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Clustering of Eigenvalues on Translation Surfaces

Luc Hillairet

Abstract. We consider Riemannian surfaces in which there are many embedded Euclidean cylinders. We show that quantitative assumptions on the number of such cylinders provide estimates for the counting function of eigenvalues. These results in particular apply to translation surfaces.

1. Introduction

Quantum chaos aims at understanding the quantum version of classically chaotic dynamical systems. Among others, one interesting question to look at is the way eigenmodes concentrate. To be more precise, consider the geodesic flow in the cotangent bundle of some Riemannian manifold M; the associated quantum problem considers the laplacian Δ in $L^2(M)$. The eigenmodes ψ_n thus satisfy

$$\Delta\psi_n = E_n\psi_n.$$

To each (normalized) eigenmode one associates a probability measure μ_n on the unit cotangent bundle (cf. [20, 6]). The concentration of a subsequence ψ_{n_j} is related with the convergence of the associated measures and the properties of the limit. This concentration depends on the properties of the classical dynamical system. For instance the celebrated quantum ergodicity theorem (see [18, 20, 6]) states that if the classical billiard flow is ergodic then almost all sequence of eigenmodes equidistributes (i.e., the corresponding μ_n's converge to the Liouville measure). The question of knowing if this statement holds for any sequence is known as the quantum unique ergodicity conjecture and is the center of an active research. From the opposite point of view, in order to prove that a system is not quantum unique ergodic, one can look for a particular sequence of eigenmodes such that the limiting measure is supported on a negligible invariant subspace, for instance on one particular periodic orbit (cf. [7]). Such a phenomenom is related to what is called scarring in the physics literature [11, 2]. Systems where such a scarring effect is expected to happen in particular include those having *bouncing ball orbits*, i.e., periodic orbits bouncing back and forth between two parallel walls as, for instance, in the rectangular part of a stadium-shaped billiard. In this setting it is expected that there exist eigenmodes concentrating on this family of periodic orbits: the so-called *bouncing ball* modes (cf. [3]). Recently, in [9], Donnelly proved that, in this setting, a weak version of quantum unique ergodicity didn't hold and in [21] Zelditch observed that good upper estimates on the remainder for Weyl's law could be used to infirm quantum unique ergodicity from Donnelly's construction. A natural question was to ask if, for a system in which many cylinders existed, these results could be improved. In this paper, we will thus consider systems having many different families of bouncing ball orbits and prove that, in these systems, the clustering of eigenvalues can be estimated from below. The natural setting in which our results will apply is that of translation surfaces (see Section 2) that are intimately linked with the billiard in a rational polygon. Finally we would like to point out that the ideas underlying [2] are very close to ours, *i.e.*, counting carefully all the possible quasimodes in all the possible cylinders. The results are however different. In particular, we do not consider the same quasimodes: here we will first fix the energy and then construct quasimodes around this precise energy (which leads to the result on the clustering of eigenvalues) whereas in [2] the energy varies (which leads to the result on the density of bouncing ball modes).

Content and results

In order to state the main result, we introduce some definitions that will be recalled and detailed in Section 2. On a Riemannian surface M and for $\alpha < 1$, we let $R_\alpha(E)$ be the number of eigenvalues of the associated positive laplacian in the interval $[E-E^\alpha, E+E^\alpha)$. This quantity describes the clustering of eigenvalues at energy E, in the scale E^α.

The aim of this paper is to link R_α with geometrical properties of M, namely with the existence of "many" Euclidean cylinders embedded in M (cf. condition (C_γ) page 692). The main result of the paper will be Theorem 3 where we prove a spectral estimate of the following form (for some α and for E going to infinity):

$$R_\alpha(E) \geq E^{e(\alpha)}, \tag{1}$$

where the exponent $e(\alpha)$ depends on the number of embedded Euclidean cylinders (see Thm. 3).

For the principal class of examples we have in mind – i.e., the so-called translation surfaces (see Def. 2) – we then have the following

Theorem 1. *On any translation surface M the following holds:*

$$\forall \alpha \in (\tfrac{1}{3}, \tfrac{1}{2}),\ \forall \varepsilon > 0,$$
$$R_\alpha(E) \geq E^{2(\alpha - \frac{1}{3}) - \varepsilon},$$

for E large enough.

We remark here that this spectral estimate does not allow us to conclude that there exists a sequence of eigenmodes concentrating on a family of bouncing ball orbits (to do this we need a sharp *upper* bound for $R_\alpha(E)$ – cf. [21]). However the general problem of getting information about R_α is known to be quite difficult. For instance, on a torus, searching a spectral estimate of the form (1) is related to the problem of the remainder in the circle problem (in [1], the reader will find in particular a historical view of the circle problem). It is thus interesting to have such an estimate for translation surfaces since very little is known about the spectrum of such surfaces although they are very interesting objects from the dynamical viewpoint. We would also like to comment on the bounds on α. The upper bound $\frac{1}{2}$ is related to the general second term in Weyl's law so that our spectral estimate is not implied by it (see remark p. 692). In the generic completely integrable case, the second term in Weyl's law can be proved to be $O(E^{\frac{1}{3}})$, (see [5]), thus yielding the following spectral estimate:

$$\forall \alpha > \frac{1}{3},\ R_\alpha(E) \sim cE^\alpha.$$

This estimate is better than ours but it is interesting to note that the lower bound $\frac{1}{3}$ is linked with the complete integrability, and that the existence of a large number of cylinders allows us to get down to this bound.

The paper is organized as follows. In the first section, we will introduce the geometrical setting and precise the kind of systems we will be considering. We will discuss the assumptions we have to make on the number of cylinders and give examples where these assumptions are fulfilled. In the second section we will recall the well-known construction of quasimodes associated to a cylinder. The spectral estimate (1) will then follow from a careful counting of the quasimodes contributing at the energy E. Actually we will prove a first spectral estimate of the form (1) (see Thm. 2) of which Theorem 3 is a refinement. The difference between these two estimates lie in the fact that, for the first one, at each energy, we only take into account one cylinder whereas for the second estimate we consider all the relevant cylinders. In particular, for the second estimate, we will need a geometric lemma to control the overlap between quasimodes that are associated with different cylinders (see Sect. 3). All the results we prove still hold when the cylinders are only immersed and not necessarily embedded. The ideas are the same and the only difference lie in technical purely geometric lemmas that we have chosen to present in the last section.

2. Geometrical setting

Let M be a compact Riemannian surface; we denote by Δ its positive laplacian. Since the applications we have in mind include surfaces with conical singularities, we allow M to have a finite number of such singularities. In this case Δ is the Friedrichs extension of the usual laplacian outside the singularities.

The sequence $\lambda_0 = 0 < \lambda_1 \leq \cdots \leq \lambda_n \leq \cdots$ denotes the eigenvalues of Δ (each eigenvalue is repeated according to its multiplicity) and we will denote by ψ_n

the n-th normalized eigenfunction. We denote by $N_w(E)$ the counting function:

$$N_w(E) = \sharp\{\lambda_j < E\},$$

and by $R_\alpha(E)$ the number of eigenvalues in the energy interval $I_\alpha(E) = [E - E^\alpha, E + E^\alpha)$.

Remark. Weyl's law states that $N_w(E) \sim \frac{Area(M)}{4\pi} E$ and in usual settings the following estimate on the remainder term holds (cf. [12, 14]):

$$N_w(E) = \frac{Area(M)}{4\pi} E + O(E^{\frac{1}{2}}).$$

Thus, for any $\alpha > \frac{1}{2}$ we have $R_\alpha(E) \sim \frac{Area(M)}{2\pi} E^\alpha$. In particular there is always at least one eigenvalue in the interval $I_\alpha(E)$. More generally any improvement in the second term in Weyl's law (such as, for instance, in the completely integrable case) will yield a similar estimate on R_α (see the discussion after Thm. 1).

We call Euclidean cylinder the set $\mathcal{C} = \mathbb{R}/L\mathbb{Z}_x \times]0, h[_s$ equipped with the metric $ds^2 + dx^2$, h is its height and L its length. We will denote by $\mathcal{A}$ the area of the cylinder ($\mathcal{A} = Lh$). In all this paper we will consider surfaces M such that many cylinders are embedded in M. We thus introduce the following counting function:

$$N_c(T, \mathcal{A}_0) = \sharp\{\text{ emb. cyl. in } M \text{ such that } L < T \text{ and } \mathcal{A} > \mathcal{A}_0\,\} \tag{2}$$

The following condition gives a quantitative meaning to the fact that "many" cylinders are embedded in M.

Definition 1. *A surface M will satisfy condition (C_γ) if the following holds*

$$\exists \mathcal{A}_0,\, \gamma\,, c > 0 \mid N_c(T, \mathcal{A}_0) \geq cT^\gamma. \tag{3}$$

Remarks.

1. We will also define the counting function $\tilde{N}_c(T, \mathcal{A}_0)$ and condition $(\tilde{C}_\gamma)$ by replacing *embedded* by *immersed*.
2. The compactness of M is not contradictory with the assumption (C_γ). Each cylinder goes in a different direction. The compactness only implies that, in general, the cylinders will have some overlap (although each one is embedded).
3. If the condition (C_γ) is satisfied, we can consider a subset of the cylinders such that the following holds:

   $$N_c^*(T, \mathcal{A}_0) \sim cT^\gamma,$$

   (the $*$ indicates that we only consider some cylinders and not all of them). Since we will never have to use all the cylinders, we can replace the inequality in (3) by the equivalence. This implies that our estimate is probably not sharp.

Examples:

1. The simplest example is the torus equipped with the Euclidean metric. Any rational direction corresponds to a cylinder of same area as the torus itself so that counting cylinders amounts to counting rational directions. This is the well-known circle problem and we have:

$$N_c(T, \mathcal{A}_0) \sim cT^2.$$

In this case, the best known remainder for the circle problem (cf. [1]) implies a better estimate than the one we prove for any translation surface.

2. A translation surface is a surface with conical singularities, and equipped with an atlas (outside the singularities) such that the transition functions are given by translations (cf. for instance [13] for a more precise definition). A well-known construction due to Katok-Zemliakov associates such a translation surface to any rational Euclidean polygon. In [19] Vorobets proves the following estimate for any translation surface M (cf. Thm. 1.8. loc. cit.):

$$\exists\, c, C > 0 \mid$$
$$cT^2 \leq \frac{N_2(T)}{Area(M)} \leq N_1(T) \leq CT^2,$$

where $N_1(T)$ denotes the number of cylinders of length bounded by T and $N_2(T)$ the sum of the areas of these cylinders. This implies that for any translation surface, there exist two constants $\mathcal{A}_0$ and c_0 such that

$$N_c(T, \mathcal{A}_0) \geq c_0 T^2.$$

For a generic translation surface this result can be improved to

$$N_c(T, \mathcal{A}_0) \sim cT^2,$$

where the constant c depends on which Teichmüller stratum the translation surface belongs (cf. [10]). We will not need this sharp estimate, only the quadratic lower bound so that our results are valid for any translation surface.

3. We will now give one example with a different exponent γ in the condition (3). Using the terminology of [15], this example is obtained by *bubbling one handle* on a surface M_0, where M_0 is constructed as in [9], i.e., M_0 is a Riemannian surface containing one embedded flat cylinder that is smoothly glued along each boundary to some non-positively curved handles. On this surface we make a geodesic slit of length α in the cylindrical part. We then consider a torus and also make on it a geodesic slit of the same length α. We now glue these two surfaces along the slits. This creates a new surface M that has three handles two of which are non-positively curved. It also has two conical singularities of angle 4π. (See Fig. 1.)

We now look for embedded cylinders in M. These cylinders have to stay in the flat part. We assume that on M_0 the direction of the slit is parallel to the periodic direction of the cylinder. We also represent the torus by $[0, 1]_u \times [0, 1]_v$ where the opposite sides are identified and we assume that

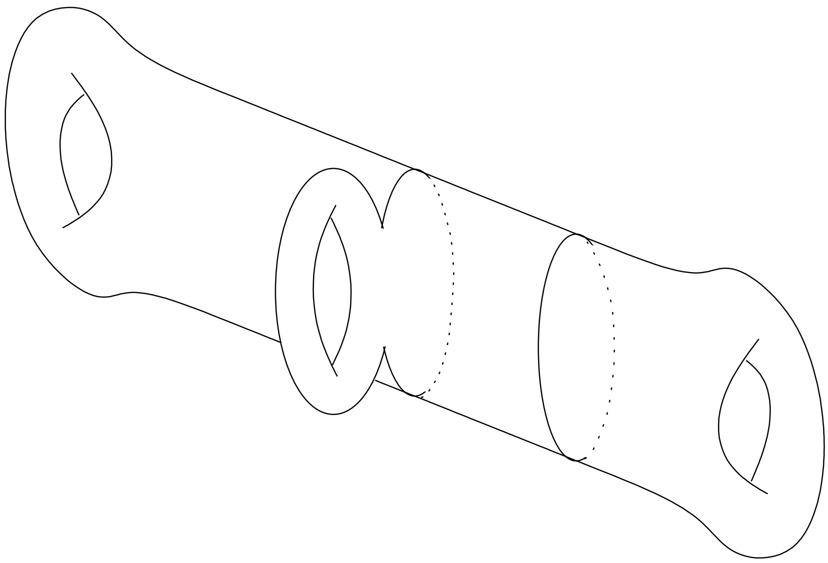

Figure 1. Bubbling a torus on M_0.

the slit corresponds to $\{0\}\times[0,\alpha]$. We now count the cylinders on this surface. There are two subcylinders of the original cylinder in M_0 that are separated by the periodic geodesic that contains the slit, and there are cylinders that are contained in the torus. These correspond to the rational directions on the torus in which there is a geodesic that avoids the slit. In any rational direction $(1,p)$, there is a cylinder of periodic geodesics that avoid the slit. This cylinder has an area of α. If $\alpha > \frac{1}{2}$ then, in any other rational direction, every periodic geodesic will cross the slit (and then pass onto M_0 and through at least one of the non-positively curved handles). This gives the following equivalent for the counting function of cylinders:

$$\exists\, \mathcal{A}_0 > 0 \mid N_c(T, \mathcal{A}_0) \sim cT,$$

i.e., this surface satisfies (C_γ) with $\gamma = 1$.

Remark. The surface M_0 can be chosen so that the geodesic flow is ergodic (cf. [9]), it would be very interesting to know if ergodicity still holds on the surface we have constructed.

2.1. Quasimode on a cylinder

We recall that a quasimode living at energy E is a function ϕ such that the following inequality holds:

$$\|(\Delta - E)\phi\| \le E^{\nu}\|\phi\|.$$

Usually (cf. [16]) one looks for quasimodes with ν largely negative (and $|\nu|$ is then called the order of the quasimode), this will not be the case here, and ν will remain positive.

Consider a cylinder of length L and of height h that is embedded in M, and take a smooth function χ supported in $(0,h)$ then the function

$$\phi_k(x,s) = \chi(s)\exp\left(\frac{2ik\pi x}{L}\right)$$

satisfy the following inequality:

$$\| \left(\Delta - \frac{4k^2\pi^2}{L^2}\right)\phi_k\| \leq C\|\phi_k\|. \tag{4}$$

In the following computations, several constants will occur. We will always denote them by the letters $c,\ C$. In particular the same letter will be used for different values as long as this constant does not depend on the relevant parameters. For instance in the preceding line, the constant C depends on χ but not on k. An explicit value for it is given by:

$$C = \|\chi''\|_{(0,h)}/\|\chi\|_{(0,h)}.$$

Remarks.

1. These quasimodes play a central role in the article [9].
2. From the explicit expression for C we can deduce that the best possible constant is given by the lowest eigenvalue of the Dirichlet problem in $(0,h)$.
3. The inequality (4) states that ϕ_k is a quasimode of order 0, and the results of [4] imply that it is not possible to do better if one looks for a sequence of quasimodes concentrating strictly in the interior of the cylinder.

Since we will consider quasimodes associated to different cylinders, it is important to tell how the constant C depends on the cylinder. This is done by choosing one function χ_1 smooth and supported in $(0,1)$ and by letting $\chi(s) = \chi_1(s/h)$ for a cylinder of height h. We now have:

$$\|(\Delta - \frac{4k^2\pi^2}{L^2})\phi_k\| \leq Ch^{-2}\|\phi_k\|, \tag{5}$$

with a constant C independent of the cylinder.

In order to localize this quasimode in the energy interval $[E-E^\alpha, E+E^\alpha)$, we first choose $\beta < \alpha$. Using the inequation (5) we can see that the two following conditions

$$|E - \frac{4k^2\pi^2}{L^2}| \leq \frac{1}{2}E^\beta \tag{6}$$

$$C_0h^{-2} \leq \frac{1}{2}E^\beta \tag{7}$$

imply

$$\| \left(\Delta - E\right)\phi_k\| \leq E^\beta\|\phi_k\|. \tag{8}$$

Condition (6) is equivalent to

$$\frac{L}{2\pi}(E-\frac{1}{2}E^{\beta})^{\frac{1}{2}} \leq k \leq \frac{L}{2\pi}(E+\frac{1}{2}E^{\beta})^{\frac{1}{2}},$$

and there exists some k satisfying this inequality provided that

$$\frac{L}{2\pi}\left[(E+\frac{1}{2}E^{\beta})^{\frac{1}{2}}-(E-\frac{1}{2}E^{\beta})^{\frac{1}{2}}\right] \geq 1.$$

We let $n_l(E)$ be the integer part of this quantity:

$$n_l(E) = \mathrm{Int}\left(\frac{L}{2\pi}\left[(E+\frac{1}{2}E^{\beta})^{\frac{1}{2}}-(E-\frac{1}{2}E^{\beta})^{\frac{1}{2}}\right]\right),$$

so that $n_l(E)$ corresponds to the number of integers n we can choose such that (6) is satisfied. When E goes to infinity we have the following equivalent:

$$n_l(E) \sim n_0(E) = \frac{L}{4\pi}E^{\beta-\frac{1}{2}}(1+O(LE^{\beta-\frac{1}{2}})).$$

Remark. In the preceding inequality and everywhere in the rest of the paper, the O's will always be universal, i.e., $O(E)$ means that there exists a universal constant c such that $|O(E)| \leq c\,E$.

We now address condition (7). It is equivalent to

$$h \geq cE^{-\frac{\beta}{2}}. \tag{9}$$

Under this assumption we can slice the cylinder into small cylinders of height $cE^{-\frac{\beta}{2}}$. Each of these small cylinders will satisfy condition (7). This will give us $n_h(E)$ cylinders where

$$n_h(E) = \mathrm{Int}\left[c^{-1}hE^{\frac{\beta}{2}}\right].$$

Eventually, the possibility of constructing at least one eigenmode localized in $\mathcal{C}$ and satisfying (8) is equivalent to the two following conditions:

$$n_l(E) \geq 1, \tag{10}$$

$$n_h(E) \geq 1. \tag{11}$$

Actually, counting all the quasimodes we have constructed we get the following proposition.

Proposition 1. *For any cylinder $\mathcal{C}$ and any energy E there exists $n_0(E)$ orthogonal quasimodes satisfying*

$$\|(\Delta - E)\phi\| \leq E^{\beta}\|\phi\|,$$

with

$$n_0(E) = c\mathcal{A}E^{\frac{3\beta-1}{2}}(1+O(LE^{\beta-\frac{1}{2}}))(1+O(hE^{\frac{\beta}{2}})),$$

where the constant c is universal.

Proof. We have seen that we could consider $\mathcal{C}$ as constituted of $n_h(E)$ small cylinders, and that we could take $n_l(E)$ values for k such that conditions (7), (6) were satisfied. This gives $n_0(E) = n_h(E) \times n_l(E)$ quasimodes satisfying condition (8). If we consider ϕ and ϕ' two such quasimodes, either ϕ and ϕ' are not located in the same small subcylinder or they are in the same but with different values of n; in any case they are orthogonal. □

The following corollary gives conditions on the cylinder so that we can get rid of the O's.

Corollary 1. *If there exists $\mathcal{A}_0 > 0$, such that, for all E there exists a cylinder $\mathcal{C}_E$ with L_E, h_E, $\mathcal{A}_E$ satisfying*

$$L_E E^{\beta-\frac{1}{2}} \underset{E\to\infty}{\longrightarrow} 0,$$
$$h_E E^{\frac{\beta}{2}} \underset{E\to\infty}{\longrightarrow} 0,$$
$$\mathcal{A}_E \geq \mathcal{A}_0,$$

then we have

$$\forall \varepsilon > 0,\ n_0(E) \geq E^{\frac{3\beta-1}{2}-\varepsilon}.$$

We denote by V_E the vector space generated by these quasimodes. In order to get an estimate on $R_\alpha(E)$ we need to project V_E on the eigenmodes.

2.2. Projection on the eigenmodes

Let $\alpha > \beta$, we denote by P_E the spectral projector on the energy interval $[E-E^\alpha, E+E^\alpha)$. For any of the quasimodes ϕ that we have constructed in the preceding section we let

$$u = P_E\phi\,;\ r = (1-P_E)\phi.$$

A straightforward computation (see for instance [9]) gives

$$\|r\| \leq E^{\beta-\alpha}\|\phi\|,\ \|u\| = \left(1+O(E^{2(\beta-\alpha)})\right)\|\phi\|.$$

We now consider two quasimodes ϕ and ϕ' associated to the cylinder $\mathcal{C}$. Since they are orthogonal, we get the following estimate on $\langle u, u'\rangle$:

$$|\langle u, u'\rangle| \leq \left(1+O(E^{2(\beta-\alpha)})\right) E^{2(\beta-\alpha)}\|u\|\|u'\|. \tag{12}$$

We recall the following elementary lemma

Lemma 1. *Let $\mathcal{H}$ be a Hilbert space and (v_i) a collection of N non-zero vectors such that*

$$\forall i \neq j,\ |\langle v_i, v_j\rangle| \leq \frac{1}{N}\|v_i\|\|v_j\|,$$

then the v_i's are linearly independent.

Remark. A slightly different version of this lemma is proved and used in [16] (see p. 362 loc. cit.)

We let $n_2(E) = E^{2(\alpha-\beta)}$, so that the inequality (12) can be rewritten:

$$\forall \varepsilon,\ |\langle u, u' \rangle| \leq \frac{E^{\varepsilon}}{n_2(E)} \|u\| \|u'\|.$$

A direct application of the lemma then shows:

$$\forall \varepsilon,\ R_\alpha(E) \geq \dim P_E(V_E) \geq E^{-\varepsilon} \min(n_1(E), n_2(E)), \tag{13}$$

where $n_1(E) = E^{\frac{3\beta-1}{2}}$.

To prove a spectral estimate of the kind (1), it then remains first to show that the geometric assumption on the number of cylinders implies that for E large enough there exists a cylinder in which we can construct the quasimodes, and then to optimize with respect to the parameters we have introduced. Following these lines, we get the theorem.

Theorem 2. *Let M be a Riemannian manifold satisfying (C_γ) then the following estimate holds:*

$$\forall \alpha \in (\tfrac{1}{2}, \tfrac{1}{3}),\ \forall \varepsilon > 0,\ \exists E_0 \mid$$
$$\forall E > E_0,\ n(E) \geq E^{\frac{6\alpha-2}{7} - \varepsilon}.$$

Proof. The first step is to show that condition (C_γ) ensures the existence of a cylinder such that the conditions of Corollary 1 are satisfied. We will consider cylinders of area bounded by below by $\mathcal{A}_0$ so that, since $h > \mathcal{A}_0 L^{-1}$, the lower bound on the height can be replaced by an upper bound on the length. If we can find δ_0 and δ_1 such that

$$\frac{1}{2} - \beta < \delta_0 < \delta_1 < \frac{\beta}{2},$$

then choosing

$$E^{\delta_0} \leq L_E \leq E^{\delta_1}$$

will give us a family of cylinders for which we can use Corollary 1. We can find δ_0 and δ_1 provided that $\beta > \frac{1}{3}$.

Condition (C_γ) now implies that the number of cylinders of area greater than $\mathcal{A}_0$ and of length between E^{δ_0} and E^{δ_1} grows like $E^{\gamma\delta_1}$ (we recall that in condition (C_γ) the inequality can be replaced by an equivalence). We choose one of these cylinders, the inequality on the δ's implies that, using Proposition 1 we have

$$\forall \varepsilon,\ n_0(E) \geq E^{\frac{3\beta-1}{2} - \varepsilon},$$

so that we can use inequality (13).

The question is now to optimize $\min[n_1(E), n_2(E)]$ with respect to β. It amounts to compare: $2(\alpha - \beta)$ and $\frac{3\beta-1}{2}$. At α fixed, the critical exponent is obtained when $2(\alpha - \beta) = \frac{3\beta-1}{2}$. This gives $\frac{6\alpha-2}{7}$ for the critical exponent. For each

α and ε, we can thus find some β so that when we compute $n_1(E)$ and $n_2(E)$ for these values of α and β we have

$$\min[n_1(E), n_2(E)] > E^{\frac{6\alpha-2}{7}-\varepsilon},$$

and Lemma 1 gives the result. □

The following corollary is straightforward.

Corollary 2. *Under the assumption of the preceding theorem, for any $\alpha > \frac{1}{3}$, and any E large enough:*

$$R_\alpha(E) > 0.$$

Remark. The lower estimate is true for any E large enough. There is no averaging procedure over the energy interval.

At a fixed energy, we only need one cylinder satisfying conditions (10) and (11) to derive the theorem. this in particular explains that the exponent in the spectral estimate doesn't depend on γ. Using the fact that actually many cylinders satisfy simultaneously these conditions. We can hope that the estimate can be sharpened. This is the aim of the following section.

3. Several cylinders

We consider ϕ_1 and ϕ_2 two quasimodes associated with cylinders $\mathcal{C}_1$ and $\mathcal{C}_2$ satisfying conditions (10) and (11). Looking at the arguments we have used in the former section we can see that the only one that doesn't hold anymore is the orthogonality of the quasimodes ϕ_1 and ϕ_2. To overcome this problem, we need to estimate the scalar product $\langle\phi_1, \phi_2\rangle$, when ϕ_i is localized in $\mathcal{C}_i$. Heuristically, since ϕ_1 and ϕ_2 are localized in different regions of the cotangent bundle, we expect $\langle\phi_1, \phi_2\rangle$ to got to 0 when the energy grows. What we need is to estimate this decay.

3.1. Estimating the scalar product

We let

$$K = \langle\phi_1, \phi_2\rangle.$$

We let (x_i, s_i) be coordinates on $\mathcal{C}_i$ so that we have:

$$K = \int_{\mathcal{C}_1} \chi_{h_1}(s_1)\chi_{h_2}(s_2) \exp\left[\frac{2i\pi k_1}{L_1}x_1 - \frac{2i\pi k_2}{L_2}x_2\right] dx_1 ds_1. \tag{14}$$

Locally near an intersection of the two cylinders we have:

$$\begin{cases} x_2 = \cos\varepsilon\, x_1 + \sin\varepsilon\, s_1 \\ s_2 = -\sin\varepsilon\, x_1 + \cos\varepsilon\, s_1, \end{cases}$$

where ε is the angle between the two cylinders (see Fig. 2).

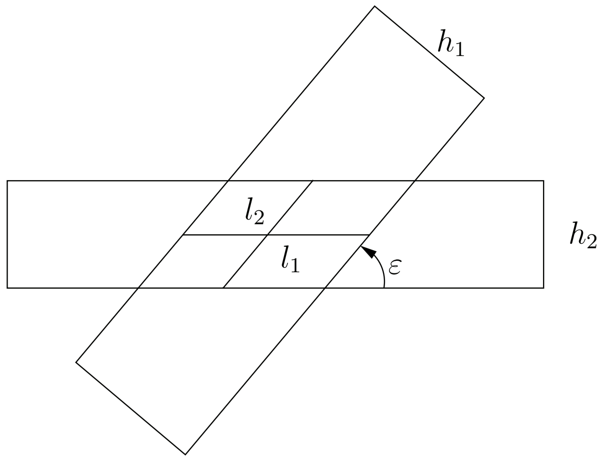

FIGURE 2. Intersection of two cylinders.

Performing N integrations by parts w.r.t. s_1 in (14) we get

$$|K| \leq \left(\frac{L_2}{2\pi k_2 \sin\varepsilon}\right)^N \int_{\mathcal{C}_1} \left| \frac{d^N}{ds_1^N} \left[\chi_{h_1}(s_1)\chi_{h_2}(s_2)\right] \right| ds_1 dx_1$$

$$|K| \leq C_N \left(\frac{L_2}{2\pi h k_2 \sin\varepsilon}\right)^N ,$$

where $h = \min(h_1, h_2)$.

We thus need a lower bound on ε in order to control K. Looking at Fig. 2 and since $\mathcal{C}_1$ is embedded in M it is clear that $l_1 \leq L_2$ and $l_2 \leq L_1$ so that:

$$\frac{1}{\sin\varepsilon} \leq \frac{L_1}{h_2} \leq \frac{L_1 L_2}{\mathcal{A}_0}. \tag{15}$$

Consequently we have:

$$|K| \leq C_N \left(\frac{L_1 L_2^2}{k_2 \mathcal{A}_0 h}\right)^N . \tag{16}$$

Remarks.

1. We remind the reader that the ϕ_i's are actually localized in subcylinders of $\mathcal{C}_{1,2}$. In estimating K, h must the height of the subcylinder (i.e., roughly $E^{-\beta/2}$), whereas in the bound on ε the height is that of the whole cylinder (h_1 or h_2). We also remark that $\mathcal{C}_1$ and $\mathcal{C}_2$ may have multiple intersections but, since the cylinders are embedded, the parallelograms (see Fig. 2) corresponding to each intersection are disjoint so that the estimate still holds.
2. When the cylinders are only immersed the inequality (15) still holds (see Lemma 2 in Section 4) but the proof is, of course, different.

3. This inequality is closely related to the *separation principle* of Delman-Galperin (cf. [8]). In particular using it, we can bound the numbers of cylinders $N(h_0)$ such that $h \geq h_0 > 0$ which is exactly the problem they consider. The same argument as theirs also shows the following proposition.

   Proposition 1. *Let M be a surface satisfying (C_γ) then $\gamma \leq 2$.*

   Proof. We consider the cylinders of length less than T and area greater than $\mathcal{A}_0$. The height of such a cylinder is thus greater than $\mathcal{A}_0/T$. Using inequality (15), we see that the angle ε of intersection between a cylinder of length L and any other such cylinder is bounded by below by $\varepsilon_L = \frac{\mathcal{A}_0}{TL}$. For each cylinder, we denote by γ the geodesic located in the middle of the cylinder and we denote by $\mathcal{V}(\mathcal{C})$ the following set of the unit tangent bundle:

$$\mathcal{V}(\mathcal{C}) = \{\, (x,\theta) \mid \exists t \mid |x - x(t)| < \frac{h}{2},\ |\theta - \theta(t)| < \frac{\varepsilon_L}{2}\,\},$$

   where $(x(t);\theta(t))$ parametrizes $(\gamma,\dot{\gamma})$ by arclength. These sets have to be disjoint so that computing their volume we get the following inequality:

$$2\pi\frac{\mathcal{A}_0^2}{T^2}\sum_{\mathcal{C}} 1 \leq \sum_{\mathcal{C}} 2\pi\varepsilon_L hL \leq 2\pi\, Area(M),$$

   so that we have:

$$N_c(T,\mathcal{A}_0) \leq \frac{Area(M)}{\mathcal{A}_0^2}T^2,$$

   and thus $\gamma \leq 2$. □

   We now need to count which quasimodes will contribute at the energy E.

3.2. Admissible quasimodes

The first thing to know is in which cylinders we can construct quasimodes contributing at the energy E. The condition on the length (10) implies

$$E^{\frac{1}{2}-\beta} \leq L.$$

Since we consider cylinders with area greater than $\mathcal{A}_0$ the condition on the height is ensured by

$$\mathcal{A}_0 L^{-1} > cE^{-\frac{\beta}{2}}.$$

We find again the a priori limitation $\beta > \frac{1}{3}$ since we must have:

$$\frac{1}{2} - \beta \leq \frac{\beta}{2} \implies \beta \geq \frac{1}{3}.$$

In the derivation of condition (10) from condition (6) we have shown that for any of the quasimodes we have constructed, we have $k = LE^{\frac{1}{2}}\left(1 + O(E^{\beta-\frac{1}{2}})\right)$. Furthermore, the height h occurring in (16) is exactly $E^{-\frac{\beta}{2}}$ since we have cut the admissible cylinders into small ones having this precise height. Consequently, the estimate (16) on the scalar product becomes:

$$|K| \leq C_N \mathcal{A}_0^{-N} \left(L_1 L_2 E^{\frac{\beta}{2}-\frac{1}{2}} \right)^N .$$

Projecting onto the energy interval we get the following estimate:

Proposition 2. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two cylinders satisfying conditions* (10) *and* (11)*, let ϕ_1 and ϕ_2 be two associated quasimodes and u_1 and u_2 their projection on the energy interval $[E-E^\alpha, E+E^\alpha[$ then, the following estimate holds:*

$$\frac{|\langle u_1, u_2\rangle|}{\|u_1\|\|u_2\|} \leq \left[|K|(L_1L_2)^{-\frac{1}{2}} E^{\frac{\beta}{2}} + E^{2(\beta-\alpha)} \right] (1 + O(E^{2(\beta-\alpha)})).$$

Proof. Using the estimate on the projection (see Sect. 2.2) we have:

$$|\langle u_1, u_2\rangle| \leq |K| + |\langle r_1, r_2\rangle|.$$

Since $\|u_i\| = \|\phi_i\|(1+O(E^{2(\beta-\alpha)}))$, and $\|r_i\| = \|\phi_i\|O(E^{\beta-\alpha})$, this gives

$$\frac{|\langle u_1, u_2\rangle|}{\|u_1\|\|u_2\|} \leq \frac{|K|}{\|\phi_1\|\|\phi_2\|}(1+O(E^{2(\beta-\alpha)})) + E^{2(\beta-\alpha)}(1+O(E^{2(\beta-\alpha)})),$$

which implies the result using

$$\|\phi_i\|^2 = CL_ih_i = CL_iE^{-\frac{\beta}{2}}.$$ □

We consider all the cylinders such that $E^{\delta_0} \leq L \leq E^{\delta_1}$, for some δ's to be specified later. We construct all the associated quasimodes and project them on the energy interval. Since each cylinder brings $E^{\frac{3\beta-1}{2}-\varepsilon}$ quasimodes, using condition (C_γ), this gives $n_0(E)$ vectors u with

$$n_0(E) \geq E^{\frac{3\beta-1}{2}+2\delta_1\gamma-\varepsilon}.$$

For any two such u's, the preceding proposition implies

$$\frac{|\langle u_1, u_2\rangle|}{\|u_1\|\|u_2\|} \leq \left(C_N E^{N(2\delta_1+\frac{\beta}{2}-\frac{1}{2})} E^{\frac{\beta}{2}-\delta_0} + E^{2(\beta-\alpha)} \right) \left(1 + O(E^{2(\beta-\alpha)}) \right).$$

If $2\delta_1 + \frac{\beta}{2} - \frac{1}{2} < 0$ we can then choose N so that, for any u_1 and u_2 we have:

$$\frac{|\langle u_1, u_2\rangle|}{\|u_1\|\|u_2\|} \leq \frac{E^\varepsilon}{n_2(E)},$$

with $n_2(E) = E^{2(\alpha-\beta)}$.

Finally, if we let α, β and δ satisfy

$$\begin{aligned} &\frac{1}{3} < \beta < \alpha < \frac{1}{2}, \\ &\frac{1}{2} - \beta < \delta < \frac{\beta}{2}, \\ &2\delta + \frac{\beta}{2} - \frac{1}{2} < 0, \end{aligned} \tag{17}$$

then, choosing $\delta_1 = \delta$ and δ_0 such that $\frac{1}{2} - \beta < \delta_0 < \delta$, the preceding construction is possible and we have as in (13)

$$R_\alpha(E) \geq \dim P_E(V_E) \geq E^{-\varepsilon} \min(n_1(E), n_2(E)),$$

but now with

$$n_1(E) = E^{\frac{3\beta-1}{2}+2\delta\gamma} \text{ and } n_2(E) = E^{2(\alpha-\beta)}.$$

It remains to choose the optimal parameters.

3.3. Optimization of the parameters

We fix α and γ and the problem is to compare $\frac{3\beta-1}{2} + \delta\gamma$ and $2(\alpha - \beta)$ under the following set of constraints:

$$\begin{cases} \frac{1}{3} < \beta < \alpha < \frac{1}{2}, \\ \frac{1}{2} - \beta < \delta < \frac{\beta}{2}, \\ 2\delta + \frac{\beta}{2} - \frac{1}{2} < 0. \end{cases} \tag{18}$$

In the (β, δ) plane the set of parameters defines a triangle $\mathcal{T}_\alpha$ bounded by the following straight lines:

- D_1 defined by $2\delta + \frac{\beta}{2} - \frac{1}{2} = 0$,
- D_2 defined by $\delta = \frac{1}{2} - \beta$,
- and the vertical line $\beta = \alpha$.

We denote by $e_\gamma(\alpha)$ the maximum on $\mathcal{T}_\alpha$ of $\min\left(\frac{3\beta-1}{2} + \gamma\delta,\, 2(\alpha - \beta)\right)$. This critical exponent is determined by the relative position of $\mathcal{T}_\alpha$ and of the straight line D_α defined by $\frac{3\beta-1}{2} + \gamma\delta = 2(\alpha - \beta)$ or equivalently by:

$$D_\alpha \;:\; \delta = -\frac{7\beta}{2\gamma} + \frac{4\alpha+1}{2\gamma}.$$

Denote by $(\beta(\gamma), \delta(\gamma))$ the intersection of D_α with D_1. Since the slope of D_α is less than the slope of D_1, D_α intersects $\mathcal{T}_\alpha$ if and only if $\frac{1}{3} \leq \beta(\gamma) \leq \alpha$. Moreover, the minimum we search is $2(\alpha - \beta)$ on the right of $\mathcal{D}_\alpha$ and $\frac{3\beta-1}{2} + \gamma\delta$ on the left. We thus have the following discussion:

- If $\beta(\gamma) \leq \frac{1}{3}$, then $e_\gamma(\alpha)$ is obtained by maximizing $2(\alpha - \beta)$ on $\mathcal{T}_\alpha$ and thus, $e_\gamma(\alpha) = 2(\alpha - \frac{1}{3})$.
- If $\beta(\gamma) \geq \alpha$, then $e_\gamma(\alpha)$ is obtained by maximizing $\frac{3\beta-1}{2} + 2\delta$ on $\mathcal{T}_\alpha$ and thus, $e_\gamma(\alpha) = 2\alpha - 1$.
- In the last case ($\frac{1}{3} \leq \beta(\gamma) \leq \alpha$), the straight line $\mathcal{D}_\alpha$ divides $\mathcal{T}_\alpha$ in two and $e_\gamma(\alpha) = 2(\alpha - \beta(\gamma))$.

This leads to the following theorem.

Theorem 3. *Let M be a Riemannian surface satisfying the condition (C_γ) for some γ $(0 < \gamma \leq 2)$ then the following estimate holds*

$$\forall\, \alpha \in (\tfrac{1}{3}, \tfrac{1}{2}),\ \forall\, \varepsilon > 0\ \exists\, E_0,\ |$$
$$\forall\, E > E_0\ R_\alpha(E) \geq E^{e_\gamma(\alpha)-\varepsilon},$$

where the critical exponent $e_\gamma(\alpha)$ *is given by*

1. *If* $\gamma = 2$ *then, on* $(\frac{1}{3}, \frac{1}{2})$, $e_\gamma(\alpha) = 2(\alpha - \frac{1}{3})$.
2. *If* $0 < \gamma < 2$ *then on* $(\frac{1}{3}, \frac{1}{2})$ *we have*

$$e_\gamma(\alpha) = \begin{cases} \frac{2((6-\gamma)\alpha+\gamma-2)}{14-\gamma} & \textit{for } \alpha > \frac{1}{3} + \frac{\gamma}{12}, \\ 2(\alpha - \frac{1}{3}) & \textit{for } \alpha \leq \frac{1}{3} + \frac{\gamma}{12}. \end{cases}$$

Proof. We analyze the discussion preceding the theorem by fixing first γ. Looking for the intersection of D_α and D_1 we find

$$\beta(\gamma) = \frac{8\alpha + 2 - \gamma}{14 - \gamma},$$

so that, since $\gamma \leq 2$, we always have $\beta(\gamma) < \alpha$ and $\beta(\gamma) > \frac{1}{3}$ is equivalent to $\alpha > \frac{1}{3} + \frac{\gamma}{12}$. This gives the exponent $e_\gamma(\alpha)$. We now fix α and ε. By definition of $e_\gamma(\alpha)$, we can choose β and δ satisfying the system of conditions (18) and such that $\min(\frac{3\beta-1}{2} + \gamma\delta, 2(\alpha - \beta)) > e_\gamma(\alpha) - \frac{\varepsilon}{2}$. We can also choose $\beta - \frac{1}{2} < \delta_0 < \delta$. We now consider all the cylinders of length bounded by E^{δ_0} and E^{δ}. This gives $n_1(E)$ quasimodes where $n_1(E) \geq E^{e_\gamma(\alpha)-\varepsilon}$. Since β and δ satisfy condition (17) we can find N such that the scalar product of two $u's$ corresponding to these quasimodes are bounded by $(E^{e_\gamma(\alpha)-\varepsilon})^{-1}$. This ends the proof, using Lemma 1. □

We can now apply this theorem to our two examples, which gives the following results.

- Translation surfaces: We have $\gamma = 2$ so that $e_\gamma(\alpha) = 2(\alpha - \frac{1}{3})$. This is the theorem that is cited in the introduction.
- Example 3: here $\gamma = 1$ so that the value of the critical exponent changes at $\alpha = \frac{5}{12}$ and we get:

$$\forall \alpha \in (\frac{1}{3}, \frac{5}{12}),\ R_\alpha(E) \geq E^{2(\alpha - \frac{1}{3}) - \varepsilon}$$
$$\forall \alpha \in (\frac{5}{12}, \frac{1}{2}),\ R_\alpha(E) \geq E^{\frac{10\alpha-2}{13} - \varepsilon}$$

4. Generalizations

The method we have presented can be generalized to slightly different settings. The main generalization we have in mind is the case of immersed cylinders. From immersed cylinders we will then be able to treat settings with boundary such as polygonal billiards.

Remark. The case of translation surfaces was completely treated in the latter section since on such a surface any cylinder is necessarily embedded (because the conical points have angle $2k\pi$). The natural setting for this section is thus a general Euclidean surface with conical singularities (for which the assumption $\tilde{C}_\gamma$ is then to be proved).

We begin by considering immersed cylinders: for each cylinder $\mathcal{C}$, there is a smooth mapping p from $\mathcal{C}$ into M that is locally an isometry. We then define ϕ as usual on $\mathcal{C}$ and transport it on M using p. This gives

$$\phi(m) = \sum_{p(s,x)=m} \chi_h(s) \exp(2ik\pi x/L).$$

As before, we have to control the norm of ϕ and the scalar products $K = \langle \phi_1, \phi_2 \rangle$, for ϕ_1 and ϕ_2 localized in the cylinders $\mathcal{C}_{1,2}$.

We let p_i be the local isometries form $\mathcal{C}_i$ into M. We lift p_i to $\mathbb{R}\times\big]-\frac{h_i}{2}, \frac{h_i}{2}\big[$ and denote by γ_i the image of $\mathbb{R} \times \{0\}$. Let m be such that $m = p_1(x_1, s_1) = p_2(x_2, s_2)$ we let ε be the angle between $\frac{\partial}{\partial_{x_1}}$ at (x_1, s_1) and $\frac{\partial}{\partial_{x_2}}$ at (x_2, s_2). We then have a parallelogram $\mathcal{L}$ on M such that the transition function $p_1 \circ p_2^{-1}$ is given by the following system (see Figs. 3 and 2):

$$\begin{cases} x_1 = \cos\varepsilon\, x_2 - \sin\varepsilon\, s_2 \\ s_1 = \sin\varepsilon\, x_2 + \cos\varepsilon\, s_2. \end{cases} \tag{19}$$

The scalar product K is then the sum on all such parallelograms of an integral K_ε which has exactly the same expression as (14). Each K_ε will thus be

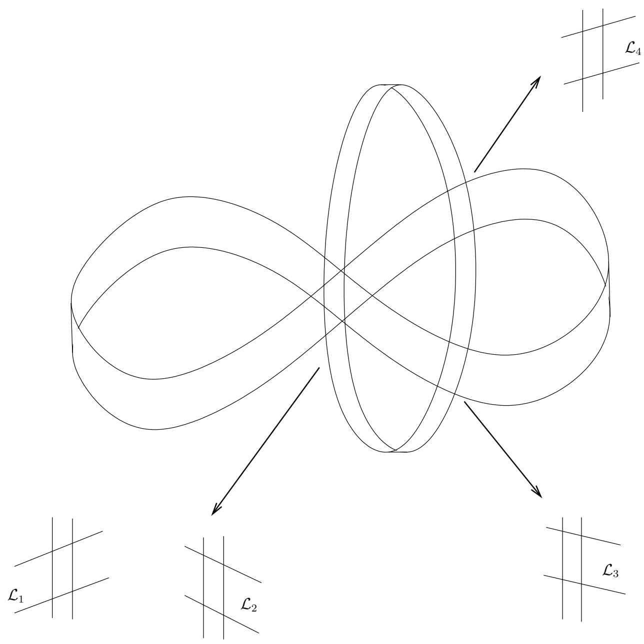

FIGURE 3. Two immersed cylinders.

bounded provided that we can estimate ε from below. To bound K it will then be enough to give an upper bound on the number of parallelograms $\mathcal{L}$.

The lower bound on ε will actually be the same as in the case of embedded cylinders. We give here a quick proof although it is not new (cf. [17], and the separation principle of [8]).

Lemma 2. *Using the preceding notations, and assuming that M isn't a torus, then we have*

$$\frac{1}{\sin\varepsilon} \leq \frac{\max(L_1, L_2)}{\max(h_1, h_2)}.$$

Proof. We consider, in the parallelogram $\mathcal{L}$, the segment representing γ_1 (or more precisely the portion of γ_1 parametrized by a neighbourhood of x_1) (see Fig. 2). This segment has length $\frac{h_2}{\sin\varepsilon}$ and we claim that it cannot project onto the whole geodesic γ_1. Otherwise, using that M is connected, a subcylinder of $\mathcal{C}_2$ would project onto M which is absurd if M isn't a torus. For the same reason, the trace of γ_2 in $\mathcal{L}$ cannot project onto the whole geodesic γ_2. □

Remark. This lemma can in particular be applied to self intersections of a cylinder so that it gives a lower bound of all the angles of self-intersections of a given geodesic in terms of the dimensions of the flat strip that surrounds it.

The term K_ε can thus be estimated as before and we now have to bound the number of parallelograms $\mathcal{L}$. Using compactness, it is clear that there is only a finite number of such parallelograms, however, we need a quantitative statement that will allow us to control K when we will let E go to infinity. Actually, we will look for uniform δ_v and δ_h such that, for any x, p is injective in the rectangle $]x-\delta_h, x+\delta_h[\times]-\delta_v, \delta_v[$. Furthermore we need to control these quantities when the cylinder varies.

We will need two geometric lemmas. The first one looks at the self intersections of γ (which, we recall, is the geodesic in the middle of the cylinder).

Lemma 3. *Let $\mathcal{C}$ be an immersed cylinder of length L and height h. Let p the associated local isometry defined from $\mathbb{R} \times \left]-\frac{h}{2}, \frac{h}{2}\right[$ into M and γ the geodesic corresponding to $s = 0$. If x_1 and x_2 are such that $p(x_1, 0) = p(x_2, 0)$ then:*

$$|x_1 - x_2| \geq \frac{h^2}{2L}.$$

Proof. We begin by choosing x_1 and x_2 that realize the minimum of $|x_1 - x_2|$, and we consider the segment $[x_1, x_2] \times \{0\}$ in the strip $\mathcal{S} = \mathbb{R} \times \left]-\frac{h}{2}, \frac{h}{2}\right[$. Near x_1, we lift the geodesic γ in two segments making the angle ε (corresponding respectively to neighbourhoods of x_1 and x_2). The lifts of γ near x_2 make the same angle ε. In $\mathbb{R}^2$ we denote by O the center of the rotation that maps the lifts of γ near x_1 to the lifts near x_2 (see Fig. 4). The segments joining O to $(x_1, 0)$ and O to $(x_2, 0)$ are mapped via p on the same geodesic of M so that O cannot be in the strip $\mathcal{S}$. Since $\alpha = \frac{\pi-\varepsilon}{2}$ (see Fig. 4) and $\varepsilon \geq \frac{h}{L}$ by the preceding lemma, this gives the bound we search. □

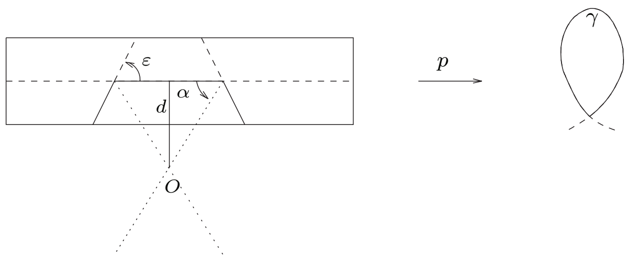

FIGURE 4. Self-intersection.

The next lemma address the injectivity in a small strip around γ.

Lemma 4. *Let* (s_1, x_1) *and* (s_2, x_2) *be two different points such that* $p(s_1, x_1) = p(s_2, x_2)$ *then if* $\max(|s_1|, |s_2|) \leq \frac{h^3}{16L^2}$ *then:*

$$|x_1 - x_2| \geq \frac{h^2}{4L}.$$

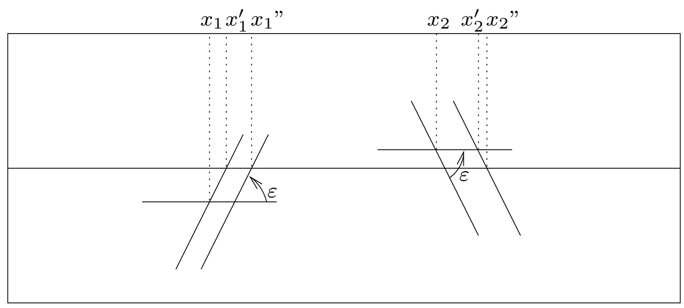

FIGURE 5. Injectivity bounds.

Proof. Since $p(x_1, s_1) = p(x_2, s_2)$ we can lift at (x_1, s_1) the geodesic $s = s_2$. This gives a straight line that makes an angle ε with the line $s = s_1$. This line meets $s = 0$ at a point $(x_1', 0)$ with $|x_1 - x_1'| = \frac{|s_1|}{\tan \varepsilon}$ (see Fig. 5). We have $p(x_1', 0) = p(x_2', s_2)$ with $|x_2 - x_2'| = \frac{|s_1|}{\sin \varepsilon}$. Considering the lift of γ at (x_2', s_2) and using the same argument, we find x_1" and x_2" such that $|x_2' - x_2"| = \frac{|s_2|}{\tan \varepsilon}$, $|x_1' - x_1"| = \frac{|s_2|}{\sin \varepsilon}$ and satisfying $p(x_1", 0) = p(x_2", 0)$. Using the triangular inequality and the preceding lemma, we find:

$$|x_1 - x_2| \geq |x_1" - x_2"| - 2\frac{|s_1| + |s_2|}{\sin \varepsilon} \geq \frac{h^2}{2L} - 4\frac{\max(|s_1|, |s_2|)}{\sin \varepsilon}.$$

Using Lemma 2 we have that $\sin \varepsilon \geq \frac{h}{L}$ so that $\max(|s_1|, |s_2|) \leq \frac{h^3}{16L^2}$ implies $|x_1 - x_2| \geq \frac{h^2}{4L}$, and the lemma thus follows. □

If we set $\delta_v = \frac{h^3}{16L^2}$ and $\delta_h = \frac{h^2}{8L}$ we thus have that p is injective on any rectangle $]x-\delta_h, x+\delta_h[\times]-\delta_v, \delta_v[$.

We now get back to the counting of parallelograms $\mathcal{L}$. In each such parallelogram there is a point m belonging to γ_1 and γ_2. We thus have to count the couples (x_1^0, x_2^0) such that $p_1(x_1^0, 0) = p_2(x_2^0, 0)$. Near this point and in this parallelogram, we can take (x_1, x_2) as coordinates. We compute the area a of the set corresponding to the rectangle $]x_1 - \delta_h(h_1, L_1), x + \delta_h(h_1, L_1)[\times] - \delta_v(h_1, L_1), \delta_v(h_1, L_1)[$ in these coordinates. Since the jacobian of the change of variables $(x_1, s_1) \leftrightarrow (x_1, x_2)$ is given by $\sin\varepsilon \geq \frac{h_1}{L_2}$, we get $a \geq \frac{h_1^5}{32L_1^3} \times \frac{h_1}{L_2}$. Eventually, each pair (x_1^0, x_2^0) is surrounded by a set of area a and two of these sets cannot intersect. This bounds the number of parallelograms by

$$32\frac{L_1^4 L_2^2}{h_1^6}.$$

When we plug in the dependence with respect to E, this gives an extra power of E in the estimate of K. Condition (17) ensures that this extra power can always be countered by choosing N large enough. so that the same set of constraints (see (18)) on the parameters implies the same estimate on the scalar products. The following theorem thus holds for immersed cylinders (see Thm. 3 and recall that condition $(\tilde{C}_\gamma)$ concerns embedded cylinders).

Theorem 4. *Let M be a Riemannian surface satisfying the condition $(\tilde{C}_\gamma)$ for some γ $(0 < \gamma \leq 2)$ then the following estimate holds*

$$\forall\,\alpha \in (\tfrac{1}{3}, \tfrac{1}{2}),\ \forall\,\varepsilon > 0\ \exists\,E_0,\ |$$
$$\forall\,E > E_0\ R_\alpha(E) \geq E^{e_\gamma(\alpha)-\varepsilon},$$

where the critical exponent $e_\gamma(\alpha)$ is given by

1. *If $\gamma = 2$ then, on $(\frac{1}{3}, \frac{1}{2})$, $e_\gamma(\alpha) = 2(\alpha - \frac{1}{3})$.*
2. *If $0 < \gamma < 2$ then on $(\frac{1}{3}, \frac{1}{2})$ we have*

$$e_\gamma(\alpha) = \begin{cases} \frac{2((6-\gamma)\alpha+\gamma-2)}{14-\gamma} & \text{for } \alpha > \frac{1}{3} + \frac{\gamma}{12}, \\ 2(\alpha - \frac{1}{3}) & \text{for } \alpha \leq \frac{1}{3} + \frac{\gamma}{12}. \end{cases}$$

There is a last step to make to state the theorem in the case of polygonal billiards. We consider a cylinder of periodic orbits in a polygonal billiard. This cylinder can be clearly seen by unfolding the billiard. We construct a quasimode in the billiard by folding back the quasimode in the cylinder. The only difference with the previous case (immersed cylinders) is some -1 prefactors, but this doesn't affect the estimate on the scalar products.

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Luc Hillairet
UMPA ENS-Lyon
UMR CNRS 5669
46, allée d'Italie
F-69364 Lyon Cedex 7
France
e-mail: lhillair@umpa.ens-lyon.fr

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