|^2.$$

By the hypothesis and Corollary 2.4, we have

$$a(u) \leq \left[1 - \varepsilon^2 \left(1 + \frac{E}{\delta}\right)^2\right]^{-1} \cdot \sup(I) \cdot \|P_a^I(u)\|^2.$$

By combining these estimates, we obtain the claim. □

Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{q_n\}_{n\in\mathbf{N}}$ be sequences of quadratic forms defined on $\mathcal{D}$. For each n, let E_n be an eigenvalue of q_n.

Proposition 2.6. *Suppose that* $\lim_{n\to\infty} E_n$ *exists and is finite. If the quadratic form* q_n *is* $1/n$*-close to* a_n *for each* n*, then there exist* $N > 0$ *and* $C > 0$ *such that for each* $n > N$ *and each eigenfunction* u *of* q_n *with eigenvalue* E_n*, we have*

$$\left| a_n\left(P^I_{a_n}(u), v\right) - E_n \cdot \langle P^I_{a_n}(u), v\rangle \right| \leq C \cdot \frac{1}{n} \cdot \left\| P^I_{a_n}(u) \right\| \cdot \|v\|. \tag{10}$$

Proof. Let $E_0 = \lim_{n\to\infty} E_n$ and let I be an open interval that contains E_0. Let δ_n be the distance from E_n to $\mathbf{R}\backslash I$. Since E_n converges to E_0 and I is open, there exists $\delta_0 > 0$ and N_0 so that if $n > N_0$, then $\delta_n > \delta_0$. Choose $N \geq \max\{N_0, 1 + 2E_0/\delta_0\}$ so that if $n > N$, then $E_n < 2E_0$. Then for each $n > N$ we have $n^{-1}(1 + E_n/\delta_n) \leq 1$, and we can apply Lemma 2.5 to obtain the claim. □

3. Asymptotic Families and Eigenvalue Convergence

Given a mapping of the form $t \mapsto f_t$, we will use $\dot{f}_t$ to denote its first derivative. More precisely, we define

$$\dot{f}_t := \left.\frac{d}{ds}\right|_{s=t} f_s.$$

Let a_t and q_t be real-analytic families of closed quadratic forms densely defined on $\mathcal{D} \subset \mathcal{H}$ for $t > 0$.[1] In this section, we show that the nonnegativity of both a_t and $\dot{a}_t$ implies that each real-analytic eigenvalue branch of a_t converges as t tends to zero. We then show that if q is asymptotic to a in the following sense then the eigenvalue branches of q_t also converge (Proposition 3.4).

Definition 3.1. *We will say that* q_t **is asymptotic to a_t at first order** *iff there exists* $C > 0$ *such that for each* $t > 0$ *and* $u, v \in \mathcal{D}$*,*

$$|q_t(u, v) - a_t(u, v)| \leq C \cdot t \cdot a_t(u)^{\frac{1}{2}} \cdot a_t(v)^{\frac{1}{2}}, \tag{11}$$

and

$$|\dot{q}_t(v) - \dot{a}_t(v)| \leq C \cdot a_t(v). \tag{12}$$

Remark 3.2. By reparameterizing the family—replacing t by say t/C—one may assume, without loss of generality, that $C = 1$. We will do so in what follows.

In what follows, we will assume that the eigenvalues and eigenfunctions of a_t and q_t vary real-analytically. To be precise, we will suppose for each $t > 0$, there exists an orthonormal collection $\{\psi_\ell(t)\}_{\ell\in\mathbf{N}}$ of eigenvectors whose span is dense in $\mathcal{H}$ such that $t \mapsto \psi_\ell(t)$ is real-analytic for each $\ell \in \mathbf{N}$. This assumption is satisfied if the operators that represent a_t and q_t with respect to $\langle\cdot,\cdot\rangle$ have a compact resolvent for each $t > 0$. See, for example, Remark 4.22 in §VII.4 of [Kato].

The following proposition is well-known:

[1] For notational simplicity, we will often drop the index t, but note that each object related to a or q will, in general, depend on t.

Proposition 3.3. *If $a_t \geq 0$ and $\dot{a}_t \geq 0$ for all small t, then each real-analytic eigenvalue branch of a_t converges to a finite limit as t tends to zero.*

Proof. Let λ_t be a real-analytic eigenvalue branch of a_t. By standard perturbation theory (see [Kato])

$$\dot{\lambda}_t \cdot \|u_t\|^2 = \dot{a}(u_t). \tag{13}$$

Thus, since $\dot{a}_t \geq 0$, the function $t \mapsto \lambda_t$ is increasing in t. Since λ_t is bounded below, the limit $\lim_{t \to 0} \lambda_t$ exists. □

If q_t is asymptotic to a_t, then the eigenvalues of q_t also converge.

Proposition 3.4. *Suppose that for each $t > 0$, the quadratic forms a_t and $\dot{a}_t$ are nonnegative. If q_t is asymptotic to a_t at first order, then each real-analytic eigenvalue branch of q_t converges to a finite limit.*

Proof. Let (E_t, u_t) be a real-analytic eigenbranch of q_t with respect to $\langle \cdot, \cdot \rangle$. We have

$$\dot{q}_t(u_t) = \dot{E}_t \cdot \|u_t\|^2. \tag{14}$$

Using (11), we have

$$q_t(v) \geq \frac{1}{2} \cdot a_t(v) \tag{15}$$

for all t sufficiently small. Since $a_t \geq 0$, we have $q_t \geq 0$ and hence $E_t \geq 0$ for small t. From (12) and Remark 3.2 we have $\dot{q}_t(u_t) \geq \dot{a}_t(u_t) - a_t(u_t)$ and hence, since $\dot{a}_t \geq 0$, we have $\dot{q}_t(u_t) \geq -a_t(u_t)$. By combining this fact with (14) and (15), we find that

$$\dot{E}_t + 2 \cdot E_t \geq 0 \tag{16}$$

for sufficiently small t.

To finish the proof, define the function f by $f(t) = E_t \cdot \exp(2t)$. By (16) we have $f'(t) \geq 0$ for $t < t_0$ and, since q_t is non-negative, f is obviously bounded from below. Therefore $\lim_{t \to 0} f(t)$ exists and is finite and so does $\lim_{t \to 0} E_t$. □

4. An Integrability Condition

Let q_t be a real-analytic family that is asymptotic to a_t at first order. In this section, we use the estimates of §2 to derive an integrability condition (Theorem 4.2) that will be used in §14 to prove that the spectrum of q_t is simple for most t under certain additional conditions.

Let E_t be a real-analytic eigenvalue branch of q_t that converges to E_0 as t tends to zero. Let V_t be the associated real-analytic family of eigenspaces. Let I be a compact interval whose interior contains E_0.

Remark 4.1. The definition of V_t implies that, for each $t > 0$, the vector space V_t is a subspace of $\ker(A_t - E_t \cdot I)$. If a distinct real-analytic eigenvalue branch crosses the branch E_t at $t = t_0$, then V_{t_0} is a proper subspace of $\ker(A_{t_0} - E_{t_0} \cdot I)$.

Theorem 4.2. *Let q_t be asymptotic to a_t at first order, and suppose that for each $t > 0$, we have*

$$0 \leq \dot{a}_t(v) \leq t^{-1} \cdot a_t(v). \tag{17}$$

If $t \mapsto u_t \in V_t$ is continuous on the complement of a countable set, then the function

$$t \mapsto \frac{\dot{a}_t\left(P^I_{a_t}(u_t)\right)}{\left\| P^I_{a_t}(u_t)\right\|^2} \tag{18}$$

is integrable on each interval of the form $(0, t^*]$.

Proof. Let $\chi_t = P^I_{a_t}(u_t)$. Since the spectrum of a_t is discrete and E_t is real-analytic, the operator family $t \mapsto P^I_{a_t}$ is real-analytic on the complement of a countable set. By combining this with the hypothesis, we find that the function $\dot{a}(P^I_{a_t}(u_t))/\|P_{a_t}(u_t)\|^2$ is locally integrable on $(0, t^*]$ for each $t^* > 0$.

By Lemma 4.3 below, there exists a constant $C > 0$ such that

$$\dot{E}_t \geq C \cdot \frac{\dot{a}_t(\chi_t)}{\|\chi_t\|^2} - C.$$

Integration then gives

$$E_{t^*} - E_t \geq C \int_t^{t^*} \frac{\dot{a}_s(\chi_s)}{\|\chi_s\|^2}\, ds - C(t^* - t).$$

Since $E_t \geq 0$ and the integrand is nonnegative, the integral on the right-hand side converges as t tends to zero. □

Lemma 4.3. *Suppose that for each $t > 0$, we have*

$$0 \leq \dot{a}_t(v) \leq t^{-1} \cdot a_t(v). \tag{19}$$

If q_t is asymptotic to a_t at first order, then there exists $t' > 0$ and a constant $C > 0$ such that for each $t \leq t'$ and each eigenvector $u \in V_t$ we have

$$\left| \dot{E}_t \cdot \|u\|^2 - \dot{a}_t\left(P^I_{a_t}(u)\right)\right| \leq C \cdot \|u\|^2 \tag{20}$$

and

$$\|P^I_a(u)\| \geq \frac{1}{C} \cdot \|u\|. \tag{21}$$

Proof. Since V_t is the real-analytic family of eigenspaces associated to E_t, for each $t > 0$ and $u \in V_t$ we have $\dot{q}(u) = \dot{E} \cdot \|u\|^2$ (see Remark 4.1). Since E_t converges to E_0, we find using (11) that there exists t_0 so that for $t < t_0$,

$$a_t(u) \leq 2q_t(u) = 2E_t \cdot \|u\|^2 \leq 2(E_0 + 1) \cdot \|u\|^2. \tag{22}$$

Thus, from (12) we find that

$$\left| \dot{E} \cdot \|u\|^2 - \dot{a}_t(u)\right| \leq 2(E_0 + 1) \cdot \|u\|^2 \tag{23}$$

for $t < t_0$.

Let $\chi_t = P^I_{a_t}(u)$. Since $\dot{a}_t$ is a nonnegative quadratic form, we have

$$\dot{a}_t(u) \leq \dot{a}_t(\chi_t) + 2\dot{a}_t(\chi_t)^{\frac{1}{2}} \cdot \dot{a}_t(u-\chi_t)^{\frac{1}{2}} + \dot{a}_t(u-\chi_t)$$

and

$$\dot{a}_t(\chi_t) \leq \dot{a}_t(u) + 2\dot{a}_t(u)^{\frac{1}{2}} \cdot \dot{a}_t(\chi_t-u)^{\frac{1}{2}} + \dot{a}_t(\chi_t-u).$$

The former estimate provides a bound on $\dot{a}_t(u) - \dot{a}_t(\chi_t)$ and the latter one gives a bound on its negation. In particular, we find that

$$|\dot{a}_t(u) - \dot{a}_t(\chi_t)| \leq 2 \cdot \max\left\{\dot{a}_t(u)^{\frac{1}{2}}, \dot{a}_t(\chi_t)^{\frac{1}{2}}\right\} \cdot \dot{a}_t(u-\chi_t)^{\frac{1}{2}} + \dot{a}_t(u-\chi_t).$$

Thus, by (19), we have

$$|\dot{a}_t(u) - \dot{a}_t(\chi_t)| \leq \frac{2}{t} \cdot \max\left\{a_t(\chi_t)^{\frac{1}{2}}, a_t(u)^{\frac{1}{2}}\right\} a_t(u-\chi_t)^{\frac{1}{2}} + \frac{a_t(u-\chi_t)}{t}. \quad (24)$$

Let δ_t be the distance from E_t to the complement $\mathbf{R}\backslash I$. Since E_0 belongs to the interior of I and $E_t \to E_0$, there exists $\delta > 0$ and $0 < t_1 \leq t_0$ so that if $t < t_1$, then $\delta_t \geq \delta$. Hence we may apply Lemma 2.3 to find that

$$a_t\,(u-\chi_t) \leq t^2 \cdot a_t(u) \cdot \left(1 + \frac{2E_0}{\delta}\right)^2$$

for $t < t_1$. Since a_t is non-negative, from (22) we have

$$a_t(\chi_t) \leq a_t(u) \leq 2(E_0+1) \cdot \|u\|^2$$

for $t \leq t_0$. By combining these estimates with (24) we find that for $t \leq t_1$,

$$|\dot{a}_t(u) - \dot{a}_t(\chi_t)| \leq 2(E_0+1) \cdot \|u\|^2 \cdot \left(1 + \frac{2E_0}{\delta}\right) \cdot \left(2 + t \cdot \left(1 + \frac{2E_0}{\delta}\right)\right). \quad (25)$$

Estimate (20) then follows from (23), (25) and the triangle inequality.

If $E_0 > 0$, then there exists $0 < t_2 \leq t_1$ such that if $t < t_2$, then

$$a_t(u) \geq \frac{1}{2} \cdot q_t(u) = \frac{1}{2} \cdot E_t \cdot \|u\|^2 \geq \frac{1}{4} \cdot E_0 \cdot \|u\|^2.$$

Thus, if $E_0 > 0$, then (21) follows from Corollary 2.4.

On the other hand, if $E_0 = 0$, then let t_1 and δ be as above. Since P^I_a is a spectral projection and the eigenspaces are orthogonal, we have

$$a\left(u - P^I_a(u)\right) \geq \delta \cdot \left\|u - P^I_a(u)\right\|^2.$$

Thus, by Lemma 2.3 and (22) we have

$$2t^2 \cdot (E_0+1) \cdot \left(1 + \frac{E_0+1}{\delta}\right) \cdot \|u\|^2 \geq \delta \cdot \left\|u - P^I_a(u)\right\|^2.$$

In particular, if $t^2 < (\delta/8) \cdot (E_0+1)^{-1} \cdot (1 + (E_0+1)/\delta)^{-1}$, then

$$\|u\|^2 \geq \frac{1}{4} \cdot \left\|u - P^I_a(u)\right\|^2.$$

Estimate (21) then follows from the triangle inequality. □

5. Definition and Basic Properties

In the sequel $\sigma : [0, \infty) \to \mathbf{R}^+$ will be a smooth positive function such that

- $\lim_{x\to\infty} \sigma(x) = 0$,
- $\sigma'(x) < 0$ for all $x \geq 0$,
- $|\sigma''|$ has at most polynomial growth on $[0, \infty)$.

For $u, v \in C_0^\infty((0, \infty))$, define

$$\langle u, v\rangle_\sigma = \int_0^\infty u(x) \cdot v(x) \cdot \sigma(x)\, dx.$$

Let $\mathcal{H}_\sigma$ denote the Hilbert space obtained by completing $C_0^\infty((0, \infty))$ with respect to the norm $\|u\|_\sigma := \sqrt{\langle u, u\rangle_\sigma}$.

Let $H^1(0, \infty)$ and $H_0^1(0, \infty)$ denote, respectively, the classical Sobolev spaces with respect to Lebesgue measure on $(0, \infty)$. For each $t > 0$ and u in $H^1(0, \infty)$, we define

$$a_t^\mu(u) = \int_0^\infty \left(t^2 \cdot |u'(x)|^2 + \mu \cdot |u(x)|^2\right) dx.$$

Remark 5.1. If $\mu > 0$, then since σ is decreasing, we have

$$\|u\|_\sigma^2 \leq \sigma(0) \int_0^\infty |u(x)|^2 dx \leq \frac{\sigma(0)}{\mu} a_t^\mu(u).$$

Let

$$\mathrm{dom}_D(a_t^\mu) = H_0^1(0, \infty) \cap \mathcal{H}_\sigma$$

and let

$$\mathrm{dom}_N(a_t^\mu) = H^1(0, \infty) \cap \mathcal{H}_\sigma.$$

Both $\mathrm{dom}_D(a_t^\mu)$ and $\mathrm{dom}_N(a_t^\mu)$ are closed form domains for a that are dense in $\mathcal{H}_\sigma$.

Definition 5.2. *The spectrum of the quadratic form a_t^μ restricted to* $\mathrm{dom}_D(a_t^\mu)$ *(resp.* $\mathrm{dom}_N(a_t^\mu)$*) with respect to $\langle\cdot,\cdot\rangle_\sigma$ will be called the* **Dirichlet** (*resp.* **Neumann**) **spectrum of a_t^μ.**

In the sequel, we will drop the subscript 'D' from $\mathrm{dom}_D(a_t^\mu)$ and the subscript 'N' from $\mathrm{dom}_N(a_t^\mu)$. In particular, unless stated otherwise, all of the results below hold for both the Neumann and Dirichlet boundary conditions. When we refer to the 'spectrum' of a_t^μ, we will mean either the Dirichlet or the Neumann spectrum.

Proposition 5.3. *If $\mu > 0$ and $t > 0$, then the quadratic form a_t^μ has discrete spectrum with respect to $\langle\cdot,\cdot\rangle_\sigma$.*

Proof. By a standard result in spectral theory—see, for example, Theorem XIII.64 [Reed-Simon]—it suffices to prove that for each $r > 0$ the set

$$A_r = \left\{u \in \mathrm{dom}(a_t^\mu) | a_t^\mu(u) \leq r,\ \|u\|_\sigma \leq 1\right\}$$

is compact with respect to $\|\cdot\|_\sigma$. To verify this, one uses Rellich's Lemma on compact sets. The decay of σ prevents the escape of mass at infinity. □

6. Estimates of Quasimodes and Eigenfunctions

In the sequel, unless otherwise stated, we assume that $\mu > 0$.

Let $r \in \mathcal{H}_\sigma$ and let $E \geq 0$. In this section, we begin our analysis of functions w in $\mathrm{dom}(a_t^\mu)$ that satisfy

$$a_t^\mu(w, v) - E \cdot \langle w, v\rangle_\sigma = \langle r, v\rangle_\sigma \tag{26}$$

for all $v \in \mathrm{dom}(a_t^\mu)$.

In applications, the function r in (26) will be negligible. For example, if $r = 0$, then w is an eigenfunction with eigenvalue E. More generally, if

$$a_{t_n}^\mu(w_n, v) - E_n \cdot \langle w_n, v\rangle_\sigma = \langle r_n, v\rangle_\sigma, \tag{27}$$

where $t_n \to 0$, $w_n \in \mathrm{dom}(a_{t_n}^\mu)$, $\lim E_n = E_0$ and $\|r_n\| = O(t_n^\rho) \cdot \|w_{t_n}\|$, then the sequence w_n is called *a quasimode of order* ρ at energy E_0. (See also Proposition 2.6 and Remark 9.2.)

Our goal is to understand the behavior of both eigenfunctions and quasimodes. Of course, in most situations, either the eigenfunction estimate will be stronger than the quasimode estimate and/or the proof will be simpler. In the following, we will first provide a general estimate—valid for any quasimode—and then, as needed, we will state and prove the stronger result for eigenfunctions.

By unwinding the definitions, Eq. (26) may be rewritten as

$$\int_0^\infty \left(t^2 \cdot w'(x) \cdot v'(x) + f_E(x) \cdot w(x) \cdot v(x)\right) dx = \int_0^\infty r(x) \cdot v(x) \cdot \sigma(x)\, dx, \tag{28}$$

where

$$f_E(x) = \mu - E \cdot \sigma(x).$$

By integrating (28) by parts, we find that w satisfies (28) for all $v \in C_0^\infty((0, \infty))$ if and only if for each $x \in (0, \infty)$,

$$-t^2 \cdot w''(x) + f_E(x) \cdot w(x) = r(x) \cdot \sigma(x). \tag{29}$$

The function w is a Dirichlet (resp. Neumann) eigenfunction of a_t^μ if and only if w is in $\mathcal{H}_\sigma \cap H^1$, satisfies Eq. (29) and $w(0) = 0$ (resp. $w'(0) = 0$).

Let $E \geq \mu/\sigma(0)$. For instance, we may choose E to be an eigenvalue of a_t^μ. Since σ is strictly decreasing, there exists a unique point $x_E \in [0, \infty)$ such that $f_E(x_E) = 0$. In particular, if $x > x_E$, then $f_E(x) > 0$ and if $x < x_E$, then $f_E(x) < 0$.

If w is an eigenfunction ($r = 0$), then one expects w to behave like an exponential function when $x >> x_E$ and to oscillate for $x << x_E$. Moreover, as t tends to zero, one expects that both types of behavior will become more and more extreme.

On the other hand, since $\lim_{x\to\infty} \sigma(x) = 0$, we do not know, for example, that r is bounded as x tends to infinity. In particular, for a non-zero r, we have no direct argument that shows that a solution w to (29) has exponential decay or is, in fact, bounded.

6.1. A general L^2 estimate. For each $E \geq \mu/\sigma(0)$ and $s \in [0, \mu)$, let $x_E^s \geq x_E$ be the unique solution to

$$f_E(x_E^s) = s. \tag{30}$$

Remark 6.1. Since the derivative of σ does not vanish, the mappings $E \mapsto x_E$ and $E \mapsto x_E^s$ are smooth from $[\mu/\sigma(0), \infty)$ to $(0, \infty)$. Note also that $\lim_{s \to 0} x_E^s = x_E$ and $\lim_{s \to \mu} x_E^s = \infty$.

The following estimate shows that if w_t satisfies (26) and

$$\lim_{t \to 0} \frac{\|r\|_\sigma}{\|w_t\|_\sigma} = 0,$$

then, for any fixed s, the L^2 mass of w_t concentrates in the region $\{x \mid x \leq x_E^s\}$ as t goes to 0. Additional work is required to prove that w_t actually concentrates in the *classically allowed region* $\{x \mid x \leq x_E\}$. See Proposition 9.1.

Lemma 6.2. *Let $K \subset [\mu/\sigma(0), \infty)$ be compact and let $s \in (0, \mu)$. There exists a constant C such that for each $E \in K$, $r \in \mathcal{H}_\sigma$, and solution w to* (29) *we have*

$$\int_{x_E^s}^{\infty} |w(x)|^2 \, dx \leq C \cdot \left(t^2 + \frac{\|r\|_\sigma}{\|w\|_\sigma} \right) \cdot \int_0^\infty |w(x)|^2 \, dx.$$

The constant C depends only upon K, μ, σ, and s.

Proof. Let $\chi : \mathbf{R} \to [0, 1]$ be a smooth function such that $\chi(x) = 0$ for all $x \leq 0$ and $\chi(x) = 1$ for all $x \geq 1$. For each $M \in \mathbf{R}$, define

$$\rho_M(x) = \chi\left(\frac{x - x_E}{x_E^s - x_E} \right) \cdot \chi(M + 1 - x).$$

Substitute $\rho_M \cdot w$ for v in (28). Since ρ_M vanishes for $x \leq x_E$, we obtain

$$\int_{x_E}^{\infty} t^2 \cdot (w')^2 \cdot \rho_M + t^2 \cdot w' \cdot w \cdot \rho_M' + f_E(x) \cdot w^2 \cdot \rho_M \, dx = \int_{x_E}^{\infty} r \cdot w \cdot \rho_M \cdot \sigma \, dx. \tag{31}$$

By integrating by parts, one finds that

$$\int_{x_E}^{\infty} w \cdot w' \cdot \rho_M' \, dx = -\frac{1}{2} \int_{x_E}^{\infty} w^2 \cdot \rho_M'' \, dx,$$

and hence (31) is equivalent to

$$\begin{aligned} t^2 \int_{x_E}^{\infty} (w')^2 \cdot \rho_M \, dx + \int_{x_E}^{\infty} f_E \cdot w^2 \cdot \rho_M \, dx \\ = \frac{t^2}{2} \int_{x_E}^{\infty} \rho_M'' \cdot w^2 \, dx + \int_{x_E}^{\infty} r \cdot w \cdot \rho_M \cdot \sigma \, dx. \end{aligned}$$

The first integral on the left-hand side is positive. Moreover, since $f_E \cdot w^2 \cdot \rho_M$ is non-negative on $[x_E, x_E^s]$, and $0 \leq \rho_M \leq 1$, we have

$$\int_{x_E^s}^{\infty} f_E \cdot w^2 \cdot \rho_M \, dx \leq \frac{t^2}{2} \int_{x_E}^{\infty} |\rho_M''| \cdot w^2 \, dx + \int_{x_E}^{\infty} |r \cdot w| \cdot \sigma \, dx. \tag{32}$$

The function $|\rho_M''|$ is bounded by a constant multiplied by $|x_E^s - x_E|^{-2}$. Therefore, since $x_E < x_E^s$, and x_E, x_E^s are smooth over the compact K (see Remark 6.1), there exists $C' > 0$ such that for each $E \in K$, $x \in [0, \infty)$, and $M \in \mathbf{R}$, we have $|\rho_M''(x)| \leq C'$.

By applying this estimate to (32) and applying the Cauchy-Schwarz inequality, we obtain

$$\int_{x_E^s}^{\infty} f_E \cdot w^2 \cdot \rho_M \, dx \leq \frac{C' \cdot t^2}{2} \int_{x_E}^{\infty} w^2 \, dx + \|r\|_\sigma \cdot \|w\|_\sigma .$$

For $x \geq x_E^s$, we have $\rho_M(x) = \chi(M + 1 - x)$. Thus, since $f_E \cdot w^2$ is integrable, by the Lebesgue dominated convergence theorem, we may let M tend to ∞ and obtain

$$\int_{x_E^s}^{\infty} f_E \cdot w^2 \, dx \leq \frac{C' \cdot t^2}{2} \int_{x_E}^{\infty} w^2 \, dx + \|r\|_\sigma \cdot \|w\|_\sigma .$$

Since σ is decreasing, the function $f_E(x)$ is increasing and

$$\|w\|_\sigma^2 \leq \sigma(0) \int_0^{\infty} w^2(x) \, dx.$$

Therefore, we find that

$$f_E(x_E^s) \cdot \int_{x_E^+}^{\infty} w^2 \, dx \leq \left(\frac{C' \cdot t^2}{2} + \sigma(0) \cdot \frac{\|r\|_\sigma}{\|w\|_\sigma} \right) \int_0^{\infty} w^2 \, dx.$$

Since $f_E(x_E^s) = s$, the lemma follows by choosing C to be $s^{-1} \max(\frac{C'}{2}, \sigma(0))$. □

6.2. An estimate of the L^2 mass of an eigenfunction. If w is an eigenfunction, then the bound given in Lemma 6.2 can be greatly improved. In particular, an eigenfunction is exponentially small in the classically forbidden region, and hence one can make L^2 estimates with polynomial weights. See Lemma 6.4.

First, we quantify the exponential decay of each eigenfunction.

Lemma 6.3. *Let w be an eigenfunction of a_t^μ with eigenvalue $\lambda \leq E$. If $x \geq y \geq x_E^s$, then*

$$w^2(x) \leq w^2(y) \cdot \exp\left(-\frac{\sqrt{2s}}{t} \cdot (x - y) \right). \tag{33}$$

Proof. The proof is a straightforward convexity estimate using the maximum principle. □

This estimate allows us to prove the following.

Lemma 6.4. *For each $\nu > 0$, there exists a function $\beta_\nu : (\mu/\sigma(0), \infty) \times (0, \mu) \to \mathbf{R}$ such that if w is an eigenfunction of a_t^μ with eigenvalue $\lambda \le E$, and $t \le 1$, then*

$$\int_{3x_E^s}^{\infty} w^2(x) \cdot (1 + x^\nu)\, dx \le \beta_\nu(E, s) \cdot t \cdot \int_{x_E^s}^{\infty} w^2(x)\, dx.$$

Proof. Let $\alpha = \sqrt{2s}/t$. By exchanging the roles of x and y in (33), we find that for all $x \in [x_E^s, y]$,

$$w^2(x) \ge w^2(y) \cdot \exp\left(\alpha \cdot (y - x)\right). \tag{34}$$

Integrating with respect to x, we obtain

$$\int_{x_E^s}^{y} w^2(x)\, dx \ge \frac{1}{\alpha} \cdot w^2(y) \cdot \big(\exp(\alpha \cdot (y - x_E^s) - 1)\big),$$

and thus

$$w^2(y) \le \frac{\alpha}{\exp(\alpha \cdot (y - x_E^s)) - 1} \cdot \int_{x_E^s}^{y} w^2(x)\, dx. \tag{35}$$

If $u \ge 0$, then $u^\nu \le c_\nu \cdot e^u$, where $c_\nu = \sup\{x^\nu e^{-u} \mid u > 0\}$. Hence, we have

$$x^\nu \le c_\nu \cdot \left(\frac{2}{\alpha}\right)^\nu \cdot e^{\alpha \cdot x/2}.$$

By combining this with (33), we find that for $x \ge y$,

$$w^2(x) \cdot x^\nu \le c_\nu \cdot \left(\frac{2}{\alpha}\right)^\nu \cdot w^2(y) \cdot \exp\left(-\alpha \cdot \left(\frac{x}{2} - y\right)\right).$$

By integrating, we find that

$$\int_{y}^{\infty} w^2(x) \cdot x^\nu\, dx \le c_\nu \cdot \left(\frac{2}{\alpha}\right)^{\nu+1} \cdot w^2(y) \cdot \exp(\alpha \cdot y/2).$$

Putting this together with (35) gives

$$\int_{y}^{\infty} w^2(x) \cdot x^\nu\, dx \le 2 \cdot c_\nu \cdot \left(\frac{2}{\alpha}\right)^\nu \cdot \left(\frac{\exp(\alpha \cdot y/2)}{\exp(\alpha \cdot y - \alpha \cdot x_E^s)) - 1}\right) \int_{x_E^s}^{y} w^2(x)\, dx. \tag{36}$$

If we let

$$c'_\nu = \sup\left\{ x \cdot \frac{\exp(3x/2)}{\exp(2x) - 1} \,\middle|\, x > 0 \right\}$$

and set $y = 3 \cdot x_E^s$, then we have

$$\frac{\exp(\alpha \cdot y/2)}{\exp(\alpha \cdot y - \alpha \cdot x_E^s)) - 1} \le \frac{c'_\nu}{\alpha \cdot x_E^s}.$$

By substituting this into (36) we obtain

$$\int_{3x_E^s}^{\infty} w^2(x)\cdot x^{\nu}\,dx \leq \frac{2c_\nu \cdot c'_\nu}{x_E^s}\cdot t^{\nu+1}\cdot s^{-\frac{\nu+1}{2}}\int_{x_E^s}^{\infty} w^2(x)\,dx. \tag{37}$$

The claim then follows by specializing (37) to the case $\nu = 0$ and adding the resulting estimate to (37). In particular, we may define

$$\beta_\nu(E,s) = 2\cdot\frac{c_0\cdot c'_0 + c_\nu\cdot c'_\nu}{x_E^s\cdot s^{\frac{\nu+1}{2}}}.$$

□

6.3. Comparing weighted L^2 inner products on eigenfunctions. Let $p : [0,\infty)\to\mathbf{R}$ be a positive continuous function of (at most) polynomial growth. That is, there exist constants C_p and ν_p such that if $x \geq 0$, then

$$0 < p(x) \leq C_p\cdot\left(1 + x^{\nu_p}\right).$$

We will regard p as a weight for an L^2-inner product.

Proposition 6.5. *Let p be as above. There exists a function $\alpha : [\mu/\sigma(0),\infty)\times(0,\mu)\to\mathbf{R}$ such that if $s\in(0,\mu)$, then*

$$\lim_{E\to\mu/\sigma(0)}\alpha(E,s) = 0 \tag{38}$$

and a function $\beta : (\mu/\sigma(0),\infty)\times(0,\mu)\to\mathbf{R}$ such that if $w_\pm$ is an eigenfunction of a_t^μ with eigenvalue $\lambda_\pm\leq E$, then

$$\left|\int_0^\infty w_+\cdot w_-\cdot p\,dx - p(0)\int_0^\infty w_+\cdot w_-\,dx\right| \leq (\alpha(E,s)+\beta(E,s)\cdot t)\int_0^\infty w^2\,dx.$$

The functions α and β depend only on p, E, σ, and μ.

Proof. Set

$$\alpha(E,s) = \sup\left\{|p(x)-p(0)| \mid 0\leq x\leq 3x_E^s\right\}.$$

Since p is continuous and $\lim_{E\to\mu/\sigma(0)} x_E^s = 0$ we have (38). Using the Cauchy-Schwarz inequality we find that

$$\left|\int_0^{3x_E^s} w_+\cdot w_-\cdot p\,dx - p(0)\int_0^{3x_E^s} w_+\cdot w_-\,dx\right| \leq \alpha(E,s)\cdot\|w_+\|\cdot\|w_-\|.$$

We also have

$$\left(\int_{3x_E^s}^\infty w_+\cdot w_-\cdot p\,dx\right)^2 \leq \left(\int_{3x_E^s}^\infty |w_+|^2\cdot p\,dx\right)\cdot\left(\int_{3x_E^s}^\infty |w_-|^2\cdot p\,dx\right).$$

By Lemma 6.4 we have

$$\int_{3x_E^s}^{\infty} |w_\pm|^2 \cdot p\, dx \le C_p \cdot \beta_{\nu_p}(E,s) \cdot t \int_0^\infty |w_\pm|^2\, dx$$

and also

$$p(0) \int_{3x_E^s}^{\infty} |w_\pm|^2\, dx \le p(0) \cdot \beta_0(E,s) \cdot t \int_0^\infty |w_\pm|^2\, dx.$$

The claim then follows from combining these estimates and using the triangle inequality. □

7. The Langer-Cherry Transform

We wish to analyse the behavior of the solutions to (29) for x near x_E and for t small. To do this, we will use a transform to put the solution into a normal form. The transform that we will use was first considered by Langer [Langer31] and Cherry [Cherry50] and is a variant of the Liouville-Green transformation. See Chapter 11 in [Olver].

As above, let $f_E = \mu - E \cdot \sigma$, where σ is smooth with $\sigma' < 0$ and $\lim_{x\to\infty} \sigma(x) = 0$. For $E \ge \mu/\sigma(0)$, there exists a unique $x_E \in [0,\infty)$ such that $f_E(x_E) = 0$. In the present context, the Langer-Cherry transform is based on the function $\phi_E : [0,\infty) \to \mathbf{R}$ defined by

$$\phi_E(x) = \operatorname{sign}(x - x_E) \cdot \left| \frac{3}{2} \int_{x_E}^{x} |f_E(u)|^{\frac{1}{2}}\, du \right|^{\frac{2}{3}}. \tag{39}$$

Before defining the Langer-Cherry transform, we collect some facts concerning ϕ_E.

Lemma 7.1. *Let* $\mathcal{U} = \left[\frac{\mu}{\sigma(0)}, \infty\right) \times [0,\infty)$.

(1) *The map* $(E,x) \mapsto \phi_E(x)$ *is smooth on* $\mathcal{U}$.
(2) $\phi_E'(x) > 0$ *for each* $(E,x) \in \mathcal{U}$.
(3) $(\phi_E')^2 \cdot \phi_E = f_E$.
(4) *The map* $(E,x) \mapsto f_E(x)/\phi_E(x)$ *defined for* $x \ne x_E$ *extends to a smooth map from* $\mathcal{U}$ *to* $\mathbf{R}^+$.
(5) *The limit*

$$\lim_{x\to\infty} x^{-\frac{2}{3}} \cdot \phi_E(x) = (3/2)^{\frac{2}{3}} \cdot \mu^{\frac{1}{3}}$$

holds uniformly for E *in each compact subset of* $\left[\frac{\mu}{\sigma(0)}, \infty\right)$.

Proof. These properties follow directly from the definition (39) or from the alternative expression (41) below that we now prove.

Since $\sigma'(x) < 0$ for all $x \in [\mu/\sigma(0), \infty)$, the map $I : \mathcal{U} \to \mathbf{R}$,

$$I(E,u) = \int_0^1 -E \cdot \sigma'(E,\; s \cdot u + (1-s) \cdot x_E)\, ds,$$

is smooth and positive on $\mathcal{U}$. The map $\pi : \mathcal{U} \to \mathbf{R}$ defined by

$$\pi(E, x) = \int_0^1 s^{\frac{1}{2}} \cdot I^{\frac{1}{2}}(E,\ s \cdot x + (1-s) \cdot x_E)\, ds \tag{40}$$

is also smooth and positive.

Since $f_E(x_E) = 0$ and $f_E'(x) = -E\sigma'(x)$, the fundamental theorem of calculus gives that

$$\mu - E \cdot \sigma(u) = (u - x_E) \cdot I(E, u).$$

Direct computation shows that

$$\phi_E(x) = (x - x_E) \cdot \left(\frac{3}{2} \cdot \pi(E, x)\right)^{\frac{2}{3}}. \tag{41}$$

□

Definition 7.2. *Let* $w : [0, \infty) \to \mathbf{R}$ *and let* $E \geq \mu/\sigma(0)$*. Define the* **Langer-Cherry transform of** w **at energy** E *to be the function*

$$W_E = \left((\phi_E')^{\frac{1}{2}} \cdot w\right) \circ \phi_E^{-1}. \tag{42}$$

It follows from Lemma 7.1 that the Langer-Cherry transform maps $C^k([0, \infty))$ to $C^k([\phi_E(0), \infty))$. The importance of this transform is due to its effect on solutions to the ordinary differential equation (29). In what follows we let

$$\rho_E = (\phi_E')^{-\frac{1}{2}}. \tag{43}$$

Proposition 7.3. *Let* $r : [0, \infty) \to \mathbf{R}$ *and let* $w \in C^2([0, \infty))$*. Let* W_E *be the Langer-Cherry transform of* w *at energy* E*. Then* w *satisfies*

$$t^2 \cdot w'' - f_E \cdot w = -r \cdot \sigma$$

if and only if W_E *satisfies*

$$t^2 \cdot W_E'' - y \cdot W_E = -t^2 \cdot (\rho_E^3 \cdot \rho_E'') \circ \phi_E^{-1} \cdot W_E - \left(\rho_E^3 \cdot r \cdot \sigma\right) \circ \phi_E^{-1}. \tag{44}$$

The proof is a straightforward but lengthy computation. See also, for example, §11.3 in [Olver], where the function $\hat{f}$ is related to h_E by $\hat{f} = h^4$.

In the analysis that follows, we will treat the right-hand side of (44) as an error term for t and r small. The following estimates will help justify this treatment.

Lemma 7.4. *Let* $K \subset [\mu/\sigma(0), \infty)$ *be compact. There exists* $C > 0$ *such that if* $x \geq 0$ *and* $E \in K$*, then*

$$\left|\frac{1}{\rho_E(x)}\right| \leq C \cdot x^{-\frac{1}{6}},$$

and

$$|\rho_E(x)| \leq C \cdot \left(1 + x^{\frac{1}{6}}\right).$$

Moreover, there exists ν *such that*

$$\left|\rho_E''(x)\right| \leq C \cdot \left(1 + x^{\nu}\right).$$

The exponent ν *depends only on* σ*. The constant* C *depends only on* μ*,* σ*, and* K*.*

Proof. By part (3) of Lemma 7.1, we have $\rho = (\phi/f)^{\frac{1}{4}}$. Hence since $\lim_{x\to\infty} f_E(x) = \mu$, we find from part (5) that

$$\lim_{x\to\infty} \rho_E \cdot x^{-\frac{1}{6}} = \left(\frac{2}{3\mu}\right)^{\frac{1}{6}}$$

uniformly for $E \in K$. The first two estimates follow.

To prove the last estimate, one computes using $f = (\phi')^2 \cdot \phi$ that

$$\rho'' = -\frac{5}{16}\frac{f^{\frac{3}{4}}}{\phi^{\frac{11}{4}}} - \frac{1}{4}\frac{\phi^{\frac{1}{4}} \cdot f''}{f^{\frac{5}{4}}} + \frac{5}{16}\frac{\phi^{\frac{3}{4}} \cdot (f')^2}{f^{\frac{9}{4}}}.$$

By part (5) of Lemma 7.1, both ϕ and $1/\phi$ have polynomial growth that is uniform for $E \in K$. By assumption, σ'' has at most polynomial growth, and hence, by integration, the function σ' also has at most polynomial growth. Therefore, f'' and f' both have polynomial growth that is uniform over K. Therefore, since $\lim_{x\to\infty} f_E(x) = \mu > 0$, we find that ρ'' has uniform polynomial growth. □

Lemma 7.5. *Let $I \subset [0, \infty)$ be a compact interval and let $K \subset [\mu/\sigma(0), \infty)$ be a compact set. There exists a constant C such that for each $E \in K$ such that if w is a solution to (29) and W_E is the Langer-Cherry transform of w at energy E, then we have*

$$\int_{\phi_E(I)} \left|t^2 \cdot W_E''(y) - y \cdot W_E(y)\right|^2 dy \le C \cdot \left(\|r\|_\sigma^2 + t^4 \int_0^\infty |w|^2\, dx\right).$$

The constant C depends only on μ, σ, I, and K.

Proof. For each continuous function $F : I \to \mathbf{R}$, let $|F|_\infty = \sup\{|F(x)| \mid x \in I\}$. We perform the change of variables $y = \phi_E(x)$. Since $\phi_E' = \rho_E^{-2}$, we have $dy = \rho_E^{-2} \cdot dx$, and thus by (42) and (43),

$$|W|^2\, dy = \rho_E^{-4} \cdot |w|^2\, dx. \tag{45}$$

Therefore

$$\int_{\phi_E(I)} \left|(\rho_E^3 \cdot \rho_E'') \circ \phi_E^{-1}\right|^2 \cdot |W|^2\, dy \le |\rho_E|_\infty^2 \cdot |\rho_E''|_\infty^2 \cdot \int_I |w|^2\, dx,$$

and

$$\int_{\phi_E(I)} \left|(\rho_E^3 \cdot r \cdot \sigma) \circ \phi_E^{-1}\right|^2\, dy \le |\rho_E^2 \cdot \sigma|_\infty \cdot \|r\|_\sigma^2.$$

The claim then follows from squaring and integrating (44) and applying the above estimates. □

Suppose w is an eigenfunction of a_t^μ, and denote by λ its eigenvalue. If we perform the Cherry-Langer transform at energy $E = \lambda$ then $r = 0$ and hence the conclusion of Lemma 7.5 is stronger. Actually, we will need the following strengthening of Lemma 7.5 which treats the case when w is an eigenfunction of a_t^μ but E is close to but not necessarily exactly the corresponding eigenvalue.

Lemma 7.6. *Let $K \subset [\mu/\sigma(0), \infty)$ be compact. There exists a constant C_K such that if $t < 1$, w is an eigenfunction of a_t^{μ} with eigenvalue $\lambda \in K$, and W is the Cherry-Langer transform of w at energy $E \in K$, then*

$$\int_{\phi_E(0)}^{\infty} \left| t^2 \cdot W'' - y \cdot W \right|^2 \, dy \leq C_K \cdot \left(|\lambda - E|^2 + t^4 \right) \int_0^{\infty} w^2 \, dx. \tag{46}$$

Proof. Since $-t^2 \cdot w'' + (\mu - \lambda \cdot \sigma) \cdot w = 0$, the function w satisfies

$$-t^2 \cdot w'' + f_E \cdot w = r.$$

with $r = (E - \lambda) \cdot \sigma \cdot w$. Therefore we may apply Proposition 7.3. In particular, it suffices to bound the integrals of the squares of the terms appearing on the right-hand side of (44).

By Lemma 7.4 there exists ν_1 and C_1 (depending only on K) such that

$$|\rho_E(x)|^{-4} \cdot |\rho_E^3 \cdot \rho_E''(x)|^2 \leq C_1 \cdot (1 + x^{\nu_1}).$$

Hence by changing variables (recall that $W^2 dy = \rho_E^{-4} w^2 dx$) we find that

$$\int_{\phi_E(0)}^{\infty} \left| (\rho_E^3 \cdot \rho_E'') \circ \phi_E^{-1} \right|^2 \cdot |W(y)|^2 \, dy \leq C_1 \int_0^{\infty} |w(x)|^2 \cdot (1 + x^{\nu_1}) \, dx.$$

Since w is an eigenfunction, we can apply Lemma 6.4. By fixing $s = \mu/2$, we obtain a constant C_2—depending only on K—such that

$$\int_{3x_E^s}^{\infty} |w(x)|^2 \cdot (1 + x^{\nu_1}) \, dx \leq C_2 \cdot t \int_{x_E^s}^{\infty} |w(x)|^2 \, dx.$$

Let $x^* = \sup\{x_E^s \mid E \in K, s = \mu/2\}$. Then

$$\int_0^{3x_E^s} |w(x)|^2 \cdot (1 + (3x)^{\nu_1}) \, dx \leq \left(1 + (3x^*)^{\nu_1}\right) \int_0^{\infty} |w(x)|^2 \, dx.$$

In sum, if $t \geq 1$, then we have a constant C_3 such that

$$\int_{\phi_E(0)}^{\infty} |\rho_E^3 \cdot \rho_E'' \circ \phi^{-1}(y)|^2 \cdot |W(y)|^2 \, dy \leq C_3 \int_0^{\infty} |w(x)|^2 \, dx.$$

A similar argument shows that there exists C_4—depending only on K—such that

$$\int_{\phi_E(0)}^{\infty} \left| (\rho_E^3 \cdot \sigma^2) \circ \phi_E^{-1}(y) \cdot W(y) \right|^2 \, dy \leq C_4 \int_0^{\infty} |w(x)|^2 \, dx.$$

By putting these estimates together we obtain the claim. □

The following lemma will allow us to control scalar products in w when they are expressed on the Cherry-Langer side, in the limit as E tends to $\mu/\sigma(0)$ and t tends to 0. It will be used in the proof of Theorem 10.4.

Lemma 7.7. *Let $q : [0,\infty) \to \mathbf{R}$ be a positive continuous function of at most polynomial growth. Given $\epsilon > 0$, there exists $\delta > 0$ such that if $t < \delta$, $E < \mu/\sigma(0) + \delta$, and $w_\pm$ is an eigenfunction of a_μ^t with eigenvalue $\lambda_\pm \le E$, then*

$$\left| \int_{\phi_E(0)}^{\infty} W_+ \cdot W_- \, dy - \frac{1}{\rho_E^4(0) \cdot q(0)} \int_0^\infty w_+ \cdot w_- \cdot q \, dx \right| \le \epsilon \cdot \|w_+\| \cdot \|w_-\|, \tag{47}$$

where $W_\pm$ is the Langer-Cherry transform of $w_\pm$ at energy E, and $\|\cdot\|$ is the standard (unweighted) L^2 norm.

Proof. Changing variables gives

$$\int_{\phi_E(0)}^{\infty} W_+ \cdot W_- \, dy = \int_0^\infty w_+ \cdot w_- \cdot \rho_E^{-4}(x) \, dx.$$

By Lemma 7.4, the function ρ_E^{-4} is bounded, and hence we can apply Proposition 6.5. In particular, choose $\delta_1 > 0$ so that if $E < \mu/\sigma(0) + \delta_1$, then $\alpha_p(E, \mu/2) < \epsilon/4$ and choose $\delta_2 \le \delta_1$ so that if $t < \delta_2$, then $\beta(\delta_1, \mu/2) \cdot t < \epsilon/4$. Thus, if $E < \mu/\sigma(0) + \delta_2$ and $t < \delta_2$, then

$$\left| \int_{\phi_E(0)}^{\infty} W_+ \cdot W_- \, dy - \rho_E^{-4}(0) \int_0^\infty w_+ \cdot w_- \, dx \right| \le \frac{\epsilon}{2} \cdot \|w_+\| \cdot \|w_-\|.$$

In a similar fashion we can apply Lemma 6.5 to find $\delta \le \delta_2$ so that if $E < \mu/\sigma(0) + \delta$ and $t < \delta$, then

$$\left| \int_0^\infty w_+ \cdot w_- \cdot q \, dy - q(0) \int_0^\infty w_+ \cdot w_- \, dx \right| \le \frac{\epsilon}{2} \cdot \rho_E^4(0) \cdot q(0) \cdot \|w_+\| \cdot \|w_-\|.$$

The claim follows. □

8. Airy Approximations

In this section we analyse solutions to the inhomogeneous equation

$$t^2 \cdot W''(y) - y \cdot W(y) = g(y). \tag{48}$$

To do this, we will use a solution operator, $\tilde{K}_t$, for the associated homogeneous equation

$$t^2 W_0'' - y \cdot W_0 = 0. \tag{49}$$

The function W_0 is a solution to (49) if and only if $\mathrm{A}(u) = W_0(t^{\frac{2}{3}} \cdot u)$ is a solution to the *Airy equation*

$$A'' - u \cdot A = 0. \tag{50}$$

Using, for example, the method of variation of constants, one can construct an integral kernel K for an 'inverse' of the operator $A(u) \mapsto A''(u) - u \cdot A(u)$ in terms of Airy functions. We give the construction of K as well as its basic properties in Appendix A. By rescaling (or by direct construction) we obtain an integral kernel for the operator $A(x) \mapsto t^2 \cdot A''(x) - x \cdot A(x)$. To be precise, define

$$\tilde{K}_t(y, z) = t^{-\frac{4}{3}} \cdot K\left(t^{-\frac{2}{3}} \cdot y,\ t^{-\frac{2}{3}} \cdot z\right),$$

where K is the integral kernel constructed in Appendix A.

Lemma 8.1. *Let $-\infty < a \leq b \leq \infty$. For each locally integrable $g : [a, b] \to \mathbf{R}$ of at most polynomial growth, the function*

$$y \mapsto \int_a^b \tilde{K}_t(y, z) \cdot g(z)\, dz \tag{51}$$

is a solution to (48).

Proof. This follows from Lemma A.1 or directly from the variation of constants construction. □

The following estimate is crucial to the proof of Proposition 9.1.

Lemma 8.2. *Let $g : \mathbf{R} \to \mathbf{R}$ be continuous. For each $-\infty < a < 0 < b$, there exist constants C and $t_0 > 0$ such that if $t < t_0$ and W satisfies* (48)*, then*

$$\int_a^0 W^2 \leq C \cdot \left(\int_a^{\frac{a}{2}} W^2 + t^{-\frac{5}{3}} \int_a^b g^2 \right), \tag{52}$$

and

$$\int_0^{\frac{b}{2}} W^2 \leq C \cdot \left(t^{\frac{1}{3}} \int_a^{\frac{a}{2}} W^2 + \int_{\frac{b}{2}}^b W^2 + t^{-\frac{5}{3}} \int_a^b g^2 \right). \tag{53}$$

The constants C and t_0 can be chosen to depend continuously upon a and b.

Proof. Define W_0 on $[a, b]$ by

$$W_0(y) = W(y) - \int_a^b \tilde{K}_t(y, z) \cdot g(z)\, dz.$$

Using Lemma 8.1 and linearity, W_0 is a solution to (49).

Using the Cauchy-Schwarz-Bunyakovsky inequality, we find that

$$|W(y) - W_0(y)|^2 \leq \left(\int_a^b \left| \tilde{K}_t(y, z) \right|^2 dz \right) \left(\int_a^b |g(z)|^2\, dz \right). \tag{54}$$

A change of variables gives

$$\int_a^b \int_a^b \left| \tilde{K}_t(y, z) \right|^2 dy\, dz = t^{-\frac{4}{3}} \int_{t^{-\frac{2}{3}} a}^{t^{-\frac{2}{3}} b} \int_{t^{-\frac{2}{3}} a}^{t^{-\frac{2}{3}} b} |K(u, v)|^2\, du\, dv. \tag{55}$$

By Lemma A.3 in Appendix A, the latter integral is less than $C_{\text{Airy}} \cdot \sqrt{\delta} \cdot t^{-\frac{1}{3}}$, where C_{Airy} is a universal constant and $\delta = \max\{|a|, b\}$. Therefore, by integrating (54) over an interval $I \subset [a, b]$ and substituting (55), we find that

$$\|W - W_0\|_I^2 \leq C_0 \cdot t^{-\frac{5}{3}} \cdot \|g\|_{[a,b]}^2, \tag{56}$$

where $C_0 = C_{\text{Airy}} \cdot \delta$ and $\|\cdot\|_J$ denotes the L^2-norm over the interval J. In particular, by the triangle inequality we have

$$\|W\|_I \leq \|W_0\|_I + C_0^{\frac{1}{2}} \cdot t^{-\frac{5}{6}} \cdot \|g\|_{[a,b]},$$

and hence

$$\|W\|_I^2 \leq 2 \cdot \|W_0\|_I^2 + 2 \cdot C_0 \cdot t^{-\frac{5}{3}} \cdot \|g\|_{[a,b]}^2. \tag{57}$$

Similarly,

$$\|W_0\|_I^2 \leq 2 \cdot \|W\|_I^2 + 2 \cdot C_0 \cdot t^{-\frac{5}{3}} \cdot \|g\|_{[a,b]}^2. \tag{58}$$

The function $u \mapsto W_0(t^{\frac{2}{3}} \cdot u)$ satisfies the Airy equation (131). Hence, it follows from Lemma A.4 (in which s is replaced by $t^{-\frac{2}{3}}$) that there exist constants M and $t_0 > 0$—depending continuously on a and b—such that if $t \leq t_0$, then

$$\int_a^0 W_0^2 \, dy \leq M \int_a^{\frac{a}{2}} W_0^2 \, dy \tag{59}$$

and

$$\int_0^{\frac{b}{2}} W_0^2 \, dy \leq M \left(t^{\frac{1}{3}} \int_a^{\frac{a}{2}} W_0^2 \, dy + \int_{\frac{b}{2}}^b W_0^2 \, dy \right). \tag{60}$$

By combining (60) with (57) and (58), we obtain (52). By combining (59) with (57) and (58), we obtain (53). □

9. A Non-Concentration Estimate

Fix μ and σ and let a_t^μ be the family of quadratic forms defined as in §5. The purpose of this section is to prove the following non-concentration estimate—see Remark 9.3—that is crucial to our proof of generic spectral simplicity.

Proposition 9.1. *Let K be a compact subset of $(\mu \cdot \sigma(0)^{-1}, \infty)$, and $C > 0$. There exist constants $t_0 > 0$ and $\kappa > 0$ such that if $E \in K$, if $t < t_0$, and if for each $v \in \text{dom}(a_t^\mu)$, the function w satisfies*

$$\left| a_t^\mu(w, v) - E \cdot \langle w, v \rangle_\sigma \right| \leq C \cdot t \cdot \|w\|_\sigma \cdot \|v\|_\sigma, \tag{61}$$

then

$$\int_0^\infty (E \cdot \sigma(x) - \mu) \cdot |w(x)|^2 \, dx \geq \kappa \cdot \|w\|_\sigma^2. \tag{62}$$

The constants t_0 and κ depend only upon K, C, μ, and σ.

In contrast to previous estimates, Proposition 9.1 is concerned with so-called *non-critical* energies, those values of E that are strictly greater than the threshold $\mu/\sigma(0)$.

Remark 9.2. Estimate (61) is a special case of an estimate of the following form: For all $v \in \text{dom}(a_{\mu,t})$,

$$\left| a_t^\mu(w, v) - E_t \cdot \langle w, v \rangle_\sigma \right| \leq t^\rho \cdot \|w\|_\sigma \cdot \|v\|_\sigma. \tag{63}$$

By the Riesz representation theorem, estimate (63) is equivalent to Eq. (26) with r such that $\|r\| \leq t^\rho \cdot \|w\|$. In other words, a sequence w_n satisfying (63) is what we have called a quasimode of order ρ at energy E_0.

Remark 9.3. Suppose that w_n is a sequence of eigenfunctions of $a^{\mu}_{t_n}$ with t_n tending to zero as n tends to infinity. Then, by Lemma 6.3, each w_n decays exponentially in the region $\{x \mid E \cdot \sigma(x) - \mu < 0\}$ and the rate of decay increases as n increases. In particular, we can use Proposition 6.4 to prove that the measure $|w_n(x)|^2\, dx$ concentrates in the 'classically allowed region' $\{x \mid E \cdot \sigma(x) - \mu \geq 0\}$. Proposition 9.1 is a twofold strengthening of this latter statement: We prove that if E is not critical then $|w_n(x)|^2\, dx$ does not concentrate solely on $\{x \mid E \cdot \sigma(x) - \mu = 0\}$, and we prove that this also holds true for a quasimode of order 1.

Estimate (62) for eigenfunctions could be obtained using a contradiction argument which is standard in the study of semiclassical measures. (See [Hillairet10] for closely related topics.) However, we believe that this method fails for first order quasimodes.

Proof of Proposition 9.1. Let $E > \mu/\sigma(0)$. Then $f_E(0) < 0$ and since $f_E = \mu - E \cdot \sigma$ is strictly increasing with $\lim_{x\to\infty} f_E(x) = \mu$, there exists a unique $x_E > 0$ such that $f_E(x_E) = 0$. Since f_E changes sign at x_E, we have

$$\int_0^{\infty} (-f_E) \cdot w^2\, dx = \int_0^{x_E} |f_E| \cdot w^2\, dx - \int_{x_E}^{\infty} |f_E| \cdot w^2\, dx. \tag{64}$$

Thus, by Lemmas 9.5 and 9.4 below, there exist constants $C^+,\ c^- > 0$ and $t^* > 0$ such that if $t < t^*$, then

$$\int_0^{\infty} (-f_E) \cdot w^2\, dx \geq c^- \int_0^{\infty} w^2\, dx - C^+ \cdot t^{\frac{1}{3}} \int_0^{\infty} w^2 dx. \tag{65}$$

Thus, if $t < t_0 = (c^-/2C^+)^3$, then we have (62) with $\kappa = c^-/(2 \cdot \sigma(0))$. □

Lemma 9.4. *There exist constants C^+ and $t^+ > 0$ so that if $t < t^+$, then*

$$\int_{x_E}^{\infty} |f_E| \cdot w^2\, dx \leq C^+ \cdot t^{\frac{1}{3}} \int_0^{x_E} w^2\, dx. \tag{66}$$

Lemma 9.5. *There exist constants $c^- > 0$, and $t^- > 0$ so that if $t < t^-$, then*

$$\int_0^{x_E} |f_E| \cdot w^2\, dx \geq c^- \int_0^{\infty} w^2\, dx. \tag{67}$$

The proofs of Lemma 9.5 and 9.4 are based on estimates provided in Sects. 6, 7, and 8. In preparation for these proofs we provide the common context.

First note that the Riesz representation theorem provides $r \in \mathcal{H}_\sigma$ so that for all $v \in \mathrm{dom}(a_t)$,

$$|a_t(w, v) - E \cdot \langle w, v\rangle_\sigma| = \langle r, v\rangle_\sigma,$$

where $\|r\|_\sigma \leq C_0 \cdot t \cdot \|w\|_\sigma$.

Let W denote the Langer-Cherry transform of w at energy E (see §7). In particular,

$$W = \left((\phi_E')^{\frac{1}{2}} \cdot w \right) \circ \phi_E^{-1},$$

where ϕ_E is defined by (39). By Proposition 7.3, the function W satisfies (48) with g equal to the right-hand side of Eq. (44).

As a last preparation for the proofs, we define the endpoints of the intervals over which we will apply the estimates from the preceding sections. Let x_E^+ be defined by $f_E(x_E^+) = \mu/2$. In other words, $x_E^+ = x_E^{\mu/2}$, where x_E^s is defined in (30). Define

$$y_E^+ = 2 \cdot \phi_E(x_E^+)$$

and

$$y_E^- = \phi_E(0).$$

Since σ is decreasing, we have $0 < x_E < x_E^+$, and hence since ϕ_E is strictly increasing, we have $y_E^- < 0 < y_E^+$. It follows from Lemma 7.1 and Remark 6.1 that y_E^+ and y_E^- depend smoothly on E.

Proof of Lemma 9.4. Since σ is decreasing, we have $\sup\{|f_E| \mid x \geq x_E\} = \mu$, and thus

$$\int_{x_E}^{\infty} |f_E| \cdot w^2 \, dx \ \leq \mu \int_{x_E}^{\infty} w^2 \, dx. \tag{68}$$

Since $\phi_E\left([x_E, x_E^+]\right) = \left[0, \frac{y_E^+}{2}\right]$ and $W^2 \cdot dy = (\phi_E')^2 \cdot w^2 \cdot dx$, we have

$$\int_{x_E}^{\infty} w^2 \, dx \leq C_1 \int_0^{\frac{y_E^+}{2}} W^2 \, dy + \int_{x_E^+}^{\infty} w^2 \, dx, \tag{69}$$

where $C_1 = \max\{(\phi_E'(x))^{-2} \mid E \in K, x \in [x_E, x_E^+]\}$.

By Lemma 8.2 there exist constants C_E and $t_E > 0$ so that if $t < t_E$, then

$$\int_0^{\frac{y_E^+}{2}} W^2 \, dy \leq C_E \left(t^{\frac{1}{3}} \int_{y_E^-}^{\frac{y_E^-}{2}} W^2 \, dy + \int_{\frac{y_E^+}{2}}^{y_E^+} W^2 \, dy + t^{-\frac{5}{3}} \int_{y_E^-}^{y_E^+} g^2 \, dx \right). \tag{70}$$

The constants C_E and t_E depend continuously on E and hence $C_2 = \sup\{C_E \mid E \in K\}$ is finite and $t_2 = \inf\{t_E \mid E \in K\}$ is positive. Since $W^2 \cdot dy = (\phi_E')^2 \cdot w^2 \cdot dx$, we have

$$t^{\frac{1}{3}} \int_{y_E^-}^{\frac{y_E^-}{2}} W^2 \, dy + \int_{\frac{y_E^+}{2}}^{y_E^+} W^2 \, dy \leq C_3 \cdot \left(t^{\frac{1}{3}} \int_0^{\infty} w^2 \, dx + \int_{x_E^+}^{\infty} w^2 \, dx \right), \tag{71}$$

where $C_3 := \sup \left\{ (\phi_E'(x))^2 \mid E \in K, \ x \in \left[0, \phi^{-1}(y_E^+)\right] \right\}$.

By Lemma 7.5, there exists a constant C^* so that

$$\int_{y_E^-}^{y_E^+} g^2 \, dy \leq C^* \cdot t^2 \int_0^{\infty} w^2 \, dx. \tag{72}$$

By substituting (71) and (72) into (70) we find that if $t < t_2$, then

$$\int_0^{\frac{y_E^+}{2}} W^2\, dy \le C_4 \cdot t^{\frac{1}{3}} \int_0^{\infty} w^2\, dx + C_5 \int_{x_E^+}^{\infty} w^2\, dx, \tag{73}$$

where $C_4 = C_2 \cdot (C_3 + C^*)$ and $C_5 = C_2 \cdot C_3$. By Lemma 6.2, there exists a constant C_6 so that if $t < 1$, then

$$\int_{x_E^+}^{\infty} w^2\, dx \le C_6 \cdot t^{\frac{1}{3}} \int_0^{\infty} w^2\, dx. \tag{74}$$

By combining (69), (73), and (74), we find that if $t < t_3 := \min\{1, t_2\}$, then

$$\int_{x_E}^{\infty} w^2\, dx \le C_7 \cdot t^{\frac{1}{3}} \int_0^{\infty} w^2\, dx, \tag{75}$$

where $C_7 = C_1 \cdot C_4 + C_1 \cdot C_5 \cdot C_6 + C_6$. Finally, split the integral on the right-hand side of (75) into the integral over $[0, x_E]$ and the integral over $[x_E, \infty)$. Then subtract the latter integral from both sides of (75). It follows that if $t < \min\{t_3, (2C_7)^{-3}\}$, then

$$\frac{1}{2} \int_{x_E}^{\infty} w^2\, dx \le C_7 \cdot t^{\frac{1}{3}} \int_0^{x_E} w^2\, dx.$$

The claim then follows by combining this with (68). □

We have the following corollary of the proof.

Corollary 9.6. *There exist constants C' and $t' > 0$ such that if $t < t'$,*

$$\int_{x_E}^{\infty} w^2\, dx \le C' \cdot t^{\frac{1}{3}} \int_0^{x_E} w^2\, dx.$$

Proof of Lemma 9.5. Since $W^2 \cdot dy = (\phi_E')^2 \cdot w^2 \cdot dx$ we have

$$\int_0^{x_E} |f_E| \cdot w^2\, dx \ge c_1 \int_{y_E^-}^{0} \left| f_E \circ \phi_E^{-1} \right| \cdot W^2\, dy,$$

where $c_1 = \inf \left\{ \phi_E'(x)^{-2} \mid E \in K,\ x \in [0, x_E] \right\}$. Since $f_E \circ \phi_E^{-1}$ is negative and increasing on $[y_E^-, y_E^-/2]$, we have

$$\int_{y_E^-}^{\frac{y_E^-}{2}} \left| f_E \circ \phi_E^{-1} \right| \cdot W^2\, dy \ge c_2 \int_{y_E^-}^{\frac{y_E^-}{2}} W^2\, dy,$$

where $c_2 = \inf \left\{ \left| f_E \circ \phi_E^{-1} \left(y_E^-/2 \right) \right| \;\middle|\; E \in K \right\}$. Putting these two estimates together we have

$$\int_0^{x_E} |f_E| \cdot w^2\, dx \ge c_1 \cdot c_2 \int_{y_E^-}^{\frac{y_E^-}{2}} W^2\, dy. \tag{76}$$

It follows from Lemma 7.1 that c_1 and c_2 are both positive.

By Lemma 8.2, there exist constants C_E and $t_E > 0$ so that if $t < t_E$, then

$$\int_{y_E^-}^{0} W^2 \leq C_E \cdot \left(\int_{y_E^-}^{\frac{y_E^-}{2}} W^2 + t^{-\frac{5}{3}} \int_{y_E^-}^{y_E^+} g^2 \right). \tag{77}$$

Moreover, C_E and t_E depend continuously on E, and hence the constants $c_3 = \sup\{1/C_E \mid E \in K\}$ and $t_1 = \inf\{t_E \mid E \in K\}$ are both positive. By manipulating (77) we find that

$$\int_{y_E^-}^{\frac{y_E^-}{2}} W^2 \geq c_3 \int_{y_E^-}^{0} W^2 - t^{-\frac{5}{3}} \int_{y_E^-}^{y_E^+} g^2 \tag{78}$$

for each $t < t_1$.

By combining (76), (78), and (72) we find that for $t < t_1$,

$$\int_0^{x_E} |f_E| \cdot w^2 \, dx \ \geq c_4 \int_{y_E^-}^{0} W^2 \, dy - C \cdot t^{\frac{1}{3}} \int_0^{\infty} w^2 \, dx, \tag{79}$$

where $c_4 = c_1 \cdot c_2 \cdot c_3$ and $C = c_1 \cdot c_2 \cdot C^*$. Since $W^2 \cdot dy = (\phi_E')^2 \cdot w^2 \cdot dx$, we have

$$\int_{y_E^-}^{0} W^2 \, dy \geq c_5 \int_0^{x_E} w^2 \, dx, \tag{80}$$

where $c_5 = \inf\{(\phi_E'(x))^2 \mid E \in K, x \in [0, x_E]\}$ is positive by Lemma 7.1. By substituting (80) into (79) and applying Corollary 9.6, we find that if $t < t_2 = \min\{t_1, t'\}$, then

$$\int_0^{x_E} |f_E| \cdot w^2 \, dx \geq c_4 \cdot c_5 \int_0^{\infty} w^2 \, dx - \left(C + c_4 \cdot c_5 \cdot C'\right) \cdot t^{\frac{1}{3}} \int_0^{\infty} w^2 \, dx. \tag{81}$$

If $t < t^- = \min\left\{t_2, \left(c_4 \cdot c_5 / 2(C + c_4 \cdot c_5 \cdot C')\right)^3\right\}$, then (67) holds with $c_- = c_4 \cdot c_5/2$. □

10. Convergence, Estimation, and Separation of Eigenvalues

Let a_t^{μ} be the family of quadratic forms defined as in §5. In this section, we will evaluate the limit to which each real-analytic eigenvalue branch converges (Proposition 10.1), estimate the asymptotic behavior of eigenvalues (Proposition 10.3), and show that if both t and $E - \mu/\sigma(0)$ are sufficiently small, then eigenvalues near energy E must be ‘super-separated’ at order t (Theorem 10.4).

10.1. Convergence. Let $t \mapsto \lambda_t$ be a real-analytic eigenvalue branch of a_t^μ with respect to $\langle \cdot, \cdot \rangle_\sigma$. Since $|w'|^2 \geq 0$ and σ is decreasing, we have

$$\lambda_t \geq \frac{\mu \int_0^\infty |w_t|^2 \, dx}{\int_0^\infty |w_t|^2 \cdot \sigma \, dx} \geq \frac{\mu}{\sigma(0)}. \tag{82}$$

The first derivative of a_t^μ,

$$\dot{a}_t^\mu(u) = 2t \int_0^\infty \left|w_t'(x)\right|^2 \, dx,$$

is nonnegative, and hence by Proposition 3.3, the eigenbranch λ_t converges as t tends to zero.

Proposition 10.1. *We have*

$$\lim_{t \to 0} \lambda_t = \frac{\mu}{\sigma(0)}.$$

Proof. Let w_t be an eigenfunction branch associated to E_t. The variational formula (13) becomes

$$\dot{\lambda} \cdot \|w_t\|_\sigma^2 = 2t \int_0^\infty \left|w_t'(x)\right|^2 \, dx. \tag{83}$$

Using the eigenvalue equation for a_t^μ with respect to $\langle \cdot, \cdot \rangle_\sigma$ we find that

$$t^2 \int_0^\infty \left|w_t'(x)\right|^2 \, dx = \int_0^\infty (\lambda_t \cdot \sigma(x) - \mu) \cdot |w_t(x)|^2 \, dx.$$

By combining this with (83) and (82) we find that

$$\dot{\lambda} \cdot \|w_t\|_\sigma^2 \geq \frac{2}{t} \cdot \int_0^\infty (\lambda_t \cdot \sigma(x) - \mu) \cdot |w_t(x)|^2 \, dx. \tag{84}$$

Suppose to the contrary that $\lambda_0 := \lim_{t \to 0} \lambda_t \neq \mu/\sigma(0)$. Then by (82), we have $\lambda_0 > \mu/\sigma(0)$. Let K be the compact interval $[\lambda_0, \lambda_0 + 1]$. Then for all t sufficiently small, $\lambda_t \in K$. Hence we can apply Proposition 9.1, with $E = \lambda_t$, and obtain a constant $\kappa > 0$ such that

$$\int_0^\infty (\lambda_t \cdot \sigma(x) - \mu) \cdot |w_t(x)|^2 \, dx \geq \kappa \cdot \|w_t(x)\|_\sigma^2.$$

By combining this with (84) we find that

$$\frac{d}{dt} \lambda_t \geq \frac{2 \cdot \kappa}{t}.$$

The left-hand side is integrable on an interval of the form $[0, t_0)$, but the right-hand side is not integrable on such an interval. The claim follows. □

10.2. Airy eigenvalues. The remainder of this section concerns quantitative estimates on the eigenvalues of a_t^μ for t small. In particular, we will use the Langer-Cherry transform to compare the eigenvalues of a_t^μ to the eigenvalues of the operator associated to the Airy equation. We first define and study the eigenvalue problem for the model operator.

For each $z \in \mathbf{R}$ and $u \in C_0^\infty[z, \infty)$ define

$$\mathcal{A}_z(u)(y) = -u''(y) + y \cdot u(y).$$

The operator $\mathcal{A}_z$ is symmetric with respect to the $L^2([z, \infty), dy)$ inner product, and we have $\langle \mathcal{A}_z(u), u\rangle \geq z \cdot \|u\|^2$. Thus, by the method of Friedrichs, we may extend $\mathcal{A}_z$ to a densely defined, self-adjoint operator on $L^2([z, \infty), dy)$ with either Dirichlet or Neumann conditions at $y = z$.

Let $A_\pm$ be the solutions to the Airy equation defined in Appendix A.

Proposition 10.2. *The real number ν is a Dirichlet (resp. Neumann) eigenvalue of $\mathcal{A}_z$ with respect to the L^2-norm if and only if $z - \nu$ is a zero of A_- (resp. A'_-). Moreover, each eigenspace of $\mathcal{A}_z$ is 1-dimensional and each eigenvalue of $\mathcal{A}_z$ is strictly greater than z.*

Proof. If ψ is an eigenfunction with eigenvalue ν, then $x \mapsto \psi(x + \nu)$ is solution to the Airy equation that decays as x tends to infinity. Sturm-Liouville theory ensures that the associated eigenspaces are one-dimensional. □

10.3. Estimation.

Proposition 10.3. *There exists δ_0 and C such that for any $t \leq \delta_0$, if $\lambda \in [\frac{\mu}{\sigma(0)}, \frac{\mu}{\sigma(0)} + \delta_0]$ is a Dirichlet (resp. Neumann) eigenvalue of a_t^μ, then there exists a zero, z, of A_- (resp. A'_-) such that*

$$\left|\phi_\lambda(0) - t^{\frac{2}{3}} \cdot z\right| \leq C \cdot t^2. \tag{85}$$

Proof. We set $\epsilon = \frac{1}{2}\min\{\rho_E^{-4}(0) \mid E \in [\frac{\mu}{\sigma(0)}, \frac{\mu}{\sigma(0)} + 1]\}$, and we choose δ_0 to be the minimum of 1 and the δ provided by Lemma 7.7 that is associated with this ϵ and q identically 1. Let K be the compact $[\frac{\mu}{\sigma(0)}, \frac{\mu}{\sigma(0)} + \delta_0]$. Let w be an eigenfunction with eigenvalue $\lambda \in K$ and $t \leq \delta_0$. Let W the Cherry-Langer transform of w at energy λ. According to Lemma 7.7 and to the choice we made of ϵ, we have

$$\frac{1}{2\rho_\lambda(0)^4}\int_0^\infty w^2 dx \leq \int_{\phi_\lambda(0)}^\infty W^2 dy \leq \frac{3}{2\rho_\lambda(0)^4}\int_0^\infty w^2 dx.$$

Combining with Lemma 7.6, (and using that $\rho_E(0)$ is uniformly bounded away from 0 over the compact K), there exists a constant C' such that

$$\int_{\phi_\lambda(0)}^\infty |t^2 \cdot W'' - y \cdot W|^2\, dy \leq C \cdot t^4 \int_{\phi_\lambda(0)}^\infty |W(y)|^2\, dy. \tag{86}$$

Setting $U(x) = W(t^{\frac{2}{3}} \cdot x)$, we have

$$\int_{t^{-\frac{2}{3}}\cdot\phi_\lambda(0)}^\infty |U'' - x \cdot U|^2\, dx \leq C \cdot t^{\frac{8}{3}} \int_{t^{-\frac{2}{3}}\cdot\phi_\lambda(0)} |U(x)|^2\, dx. \tag{87}$$

Let $z_t = t^{-\frac{2}{3}} \cdot \phi_\lambda(0)$. Then $U(z) = 0$ (resp. $U'(z) = 0$) if λ is a Dirichlet (resp. Neumann) eigenvalue. In particular, U belongs to the domain of $\mathcal{A}_{z_t}$. Moreover, from (87) we have that

$$\left\|\mathcal{A}_{z_t}(U)\right\|^2 \leq C \cdot t^{\frac{8}{3}} \cdot \|U\|^2. \tag{88}$$

Thus, since $\mathcal{A}_{z_t}$ is self-adjoint,

$$\left\langle \mathcal{A}^2_{z_t}(U), U \right\rangle \leq C \cdot t^{\frac{8}{3}} \cdot \|U\|^2. \tag{89}$$

Thus, by the minimax principle, $\mathcal{A}^2_{z_t}$ has an eigenvalue in the interval $[0, Ct^{\frac{8}{3}}]$. Hence $\mathcal{A}_{z_t}$ has an eigenvalue in the interval $[-C^{\frac{1}{2}}t^{\frac{4}{3}}, C^{\frac{1}{2}}t^{\frac{4}{3}}]$, and the claim follows from Proposition 10.2. □

10.4. Separation. We next show that, as t tends to zero, the eigenvalues of a^μ_t with respect $\langle \cdot, \cdot \rangle_\sigma$ are separated at order greater than t. More precisely, we have the following.

Theorem 10.4. *Let $t_1, t_2, t_3, \ldots$ be a sequence of positive real numbers such that $\lim_{n\to\infty} t_n = 0$. For each $n \in \mathbb{Z}^+$, let λ^+_n and λ^-_n be distinct eigenvalues of the quadratic form $a^\mu_{t_n}$. If $\lim_{n\to\infty} \lambda^\pm_n = \mu/\sigma(0)$, then*

$$\lim_{n\to\infty} \frac{1}{t_n} \cdot \left|\lambda^+_n - \lambda^-_n\right| = \infty.$$

This fact may be understood by using the following semiclassical heuristics: The threshold $\frac{\mu}{\sigma(0)}$ is the bottom of the potential, and the eigenvalues near it are driven by the shape of this minimum. Since $\sigma'(0) \neq 0$, the asymptotics are given by the eigenvalues of the model problem $P_t u = -t^2 \cdot u'' + x \cdot u = 0$ on $(0, \infty)$. Denote by $e_n(t)$ the n^{th} eigenvalue of the model operator. Using homogeneity, $e_n(t)$ behaves like $e_n(1) \cdot t^{\frac{2}{3}}$ (and $e_n(1)$ actually is some zero of the Airy function see Proposition 10.2). For fixed n, the separation between two eigenvalues is thus of order $t^{\frac{2}{3}}$.

It would be relatively straightforward to make the preceding reasoning rigorous in the case of a finite number of real-analytic eigenvalue branches. (For instance we could use [FrdSlm09]). Unfortunately, this is not enough for our purposes. In Sect. 14 we will need the result for a sequence of eigenvalues that may belong to an infinite number of distinct branches.

Remark 10.5. The same semiclassical heuristics show that this super-separation does not hold near an energy strictly greater than $\frac{\mu}{\sigma(0)}$. Indeed, near a non-critical energy, the spectrum is separated at order t.

Proof of Theorem 10.4. Suppose to the contrary that there exists a subsequence—that we will abusively call t_n—such that $|\lambda^-_n - \lambda^+_n|/t_n$ is bounded. Let $w^\pm_n$ denote a sequence of eigenfunctions associated to $\lambda^\pm_n$ with $\|w^\pm_n\|_\sigma = 1$. Since $\lambda^-_n \neq \lambda^+_n$, we have $\langle w^-_n, w^+_n \rangle_\sigma = 0$.

Let $W^\pm_n$ denote the Langer-Cherry transform of $w^\pm_n$ at the energy $E_n = \sup\{\lambda^-_n, \lambda^+_n\}$. By hypothesis $\lim_{n\to\infty} = \mu/\sigma(0)$. By Lemma 7.6 and Lemma 7.7, we find that there exist N_1 and C such that if $n > N_1$, then

$$\left\|\left(-t_n^2 \cdot \partial_y^2 - y\right) W^\pm_n\right\|^2 \leq C \cdot t_n^2 \cdot \left\|W^\pm_n\right\|^2. \tag{90}$$

Since $\langle w_n^-, w_n^+\rangle_\sigma = 0$ and $\|w_n^\pm\|_\sigma = 1$, it follows from Lemma 7.7 that there exists $N_2 > N_1$ such that if $n > N_2$, then

$$\left|\langle W_n^-, W_n^+\rangle\right| \leq \frac{1}{2} \cdot \|W_n^-\| \cdot \|W_n^+\|.$$

This implies that for any linear combination of W_n^+ and W_n^- we have

$$|\alpha_+|^2\|W_n^+\|^2 + |\alpha_-|^2\|W_n^-\|^2 \leq 2\|\alpha_+ W_n^+ + \alpha_- W_n^-\|^2.$$

Therefore, it follows from (90) that if W belongs to the span, $\mathcal{W}_n$, of $\{W_n^-, W_n^+\}$, then

$$\left\|\left(-t_n^2 \cdot \partial_y^2 - y\right) W\right\|^2 \leq 4 \cdot C \cdot t_n^2 \cdot \|W\|^2.$$

Let $U(x) = W(t^{\frac{2}{3}} \cdot x)$ and let $\mathcal{U}_n$ denote the vector space corresponding to $\mathcal{W}_n$. If $U \in \mathcal{U}_n$, then

$$\left\|\left(\partial_x^2 - x\right) U\right\|^2 \leq 4 \cdot C \cdot t_n^{\frac{2}{3}} \cdot \|U\|^2. \tag{91}$$

Since $w_n^\pm$ satisfies the boundary condition at 0, the Langer-Cherry transform $W_n^\pm$ at energy E_n satisfies the boundary condition at $\phi_{E_n}(0)$. It follows that $\mathcal{U}_n \subset \mathrm{dom}(\mathcal{A}_{z_n})$, where $z_n = t_n^{-\frac{2}{3}} \cdot \phi_{E_n}(0)$. By (91) we have

$$\langle \mathcal{A}_z^2(U),\ U\rangle \leq 4 \cdot C \cdot t_n^{\frac{2}{3}} \cdot \|U\|^2$$

for each $U \in \mathcal{U}_n$. Hence, by the minimax principle, $\mathcal{A}_{z_n}^2$ has at least two independent eigenvectors with eigenvalues in the interval $[0, 4C \cdot t_n^{\frac{2}{3}}]$. Thus, $\mathcal{A}_{z_n}$ has at least two independent eigenvectors with eigenvalues in the interval $[-2\sqrt{C} \cdot t_n^{\frac{1}{3}}, 2\sqrt{C} \cdot t_n^{\frac{1}{3}}]$. By Proposition 10.2, the eigenvalues of $\mathcal{A}_{z_n}$ are simple, and hence $\mathcal{A}_{z_n}$ has at least two distinct eigenvalues, $\nu_n^+ < \nu_n^-$ lying in $[-2\sqrt{C} \cdot t_n^{\frac{1}{3}}, 2\sqrt{C} \cdot t_n^{\frac{1}{3}}]$. By Proposition 10.2, the number $a_n^\pm = z_n - \nu_n^\pm$ is a zero of the funtion A_-. Note that

$$|a_n^+ - a_n^-| \leq 4\sqrt{C} \cdot t_n^{\frac{1}{3}}. \tag{92}$$

Since A_- is real-analytic and $A_-(x) \neq 0$ for x nonnegative, the zeroes Z of A_- are a countable discrete subset of $(-\infty, 0)$. In particular, there is a unique bijection $\ell : Z \to \mathbb{Z}^+$ such that $a < a'$ implies $\ell(a) > \ell(a')$ and $\lim_{k\to\infty} \ell^{-1}(k) = -\infty$. From the asymptotics of A_-—see Appendix A—one finds that there exists a constant $c > 0$ so that

$$\lim_{k\to\infty} k^{-\frac{2}{3}} \cdot \ell^{-1}(k) = -c, \tag{93}$$

$$\lim_{k\to\infty} k^{\frac{1}{3}} \cdot \left|\ell^{-1}(k) - \ell^{-1}(k+1)\right| = \frac{2}{3} \cdot c. \tag{94}$$

Since $\lim_{n\to\infty} t_n = 0$, estimate (92) implies that $\lim_{n\to\infty} a_n^\pm = -\infty$, and hence $\lim_{n\to\infty} \ell(a_n^\pm) = \infty$. Therefore, since $a_n^+ \neq a_n^-$ for all n, we have from (94) that there exists N such that if $n > N$ then

$$\lim_{k\to\infty} \left|a_n^+ - a_n^-\right| \geq \frac{c}{2} \cdot \ell(a_n^+)^{-\frac{1}{3}}.$$

By combining this with (92) we find that

$$(\ell(a_n) \cdot t_n)^{\frac{1}{3}} \geq \frac{c}{4\sqrt{C}}. \tag{95}$$

But since $\lim_{n\to\infty} E_n = \mu/\sigma(0)$, we have $\lim_{n\to\infty} \phi_{E_n}(0) = 0$. Therefore, by Proposition 10.3 we have $\lim_{n\to\infty} t^{\frac{2}{3}} \cdot a_n^{\pm} = 0$. By (93) we have

$$\lim_{n\to\infty} \frac{a_n}{\ell(a_n)^{\frac{2}{3}}} = -c.$$

Thus, $\lim_{n\to\infty} t_n^{\frac{2}{3}} \cdot \ell(a_n)^{\frac{2}{3}} = 0$. This contradicts (95). □

11. Separation of Variables in the Abstract

Recall that the first step in our method for proving generic simplicity consists of finding a family a_t such that q_t is asymptotic to a_t and such that a_t decomposes as a direct sum of '1-dimensional' quadratic forms a_t^{μ} of the type considered in the previous sections. In the present section we discuss the decomposition of a_t into forms a_t^{μ}. Although the content is very well-known, we include it here for the purpose of establishing notation and context.

Let $\langle \cdot, \cdot \rangle_{\sigma}$ be the inner product on $\mathcal{H}_{\sigma}$ defined in §5. Let $\mathcal{H}'$ be a real Hilbert space with inner product $(\cdot, \cdot)$. Consider the tensor product $\mathcal{H} := \mathcal{H}_{\sigma} \bigotimes \mathcal{H}'$ completed with respect to the inner product $\langle \cdot, \cdot \rangle$ determined by

$$\langle u_1 \otimes \varphi_1, u_2 \otimes \varphi_2 \rangle := \langle u_1, u_2 \rangle_{\sigma} \cdot (\varphi_1, \varphi_2). \tag{96}$$

Let b be a positive, closed, densely defined quadratic form on $\mathcal{H}'$. We will assume that the spectrum of b with respect to $(\cdot, \cdot)$ is discrete and the eigenspaces are finite dimensional. For each $t > 0$ and $u \otimes \varphi \in \mathcal{C}_0^{\infty}([0, \infty)) \bigotimes \mathrm{dom}(b)$, define

$$a_t(u \otimes \varphi) = t^2 \cdot (\varphi, \varphi) \int_0^{\infty} |u'(x)|^2 \, dx + b(\varphi) \int_0^{\infty} |u(x)|^2 \, dx. \tag{97}$$

Let $Y \subset \mathcal{C}_0^{\infty}([0, \infty))$ be a subspace. The restriction of a_t to $Y \otimes \mathcal{H}'$ is a nonnegative real quadratic form. By Theorem 1.17 in Chap. VI of [Kato], this restriction has a unique minimal closed extension. In particular, let $\mathrm{dom}(a_t)$ be the collection of $u \in \mathcal{H}_{\sigma} \otimes \mathcal{H}'$ such that there exists a sequence $u_n \in Y \otimes \mathrm{dom}(b)$ such that $\lim_{n\to\infty} \|u_n - u\| = 0$ and u_n is Cauchy in the norm

$$[u]_t := a_t(u) + \|u\|_{\mathcal{H}}.$$

For each $u \in \mathrm{dom}(a_t)$ define

$$a_t(u) := \lim_{n\to\infty} a_t(u_n),$$

where u_n is a sequence as above. For $t, t' > 0$ the norms $[\cdot]_t$ and $[\cdot]_{t'}$ are equivalent, and hence $\mathrm{dom}(a_t)$ does not depend on t.

Remark 11.1. In applications, either $Y = \mathcal{C}_0([0, \infty))$ or Y consists of smooth functions whose support is compact and does not include zero. In the former case, eigenfunctions of a_t will satisfy a Neumann condition at $x = 0$ and in the latter case they will satisfy a Dirichlet condition at $x = 0$.

Proposition 11.2. *The family $t \mapsto a_t$ is a real-analytic family of type (a) in the sense of Kato.*[2]

Proof. For each t the form a_t is closed with respect to $\langle \cdot, \cdot \rangle$, the domain $\mathrm{dom}(a_t)$ is constant in t, and for each $u \in \mathrm{dom}(a_t)$, the function $t \mapsto a_t(u)$ is analytic in t. □

Example 11.3. Let $\mathcal{H}'$ be the space of square integrable functions on a compact Lipschitz domain $U \subset \mathbf{R}^n$ with the usual inner product. Then $\mathcal{H} \bigotimes \mathcal{H}'$ is isomorphic to the completion of $C_0^\infty((0, \infty) \times U)$ with respect to the inner product

$$\langle f, g \rangle = \int_U \int_0^\infty f(x, y) \cdot g(x, y) \cdot \sigma(x)\, dx\, dy.$$

Let $\tilde{b}$ be the quadratic form defined on $H^1(U)$ by

$$\tilde{b}(\phi) = \int_U |\nabla \phi|^2\, dx\, dy. \tag{98}$$

We define b to be the restriction of $\tilde{b}$ to any closed subset of $H^1(U)$ on which it defines a positive quadratic form. In this case the quadratic form a_t is equivalent to the form

$$\overline{a}_t(u) = \int_{\mathbf{R}^+ \times U} \left(t^2 \cdot |\partial_x u|^2 + |\nabla_y u|^2 \right) dx\, dy. \tag{99}$$

For each $\mu > 0$ and $t > 0$, we define the quadratic form a_t^μ as in §5. The form a_t^μ is equivalent to the construction above with $\mathcal{H}' = \mathbf{R}$ with its standard inner product and $b(s) = \mu \cdot s^2$. The norms $[\cdot]_{t,\mu}$ and $[\cdot]_{t',\mu'}$ that are used to extend a_t^μ and $a_{t'}^{\mu'}$ are equivalent. Hence $\mathrm{dom}(a_t^\mu)$ is independent of t and μ. We will denote this common domain by $\mathcal{D}$.

Proposition 11.4. *If ϕ is a μ-eigenvector for b with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H}'}$, and v is a λ-eigenvector of a_t^μ with respect to $\langle \cdot, \cdot \rangle_\sigma$, then $v \otimes \phi$ is a λ-eigenvector of a_t with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. Conversely, if u is a λ-eigenvector of a_t with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H}}$, then u is a finite sum $\Sigma v_\mu \otimes \phi_\mu$, where v_μ is a λ-eigenvector of a_t^μ with respect to $\langle \cdot, \cdot \rangle_\sigma$ and ϕ_μ is a μ-eigenvector of b with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H}'}$.*

Proof. Straightforward. □

Proposition 11.5. *For each analytic eigenvalue branch λ_t of a_t, there exists a unique $\mu \in \mathrm{spec}(b)$ such that λ_t is an analytic eigenvalue branch of a_t^μ. In particular, λ_t decreases to $\frac{\mu}{\sigma(0)}$ as t tends to 0.*

Proof. Let $t_0 > 0$. For each $\mu \in \mathrm{spec}(b)$, consider the set A_μ of $t \in (0, t_0)$ such that $\lambda_t \in \mathrm{spec}(a_t^\mu)$. By Proposition 11.4, the union $\bigcup_\mu A_\mu$ equals $(0, t_0)$. Since $\mathrm{spec}(b)$ is countable, the Baire Category Theorem implies that there exists $\mu \in \mathrm{spec}(b)$ such that A_μ has nonempty interior A_μ^0. For each real-analytic eigenvalue branch ν_t of a_t^μ, let $B_\nu \subset A_\mu^0$ be the set of t such that $\nu_t = \lambda_t$. Since there are only countably many eigenvalue branches, the Baire Category Theorem implies that there exists an eigenvalue branch ν_t of a_t^μ such that B_μ has nonempty interior B_μ^0. Since λ_t and ν_t are real-analytic functions that coincide on a nonempty open set, they agree for all t.

The latter statement then follows from Proposition 10.1. □

[2] See Chap. VII §4.2 in [Kato].

Corollary 11.6. *If each eigenspace of b is 1-dimensional, then for each t belonging to the complement of a countable set, each eigenspace of a_t with respect to $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ is 1-dimensional.*

Proof. Use the assumption that b is simple and the fact that the eigenbranches are analytic. □

We end this section by establishing some notation that will be useful in the sections that follow. For each eigenvalue μ of b, let $\mathcal{V}_\mu$ denote the associated eigenspace and let $P_\mu : \mathcal{H}' \to \mathcal{V}_\mu$ denote the associated orthogonal projection. Define $\Pi_\mu : \mathcal{H}_\sigma \bigotimes \mathcal{H}'$ by

$$\Pi_\mu(v \otimes w) = v \otimes P_\mu(w).$$

If $\mathcal{M}$ is a collection of eigenvalues μ of b, then we define $\Pi_{\mathcal{M}}$ to be the orthogonal projection onto the direct sum of μ-eigenspaces. That is,

$$\Pi_{\mathcal{M}} = \sum_{\mu \in \mathcal{M}} \Pi_\mu.$$

The subscript for Π may represent either an eigenvalue or a set of eigenvalues.

Assumption 11.7. In what follows we assume that each eigenspace of b with respect to $\langle\cdot,\cdot\rangle$ is 1-dimensional.

One convenient consequence of this assumption is that for each $w \in \mathcal{H}$, there exists $\tilde{w}_\mu \in \mathcal{H}_\sigma$ and a unit norm eigenvector ϕ_μ of b such that

$$\Pi_\mu(w) = \tilde{w}_\mu \otimes \phi_\mu. \tag{100}$$

Indeed, for each $\mu \in \text{spec}(b)$, let $\phi_\mu \in \mathcal{V}_\mu$. Since $\dim(\mathcal{V}_\mu) = 1$, each vector in $\mathcal{H}_\sigma \otimes \mathcal{V}_\mu$ is of the form $v \otimes \phi_\mu$. In particular, there exists $\tilde{w}_\mu$ so that (100) holds. Note that

$$w = \sum_{\mu \in \text{spec}(b)} \Pi_\mu(w) = \sum_{\mu \in \text{spec}(b)} \tilde{w}_\mu \otimes \phi_\mu.$$

12. Projection Estimates

In this section, q_t will denote a family of quadratic forms densely defined on $\mathcal{H}$ that is asymptotic at first order[3] to the family a_t defined in the preceding section. Let $P^I_{a_t}$ be the orthogonal projection onto the direct sum of eigenspaces of a_t associated to the eigenvalues of a_t that belong to the interval I (see §2). We will provide some basic estimates on

$$w := P^I_{a_t}(u) \tag{101}$$

We begin with the following quasimode type estimate. In the sequel ϕ_μ will denote a unit norm eigenvector of b with eigenvalue μ. By Assumption 11.7, ϕ_μ is unique up to sign.

[3] See Definition 3.1.

Lemma 12.1. *Let $J \subset I$ be a proper closed subinterval of a compact interval I. There exist constants $C > 0$ and $t_0 > 0$ such that if $\mu \in \mathrm{spec}(b)$, $t < t_0$, u is an eigenfunction of q_t with eigenvalue $E \in J$, $z \in \mathcal{D}$, then the projection $w = P^I_{a_t}(u)$ satisfies*

$$\left|a_t\left(\Pi_\mu w, z \otimes \phi_\mu\right) - E \cdot \left\langle \Pi_\mu w, z \otimes \phi_\mu\right\rangle\right| \leq C \cdot t \cdot \|z\|_\sigma \cdot \|w\|. \tag{102}$$

Proof. Since q_t and a_t are asymptotic at first order, Lemma 2.5 applies. In particular, by letting $\delta = \mathrm{dist}(J, \partial I)$, $t_0 = \frac{1}{2}(1 + E/\delta)^{-1}$, and $C = (4/3) \cdot \sup(I)$, we have for $t < t_0$ and $v \in \mathcal{D} \bigotimes \mathrm{dom}(b)$,

$$|a_t(w, v) - E \cdot \langle w, v\rangle| \leq C \cdot t \cdot \|v\| \cdot \|w\|. \tag{103}$$

For each $\mu' \in \mathrm{spec}(b)$, there exists $\tilde{w}_{\mu'} \in \mathcal{D}$ so that

$$w = \sum_{\mu' \in \mathrm{spec}(b)} \tilde{w}_{\mu'} \otimes \phi_{\mu'} \tag{104}$$

and $v = \tilde{v}_\mu \otimes \phi_\mu$. If $\mu' \neq \mu$, then $b(\phi_\mu, \phi_{\mu'}) = 0$ and $\langle \phi_\mu, \phi_{\mu'}\rangle = 0$, and hence using (96) and (97) we find that

$$a_t(\tilde{w}_{\mu'} \otimes \phi_{\mu'}, z \otimes \phi_\mu) - E \cdot \left\langle \tilde{w}_{\mu'} \otimes \phi_{\mu'}, z \otimes \phi_{\mu'}\right\rangle = 0.$$

Thus,

$$a_t(w, v) - E \cdot \langle w, v\rangle = a_t\left(\Pi_\mu w, v\right) - E \cdot \left\langle \Pi_\mu w, v\right\rangle.$$

The claim then follows from substituting this into (103). □

Lemma 12.2. *Let $J \subset I$ be a proper closed subinterval of a compact interval I. Let $\mu \in \mathrm{spec}(b)$ with $\mu < \sigma(0) \cdot \inf(I)$ and let $\epsilon > 0$. There exist constants $\kappa > 0$ and $t_0 > 0$ such that if $t < t_0$, u is an eigenfunction of q_t with eigenvalue $E \in J$, and*

$$\left\|\Pi_\mu w\right\| \geq \epsilon \cdot \|w\|,$$

where $w = P^I_{a_t}(u)$, then we have

$$\dot{a}_t\left(\Pi_\mu(w)\right) \geq \frac{\kappa}{t} \cdot \left\|\Pi_\mu(w)\right\|^2_\sigma. \tag{105}$$

Proof. We have $\Pi_\mu w = \tilde{w}_\mu \otimes \phi_\mu$ for some $\tilde{w}_\mu \in \mathcal{D}$. Since, by assumption, $\|\phi_\mu\| = 1$, we have $\|\Pi_\mu(w)\| = \|\tilde{w}_\mu\|$ and hence the assumption becomes

$$\|\tilde{w}_\mu\|_\sigma \geq \epsilon \cdot \|w\|.$$

Therefore, Lemma 12.1 gives

$$\left|a^\mu_t\left(\tilde{w}_\mu, z\right) - E_t \cdot \langle \tilde{w}_\mu, z\rangle_\sigma\right| \leq C \cdot t \cdot \|z\|_\sigma \cdot \frac{\|\tilde{w}_\mu\|_\sigma}{\epsilon} \tag{106}$$

for all sufficiently small t. Since $\mu/\sigma(0) < \inf(I)$, the compact set I is a subset of $(\mu/\sigma(0), \infty)$. Hence we may apply Proposition 9.1 to obtain $\kappa > 0$ and $t_1 > 0$ so that if $t < t_1$, then

$$\int_0^\infty (E_t \cdot \sigma(x) - \mu) \cdot |\tilde{w}_\mu|^2\, dx \geq \kappa \cdot \|\tilde{w}_\mu\|^2_\sigma. \tag{107}$$

Inspection of (97) gives that

$$\dot{a}_t(\tilde{w}_\mu \otimes \phi_\mu, \tilde{w}_{\mu'} \otimes \phi_{\mu'}) = 2t \cdot \langle \phi_\mu, \phi_{\mu'} \rangle \int_0^\infty \left(\partial_x \tilde{w}_\mu \cdot \partial_x \tilde{w}_{\mu'} \right). \tag{108}$$

In particular

$$\dot{a} \left(\tilde{w}_\mu \otimes \phi_\mu \right) = 2t \int_0^\infty \left| \partial_x \tilde{w}_\mu \right|^2 \, dx.$$

Thus, by using the definition of a_t^μ and estimates (106) and (107) we find that

$$\begin{aligned} \dot{a} \left(\Pi_\mu(w) \right) &= \frac{2}{t} \left(a_t^\mu(\tilde{w}_\mu) - \mu \int_0^\infty |\tilde{w}_\mu|^2 \, dx \right) \\ &\geq \frac{2}{t} \left(\int_0^\infty (E_t \cdot \sigma - \mu) \, |\tilde{w}_\mu|^2 \, dx \right) - \frac{2C}{\epsilon} \cdot \|\tilde{w}_\mu\|_\sigma^2 \\ &\geq 2 \left(\frac{\kappa}{t} - \frac{C}{\epsilon} \right) \cdot \|\tilde{w}_\mu\|_\sigma^2. \end{aligned}$$

By choosing $t_0 = \min\{t_1, C/(\epsilon \cdot \kappa)\}$ we obtain the claim. □

Remark 12.3. In the preceding lemma the constants t_0 and κ a priori depend on the chosen μ. However, since there is only a finite number of eigenvalues of b that satisfy $\mu \leq \sigma(0) \inf I$, we can choose t_0 and κ depending only on I and not on the eigenvalue μ.

It will be convenient to introduce the following notation. Given $\mu \in \text{spec}(b)$, define

$$\tilde{\mu} = \frac{\mu}{\sigma(0)},$$

where σ is as in §11. For each compact interval I, define

$$\begin{aligned} \mathcal{M}_I &= \{\mu \in \text{spec}(b) | \ \tilde{\mu} \in I\}, \\ \mathcal{M}_I^- &= \{\mu \in \text{spec}(b) | \ \tilde{\mu} < \inf I\}, \\ \mathcal{M}_I^+ &= \{\mu \ \in \text{spec}(b) | \ \tilde{\mu} > \sup I\}. \end{aligned}$$

The spectrum spec(b) equals the disjoint union of $\mathcal{M}_I^-$, $\mathcal{M}_I$, and $\mathcal{M}_I^+$, and in particular, each $v \in \mathcal{H}$ can be orthogonally decomposed as

$$v = \Pi_{\mathcal{M}_I^-}(v) + \Pi_{\mathcal{M}_I}(v) + \Pi_{\mathcal{M}_I^+}(v).$$

The following lemma is crucial to our proof of generic simplicity. The proof uses both Theorem 4.2 and—by way of Lemma 12.2—Proposition 9.1.

Lemma 12.4. *Let E_t be a real-analytic eigenvalue branch q_t, and let V_t be the associated family of eigenspaces. Let $t \mapsto u_t$ be a map from* $(0, t_0]$ *to V_t that is continuous on the complement of a countable set. If $w_t = P_{a_t}^I(u_t)$, then*

$$\liminf_{t \to 0} \frac{\|\Pi_{\mathcal{M}_I^-}(w_t)\|}{\|w_t\|} = 0. \tag{109}$$

Here if $w_t = 0$, then we interpret the ratio to be equal to 1.

Proof. Suppose that (109) is false. We have the orthogonal decomposition

$$\Pi_{\mathcal{M}_I^-}(w_t) = \sum_{\mu \in \mathcal{M}_I^-} \Pi_\mu(w_t),$$

and hence there exists $\epsilon > 0$ and $t_0 > 0$ such that for each $t < t_0$ there exists $\mu_t \in \mathcal{M}_I^-$ such that

$$\|\Pi_{\mu_t}(w_t)\| \geq \epsilon \cdot \|w_t\|. \tag{110}$$

Using the orthogonal decomposition of w as in (104) we find that

$$\dot{a}_t\,(w_t) = \sum_{\mu \in \mathrm{spec}(b)} \dot{a}_t\left(\Pi_{\mu_t}(w_t)\right).$$

(See also (108).) In particular, since the quadratic form $\dot{a}_t$ is nonnegative, we have that $\dot{a}_t(w_t) \geq \dot{a}_t(\Pi_{\mu_t}(w_t))$. Thus we may apply Lemma 12.2 with $J = E((0, t_0])$ as well as (110) to find that

$$\dot{a}_t\,(w_t) \geq \frac{\epsilon \cdot \kappa}{t} \cdot \|w_t\|^2$$

for all t sufficiently small with some κ independent of t (according to Remark 12.3). Thus, it follows from Theorem 4.2 that the function $1/t$ is integrable on an interval whose left endpoint is zero. This is absurd. □

Lemma 12.5. *Let I be a compact interval. If w belongs to the range of $P^I_{a_t}$, then*

$$\Pi_{\mathcal{M}_I^+}(w) = 0.$$

In particular,

$$\|w\|^2 = \left\|\Pi_{\mathcal{M}_I}(w)\right\|^2 + \left\|\Pi_{\mathcal{M}_I^-}(w)\right\|^2.$$

Proof. By definition, w is a linear combination of eigenfunctions of a_t whose eigenvalues belong to I. Hence by Proposition 11.4, we have

$$w = \sum_{\mu \in \mathrm{spec}(b)} \sum_{\lambda \in I \cap \mathrm{spec}(a_t^\mu)} v_{\lambda,\mu} \otimes \phi_\mu,$$

where $v_{\lambda,\mu}$ belongs to the λ-eigenspace of a_t^μ and ϕ_μ belongs to the μ-eigenspace of b. Hence

$$\Pi_{\mathcal{M}_I^+}(w) = \sum_{\mu \in \mathcal{M}_I^+} \sum_{\lambda \in I \cap \mathrm{spec}(a_t^\mu)} v_{\lambda,\mu} \otimes \phi_\mu. \tag{111}$$

According to Proposition 10.1, each eigenvalue λ of a_t^μ satisfies $\lambda \geq \tilde{\mu}$. If $\mu \in \mathcal{M}_I^+$, then $\tilde{\mu} \geq \sup(I)$. Hence each term in (111) vanishes. □

13. The Limits of the Eigenvalue Branches of q_t

Proposition 3.4 implies that each real-analytic eigenvalue branch E_t of q_t converges as t tends to zero. In this section we use the results of the previous section to show that each limit belongs to the set

$$\widetilde{\operatorname{spec}(b)} = \{\tilde{\mu} \mid \mu \in \operatorname{spec}(b)\}.$$

Theorem 13.1. *For each real-analytic eigenvalue branch E_t of q_t, we have*

$$\lim_{t \to 0} E_t \in \widetilde{\operatorname{spec}(b)}.$$

Proof. Suppose to the contrary that the limit, E_0, does not belong to $\widetilde{\operatorname{spec}(b)}$. Since $\widetilde{\operatorname{spec}(b)}$ is discrete, there exists a nontrivial compact interval I such that $E_0 \in J$, such that

$$J \cap \widetilde{\operatorname{spec}(b)} = \emptyset. \tag{112}$$

Since J is nontrivial and E_t is continuous, there exists t_0 such that if $t < t_0$, then $E_t \in J$. Let I be a compact interval such that $J \subset I \subset \left(\mathbf{R} \backslash \widetilde{\operatorname{spec}(b)}\right)$.

Let u_t be a real-analytic eigenfunction branch associated to E_t and let $w_t = P^I_{a_t}(u_t)$. We have chosen I so that $\mathcal{M}_I = \emptyset$. Thus, by Lemma 12.5,

$$\left\| \Pi_{\mathcal{M}_I^-}(w_t) \right\|^2 = \|w_t\|^2.$$

This contradicts Lemma 12.4. □

14. Generic Simplicity of q_t

In this section, we prove that the spectrum of q_t is generically simple. We will make crucial use of the 'super-separation' of the eigenvalues of a_t for small t (see Theorem 10.4).

Before providing the details of the proof, we first illustrate how super-separation can be useful in proving simplicity. Suppose that there exists an eigenvalue branch E_t of q_t such that $E_t \to \tilde{\mu}$ and the associated real-analytic family of eigenspaces V_t is at least two dimensional. If for each $u_t \in V_t$ we knew that $\|\Pi_\mu u_t\|$ were uniformly bounded away from 0, then, arguing as in the beginning of the proof of Lemma 12.2, we would find that $\Pi_\mu u_t$ is a first order quasimode of a_t^μ at energy $\tilde{\mu}$. Then, since $\dim(V_t) \geq 2$, we would have a sequence t_n tending to zero and two distinct eigenvalues λ, λ' of $a_{t_n}^\mu$ such that $\left(\lambda - \lambda'\right)/t_n$ is bounded. This would contradict super-separation.

Theorem 14.1. *Let E_t be a real-analytic eigenvalue branch E_t of q_t, and let V_t be the associated real-analytic family of eigenspaces* (*see Remark* 4.1). *For each $t \in (0, t_0]$ we have* $\dim(V_t) = 1$.

Since each eigenvalue branch of q_t is real-analytic and the spectrum of each q_t is discrete with finite dimensional eigenspaces, we have the following corollary.

Corollary 14.2. *Let E_t be a real-analytic eigenbranch, then E_t is a simple eigenvalue of q_t for all t in the complement of a discrete subset of* $(0, t_0]$.

Proof of Theorem 14.1. Suppose that the conclusion does not hold. Since V_t is a real-analytic family of vector spaces, its dimension is constant and so for each $t \in (0, t_0]$, we have $\dim(V_t) > 1$.

By Theorem 13.1 there exists $\mu \in \operatorname{spec}(b)$ such that E_t tends to $\tilde{\mu} = \mu/\sigma(0)$ as t tends to zero. Let I be a compact interval so that $I \cap \widetilde{\operatorname{spec}(b)} = \{\tilde{\mu}\}$. By Lemma 14.3 below, there exists $t_3 \leq t_0$ and a map $t \mapsto u_t$ from $(0, t_3]$ into V_t that is continuous on the complement of a discrete set so that if $t \in (0, t_3]\backslash Z'$, then

$$\|\Pi_\mu(w_t)\| < \frac{1}{2} \cdot \|w_t\|,$$

where $w_t = P_a^I(u_t)$. Thus, since $\{\mu\} = \mathcal{M}_I$, Lemma 12.5 gives that

$$\|\Pi_{\mathcal{M}_I^-}(w_t)\| \geq \frac{1}{2} \cdot \|w_t\|.$$

This contradicts Lemma 12.4. □

Lemma 14.3. *Let E_t be a real-analytic eigenvalue branch of q_t such that for each $t > 0$ we have $\dim(V_t) > 1$. Let $\mu \in \operatorname{spec}(b)$ be such that $\lim_{t \to 0} E_t = \tilde{\mu}$, and let I be a compact interval such that*

$$I \cap \widetilde{\operatorname{spec}(b)} = \tilde{\mu}.$$

There exists $t_0 > 0$ and a function $t \mapsto u_t$ that maps $(0, t_0]$ to V_t, is continuous on the complement of a discrete set, and satisfies

$$\left\|\Pi_\mu(w_t)\right\| \leq \frac{1}{2} \cdot \|w_t\| \tag{113}$$

where $w_t = P_{a_t}^I(u_t)$.

To prove Lemma 14.3, we will use the following well-known fact.

Lemma 14.4. *Let $\{g_k : (a, b) \to \mathbf{R} \mid k \in \mathbf{N}\}$ be a collection of real-analytic functions. If for each $k \in N$ and $t \in (a, b)$ we have $g_{k+1}(t) > g_k(t)$ then the set*

$$\{t \in (a, b) \mid g_k(t) = 0,\ k \in \mathbf{N}\}$$

is a discrete subset of (a, b).

Proof. Suppose that $g_k(t) = 0$ for some $k \in \mathbf{N}$ and $t \in (a, b)$. Since g_k is real-analytic there exists an open set $U \ni t$ such that if $t' \in U\backslash\{t\}$, then $g_k(t) = 0$. Since $k' > k''$ implies $g_{k'}(t) > g_{k''}(t)$ we have

$$t \in g_{k+1}^{-1}(0, \infty) = \bigcup_{k' > k} g_{k'}^{-1}(0, \infty)$$

and

$$t \in g_{k-1}^{-1}(-\infty, 0) = \bigcup_{k' < k} g_{k'}^{-1}(-\infty, 0).$$

It follows that if

$$t' \in W := U \cap g_{k+1}^{-1}(0, \infty) \cap g_{k-1}^{-1}(-\infty, 0),$$

$t' \neq t$, and $k' \in \mathbf{N}$, then $g_{k'}(t) \neq 0$. Since W is open, we have the claim. □

Proof of Lemma 14.3. By Lemma 12.1, there exist C and $t_1 > 0$ such that if $t \leq t_1$, $z \in \mathcal{D}$, and u is an eigenfunction with eigenvalue E_t, then

$$\left|a_t^{\mu}\left(\tilde{w}_{\mu}, z\right) - E_t \cdot \left\langle \tilde{w}_{\mu}, z\right\rangle_{\sigma}\right| \leq C \cdot t \cdot \|w\| \cdot \|z\|_{\sigma}, \tag{114}$$

where $w = P^I_{a_t}(u)$ and $\tilde{w} \otimes \varphi_{\mu} = \Pi_{\mu} w$.

Since a_t^{μ} is a real-analytic family of type (a) in the sense of [Kato], for each $k \in \mathbf{N}$, there exists a real-analytic function $\lambda_k : (0, t_1] \to \mathbf{R}$ so that for each $t \in (0, t_1]$, we have $\text{spec}(a_t^{\mu}) = \{\lambda_k(t) \mid k \in \mathbf{N}\}$. Since each eigenspace of a_t^{μ} is 1-dimensional, we may assume that $k > k'$ implies $\lambda_k(t) > \lambda_{k'}(t)$ for all $t \in (0, t_1]$.

By Theorem 10.4, there exists $t_0 \in (0, t_1]$ such that if $t < t_0$, then $k \neq k'$, then

$$|\lambda_k(t) - \lambda_{k'}(t)| > 4C \cdot t. \tag{115}$$

For each $k \in \mathbf{N}$ and $t \in (0, t_0)$, define

$$g_k^{\pm}(t) = \lambda_k(t) - E_t \pm 2C \cdot t.$$

Thus, by Lemma 14.4, the set

$$Z = \bigcup_{k \in \mathbf{N}} \left((g_k^{+})^{-1}\{0\} \bigcup (g_k^{-})^{-1}\{0\} \right)$$

is discrete in $(0, t_0]$. On each component J of the complement $(0, t_0] \backslash Z$, we have either

- for all $t \in J$, we have $\text{dist}\left(E_t, \text{spec}(a_t^{\mu})\right) \geq 2C \cdot t$, or
- for all $t \in J$, we have $\text{dist}\left(E_t, \text{spec}(a_t^{\mu})\right) < 2C \cdot t$.

It suffices to construct in each of these cases a continuous map $t \mapsto u_t$ from J to V_t that satisfies (113). Without loss of generality, each interval J is precompact in $(0, t_0]$, for otherwise we may, for example, add the discrete set $\{1/n \mid n \in \mathbf{N}\}$ to Z.

We consider the first case. Let u_t be a real-analytic eigenfunction branch of q_t associated to E_t. By estimate (114), we may apply Lemma 2.1 with $\epsilon = C \cdot t \cdot \|w_t\|$ and find that

$$\|\tilde{w}_t\|_{\sigma} \leq \frac{1}{2} \cdot \|w_t\|. \tag{116}$$

Since $\|\Pi_{\mu} w\| = \|\tilde{w}_{\mu}\|_{\sigma}$, the desired (113) follows.

We consider the second case. By (115) and since $J \subset (0, t_0)$ there exists a unique k such that if $t \in J$, then

$$|E_t - \lambda_k(t)| < 2C \cdot t. \tag{117}$$

Let $t \mapsto \tilde{v}_t$ be the unique eigenfunction branch of a_t^{μ} associated to the eigenvalue branch λ_k. Since $\dim(V_t) > 1$ and V_t is an analytic family of vector spaces, there exist analytic eigenfunction branches $x_t, x_t' \in V_t$ so that for each t, the eigenvectors x_t and x_t' are independent.

The function $t \mapsto \langle x_t, \tilde{v}_t \otimes \phi_{\mu}\rangle$ is real-analytic, and thus it vanishes on at most a finite subset $Z_J \subset J$. Away from Z_J, set

$$c(t) = -\frac{\langle x_t', \tilde{v}_t \otimes \phi_{\mu}\rangle}{\langle x_t, \tilde{v}_t \otimes \phi_{\mu}\rangle}.$$

Then $u_t = c(t) \cdot x_t + x_t'$ depends real-analytically on t and satisfies

$$\langle u_t, \tilde{v}_t \otimes \phi_\mu \rangle = 0.$$

For each $t \in J \backslash Z_J$, let r_t denote the restriction of the quadratic form a_t^μ to the orthogonal complement of $\tilde{v}_t \otimes \phi_\mu$ in $\mathcal{D} \bigotimes \mathrm{dom}(b)$. Let $w_t = P_{a_t}^I(u_t)$ and let $\tilde{w}_{\mu,t} \in \mathcal{D}$ such that $\Pi_\mu w_t = \tilde{w}_{\mu,t} \otimes \phi_\mu$. From (114), we have

$$\left| r_t\left(\tilde{w}_{\mu,t}, z\right) - E_t \cdot \left\langle \tilde{w}_{\mu,t}, z \right\rangle_\sigma \right| \leq C \cdot t \cdot \|w_t\| \cdot \|z\|_\sigma .$$

It follows from (115) that $\mathrm{dist}(E_t, \mathrm{spec}(r_t)) \geq 2C \cdot t$. Hence Lemma 2.1 applies with $\epsilon = 2C \cdot t \cdot \|w\|$ to give (113).

Therefore, on the complement of $Z \cup \bigcup_J Z_J$, we have constructed a real-analytic function $t \mapsto V_t$ so that (113) holds. □

15. Stretching Along an Axis

In this section, we consider a family of quadratic forms q_t obtained by 'stretching' certain domains in Euclidean space $\mathbf{R}^{n+1}$ that fiber over an interval. To be precise, let $I = [0, c]$ be an interval, let $Y \subset \mathbf{R}^n$ be a compact domain with Lipschitz boundary, and let $\rho : [0, c] \to \mathbf{R}$ be a smooth nonnegative function. For $t > 0$, define $\phi_t : I \times Y \to \mathbf{R}^{n+1}$ by

$$\phi_t(x, y) = (x/t, \rho(x) \cdot y). \tag{118}$$

We will consider the Dirichlet Laplacian associated to the domain $\Omega_t = \phi_t(I \times Y)$.

Example 15.1 (Triangles and simplices). Let $Y = [0, a]$ and $\rho(x) = x$. Then Ω_t is the right triangle with vertices $(0, 0)$, $(c/t, 0)$, $(c/t, c)$. More generally, if $\rho(x) = x$ and Y is a n-simplex, then Ω_t is a $n + 1$-simplex.

Theorem 15.2. *If* $\rho : [0, a] \to \mathbf{R}$ *is smooth,* $\rho(0) = 0$, $\rho' > 0$,

$$\lim_{\epsilon \to 0} \int_\epsilon^c \frac{dx}{\rho(x)} = \infty,$$

and each eigenspace of the Dirichlet Laplacian acting on $L^2(Y)$ *is* 1*-dimensional, then for all but countably many* t*, each eigenspace of the Dirichlet Laplacian acting on* $L^2(\Omega_t)$ *is* 1*-dimensional.*

Proof. In order to apply Theorem 14.1, we make the following change of variables. Define $\psi : (0, c] \to [0, \infty)$ by

$$\psi(x) = \int_x^c \frac{dx}{\rho(x)}.$$

By hypothesis, ψ is an orientation reversing homeomorphism. Define $\Phi_t : C^\infty([0, \infty) \times Y) \to C^\infty(\Omega_t)$ by

$$\Phi_t(u) = \left(\rho^{\frac{n-1}{2}} \cdot u \circ (\psi \times \mathrm{Id})\right) \circ \phi_t,$$

where ϕ_t is defined by (118). We will use Φ_t to pull-back the L^2 inner product and the Dirichlet energy form.

First note that the Jacobian matrix of ϕ_t is

$$J\phi = \begin{pmatrix} 1/t & 0 \\ \partial_x \rho \cdot y & \rho \cdot \mathrm{Id} \end{pmatrix}, \tag{119}$$

where Id is the $n \times n$ identity matrix, and hence the Jacobian determinant $|J\phi_t|$ equals $t^{-1} \cdot \rho^n$. The Jacobian determinant of $\psi \times \mathrm{Id}$ is ρ^{-1}. It follows that

$$\int_{\Omega_t} (\Phi_t(u) \cdot \Phi_t(v))\, dV = \frac{1}{t} \int_0^\infty \int_Y u \cdot v\, \sigma(x)\, dx\, dy, \tag{120}$$

where $\sigma = \rho^2 \circ \psi^{-1}$ and where dy denotes Lebesgue measure on $Y \subset \mathbf{R}^n$. In order to have an inner product that does not depend on t, we rescale by t. Define

$$\langle u, v \rangle = \int_0^\infty \int_Y u \cdot v\, \sigma(x)\, dx\, dy.$$

Define a family of quadratic forms on $C^\infty([0, \infty) \times Y)$ by

$$q_t(u) = t \cdot \int_{\Omega_t} |\nabla (\Phi_t(u))|^2 \, dx\, dy.$$

The map Φ_t defines an isomorphism from each eigenspace of q_t with respect to $\langle \cdot, \cdot \rangle$ to the eigenspaces of the Dirichlet energy form on Ω_t with respect to the L^2-inner product on Ω_t. In particular, it suffices to show that each eigenspace of q_t with respect to $\langle \cdot, \cdot \rangle$ is 1-dimensional.

Define

$$a_t(u) = \int_0^\infty \int_Y \left(t^2 \cdot |\partial_x u|^2 + \left|\nabla_y u\right|^2 \right) dx\, dy.$$

By Theorem 14.1, it suffices to show that q_t is asymptotic to a_t at first order.

Let $\tau = \rho' \circ \psi^{-1}$. A straightforward calculation of moderate length shows that

$$\begin{aligned} q_t(u, v) - a_t(u, v) = t \cdot (&I_1(u, v) + I_2(u, v) + I_3(u, v) + I_4(u, v) + I_5(u, v) \\ &+ I_3(v, u) + I_4(v, u) + I_5(v, u)), \end{aligned}$$

where

$$\begin{aligned} I_1(u, v) &= t \cdot \frac{(n-1)^2}{4} \int_0^\infty \int_Y \tau^2 \cdot u \cdot v\, dx\, dy, \\ I_2(u, v) &= t \int_0^\infty \int_Y \tau^2 \cdot \left(y \cdot \nabla_y u\right) \cdot \left(y \cdot \nabla_y v\right) dx\, dy, \\ I_3(u, v) &= t \cdot \frac{n-1}{2} \int_0^\infty \int_Y \tau^2 \cdot u \cdot \left(y \cdot \nabla_y v\right) dx\, dy, \\ I_4(u, v) &= t \int_0^\infty \int_Y \tau \cdot \partial_x u \cdot \left(y \cdot \nabla_y v\right) dx\, dy, \\ I_5(u, v) &= t \cdot \frac{n-1}{2} \cdot \int_0^\infty \int_Y \tau \cdot u \cdot \partial_x v\, dx\, dy. \end{aligned}$$

To get (11), it suffices to show that for each $k = 1, \ldots, 5$, there exists a constant C_k such that $|I_k(u, v)| \le C_k \cdot a_t(u)^{\frac{1}{2}} \cdot a_t(v)^{\frac{1}{2}}$ for $t < 1$.

First note that by assumption $|\rho'|$—and hence $|\tau|$—is bounded by a constant C. Second, note that if $\lambda_0 > 0$ is the smallest eigenvalue of the Dirichlet Laplacian on $L^2(Y)$, then for each $u \in C^\infty([0,\infty) \times Y)$ we have

$$\int_0^\infty \int_Y u^2 \, dx \, dy \leq \frac{1}{\lambda_0} \int_0^\infty \int_Y \left|\nabla_y u\right|^2 \, dx \, dy. \tag{121}$$

If $n = 1$, then $|I_1(u,v)|$ is trivial. Otherwise, apply the Cauchy-Schwarz inequality and estimate (121). More precisely

$$\begin{aligned} \frac{4}{C^2(n-1)^2} \cdot |I_1(u,v)| &\leq t \cdot \left(\int_0^\infty \int_Y u^2 \, dx \, dy\right)^{\frac{1}{2}} \cdot \left(\int_0^\infty \int_Y v^2 \, dx \, dy\right)^{\frac{1}{2}} \\ &\leq \frac{t}{\lambda_0} \cdot \left(\int_0^\infty \int_Y \left|\nabla_y u\right|^2 \, dx \, dy\right)^{\frac{1}{2}} \cdot \left(\int_0^\infty \int_Y \left|\nabla_y v\right|^2 \, dx \, dy\right)^{\frac{1}{2}} \\ &\leq \frac{t}{\lambda_0} \cdot a_t(u)^{\frac{1}{2}} \cdot a_t(v)^{\frac{1}{2}}. \end{aligned}$$

To bound $|I_2(u,v)|$, note that $|y \cdot \nabla_y u|^2 \leq |y|^2 \cdot |\nabla_y u|^2$ and that $|y|^2$ is bounded since Y is compact. The desired bound of $|I_2(u,v)|$ then follows from an application of the Cauchy-Schwarz inequality.

If $n = 1$, then $|I_3(u,v)|$ is trivial. Otherwise, we apply the Cauchy-Schwarz inequality and estimate (121) as in the bound of $|I_1(u,v)|$.

To bound $|I_4(x,y)|$ we apply Cauchy-Schwarz as follows:

$$\int |t \cdot \partial_x u| \cdot \left|y \cdot \nabla_y v\right| \leq \left(\int |t \cdot \partial_x u|^2\right)^{\frac{1}{2}} \left(\int \left|y \cdot \nabla_y v\right|^2\right)^{\frac{1}{2}}.$$

It then follows that

$$|I_4(u,v)| \leq C \cdot a_t(u)^{\frac{1}{2}} \cdot a_t(u)^{\frac{1}{2}}.$$

To bound $|I_5(u,v)|$ apply Cauchy-Schwarz and argue in a fashion similar to the above. Condition (12) also follows using that $(tI_k)' = 2I_k$. □

15.1. Changing the boundary condition. Theorem 15.2 extends to a more general boundary condition that we describe here. Inspecting the proof, the only thing we have used from the Laplace operator on Y is that it satisfies the Poincaré inequality (121). This fact is true for any mixed Dirichlet-Neumann boundary condition except Neumann on all faces.

As a consequence we may take on the faces of Ω_t of the form $I \times \partial Y$ any kind of boundary condition except full Neumann.

On the face $\{1\} \times Y$ we may take Dirichlet or Neumann as we want since we have allowed Dirichlet or Neumann at 0 for the one-dimensional model operators a_t^μ.

16. Domains in the Hyperbolic Plane with a Cusp

Recall that the hyperbolic metric on the upper half-plane $\mathbf{R} \times \mathbf{R}^+$ is defined by $(dx^2 + dy^2)/y^2$. The associated Riemannian measure is given by $d\mu = y^{-2} dx\, dy$ and the gradient is given by $\nabla f = y^2(\partial_x f \cdot \partial_x + \partial_y f \cdot \partial_y)$.

Let $h : (-\eta, \eta) \to \mathbf{R}$ be a positive real-analytic function such that $h'(0) = 0$. For each $t < \eta$, define Ω_t by

$$\Omega_t = \big\{(x, y) \in \mathbf{R} \times \mathbf{R}^+ | -t \le x \le t \text{ and } y \ge h(x)\big\}.$$

The domain Ω_t is unbounded but has finite hyperbolic area. It is known that the hyperbolic Dirichlet Laplacian acting on $L^2(\Omega_t, d\mu)$ is compactly resolved and hence has discrete spectrum (see e.g. [LaxPhl]).[4]

Example 16.1. Let $h : (-1, 1) \to \mathbf{R}$ be defined by $h(x) = \sqrt{1 - x^2}$. For each $t < 1$, the domain Ω_t is a hyperbolic triangle with one ideal vertex. In particular, $\Omega_{1/2}$ is a fundamental domain for the modular group $SL(2, \mathbb{Z})$ acting on $\mathbf{R} \times \mathbf{R}^+ \subset \mathbb{C}$ as linear fractional transformations.

Theorem 16.2. *For all but countably many t, each eigenspace of the Dirichlet Laplacian acting on $L^2(\Omega_t, d\mu)$ is 1-dimensional.*

The remainder of this section is devoted to the proof of Theorem 16.2.

The spectrum of the hyperbolic Laplacian on Ω_t coincides with the spectrum of the Dirichlet energy form

$$\mathcal{E}(u) = \int_{\Omega_t} \Big(|\partial_x u|^2 + |\partial_y u|^2\Big)\, dx\, dy, \tag{122}$$

with respect to the inner product

$$\langle u, v\rangle_\mu = \int_{\Omega_t} u \cdot v\, \frac{dx\, dy}{y^2}. \tag{123}$$

In order to study the variational behavior of the eigenvalues, we first adjust the domains by constructing a family of diffeomorphisms ϕ_t from the fixed set $\mathcal{U} = [-1, 1] \times [h(0), \infty[$ onto Ω_t. In particular, define

$$\phi_t \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} t \cdot a \\ b + h(t \cdot a) - h(0) \end{pmatrix}.$$

For each $u \in \mathcal{C}_0^\infty(\mathcal{U})$, we define

$$\tilde{u} = \psi \cdot u \circ \phi_t^{-1},$$

where

$$\psi(x, y) = \frac{y}{y - h(x) + h(0)}.$$

Since ϕ_t is a smooth diffeomorphism from $\mathcal{U}$ onto Ω_t and ψ is smooth on Ω_t, the mapping $u \mapsto \tilde{u}$ is a bijection from $\mathcal{C}_0^\infty(\mathcal{U})$ onto $\mathcal{C}_0^\infty(\Omega_t)$.

[4] The Neumann Laplacian is not compactly resolved, and in fact, has essential spectrum.

Since the Jacobian of ϕ_t is

$$J(\phi_t)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} t & 0 \\ t \cdot h'(t \cdot a) & 1 \end{pmatrix} \tag{124}$$

and $\psi \circ \phi_t = (y \circ \phi_t)/b$, we find that, for any smooth u and v compactly supported in $\mathcal{U}$,

$$t^{-1} \int_{\Omega_t} \tilde{u} \cdot \tilde{v} \, \frac{dxdy}{y^2} = \int_{\mathcal{U}} u \cdot v \, \frac{da\, db}{b^2}. \tag{125}$$

In particular, the mapping $u \mapsto \tilde{u}$ extends to an isometry of $\mathcal{H} := L^2(\mathcal{U},\ da \cdot db/b^2)$ onto $L^2(\Omega_t, t^{-1} d\mu)$.

We now pull-back the Dirichlet energy form from Ω_t to $\mathcal{U}$. In particular, we define $q_t : \mathcal{C}_0^\infty(\mathcal{U}) \to \mathbf{R}$ by

$$q_t(u) = t \cdot \mathcal{E}(\tilde{u}).$$

The form extends to a closed densely defined form on $\mathcal{H}$. By construction, λ belongs to the spectrum of q_t if and only if $t^{-2} \cdot \lambda$ belongs to the Laplace spectrum of the hyperbolic triangle Ω_t. Because h is real-analytic, $t \mapsto \phi_t$ is a real-analytic family of bi-Lipschitz homeomorphisms. It follows that q_t is a real-analytic family of quadratic forms of type (a) in the sense of Kato [Kato].

On $\mathcal{C}_0^\infty(\mathcal{U})$, we also define

$$a_t(u) = \int_{\mathcal{U}} \left(t^2 \cdot |\partial_b u|^2 + |\partial_a u|^2 \right) da\, db.$$

Theorem 16.2 follows from Theorem 14.1 and the following proposition.

Proposition 16.3. *q_t is asymptotic to a_t at first order.*

Proof. Let $\bar{u} = (\psi \circ \phi_t) \cdot u$. One computes that

$$\begin{aligned} \left(\partial_y \tilde{u}\right) \circ \phi_t &= \partial_b \bar{u}, \\ (\partial_x \tilde{u}) \circ \phi_t &= \frac{1}{t} \cdot \partial_a \bar{u} - h'(ta) \cdot \partial_b \bar{u}. \end{aligned}$$

Thus, by making a change of variables in the integral that defines $\mathcal{E}$, we find that

$$q_t(u) = \int_{\mathcal{U}} |\partial_a \bar{u}|^2 - 2t \cdot h'(ta) \cdot \partial_a \bar{u} \cdot \partial_b \bar{u} + t^2 \cdot (1 + h'(ta)^2)\, |\partial_b \bar{u}|^2 \ da\, db, \tag{126}$$

where $\bar{u} = \bar{\psi} \cdot u$. To aid in computation we define a weighted gradient

$$\bar{\nabla} w = [\partial_a w,\ t \cdot \partial_b w],$$

and we define

$$A_t = \begin{pmatrix} 1 & -h'(t \cdot a) \\ -h'(t \cdot a) & 1 + h'(t \cdot a)^2 \end{pmatrix}.$$

Thus, (126) becomes

$$q_t(u, v) = \int_{\mathcal{U}} \bar{\nabla} \bar{u} \cdot A_t \cdot \bar{\nabla} \bar{v} \, da\, db$$

and

$$a_t(u, v) = \int_{\mathcal{U}} \bar{\nabla} u \cdot \bar{\nabla} v \, da \, db.$$

Letting $\bar{\psi} = \psi \circ \phi$, we have

$$\bar{\nabla} \bar{w} = \bar{\psi} \cdot \bar{\nabla} w + w \cdot \bar{\nabla} \psi,$$

and hence $q_t(u, v) - a_t(u, v)$ is the sum of four terms:

$$\int_{\mathcal{U}} \bar{\nabla} u \cdot (\bar{\psi}^2 \cdot A_t - I) \cdot \bar{\nabla} v \, da \, db, \tag{127}$$

$$\int_{\mathcal{U}} \bar{\psi} \cdot v \cdot (\bar{\nabla} \bar{\psi} \cdot A_t \cdot \bar{\nabla} u) \, da \, db, \tag{128}$$

$$\int_{\mathcal{U}} \bar{\psi} \cdot u \cdot (\bar{\nabla} \bar{\psi} \cdot A_t \cdot \bar{\nabla} v) \, da \, db, \tag{129}$$

$$\int_{\mathcal{U}} (\nabla \bar{\psi} \cdot A_t \cdot \bar{\nabla} \bar{\psi}) \cdot u \cdot v \, da \, db, \tag{130}$$

where I denotes the 2×2 identity matrix. To finish the proof, it suffices to show that each of these terms is bounded by $O(t) \cdot a_t(u)^{\frac{1}{2}} \cdot a_t(v)^{\frac{1}{2}}$, where $O(t)$ represents a function that is bounded by a constant times t for t small.

In order to estimate these terms, we use elementary estimates of $h(t \cdot a)$, $h'(t \cdot a)$, $\bar{\psi}$, and $\bar{\nabla} \bar{\psi}$. In particular, since $h'(0) = 0$ we have that $|h(t \cdot a) - h(0)| = O(t)$ and $|h'(t \cdot a)| = O(t)$ uniformly for $a \in [-1, 1]$. Thus, since

$$\bar{\psi}(a, b) = 1 - \frac{h(t \cdot a) - h(0)}{b},$$

we find that $|\bar{\psi}^2(a, b) - 1| = O(t)$ and $|\nabla \bar{\psi}| = O(t)$ uniformly for $(a, b) \in \mathcal{U}$.

To bound (127), note that

$$\mathrm{tr}(\bar{\psi}^2 \cdot A - I) = 2(\bar{\psi}^2 - 1) + \bar{\psi}^2 \cdot h'(t \cdot a)^2$$

and

$$\det(\bar{\psi}^2 \cdot A - I) = (\bar{\psi}^2 - 1)^2 - h'(t \cdot a)^2.$$

Hence $\mathrm{tr}(\bar{\psi}^2 \cdot A - I) = O(t)$ and $\det(\bar{\psi}^2 \cdot A - I) = O(t^2)$. It follows that the eigenvalues of $\bar{\psi}^2 \cdot A - I$ are $O(t)$. Therefore,

$$\int_{\mathcal{U}} \bar{\nabla} u \cdot (\bar{\psi}^2 \cdot A_t - I) \cdot \bar{\nabla} v \, da \, db = O(t) \cdot \int_{\mathcal{U}} \bar{\nabla} u \cdot \bar{\nabla} v \, da \, db.$$

To estimate (128) we first note that the eigenvalues of A_t are $O(1)$. Then we apply Cauchy-Schwarz

$$|\bar{\nabla} \bar{\psi} \cdot \bar{\nabla} u| \leq |\bar{\nabla} \bar{\psi}| \cdot |\bar{\nabla} u|,$$

and then the elementary estimate on $|\bar{\nabla} \bar{\psi}|$ to find that

$$\int_{\mathcal{U}} |\bar{\psi}| \cdot |v| \cdot |\bar{\nabla} \bar{\psi} \cdot \bar{\nabla} u| \, da \, db \leq O(t) \int_{\mathcal{U}} v \cdot |\bar{\nabla} u| \, da \, db.$$

Cauchy-Schwarz applied to the latter integral gives

$$\int_{\mathcal{U}} |v| \cdot |\bar{\nabla} u| \, da \, db \leq \left(\int_{\mathcal{U}} |v|^2 \, da \, db \right)^{\frac{1}{2}} \cdot \left(\int_{\mathcal{U}} |\bar{\nabla} u|^2 \, da \, db \right)^{\frac{1}{2}}.$$

From a Poincaré inquality—Lemma 16.4 below—we find that

$$\int_{\mathcal{U}} |v|^2 \, da \, db \leq \pi^2 \int_{\mathcal{U}} |\bar{\nabla} v|^2 \, da \, db.$$

In sum we find that the expression in (128) is bounded by $O(t) \cdot a_t(u)^{\frac{1}{2}} \cdot a_t(v)^{\frac{1}{2}}$. Switching the rôles of u and v, we obtain the same bound for the expression in (129).

To estimate (130) we use the fact that the norm of the eigenvalues of A_t are $O(1)$ and the fact that $|\bar{\nabla}\bar{\psi}|^2 = O(t^2)$ to find that

$$\int_{\mathcal{U}} |\nabla \bar{\psi} \cdot A_t \cdot \bar{\nabla}\bar{\psi}| \cdot |u| \cdot |v| \, da \, db = O(t) \cdot \int_{\mathcal{U}} |u| \cdot |v| \, da \, db.$$

By applying Cauchy-Schwarz and the Poincaré inequality of Lemma 16.4 below we obtain the claim. Condition (12) follows using the same kind of arguments. □

Lemma 16.4. *Any* $u \in \mathcal{C}_0^\infty(\mathcal{U})$ *satisfies*:

$$\int_{\mathcal{U}} |u|^2 \, da \, db \leq \pi^2 \int_{U} |\partial_a u|^2 \, da \, db.$$

Proof. We decompose $u = \sum_k u_k(b) \sin(k\pi a)$. Then we have

$$\begin{aligned} \int_U |\partial_a u|^2 &= \sum_k k^2 \pi^2 \int_{h(0)}^{\infty} u_k(b)^2 \, db \\ &\geq \pi^2 \sum_k \int_{h(0)}^{\infty} u_k(b)^2 db = \pi^2 \int_{\mathcal{U}} |u|^2 \, da \, db. \end{aligned}$$

□

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Appendix A. Solutions to the Airy Equation

Here we consider solutions to Airy's differential equation

$$A''(u) = u \cdot A(u) \tag{131}$$

for $u \in \mathbf{R}$. It is well-known that there exist unique solutions A_+ and A_- that satisfy[5]

$$A_{\pm}(u) = \frac{u^{-\frac{1}{4}}}{2^{\frac{1\pm 1}{2}}} \cdot \exp\left(\pm \frac{2}{3} \cdot u^{\frac{3}{2}} \right) \left(1 + O\left(u^{-\frac{3}{2}} \right) \right) \tag{132}$$

[5] The functions $\pi^{-\frac{1}{2}} \cdot A_{\pm}$ are the classical Airy functions Ai and Bi. See, for example, [Olver] Chap. 11.

and

$$A_{\pm}(-u) = u^{-\frac{1}{4}} \left(\cos\left(\frac{2}{3} \cdot u^{\frac{3}{2}} \mp \frac{\pi}{4} \right) + O\left(u^{-\frac{3}{2}} \right) \right), \tag{133}$$

where $u^{\frac{3}{2}} \cdot O(u^{-\frac{3}{2}})$ is bounded on $[1, \infty)$.

Let W denote the Wronskian of $\{A_+, A_-\}$. Define $K : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ by

$$K(u, v) = W^{-1} \cdot \begin{cases} A_+(u) \cdot A_-(v) & \text{if } v \geq u \geq 0 \text{ or } v \geq 0 \geq u \\ A_-(u) \cdot A_+(v) & \text{if } u \geq v \geq 0 \\ A_+(u) \cdot A_-(v) - A_-(u) \cdot A_+(v) & \text{if } u \leq v \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Lemma A.1. *Let $-\infty < \alpha \leq 0 \leq \beta \leq \infty$. For each locally integrable function $g : [\alpha, \beta] \to \mathbf{R}$ of at most polynomial growth, we have*

$$(\partial_u^2 - u) \int_{\alpha}^{\beta} K(u, v) \cdot g(v)\, dv = g(u), \tag{134}$$

Proof. The Wronskian W is constant and hence by, for example, variation of parameters we find that the function

$$P(u) = W^{-1} \cdot A_+(u) \int_u^{\beta} A_-(v) \cdot g(v)\, dv + W^{-1} \cdot A_-(u) \int_0^u A_+(v) \cdot g(v)\, dv$$

is a solution to $P''(u) - u \cdot P(u) = g(u)$. Hence K satisfies (134). □

Lemma A.2. *There exists a constant C_{Airy} so that*

$$|K(u, v)| \leq C_{\text{Airy}} \cdot \begin{cases} \exp(-|v - u|) & \text{if } u, v \geq 0 \\ |u \cdot v|^{-\frac{1}{4}} & \text{if } u \leq v \leq 0 \\ |u|^{-\frac{1}{4}} \cdot \exp(-v) & \text{if } u \leq 0 \leq v \end{cases} \tag{135}$$

and

$$|\partial_u K(u, v)| \leq C_{\text{Airy}} \cdot \begin{cases} \exp(-|v - u|) & \text{if } u, v \geq 0 \\ |u \cdot v|^{\frac{1}{4}} & \text{if } u \leq v \leq 0 \\ |u|^{\frac{1}{4}} \cdot \exp(-v) & \text{if } u \leq 0 \leq v. \end{cases} \tag{136}$$

Proof. Straightforward using definition of K and the asymptotic behavior of the Airy functions [Olver]. □

Lemma A.3. *There exists a constant C so that*

$$\int_{-\alpha}^{\alpha} \int_{-\alpha}^{\alpha} |K(u, v)|^2\, du\, dv \leq C \cdot \sqrt{\alpha}. \tag{137}$$

Proof. This follows directly from Lemma A.2. □

Lemma A.4. *Let $b^- < a^- < 0 < b^+ < a^+$. There exist constants C and s_0 such that if $s > s_0$ and A is a solution to (131), then*

$$\int_{s\cdot a^-}^{0} A^2\,du \le C\int_{s\cdot a^-}^{s\cdot b^-} A^2\,du, \tag{138}$$

and

$$\int_{0}^{s\cdot b^+} A^2\,du \le C\left(s^{-\frac{1}{2}}\int_{s\cdot b^-}^{s\cdot a^-} A^2\,du + \int_{s\cdot b^+}^{s\cdot 2b_+} A^2\,du\right). \tag{139}$$

The constants C and s_0 may be chosen to depend continuously on a^-, b^-, a^+, and b^+.

Proof. Let $0 < \alpha < \beta$. By using (133) and the identity $\cos^2(\xi) = 2^{-1}\cdot(1+\cos(2\xi))$, we have

$$\int_{-\beta}^{-\alpha} A_{\pm}^2\,du = \frac{1}{2}\int_{\alpha}^{\beta} u^{-\frac{1}{2}}\,du + \frac{1}{2}\int_{\alpha}^{\beta} u^{-\frac{1}{2}}\cdot\cos(2\xi)\,du + \int_{\alpha}^{\beta} O\left((1+u)^{-2}\right)du,$$

where $\xi = (2/3)\cdot u^{\frac{3}{2}} \mp \pi/4$. Integration by parts gives

$$\int_{\alpha}^{\beta} u^{-\frac{1}{2}}\cdot\cos(2\xi)\,du = \frac{1}{2}\cdot u^{-1}\cdot\sin(2\xi)\Big|_{\alpha}^{\beta} + \frac{1}{2}\int_{\alpha}^{\beta} u^{-2}\cdot\sin(2\xi)\,du,$$

and hence we have

$$\int_{-\beta}^{-\alpha} A_{\pm}^2\,du = \beta^{\frac{1}{2}} - \alpha^{\frac{1}{2}} + O\left(\beta^{-1}+\alpha^{-1}\right). \tag{140}$$

Since $A_{\pm}$ is bounded on $[-1,0]$ we also have

$$\int_{-\beta}^{0} A_{\pm}^2\,du = \beta^{\frac{1}{2}} + O(1). \tag{141}$$

Using (133) and the fact that $2\cos(\xi+\frac{\pi}{4})\cos(\xi-\frac{\pi}{4}) = \cos(2\xi)$, we find that for $0 < \alpha < \beta$, we have

$$\int_{-\beta}^{-\alpha} A_+\cdot A_-\,du = O\left(\beta^{-1}+\alpha^{-1}\right). \tag{142}$$

Since $A_{\pm}$ is bounded on $[-1,0]$, it follows that

$$\int_{-\beta}^{0} A_+\cdot A_-\,du = O(1). \tag{143}$$

We now specialize to the case $\alpha = -s\cdot a^-$ and $\beta = -s\cdot b^-$. By (140) and (141), there exists s_1—depending continuously on $b^- < a^- < 0$—such that for $s > s_1$,

$$\int_{s\cdot b^-}^{s\cdot a^-} A_{\pm}^2\,du \ge m\int_{s\cdot a^-}^{0} A_{\pm}^2\,du, \tag{144}$$

where

$$m = \frac{1}{2} \cdot \left(1 - \left(\frac{a^-}{b^-}\right)^{\frac{1}{2}}\right).$$

By (141) and (142), there exists a constant s_2—depending continuously on b^-, $a^- < 0$—such that if $s > s_2$, then

$$\left| \int_{s \cdot b^-}^{s \cdot a^-} A_+ \cdot A_- \, du \right| \leq \frac{m}{2} \cdot \int_{s \cdot a^-}^{0} A_\pm^2 \, du. \tag{145}$$

If A is a general solution to (131), then there exist $c_+, c_- \in \mathbf{R}$ such that

$$A = c_+ \cdot A_+ + c_- \cdot A_-.$$

Using (145) we find that

$$2|c_+ \cdot c_-| \cdot \left| \int_{s \cdot b^-}^{s \cdot a^-} A_+ \cdot A_- \, du \right| \leq \frac{m}{2} \cdot \left(c_+^2 \int_{s \cdot a^-}^{0} A_+^2 \, du + c_-^2 \int_{s \cdot a^-}^{0} A_-^2 \, du \right).$$

By combining this with (144) we find that if $s > \max\{s_1, s_2\}$, then

$$\int_{s \cdot b^-}^{s \cdot a^-} A^2 \, du \geq \frac{m}{4} \int_{s \cdot a^-}^{0} A^2 \, du. \tag{146}$$

This finishes the proof of the first estimate.

To prove the second estimate, first define $f(u) = \exp((2/3) \cdot u^{\frac{3}{2}})$ and let $0 < \alpha < \beta$. By using (132) and integrating by parts we find that, for β large,

$$\int_0^{\beta} A_+^2 \, du = \frac{1}{4} \cdot \beta^{-1} \cdot f(\beta) \cdot \left(1 + O(\beta^{-1})\right).$$

It follows that there exists s_3 so that for $s > s_3$,

$$\int_{s \cdot b^+}^{s \cdot 2b^+} A_+^2 \, du \geq \frac{1}{2} \cdot \int_0^{s \cdot b^+} A_+^2 \, du. \tag{147}$$

Equation (132) also implies that

$$\int_{\alpha}^{\beta} A_+ \cdot A_- \, du = \beta^{\frac{1}{2}} - \alpha^{\frac{1}{2}} + O\left(\beta^{-1} + \alpha^{-1}\right).$$

In particular, there exists $s_4 > 0$ so that if $s > s_4$, then

$$\int_{s \cdot b^+}^{s \cdot 2b^+} A_+ \cdot A_- \, du \geq 0. \tag{148}$$

By (132), the function A_-^2 is integrable on $[0, \infty)$. Let I be the value of this integral. Using (140) we find that there exists s_5 such that if $s > s_5$, then

$$\int_0^{s \cdot b^+} A_-^2 \, du \leq M \cdot s^{-\frac{1}{2}} \int_{s \cdot b_-}^{s \cdot a^-} A_-^2 \, du, \tag{149}$$

where $M = 2I/\left((b^-)^{\frac{1}{2}} - (a^-)^{\frac{1}{2}}\right)$. From (140) and (142) we find that there exists s_6 such that if $s > s_6$, then

$$\left|\int_{s\cdot b^-}^{s\cdot a^-} A_+ \cdot A_-\, du\right| \leq \frac{1}{2}\int_{s\cdot b^-}^{s\cdot a^-} A_\pm^2\, du. \tag{150}$$

Let $A = c_+ A_+ + c_- A_-$ be a general solution to the Airy equation. From (147) and (148) it follows that if $s > \max\{s_3, s_4\}$, then

$$c_+^2 \int_0^{s\cdot b^+} A_+^2\, du \leq 2\int_{s\cdot b^+}^{2s\cdot b^+} A^2\, du. \tag{151}$$

From (150) we have that if $s > s_6$, then

$$2|c_+ \cdot c_-| \cdot \left|\int_{s\cdot b^-}^{s\cdot a^-} A_+ \cdot A_-\, du\right| \leq \frac{1}{2} \cdot \left(c_+^2 \int_{s\cdot b^-}^{s\cdot a^-} A_+^2\, du + c_-^2 \int_{s\cdot b^-}^{s\cdot a^-} A_-^2\, du\right).$$

It follows that for $s > s_6$,

$$c_-^2 \int_{s\cdot b^-}^{s\cdot a^-} A_-^2\, du \leq 2\int_{s\cdot b^-}^{s\cdot a^-} A^2\, du.$$

Putting this together with (149) gives

$$c_-^2 \int_0^{s\cdot b^+} A_-^2\, du \leq 2M \cdot s^{-\frac{1}{2}} \int_{s\cdot b^-}^{s\cdot a^-} A^2\, du. \tag{152}$$

By combining (151) and (152) we find that

$$\frac{1}{2}\int_0^{s\cdot b^+} A^2\, du \leq 2M \cdot s^{-\frac{1}{2}} \int_{s\cdot b^-}^{s\cdot a^-} A^2\, du + 2\int_{s\cdot b^+}^{2s\cdot b^+} A^2\, du.$$

This completes the proof of the second estimate. □

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