

GENERIC SPECTRAL SIMPLICITY OF POLYGONS

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ABSTRACT. We study the Laplace operator with Dirichlet or Neumann boundary conditions on polygons in the Euclidean plane. We prove that almost every simply connected polygon with at least four vertices has a simple spectrum. We also address the more general case of geodesic polygons in a constant curvature space form.

1. INTRODUCTION

Let \mathcal{P}_n be the space of polygons in \mathbb{R}^2 having n vertices and boundary equal to a simple closed curve. By labeling the vertices, one obtains a local parametrization of \mathcal{P}_n by \mathbb{R}^{2n} . This defines an affine structure on \mathcal{P}_n as well as a notion of sets of measure zero.

Theorem 1.1. *If $n \geq 4$, then for almost every $P \in \mathcal{P}_n$, the Dirichlet (resp. Neumann) Euclidean Laplacian on P has one-dimensional eigenspaces.*

As of yet, we do not know whether or not the generic triangle has a simple spectrum. The methods of this paper show that either the generic triangle has a simple spectrum or no triangle has a simple spectrum.

We also generalize Theorem 1.1 to

Theorem 1.2. *Almost every simply connected geodesic polygon belonging to a hemisphere of the standard sphere has a simple Dirichlet (resp. Neumann) spectrum. Almost every simply connected geodesic polygon in the hyperbolic plane has a simple Dirichlet (resp. Neumann) spectrum.*

It is conjectured by number theorists that the geodesic triangle in the hyperbolic plane \mathbb{H}^2 with angles $\pi/2$, $\pi/6$, and 0 has a simple spectrum [Srn]. The (joint) spectrum of this triangle coincides with the spectrum of the hyperbolic surface $\mathbb{H}^2/\mathrm{SL}(2, \mathbb{Z})$. It is not known whether there exists a closed hyperbolic surface with a simple Laplace spectrum.

We note that K. Uhlenbeck [U], building upon earlier work of J. Albert [A], proved the simplicity of the Laplace spectrum for the generic compact domain in \mathbb{R}^d and the generic metric on a compact manifold. Such moduli spaces are infinite dimensional whereas \mathcal{P}_n is only finite dimensional.

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2. ANALYTIC BI-LIPSCHITZ BOUNDARY PERTURBATIONS

The proofs of Theorems 1.1 and 1.2 are based upon analytic perturbation theory. We first show that the Laplace spectrum of a real-analytic path P_t of polygons depends real-analytically on t . This will follow from

Lemma 2.1. *Let $\Omega_t, 0 \leq t \leq 1$, be a family of Lipschitz domains such that there exists a real-analytic family of homeomorphisms $f_t : \Omega_0 \rightarrow \Omega_t$ and a constant $C > 0$ such that*

$$(2.1) \quad C \cdot |x - y| \leq |f_t(x) - f_t(y)| \leq \frac{1}{C} \cdot |x - y|.$$

Then the eigenvalues of the Neumann (resp. Dirichlet) Laplacian, Δ_t , vary real-analytically in t .

*Proof.*¹ For each t_0 , there exists an interval neighborhood $I \ni t_0$ and Lipschitz functions $f^{(k)}$ such that $f_t(x) = \sum_k f^{(k)}(x)(t - t_0)^k$. By Rademacher's theorem, for each k , there exists a set of full measure $A_k \subset \Omega_0$ such that for each $x \in \Omega_0$, the differential $D_x f^{(k)}$ exists. Letting $A = \bigcap_k A_k$ we obtain a set of full measure such that $D_x f_t$ exists for all $x \in A$ and $t \in I$.

From (2.1) we have $C \leq |D_x f_t| \leq C^{-1}$ for $x \in A$. It follows that the pull-back operator $f_t^* : C^\infty(\Omega_t) \rightarrow C^\infty(\Omega_0)$ extends continuously in the H^1 norm to an isomorphism from $H^1(\Omega_t)$ to $H^1(\Omega_0)$. Also, it follows that the measure $(f_t)_*^{-1}(\mu)$ is absolutely continuous with respect to Lebesgue measure, and hence equals $h_t d\mu$ for some function $h_t : \Omega_0 \rightarrow \mathbb{R}$ that is positive almost everywhere.

For any $v \in H^1(\Omega_t)$, we then have that

$$(2.2) \quad \int_{\Omega_t} |v(X)|^2 d\mu(X) = \int_{\Omega_0} |v \circ f_t(x)|^2 h_t(x) d\mu(x),$$

$$(2.3) \quad \nabla_x(v \circ f_t) = D_x f_t \cdot [(\nabla_X v) \circ f_t].$$

It follows that the operator H_t defined by

$$H_t(u) = (u \circ f_t)$$

is unitary as a map from $L^2(\Omega_t, d\mu)$ onto $L^2(\Omega_0, h_t d\mu)$.

Let us consider first the Neumann eigenvalue problem: Δ_t is the self-adjoint operator associated with the quadratic form q_t defined on $H^1(\Omega_t)$ by

$$q_t(u) = \int_{\Omega_0} |\nabla_x u|^2 d\mu.$$

For each t , define the \tilde{q}_t on $H^1(\Omega_0)$ by $\tilde{q}_t = q_t \circ H_t^{-1}$. Using (2.2) and (2.3) we find that

$$\tilde{q}_t(u) = \int_{\Omega_0} |(D_x f_t)^{-1} \nabla_x u|^2 h_t d\mu.$$

Denote by $\tilde{\Delta}_t$ the associated self-adjoint operator (with respect to $L^2(\Omega_0, h_t d\mu)$). Since H_t defines a unitary equivalence, the operators $\tilde{\Delta}_t$ and Δ_t have the same Neumann spectrum.

Using the power series expansion for f_t in t , we find that, for each $u \in H^1(\Omega_0)$, the maps $t \mapsto \tilde{q}_t(u)$ and $t \mapsto \int_{\Omega_0} |u|^2 h_t d\mu$ are real-analytic. The spectrum of $\tilde{\Delta}_t$ is

¹In the case where f_t is a real-analytic family of smooth diffeomorphisms, this result is standard in the theory of *boundary perturbations*. See, for instance, [K, §VII.6.5]. Our proof will follow the same lines.

thus determined by a generalized eigenvalue problem in the sense of Kato (see [K, §VII.6]). The conclusion thus holds for the Neumann problem.

In the case of Dirichlet eigenvalues, Δ_t is the self-adjoint operator associated with q_t defined on $H_0^1(\Omega_t)$, the closure in $H^1(\Omega_t)$ of the smooth functions on Ω_t that vanish on the boundary of Ω_t . Since H_t maps $H_0^1(\Omega_t)$ onto $H_0^1(\Omega_0)$ unitarily, the same method of proof applies. \square

Corollary 2.2. *If for some t_0 , the spectrum of Δ_{t_0} is simple, then for all but countably many t , the spectrum of Δ_t is simple.*

Proof. Let $\lambda_n(t)$ be the real-analytic functions corresponding to the eigenvalues of Δ_t . Since the spectrum of Δ_{t_0} is simple, $\lambda_i(t_0) \neq \lambda_j(t_0)$ if $i \neq j$. Thus, since they are real-analytic, the set $\{t \mid \lambda_i(t) = \lambda_j(t)\}$ is at most countable for every $i \neq j$. A countable union of countable sets is countable. \square

In what follows, we will consider linear families of piecewise linear homeomorphisms. These are real-analytic families of uniformly bi-Lipschitz homeomorphisms that describe all deformations of polygons. Corollary 2.2 thus implies the following proposition.

Proposition 2.3. *If the Laplace spectrum of $P \in \mathcal{P}^n$ is simple, then there exists an open ball neighborhood \mathcal{B} of $P \in \mathcal{P}^n \subset \mathbb{R}^{2n}$ such that almost every polygon in $Q \in \mathcal{B}$ has a simple spectrum.*

Proof. Triangulate P so that the vertices of the triangulation that lie in ∂P are exactly the vertices of P . Let v_{n+1}, \dots, v_{n+k} denote the additional vertices. Let $\mathcal{B} \subset \mathbb{R}^{2n}$ be a ball neighborhood of $P = (v_1, v_2, \dots, v_n)$ such that for each $P' = (v'_1, v'_2, \dots, v'_n) \in \mathcal{B}$, we have that P' is a simply connected polygon and $v'_1, v'_2, \dots, v'_n, v_{n+1}, \dots, v_{n+k}$ are the vertices of a triangulation of P' . Using the triangulation, we can define a piecewise linear homeomorphism f such that $P' = f(P)$. By letting $f_t = (1 - t)\text{Id} + tf$ and $P_t = f_t(P)$, we obtain a linear family of piecewise linear homeomorphisms corresponding to the segment in \mathcal{P}^n that joins P to P' .

Thus, by Corollary 2.2, for all but countably many t , the spectrum of Δ_{P_t} is simple. In other words, for each radial line segment σ based on P and lying in \mathcal{B} , the set of $Q \in \sigma$ that do not have a simple spectrum is countable and hence this set has measure zero. The claim then follows from the use of polar coordinates based at P and Fubini's theorem. \square

3. PROOF OF THEOREM 1.1

Proposition 2.3 reduces the proof of Theorem 1.1 to verifying that for each $n \geq 4$, the set $\mathcal{P}^n \subset \mathbb{R}^{2n}$ is connected and contains at least one polygon P with a simple spectrum. We first show that for each $n \geq 4$ there exists an n -gon whose Laplace spectrum is simple.

Let R be a rectangle with sidelengths s_1 and s_2 . An $L^2(R)$ basis of Neumann (resp. Dirichlet) Laplace eigenfunctions of R is given by $\cos(\pi mx/s_1) \cdot \cos(\pi nx/s_2)$ (resp. $\sin(\pi mx/s_1) \cdot \sin(\pi nx/s_2)$), where m and n vary over the nonnegative integers (resp. positive integers). In particular, we have multiple eigenvalues iff there exist pairs of integers (m, n) and (\bar{m}, \bar{n}) so that

$$\frac{m^2}{s_1^2} + \frac{n^2}{s_2^2} = \frac{\bar{m}^2}{s_1^2} + \frac{\bar{n}^2}{s_2^2}.$$

Thus if $(s_1/s_2)^2 \notin \mathbb{Q}$, then the spectrum of the rectangle R is simple.

By adding $n - 4$ vertices to a side of a rectangle, one may regard any rectangle as an n -gon. The spectrum is unchanged by adding such ‘false’ vertices. Thus, for any $n \geq 4$, there exists an n -gon with a simple spectrum.

The (path) connectedness of \mathcal{P}^n can be verified by induction on the number of vertices.² The case of three vertices can be verified in many ways. For example, the connected Lie group consisting of affine homeomorphisms of \mathbb{R}^2 acts transitively and continuously on triples of points in \mathbb{R}^2 that are the vertices of a nondegenerate triangle.

Recall that an $(n+1)$ -gon may be regarded as an n -gon if three successive vertices lie on the same line segment. We will show that any $(n+1)$ -gon, $n \geq 3$, can be linearly deformed into an n -gon. Thus, the claim will follow from the induction hypothesis.

Any $(n+1)$ -gon $P = (v_1, \dots, v_{n+1})$ can be triangulated so that the vertices of the triangulation are exactly the vertices v_1, \dots, v_{n+1} of the polygon.³ Since P is simply connected and $n+1 \geq 4$, the dual graph of this triangulation is a tree with at least two vertices. Let T be a triangle corresponding to an end of the dual graph. Then there is a vertex v of T such that both of the sides of T adjacent to v are also sides of P . Without loss of generality, $v = v_1$.

Let m be the midpoint (for example) of the side of T that is opposite to v_i . Define

$$P_t = ((1-t) \cdot v_1 + t \cdot m, v_2, \dots, v_{n+1}).$$

Note that the vertex m lies in a segment joining two other vertices of $P_1 = (m, v_2, \dots, v_{n+1})$. Hence P_1 can be regarded as an n -gon.

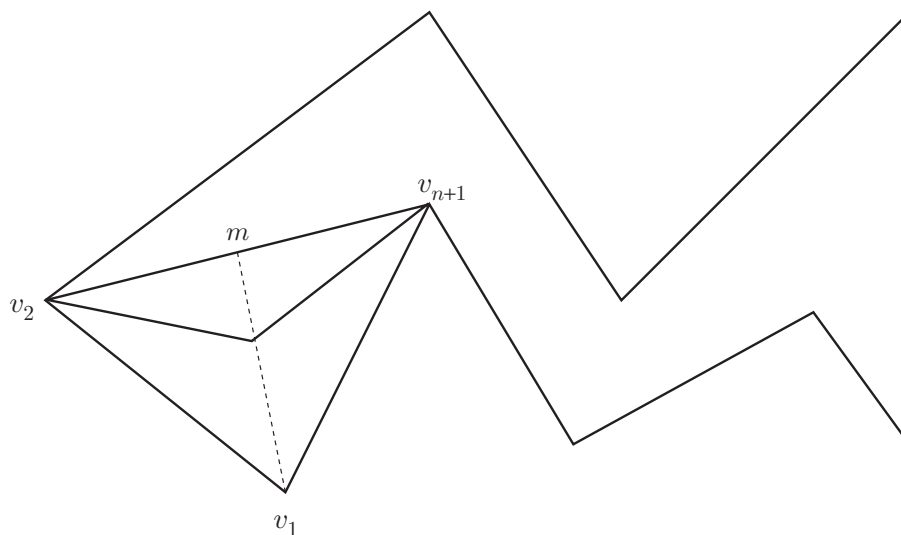


FIGURE 1. Deleting a vertex

²We remark that any star-shaped n -gon can be easily connected to a rectangle with $n - 4$ ‘false’ vertices.

³For example, the Delaunay triangulation of P has this property [Thr].

4. PROOF OF THEOREM 1.2

In this section we prove Theorem 1.2. The key is to construct, for each Euclidean polygon P_0 , a natural real-analytic family of polygons $\kappa \rightarrow P_\kappa$ such that P_κ is a geodesic polygon in the constant curvature κ space form M_κ . To make the construction transparent, we make a convenient choice of model for M_κ .

The following constructions are standard. (See for example [T].) For $R > 0$, let

$$\tilde{M}_R^\pm = \{(X, Y, Z) \mid X^2 + Y^2 \pm Z^2 = \pm R^2\},$$

and let M_R^\pm be the connected component of \tilde{M}_R^\pm that contains the point $(0, 0, R)$. The quadratic form $X^2 + Y^2 \pm Z^2$ defines a Riemannian metric of constant curvature $\pm R^{-2}$ on M_R^\pm . For example, M_1^- is isometric to the hyperbolic plane and M_1^+ is the unit sphere.

We construct a projective (Klein) model as follows: Given a point $P \in M_R^\pm$, there exists a unique pair (X, Y) such that (X, Y, R) lies in the line containing P and the origin. This defines an injective map $F_R^- : M_R^- \rightarrow \mathbb{R}^2$ and a two-to-one map $F_R^+ : M_R^+ \rightarrow \mathbb{R}^2$. Henceforth, we will restrict F_R^+ to the upper hemisphere of M_R^+ . By pulling the constant curvature metrics back by the diffeomorphism $(F_R^\pm)^{-1}$ and by setting $\kappa = \pm R^{-2}$, we obtain the following model for M_κ :

Proposition 4.1. *Let (ρ, θ) be polar coordinates on \mathbb{R}^2 and let*

$$(4.1) \quad g_\kappa = \frac{1}{(1 + \kappa \cdot \rho^2)^2} d\rho^2 + \frac{\rho^2}{1 + \kappa \cdot \rho^2} d\theta^2.$$

If $\kappa < 0$, then g_κ is a complete Riemannian metric of constant curvature κ on the Euclidean disc of radius $R = |\kappa|^{-\frac{1}{2}}$. If $\kappa \geq 0$, then g_κ is a complete Riemannian metric of constant curvature κ on the Euclidean plane.

Remark 4.2. Note that a path is geodesic with respect to g_κ if and only if it is a Euclidean line segment. (Indeed, each geodesic in M_R^\pm is the intersection of a plane through the origin and F_R^\pm is defined via radial projection.) In particular, given $\kappa, \kappa' \in \mathbb{R}$, a curve in \mathbb{R}^2 is a geodesic polygon with respect to g_κ if and only if it is a polygon with respect to $g_{\kappa'}$ and the obvious containment conditions are satisfied.

Let P be a Euclidean polygon belonging to the Euclidean ball $B(0, R)$. Let ∇_κ (resp. dv_κ) denote the gradient (resp. volume form) associated to g_κ . Define the quadratic form

$$q_\kappa(u) = \int_P |\nabla_\kappa u|^2 dv_\kappa.$$

Let Δ_κ denote the self-adjoint operator associated with q_κ defined either on $H_0^1(P)$ (Dirichlet boundary condition) or on $H^1(P)$ (Neumann boundary condition).

Proposition 4.3. *The eigenvalues of Δ_κ vary analytically for $\kappa \in (-R^{-2}, \infty)$.*

Proof. For each compact interval in $(-R^{-2}, \infty)$, uniform estimates on the coefficients of g_κ on P provide $C > 0$ such that $C^{-1}|\nabla_0 f|_0 \leq |\nabla_\kappa f|_\kappa \leq C|\nabla_0 f|_0$ and $C^{-1} \leq dv_\kappa/dv_0 \leq C$. It follows that the form domain of q_κ is independent of κ . Inspection of (4.1) shows that the metric g_κ depends analytically on $\kappa \in (-R^{-2}, \infty)$. It follows that both $q_\kappa(u)$ and the quadratic form $u \rightarrow \int |u|^2 dv_\kappa$ depend analytically on κ . Kato-Rellich theory thus applies, yielding the proposition. \square

Theorem 1.2 will follow from

Theorem 4.4. *For any κ , almost every simply connected, geodesic polygon in M_κ has a simple Neumann (resp. Dirichlet) spectrum.*

Proof. By Remark 4.2 we can identify the set of geodesic n -gons in M_κ with a set of Euclidean polygons. In particular, the set of all geodesic polygons in M_κ with $\kappa \leq 0$ can be identified with the set $U \subset \mathcal{P}_n \times (-\infty, 0]$ defined by

$$U = \left\{ (P, \kappa) \mid \kappa \leq 0, P \subset B\left(0, |\kappa|^{-\frac{1}{2}}\right) \right\}.$$

By Theorem 1.1 and Proposition 4.3, for almost every $P \in \mathcal{P}_n$, the set of κ such that $(P, \kappa) \in U$ has a simple spectrum is at most countable. (See the proof of Corollary 2.2.) It follows that, with respect to Lebesgue measure on $\mathcal{P}_n \times \mathbb{R}$, the spectrum of almost every $(P, \kappa) \in U$ is simple.

Given $\kappa \leq 0$, let A_κ be the set of $P \in \mathcal{P}_n$ such that $(P, \kappa) \in U$ does not have a simple spectrum. Then it follows from above that for almost every $\kappa \in \mathbb{R}$, the set A_κ has measure zero.

To conclude that the measure of A_κ equals zero for every κ , we use the fact that if a metric g is rescaled by a constant k , then the Laplacian is rescaled by k^{-1} . In particular, the multiplicity of each eigenvalue does not change if the metric is rescaled.

On the other hand, the curvature of the metric does change under rescaling if the initial curvature is nonzero. In particular, the metric $(\kappa/\kappa') \cdot g_\kappa$ on $B(0, |\kappa|^{-\frac{1}{2}})$ has curvature κ' and serves as another model for $M_{\kappa'}$. It follows that if A_κ has measure zero, then so does $A_{\kappa'}$.

The case of nonnegative curvature is proven similarly. \square

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