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Contribution of periodic diffractive geodesics

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Abstract

On an euclidean surface with conical singularities, the wave-trace is expected to be singular at L where L is the length of some diffractive periodic geodesic. In this paper, we compute the leading term of the singularity brought to the trace by a regular, isolated diffractive geodesic and by a regular family of periodic non-diffractive geodesic. These results can be applied to polygons.

Keywords: Wave-trace; Conical singularities; Polygonal billiard

1. Introduction

The billiard in a domain Ω of the euclidean plane consists in considering the evolution of a particle reflecting at the boundary according to Snell–Descartes' law of equal-angle reflection. When the boundary of Ω is smooth, this dynamical system is associated with the propagation of waves inside Ω in the following way. The singularities of a solution of the wave equation in Ω travel along billiard trajectories (cf. [19]). One can further prove a so-called *trace formula*. Rather than a formula in the usual sense, the expression *trace formula* covers a set of results that in the former setting include the

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following:

- Let $\sigma(t)$ be the trace of the propagator for the wave equation $\left(\text{i.e. } \sigma(t) = \mathrm{Tr}(e^{it\sqrt{\Delta}})\right)$, and let $\mathbb{L}$ be the set of lengths of periodic billiard trajectories, then:

$$\text{sing.supp.}(\sigma) \subset \{0\} \cup \pm\mathbb{L}. \tag{1}$$

- Let g be a periodic billiard orbit of length L_0, one can describe the singularity of σ at L_0. In particular one has the (Sobolev-)order of the singularity and the leading part.

In the smooth boundary case, these results are proved by Guillemin and Melrose in [15] where they generalize the results of Duistermaat and Guillemin [8], Colin de Verdière [6] and Chazarain [3] in the boundaryless case. The expression *trace formula* refers to some very particular cases where σ can be completely written as a sum over the periodic orbits of explicit distributions (Poisson–Selberg trace formula). This article aims at generalizing these results to the case where Ω is an euclidean polygon. Actually, we are led to study the wave equation on an euclidean surface with conical singularities. The inclusion (1) on such surfaces is the object of the note [17] and in the present paper, we will focus on the contribution brought in σ by a periodic diffractive orbit (or by a family of periodic orbits).

Polygonal billiards is a widely spread subject as far as dynamics is concerned (cf. [27] for instance). One reason for this is that, in some sense, any dynamic behaviour can be achieved by a polygonal billiard, from integrable to chaotic. It is thus interesting to see what geometrical or dynamical information is contained in the trace formula. Furthermore, there is also an intrinsic interest in studying the trace formula for manifolds with conical singularities (see [24]). Among these, euclidean surfaces with conical singularities are the simplest examples. One can thus expect to get for these the flavour of what happens in the general case, with less technical difficulty. We will thus restrict ourselves to euclidean conical singularities, although the theorem concerning the propagation of singularities (and also the Poisson relation—see [28]) is true in much more generality.

This work is directly inspired by those on the trace formula that are mentioned above, in particular we use the propagator for the wave equation and the associated propagation of singularities. As expected, it is essential to understand what happens when a singularity hits a conical point. The study of the wave equation in presence of conical singularities goes back to Sommerfeld and has been developed by many authors (cf. [4,11,13,24]). Concerning the trace formula, our work is linked with [2,9]. In [9] the author is concerned with the contribution of the smallest altitude in a triangle. The results we propose here are generalizations of hers. In [2], the approach is slightly different since the authors use the resolvent instead of the wave propagator but their results are consistent with ours. One peculiar feature of our approach is the constant use of the theory of Fourier integral operators (FIO).

1.1. Content and results

In all the article M will denote an euclidean surface with conical singularities. In the first section, we will recall what it means and we will gather the basic facts concerning both the geometry of M and the functional ingredients needed to analyse the wave equation on M. In particular, we will recall the definition of diffractive geodesics, and the results concerning the propagation of singularities on M. We will also introduce the microlocalized propagator along a geodesic g (denoted by U_g). In the second section, we will study these microlocalized propagators into more details. The main result of this section is the following (cf. Theorem 5)

Theorem 1. *Let g be a regular diffractive geodesic, then U_g is a FIO with explicit phase and symbol.*

We will also give a description of the microlocalized propagator near the so-called optical boundary, which is the simplest case where it fails to be a FIO. To derive these results, we will use the construction of the propagator on a cone due to Friedlander (cf. [11]). The last section will be devoted to computing the leading term of the trace of these microlocalized propagator in the two following cases:

- A regular diffractive periodic orbit, i.e. a periodic diffractive orbit such that all the diffraction angles β_i satisfy $\beta_i \neq \pm\pi \bmod \alpha_i$ (see Definition 1).
- A regular family of periodic orbits, i.e. a family of non-diffractive periodic orbits such that the limiting diffractive geodesics have only one diffraction.

In these two cases, we can make intensive use of the theory of FIO and of the fact that all the trace operations can be made away from the conical points (cf. [17]). We will finally be able to derive the following theorem (cf. Theorems 6 and 7). In this theorem, we call contribution of the (family) orbit the quantity:

$$I(s) = \langle \sigma_g(t), h(t)\exp(-ist)\rangle,$$

where σ_g is the trace of the propagator microlocalized near the orbit and h is a test function localizing near the length of the orbit (see the beginning of Section 4 and the definition of I_{tot}).

Theorem 2.

- *The contribution of a regular periodic diffractive orbit g of period L, and of primitive length L_0 is given at leading order by*

$$s^{-\frac{n}{2}} c_g h(L) e^{-isL} L_0,$$

with $c_g = (2\pi)^{\frac{n}{2}} e^{-\frac{ni\pi}{4}} \frac{D_g}{P_g^{\frac{1}{2}}}$, *where* n *is the number of diffractions, and* D_g *and* P_g *depend on the angles of diffraction and of the length of the different geodesic segments of* g. (*see Section 4.1*).

- *The contribution of a regular family of periodic orbits is*

$$\frac{e^{i\frac{\pi}{4}}}{\sqrt{2\pi}} s^{\frac{1}{2}} \frac{1}{\sqrt{L}} h(L) e^{-isL} |\mathcal{A}_g|,$$

where $|A_g|$ *is the area swept out by the family and* L *is the* (*necessarily*) *primitive length of the family*.

This theorem shows in particular that the leading contribution of a regular family of periodic of periodic orbits is the same as in the integrable case. The effect of the boundary of such a family only appear at a corrective order.

2. Generalities

2.1. Diffractive geodesics

In this section, we will recall the notion of diffractive geodesic of an euclidean surface with conical singularities. All this material is explained in greater detail in [18].

Let M be a compact euclidean surface with conical singularities (e.s.c.s.). By definition, the surface M can be partitioned in two pieces: M_0 on which there is an euclidean metric, and P which contains a finite number of points p_i such that in the neighbourhood of p_i, M is locally isometric to the euclidean cone of total angle α_i. The conical points such that $\alpha_i = 2\pi/k$, $k \in \mathbb{N}$ are called non-diffractive and P_d will be the set of diffractive conical points.

An interesting way of producing such surfaces is by gluing polygons along sides of same length. For instance we can create an euclidean surface with conical singularities by taking two copies of the same polygon Q and gluing each side of the first polygon with the corresponding one of the second: a surface obtained by this procedure will be called the double of the polygon Q. The so-called Katok–Zemliakov procedure (cf. [22]) also associates an e.s.c.s. with any rational polygon.

We have in M_0 a notion of geodesic that is clearly defined. Any geodesic starting in M_0 either can be extended indefinitely or ends at a conical point in finite time. In order to state the theorem concerning the propagation of singularities on M, we have to extend geodesics that end at a conical point. This extension is unique at a non-diffractive conical point since there the Euclidean plane is a finite covering of the cone of angle $2\pi/k$. We thus have a clearly defined notion of *non-diffractive* geodesic. At a diffractive point, the geodesic can make any angle. This leads to the following definition of (possibly) *diffractive* geodesics.

Definition 1. A geodesic of length L will be a continuous mapping g from $[0, L]$ to M such that:

- if $g(t) \notin P_d$ then there exists ε such that on $]t-\varepsilon, t+\varepsilon[\cap[0, L]$, g parametrizes a non-diffractive geodesic by arclength,
- the set $g^{-1}(P_d)$ is discrete.

Assume M is oriented and consider polar coordinates (R, x) at a diffractive point p (such that $\frac{\partial}{\partial R}, \frac{\partial}{\partial x}$ is a direct basis), a diffractive geodesic at p is parametrized by (t, x_i) on an interval $[0, t_0]$ before the diffraction and by $(t-t_0, x_o)$ after. The difference $x_o - x_i$ is the angle of diffraction and is denoted by β. It belongs to $\mathbb{R}/\alpha\mathbb{Z}$, where α is the cone angle corresponding to p.

Remark. With this definition, a geodesic *does not minimize* the distance locally near a diffractive point such that $|\beta| < \pi$. (cf. [18]).

Along a geodesic g the subscript (g, i) will refer to the ith diffraction. We will then speak of $t_{g,i}$, $p_{g,i}$, $\alpha_{g,i}$, $\beta_{g,i}$ etc... In view of the trace formula, the interesting objects are the periodic geodesics, we recall from [18] the following classification of periodic geodesics on an e.s.c.s.

Proposition 1. *Let g be a periodic geodesic of length T of an oriented e.s.c.s. then one of the following occurs.*

1. *The geodesic g is non-diffractive, it is then interior to a family of non-diffractive periodic geodesics of same length.*
2. *All the angles of diffraction are π (or $-\pi$), g is then the boundary of a family described in the first case.*
3. *In any other case, g is isolated in the set of periodic geodesics.*

We will call *regular* a geodesic such that all its angles of diffraction are different from $\pm\pi$. For a family of periodic orbits, regular will mean that both diffractive orbits bounding the family only have one diffraction (this implies in particular that the family is primitive).

2.2. Propagation of waves

This section is devoted to introduce the basic facts concerning analysis on a e.s.c.s. that we need to study the wave equation. We begin by defining the laplacian on M. The euclidean metric on M_0 gives us a positive symmetric operator defined on $\mathcal{C}_0^\infty(M_0)$. The Friedrichs procedure provides us with a self-adjoint extension Δ (cf. [25]). We will not really need the spectral theory of this operator. Let us just mention that there is a Rellich type theorem implying that the spectrum is discrete (cf. [4]). Denoting by λ_n

the nth eigenvalue of Δ we have the following Weyl estimate:

$$\sharp\{\lambda_n < \lambda\} \sim \frac{\text{area}(M)}{4\pi}\lambda.$$

Remark. When M is the double of the polygon Q, there is a natural involution defined on M. Restricting Δ to functions that are even (resp. odd) with respect to this involution amounts to consider the Neumann (resp. Dirichlet) problem in Q. As a consequence, the spectrum of Δ is the union of the spectra for the Dirichlet and Neumann problems in Q.

We recall the way the following objects are constructed from Δ:

1. The Sobolev space of order s, $H^s(M)$, is the domain of $\Delta^{\frac{s}{2}}$.
2. The space of "smooth" functions, $H^\infty(M) = \bigcap_s H^s(M)$. A distribution will be an element of $H^{-\infty}(M) = \bigcup H^s(M)$. An operator A is smoothing (or regularizing) if for all m, n, $\Delta^m A \Delta^n$ is bounded.
3. The propagator for the (half-)wave equation is $U(t) = e^{it\sqrt{\Delta}}$. It is constructed via the functional calculus. Its kernel will be denoted by $U(t, m_1, m_0)$.

We now have to introduce the notion of singularities. Let u be a distribution on M, we define $\mathrm{WF}_0(u)$ to be its wave-front, seen as a distribution on M_0. This makes of $\mathrm{WF}_0(u)$ a subset of $T^*(M_0)$. To understand what happens above the diffractive conical points we recall from [4] (see also Theorem 3) that a singularity hitting the tip of a cone will be reemitted in all the possible directions. It is therefore possible to forget the information about direction above P_d. We thus complete $T^*(M_0)$ by adding a point above each element of P_d and we define

$$\mathrm{WF}(u) = \mathrm{WF}_0(u) \bigcup \{p \in P_d \mid u \text{ is not smooth near } p\},$$

with the notion of smoothness that is defined right above (see point 2).

Remarks.

1. It is possible to be more precise about the definition of wave-front above the conical points (cf. [24]). However, the smoothness near p is easier to check, and since it is enough as long as solutions of the wave equation are concerned, we have chosen the former, simpler definition.
2. The former definition is also convenient because a singularity hitting a conical point cannot stay at this point (cf. [4]).
3. For each geodesic g, the startpoint and endpoint of g are well-defined in T^*M. We thus get a relation Λ_t in $T^*M \times T^*M$ by considering the pairs $(\tilde{m}_1, \tilde{m}_0)$ for which there exists a geodesics of length t starting at $\tilde{m}_0$ and ending at $\tilde{m}_1$. This set is described in detail in [18].

We then have the theorem of propagation of singularities on an e.s.c.s.

Theorem 3. *For any distribution u_0, and any time t, we have the following inclusion*

$$\mathrm{WF}(e^{it\sqrt{\Delta}}u_0) \subset \Lambda_t \circ \mathrm{WF}(u_0).$$

The proof follows directly from the propagation of singularities on a cone [4] and a finite propagation speed argument.

Remarks.

1. The definitions of geodesics and singularities are taken so that this theorem says: "The singularities propagate along the geodesics".
2. Writing $U(t_0 + s) = U(s)U(t_0)$ and letting s (small) vary gives the time-dependent wave-front relation associated with the propagator. The Poisson relation (1) derives from that remark, provided that one can take the trace above the conical points (see [17]).

2.3. Microlocalized propagator

As usual for the kind of trace formula we are considering, we will not exactly compute the singularity of σ that is located at L (the length of a periodic orbit). What we compute is the singularity created by a particular orbit of length L, the total singularity of σ being then the sum of all these contributions. Due to possible cancellations, the exact singularity at L is actually not known (cf. [14] for examples where such cancellations occur).

To address the singularity created by one particular periodic orbit g we need to microlocalize the propagation along g. In the smooth case, near a geodesic g of length L, this is done by using cut-offs Π_0 and Π_1 microlocalizing, respectively, near the startpoint and the endpoint of the geodesic so that the wave-front relation associated with $\Pi_1 U(L)\Pi_0$ only takes into account the geodesics of length L close to g. In the diffractive case, this feature can only be achieved by using microlocal cut-offs after every diffraction.

We first extend the notion of microlocal cut-off in order to take into account the conical points. The following defines a microlocal cut-off Π near a point of T^*M.

1. Near a diffractive point p, Π is the multiplication by some function ρ that is identically 1 near p.
2. Near a regular point (m_0, μ_0) in T^*M_0, Π is identically 0 near the conical points, and in M_0, Π is a pseudo-differential operator whose symbol is identically 1 in a conical neighbourhood of (m_0, μ_0).

In both cases the notion of support of Π is clearly defined as subset of T^*M, since near a diffractive point, a microlocal cut-off is identically 0 or Id.

Remark. In the following, each time a pair $\mathcal{V}$, $\mathcal{W}$ will denote two conical neighbourhoods of the same point in T^*M_0, it will tacitly imply that $\mathcal{V}$ is relatively compact in $\mathcal{W}$ (when restricted to the unit cotangent bundle). This implies that there exists a microlocal cut-off that is identically 1 in $\mathcal{V}$ and identically 0 outside $\mathcal{W}$.

We consider a geodesic g of length t, and a subdivision $(t_i)_{0\leqslant i\leqslant N+1}$ of $[0,t]$. We ask that $0=t_0<t_1<\cdots<t_N<t_{N+1}=t$, and that each conical point corresponds to some t_k (i.e. $\forall i \exists k \mid t_{g,i}=t_k$). We also let $s_i=t_{i+1}-t_i$. For each k, we choose some microlocal cut-off Π_k microlocalizing near the point of T^*M corresponding to $g(t_k)$. We will call *microlocalized propagator* the operator:

$$U_g(t)=\Pi_{N+1}U(s_{N+1})\Pi_N\cdots\Pi_1U(s_1)\Pi_0. \tag{2}$$

The main aim of this section is the following proposition.

Proposition 2. *For every $m_0\in M_0$, for every t in $\mathbb{R}$, there exists $r_0>0$, a finite collection of microlocalized propagators associated with geodesics g_i of length t emanating from m_0, and a smoothing operator $R(t)$ such that*

$$U(t)=\sum_i U_{g_i}(t)+R(t),$$

when restricted to distributions with support in $B(m_0,r_0)$.

Remark. This expansion is not unique. Only the diffractive points occurring in the g_i's are prescribed.

There are two steps in the proof of the preceding proposition; the first one is purely geometric and consists in constructing the geodesics g_i and well-chosen microlocal neighbourhoods along them. In the second step, we will construct the microlocalized propagators that are associated with this geometric information (geodesics and neighbourhoods) and the end of the proof will follow from the propagation of singularities.

Geometric construction: It is convenient to describe the set of all the geodesics emanating from m_0 using a tree that we construct in the following way. We start with the point m_0. The branches emanating from m_0 correspond to the rays that reach a diffractive point. We end each branch by the corresponding conical point and label the branch by its length. Iterating the construction, we end up with a tree whose vertices correspond to m_0 and diffractive points and whose edges correspond to non-diffractive geodesics joining its vertices. Each edge is labelled by its length, and to each vertex p we associate $L(p)$ the total length of the diffractive geodesic joining it to m_0, and $N(p)$ the number of diffractions along it. This tree is infinite but by considering only the vertices such that $L(p)\leqslant t$ we get a finite sub-tree. For each vertex, we will also choose ε_v such that $\mathcal{B}(p,3\varepsilon_v)$ is isometric to the same ball on the corresponding cone (resp. plane for m_0). We will also denote by $N^*_+(\mathbb{S}(p,\varepsilon))$ the set of conormal vectors to the sphere centred at p and of radius ε pointing outwards.

In order to construct the microlocalized propagators, we will need microlocal cut-offs after each diffraction and near the endpoints of the geodesic g_i. To simplify the exposition, we assume that $t - L(v) > 0$ for all the vertices so that we can assume that $\varepsilon_v < t - L(v)$ (but the construction can be adapted to rule this restriction out). We begin by choosing, for each vertex v, and for each non-diffractive geodesic g of length $t - L(v)$ emanating from v, two microlocal neighbourhoods $\mathcal{V}^r(v, t - L(v), g)$ and $\mathcal{W}^r(v, t - L(v), g)$ near the endpoint. We now address the vertices p such that $N(p)$ is maximal. Since $N(p)$ is maximal, each geodesic of length $t - L(p)$ emanating from p is non-diffractive. On each such geodesic g, we denote by m the point at length ε_p of p. We can find $\mathcal{V}(p, \varepsilon_p, g)$ and $\mathcal{W}(p, \varepsilon_p, g)$ near m such that all the geodesics emanating from $\mathcal{W}(p, \varepsilon_p, g)$ ends in the chosen neighbourhood $\mathcal{V}^r(p, t - L(p), g)$. By compactness we only have to consider a finite number of geodesics $g_{p,i}$ to cover $N_+^*(\mathbb{S}(p, \varepsilon_p))$. We can then find a radius η_p so that every geodesic of length ε_p emanating from $\mathcal{B}(p, \eta_p)$ ends in the microlocal neighbourhood $\cup\mathcal{V}(p, \varepsilon_p, g_{p,i})$ that we have just constructed.

We can now construct η_p, $\mathcal{V}(p, \varepsilon_p, g_{p,i})$, $\mathcal{W}(p, \varepsilon_p, g_{p,i})$ by induction on decreasing $N(p)$. As in the preceding step, we look for a microlocal neighbourhood of $N_+^*(\mathbb{S}(p, \varepsilon_p))$. For each non-diffractive geodesic emanating from p we do the same thing as before. We now have to deal with the diffractive geodesics emanating from p. Each such geodesic g hits a diffractive point p_1 in time $l = L(p_1) - L(p)$. Since $N(p_1) = N(p) + 1$, we have already constructed η_{p_1}, we can thus construct $\mathcal{V}(p, \varepsilon_p, g)$, $\mathcal{W}(p, \varepsilon_p, g)$ such that each geodesic of length l emanating from $\mathcal{W}(p, \varepsilon_p, g)$ ends in $\mathcal{B}(p_1, \frac{\eta_{p_1}}{2})$. We now get a covering of $N_+^*(\mathbb{S}(p, \varepsilon_p))$, of which we can extract a finite covering, and we construct η_p as before.

Eventually, for each vertex v, we have constructed a ball $\mathcal{B}(v, \eta_v)$, a finite number of geodesics $g_{v,i}$ and neighbourhoods $\mathcal{V}(v, \varepsilon_v, g_{v,i})$, $\mathcal{W}(v, \varepsilon_v, g_{v,i})$ along $g_{v,i}$. By construction, all the edges of the tree correspond to one $g_{v,i}$ and the other $g'_{v,i}s$ are non-diffractive and of length $t - L(v)$. We denote by $\{g_i\}$ all the geodesics emanating from m_0 that can be obtained by following some $g_{v,i}$ one after another.

Construction of the microlocalized propagators: Using a partition of the unity, we can construct microlocal cut-offs $\Pi(v, \varepsilon_v, g_{v,i})$ that are identically 0 outside $\mathcal{W}(v, \varepsilon_v, g_{v,i})$. Near each vertex, we use a microlocal cut-off that is 1 in $\mathcal{B}(v, \frac{\eta_v}{2})$ and 0 outside $\mathcal{B}(v, \eta_v)$. The microlocal cut-offs we have constructed along a geodesic g_i correspond to the times $t_{g_i,k}$ and $t_{g_i,k} + \varepsilon_v$ for some ε_v that we rewrite ε_k in this context. We denote them by Π_k and $\Pi_{k,+}$, respectively. The following thus defines a microlocalized propagator along g_i (compare with (2))

$$\begin{aligned} U_g(t) = {} & \Pi_{N+1} U(t_{N+1}) \Pi_{N,+} U(\varepsilon_N) \Pi_N U(t_N) \Pi_{N-1,+} \\ & \cdots U(t_2) \Pi_{1,+} U(\varepsilon_1) \Pi_1 U(t_0) \Pi_{0,+} U(\varepsilon_0), \end{aligned} \tag{3}$$

with $t_{g,k} = \sum_{i \leqslant k} t_i + \varepsilon_i$.

End of the proof: Using the propagation of singularities, the result of the proposition follows for $r_0 = \eta_{m_0}$. Indeed, the only thing to be careful about is that we have taken into account all the geodesics emanating from this neighbourhood and this is ensured by construction.

Expression (3) can be simplified in two ways. First we suppress the cut-offs near p: this is possible since by construction all the geodesics of length t_k emanating from the support of $\Pi_{k-1,+}$ ends where Π_k is identically 1. The second simplification amounts to replacing U by some $U_{\alpha_{g,i}}$: this is possible using the propagation of singularities and the finite speed of propagation. Eventually we came up with the following expression:

$$U_g(t) = \Pi_{N+1,+}U_{\alpha_N}(s_N)\Pi_{N-1,+}U_{\alpha_{N-1}}(s_{N-1})\cdots\Pi_{1,+}U_{\alpha_1}(s_1)\Pi_0, \tag{4}$$

with, for each k, $\alpha_k = \alpha_{g,k}$, and $s_k = t_{k-1} + \varepsilon_k$ (we have made a last simplification replacing the cut-offs $\Pi_{0,+}$ and $\Pi_{N,+}$ by cut-offs, respectively, near the origin and near the end of the geodesic).

These are the microlocalized propagators we will be working with in the rest of the paper. In particular, we will compute the trace of such objects. Proposition 2 shows that this information will be enough to recover the trace of the complete propagator (see Section 4.1 below).

3. Singularities of the microlocalized propagator

In this way of deriving the trace formula (cf. [3,8]) the first step is to study quantitatively the propagation of singularities. Since the microlocalized propagators are expressed in terms of the propagation on the euclidean cone, we will begin by addressing this setting. The propagator on an euclidean cone is known for a long time (cf. [26,11]) and the microlocalized propagator along a regular diffractive geodesic is given by the so-called "Geometric Theory of Diffraction" (cf. [2]). The results that we present are thus not new (and for some are even quite old!). However, we think that our approach is more convenient when aiming at a trace formula. It consists in using intensively the theory of FIO. In particular, we will show that the microlocalized propagator along a regular diffractive geodesic is a FIO (with explicit phase function and symbol). We begin by revisiting Friedlander's construction of the wave propagator on a cone.

Remark. All the statements concerning FIO will be given using a specified representation by some oscillatory integral and not using the invariant form involving half-densities. We have found that this way the computations occurring when making compositions or taking the trace were a little simpler to describe.

3.1. Friedlander's construction

This construction is the main issue in the article [11]. Here, we want to interpret it in the language of FIOs. The following steps are followed. We first use the article [11] to address the operator $\frac{\sin(t\sqrt{\Delta})}{\sqrt{\Delta}}$ (cf. Proposition 3). Using some pseudo-differential techniques, we then derive the corresponding result for $e^{it\sqrt{\Delta}}$ and then for $\chi(\Delta)\Delta^{-N}e^{it\sqrt{\Delta}}$. Since we will only have to take the trace above M_0 all the operators we consider here act on $\check{\mathcal{C}}_\alpha$ $(= \mathcal{C}_\alpha\backslash\{p\})$.

We recall the following notation for homogeneous distributions (see [12]): for $\mathrm{Re}(\lambda) > -1$, θ_+^λ, and θ_-^λ denotes the distributions defined by the following L^1_{loc} functions:

$$\theta_+^\lambda = \begin{cases} \theta^\lambda & \text{if } \theta > 0, \\ 0 & \text{elsewhere,} \end{cases}$$

$$\theta_-^\lambda = \begin{cases} |\theta|^\lambda & \text{if } \theta < 0, \\ 0 & \text{elsewhere.} \end{cases}$$

This family can be analytically continued to $\lambda \in \mathbb{C}$ except for negative integers.

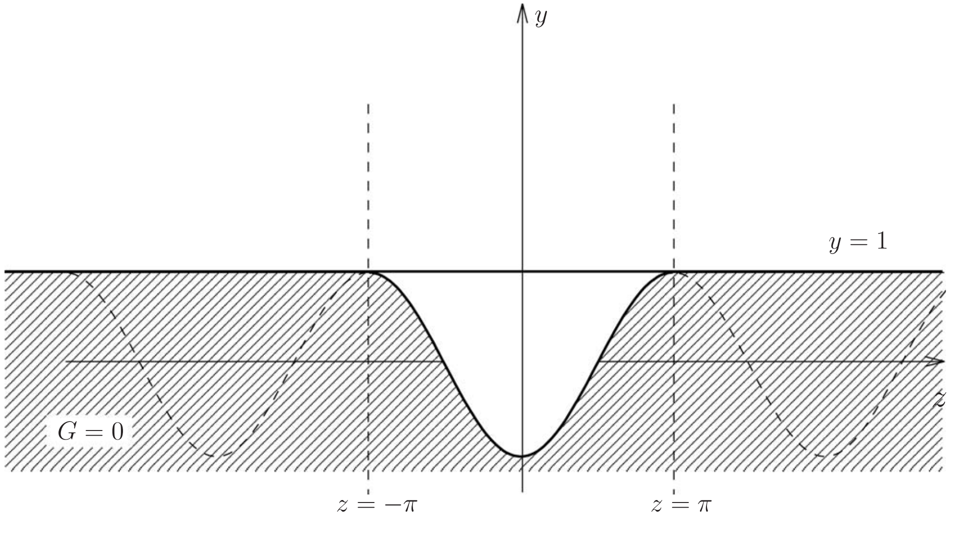
Friedlander's construction is as follows. We begin with $G(y, z)$ defined by the following L^∞ function in $\mathbb{R}^2$:

$$G(y, z) = \begin{cases} H(y + \cos z)H(\pi - |z|) & \text{if } y < 1, \\ \dfrac{1}{\pi}\left[\arctan\left(\dfrac{\pi - z}{\mathrm{ch}^{-1} y}\right) + \arctan\left(\dfrac{\pi + z}{\mathrm{ch}^{-1} y}\right)\right] & \text{if } y > 1, \end{cases}$$

where H is the Heaviside function. We have the following alternative way of writing G:

$$G(y, z) = H(y + \cos z)H(\pi - |z|) - \frac{H(y-1)}{\pi}\left[\arctan\left(\frac{\mathrm{ch}^{-1} y}{\pi - z}\right) + \arctan\left(\frac{\mathrm{ch}^{-1} y}{\pi + z}\right)\right]. \quad (5)$$

We can find the singular support of G by inspection. It is represented in the following figure.



The wave-front of G can be easily derived from the fact that G is a solution of the following partial differential equation (see Eq. (19) of [11]):

$$(1-y^2)\frac{\partial^2}{\partial y^2}G - \frac{\partial^2}{\partial z^2}G + y\frac{\partial}{\partial y}G = 0. \tag{6}$$

Lemma 1. *The following inclusion holds*:

$$\mathrm{WF}(G) \subset \left[N^*\{y+\cos z = 0\} \cap \{|z| \leqslant \pi\}\right] \cup N^*\{y=1\}.$$

In particular, we also have the following inclusion:

$$\mathrm{WF}(G) \subset \{(y,z,\eta,\zeta) \mid |\zeta| \leqslant |\eta|\}.$$

Proof. The distribution G is $\mathcal{C}^\infty$ outside $\{y = 1\} \cup \{y + \cos z = 0\} \cap \{|z| \leqslant \pi\}$. The only thing to show is that above a point of this set, only a conormal covector can be in $\mathrm{WF}(G)$. This is ensured by Hörmander's theorem on solutions of PDE (cf. [19, Theorem 8.3.1]) which we apply to G, solution of (6). The second point is then straightforward. □

Remark. The two conormal sets that define $\mathrm{WF}(G)$ intersect cleanly along $\Sigma = \{(y=1, z=\pm\pi, \eta, \zeta=0)\}$. After being transported on the cone this set corresponds to the diffractive rays that are limits of non-diffractive geodesics. The expression "optical boundary" will refer either to Σ or to the corresponding set on the cone (or on M).

The kernel of the wave propagator on the euclidean cone $\mathcal{C}_\alpha$ is then obtained by applying successively to G the following operations:

- Periodization w.r.t. $(y,z) \to (y, z+\alpha)$.
- Half-derivation w.r.t. y; i.e. action of the operator defined by

$$\left[D^{\frac{1}{2}}\right]u(y) = \partial_y \int (y-y')_+^{-\frac{1}{2}} u(y',z)\,|dy'|.$$

- Pull-back by the map:

$$F : (t, R_1, x_1, R_0, x_0) \to (y = f(t, R_1, R_0), z = x_1 - x_0),$$
$$\text{with } (t, R_1, R_0) = \frac{t^2 - R_1^2 - R_0^2}{2R_1R_0}.$$

This is well-defined since F is a submersion for $t \neq 0$.

- Multiplication by $C(R_1R_0)^{-\frac{1}{2}}$, for some constant C. The constant C will be fixed later (see Remarks 3 in the following page).

The main result of [11] is that we have constructed this way the kernel of $\frac{\sin(t\sqrt{\Delta_\alpha})}{\sqrt{\Delta_\alpha}}$ acting on $L^2(C_\alpha)$. This is summed up in the following proposition:

Proposition 3. *The distributional kernel* E_α *of* $\frac{\sin(t\sqrt{\Delta_\alpha})}{\sqrt{\Delta_\alpha}}$ *is given by*:

$$E_\alpha = AG_\alpha,$$

where G_α *is an element of* $\mathcal{D}'(\mathbb{R}\times\mathbb{R}/\alpha\mathbb{Z})$, *and* A *is a FIO acting from* $\mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{R}/\alpha\mathbb{Z})$ *to* $\mathcal{D}'(\mathbb{R}\times\check{C}_\alpha\times\check{C}_\alpha)$ (*such that the composition* AG_α *is well defined*). *The following description holds*:

1. *The distribution* G_α *is obtained by making* G α*-periodic*:

$$G_\alpha(y,z) = \sum_{k\in\mathbb{Z}} G(y, z+k\alpha).$$

2. *The operator* A *is associated with the lagrangian manifold*

$$\Lambda_A = N^*\{(y,z) = F(t, R_1, x_1, R_0, x_0)\}$$

and we have the following representation of the kernel of A *as an oscillatory integral*:

$$A(t, m_1, m_0, y, z) = \int e^{i\theta[f(t,R_1,R_0)-y]} e^{i\sigma[(x_1-x_0)-z]} a(t, m_0, m_1, y, \theta)\,|d\theta\, d\sigma|,$$

in which the symbol a *is given by*

$$a = \frac{e^{i\frac{\pi}{4}}}{4\pi\sqrt{2\pi}} \times \frac{\theta_+^{\frac{1}{2}} - i\theta_-^{\frac{1}{2}}}{(R_0R_1)^{\frac{1}{2}}}.$$

Remark. The distribution AG_α is well-defined because the wave-fronts appearing in the composition are transversal. The fact that we know that

$$\mathrm{WF}(G) \subset \{|\zeta| \leqslant |\eta\|\}$$

is at this stage important.

Proof. There are two things to prove:

1. the periodization of G (i.e. G_α) is well-defined,
2. the composition of the half-derivation, the pull-back, then the multiplication by $C(R_0R_1)^{\frac{1}{2}}$ is an FIO given by A.

The periodization is mainly technical. We refer to Appendix A, where we derive the needed estimates both for the periodization and for the expansion of G_α as a lagrangian distribution far from the optical boundary. We then have to study the operator A defined by

$$Au(t, m_1, m_0) = C(R_0R_1)^{-\frac{1}{2}} F^* D_y^{\frac{1}{2}} u$$

on $\mathcal{C}_0^\infty(\mathbb{R} \times \mathbb{R}/\alpha\mathbb{Z})$. We take (y, z, η, ζ) some local coordinates on $T^*(\mathbb{R} \times \mathbb{R}/\alpha\mathbb{Z})$. The half-derivation is not a pseudo-differential operator everywhere but it is one when acting on distributions u satisfying

$$\mathrm{WF}(u) \subset \{(y, z, \eta, \zeta) \mid |\zeta| \leqslant c|\eta|\}$$

for some c. Using the Fourier transform of y_+^α (cf. [12]) this pseudo-differential operator is easily written as an oscillatory integral. The action of A on the distributions satisfying the latter condition is thus a FIO whose description as an oscillatory integral follows from standard computations. The remark after Lemma 1 then shows that this gives us the behaviour near of E_α near the diffracted front. □

Remarks.

1. This proposition is weaker than the results of [11] that tells us that the description we obtain is also true near the tip of the cone. However, we have chosen not to extend the notion of FIO there and that prevents us from stating the former proposition near the tip of the cone. This choice is motivated by the fact that we have seen that the microlocalized propagators involve only what happens far from the vertices.
2. This description is quite simple since it only involves some “special function” G_α and the rest is given by a FIO.
3. The constant C can be easily fixed by looking at the primary front where we should get the free propagation (this amounts to make A act on $H(y + \cos z)$). This gives $C = (2\pi\sqrt{2})^{-1}$ (cf. [11]).

The former proposition gives us a description of the distributional kernel of $\frac{\sin(t\sqrt{\Delta_\alpha})}{\sqrt{\Delta_\alpha}}$. Restricting the FIO $\partial_t A$ to the part of the wave front such that $\tau > 0$ gives us the kernel of $e^{it\sqrt{\Delta_\alpha}}$. We denote by A_0 the operator that we obtain this way. We get the expression of A_0 as an oscillatory integral:

$$A_0(t, m_1, m_0, y, z) = \int_{\theta>0} e^{i\theta[f(t,R_1,R_0)-y]} e^{i\sigma[(x_1-x_0)-z]} a_0(t, m_0, m_1, y, \theta)\, |d\theta\, d\sigma|, \quad (7)$$

in which the principal part of the symbol a_0 is given by

$$a_0(t, m_1, m_0, y, \theta) \sim e^{i\frac{\pi}{4}}(2\pi)^{-\frac{3}{2}} \frac{1}{(R_0 R_1)^{\frac{1}{2}}} \left(\frac{it}{R_0 R_1}\right) \theta^{\frac{3}{2}}.$$

We have just proved the following corollary of Proposition 3.

Lemma 2. *The distributional kernel U_α of $e^{it\sqrt{\Delta_\alpha}}$ is given by $A_0 G_\alpha$, where G_α is the distribution defined in Proposition* 3 *and A_0 is a FIO given by the oscillatory integral* (7).

Remark. In order to deal with the contribution given by periodic orbits on the optical boundary, we will have to quit the theory of FIO, and it will thus be more convenient to have convergent oscillatory integrals. This can be done by dealing with $\chi(\Delta)\Delta^{-N}e^{it\sqrt{\Delta_\alpha}}$ instead of $e^{it\sqrt{\Delta_\alpha}}$ (for some function χ cutting off away 0). Denoting by $A_n = \chi(\Delta)\Delta^{-N}A_0$ we get a FIO represented by the following oscillatory integral:

$$A_N(t, m_1, m_0, y, z) = \int_{\theta>0} e^{i\theta[f(t,R_1,R_0)-y]} e^{i\sigma[(x_1-x_0)-z]} a_N(t, m_0, m_1, y, \theta)\, |d\theta\, d\sigma|, \quad (8)$$

in which a_N is a symbol of order $\frac{3}{2} - N$, that can be derived from a_0.

A particularly interesting consequence of Proposition 3 and Corollary 2 is that, to study the singularities of $e^{it\sqrt{\Delta_\alpha}}$, one only has to study G_α and then use the theory of FIO. In fact, we will shortly prove that G_α is a lagrangian distribution away from the optical boundary so that $e^{it\sqrt{\Delta_\alpha}}$ will be a FIO there. By composition, this will give us a precise description of the microlocalized propagator along a regular diffractive geodesic.

3.2. Regular diffractive geodesics

As we have just said, the key ingredient is to prove that G_α is lagrangian away from the "optical boundary". We recall expression (5):

$$G(y, z) = H(y + \cos z)H(\pi - |z|) - \frac{H(y-1)}{\pi}\left[\arctan\left(\frac{\mathrm{ch}^{-1} y}{\pi - z}\right) + \arctan\left(\frac{\mathrm{ch}^{-1} y}{\pi + z}\right)\right]$$

$$\text{and} \quad G_\alpha(y, x) = \sum_{\mathbb{Z}} G(y, x + k\alpha).$$

The singularities corresponding to $y + \cos z = 0$ are transported by f to the primary wave-front. The function $H(y + \cos z)$ is a lagrangian distribution and $e^{it\sqrt{\Delta_\alpha}}$ is a FIO near the primary front (given by free propagation). The optical boundary corresponds to $(y = 1, z = \pm\pi)$, we will deal with it in the next section. Here, we are interested in the behaviour of G_α near $y = 1,\ x \neq \pm\pi \bmod \alpha$.

Near $(1, x)$ the periodization of $H(y + \cos z)H(\pi - |z|)$ is $\mathcal{C}^\infty$ so that we only have to address the periodization of G^1 defined by

$$G^1(y, z) = \arctan\left(\frac{\mathrm{ch}^{-1} y}{\pi - z}\right) + \arctan\left(\frac{\mathrm{ch}^{-1} y}{\pi + z}\right).$$

This is done in the following lemma.

Lemma 3. *Let G^1 be defined as above, the following function G^1_α:*

$$G^1_\alpha(y, x) = \sum_{k\in\mathbb{Z}} G^1(y, x + k\alpha)$$

is well defined on $\mathbb{R} \times \mathbb{R}/\alpha\mathbb{Z}$. In the neighbourhood of $(1, x_0)$ with $x_0 \neq \pm\pi$, the following asymptotic expansion holds:

$$G^1_\alpha(y, x) \sim \sum_{k=0}^{\infty} g_{\alpha,k}(x)(y - 1)_+^{\frac{1}{2}+k}, \tag{9}$$

where the $g_{\alpha,k}$'s are smooth on $[1, \infty[\times[\mathbb{R}/\alpha\mathbb{Z}\backslash\{\pm\pi\}])$, and $g_{\alpha,0}(1, x) = 2\sqrt{2}d_\alpha(x)$ with

$$d_\alpha(x) = -\sum_k \frac{1}{\pi^2 - (x + k\alpha)^2}.$$

We will here only sketch the proof. The estimates allowing the following are made in Appendix A. Developing arctan near 0 gives

$$G^1(y, x) = \sum_k a_k \left[\frac{1}{(\pi - x)^k} + \frac{1}{(\pi + x)^k}\right] \left(\mathrm{ch}^{-1}(y)\right)^k,$$

that we can periodize term by term:

$$G_{1,k}(y, x) = \sum_k a_{k,\alpha}(x) \left(\mathrm{ch}^{-1}(y)\right)^k.$$

We can then use the following asymptotic expansion:

$$\mathrm{ch}^{-1}(y) = \sum c_k (y-1)_+^{k+\frac{1}{2}}$$

and reorder the terms according to the powers of $(y-1)_+$. The formula for d_α follows by inspection.

Remarks.

1. The function d_α can be expressed differently:

$$d_\alpha(z) = -\frac{\sin\left(\dfrac{2\pi^2}{\alpha}\right)}{\alpha \sin\left[\dfrac{\pi}{\alpha}(\pi - z)\right] \sin\left[\dfrac{\pi}{\alpha}(\pi + z)\right]}$$

   (see [10, Example 2, p. 112]). This expression is denoted by $L(0,z)$ in Durso's paper [9]. The two following facts are straightforward:
   - If $\alpha = \frac{2\pi}{k}$, d_α is identically 0.
   - If not, $\forall x$, $d_\alpha(x) \neq 0$.
2. Expansion (9) is clearly that of a lagrangian distribution associated with the lagrangian submanifold $N^*\{y-1\}$.

To have the expression of U_α near the diffracted wave-front and away from the optical boundary, we now have to apply the FIO A_0 to G_α. This gives the following theorem.

Theorem 4 (*In the neighbourhood of* $\Lambda_d\backslash\Sigma$). *In the neighbourhood of* $\Lambda_d\backslash\Sigma$, $e^{it\sqrt{\Delta_\alpha}}$ *is a FIO associated with the lagrangian manifold* $N_+^*\{t = R_0 + R_1\}$. *Its kernel* U_α *can be written as the following oscillatory integral*:

$$U_\alpha(t, R_1, x_1, R_0, x_0) = \int_{\theta>0} \exp\left[i\theta(t - R_1 - R_0)\right] k_\alpha(t, R_1, x_1, R_0, x_0, \theta)\, |d\theta|, \quad (10)$$

in which the principal part of k_α *is given by*

$$k_\alpha \sim_p \frac{1}{2\pi} \frac{d_\alpha(x_1 - x_0)}{(R_1 R_0)^{\frac{1}{2}}}.$$

Proof. We pick up some neighbourhood $\mathcal{V}$ of a point in $\Lambda_d\backslash\Sigma$. We can then find a cut-off ρ so that

- applying A_0 to $(1-\rho)G_\alpha$ gives a smooth function,
- the function $G_\alpha\rho$ is given by expansion (9) (multiplied by ρ).

We apply now A_0 to the lagrangian distribution $G_\alpha \rho$; this gives the oscillatory integral

$$\int_{\theta>0,\sigma,y} e^{i\theta[f(t,R_0,R_1)-y]} a_0(t, R_0, R_1, y, x_1 - x_0, \theta) G_\alpha^1(y, x_1 - x_0) \times \rho(y, x_1 - x_0)\, |d\theta\, d\sigma\, dy|.$$

We plug into it the expansion for G_α^1 and apply the theorem of composition of FIO (see [7]). The fact that U_α represents a FIO associated with $N_+^*\{t = R_0 + R_1\}$ follows directly, the representation as an oscillatory integral also, since the phase function $\theta(t - R_1 - R_0)$ generates $N_+^*\{t = R_0 + R_1\}$. To have simply the principal symbol, we write G_α^1 as an oscillatory integral (using the Fourier transform of $\theta^{\frac{1}{2}}$):

$$G_\alpha^1 = \int \exp\left[i(y-1)\theta\right] g_{1,\alpha}(x, \theta)|\, d\theta|,$$

with

$$g_{1,\alpha}(x, \theta) \sim \frac{1}{\sqrt{2\pi}} d_\alpha(x) \left[e^{-\frac{3i\pi}{4}} \theta_+^{-\frac{3}{2}} + e^{\frac{3i\pi}{4}} \theta_-^{-\frac{3}{2}} \right].$$

We plug this expression into (10) and perform a stationary phase argument with respect to (R, x) variables. This is the usual procedure for the composition of FIOs. □

Remarks.

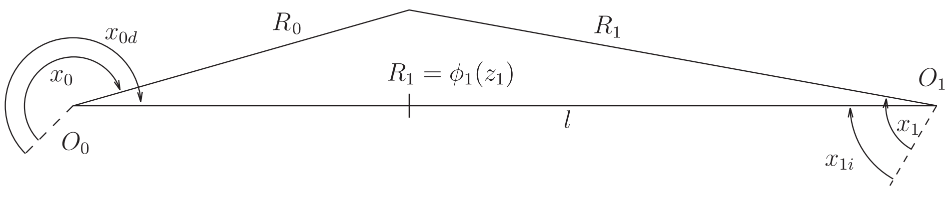
1. This result can be referred to what is called "Geometrical Theory of Diffraction" (cf. [2]). As it was already mentioned, we have found it interesting to restate such a result using the language of FIO.
2. The fact that away from the optical boundary the propagator is a FIO can be proved in much more general settings. In particular, it results from the study of Melrose and Wunsch [24] and one does not need to know an explicit expression for the propagator. However, such an explicit expression gives an alternative proof in this simpler setting.
3. The diffraction coefficient d_α blows up near the optical boundary.
4. Looking at the magnitude order of U_α, we recover the fact that can roughly be stated as: " the diffracted wave is $\frac{1}{2}$-times more regular than the primary wave" (cf. [2,21]). This statement has to be understood carefully since it is not true in general (see [24]). It is only valid provided that one can apply some stationary phase argument to the incoming wave. In particular the incoming wave should not focus on the conical point.

We can now easily derive the expression of the microlocalized propagator along a regular geodesic. Indeed, such a propagator only involves the expression of the propagator on a cone near the regular part of the diffracted wave-front. There, it is a FIO, and so we get the microlocalized propagator by applying the theorem on the composition of FIOs.This can be specified further by remarking that the theorem of composition will always be used in the following situation. We have two different points $O_0,\ O_1$ in $\mathbb{R}^2$ and two FIO $B_0,\ B_1$ acting, respectively, from some manifold Z_0 in $\mathbb{R}^2$ and from $\mathbb{R}^2$ in a manifold Z_1. We choose (R_i, x_i) the polar coordinates centred at O_i. We suppose that the kernels of B_0 and B_1 can be written as

$$B_0(m, z_0) = \int e^{i\theta_0[\phi_0(z_0)-R_0]}\tilde{b}_0(z_0, R_0, x_0, \theta_0)\,|d\theta_0|,$$

$$B_1(z_1, m) = \int e^{i\theta_1[\phi_1(z_1)-R_1]}\tilde{b}_1(z_1, R_1, x_1, \theta_1)\,|d\theta_1|.$$

Geometrically, we have the following picture:



Lemma 4. *Let B_0 and B_1 satisfy the preceding hypotheses, then the composition B_1B_0 is well-defined and results in a FIO C that can be written as*

$$C(z_1, z_0) = \int e^{i\theta[\phi_0(z_0)+\phi_1(z_1)-l]}c(z_0, z_1, \theta)\,|d\theta|,$$

where l is the euclidean distance between O_1 and O_0. Furthermore, if the principal symbols of B_0 and B_1 are $b_i(z_i, R_i, x_i)\theta^{\alpha_i}$, respectively, then the leading term of the symbol c is

$$c(z_0, z_1, \theta) \sim (2\pi)^{\frac{3}{2}} e^{-i\frac{\pi}{4}} \left[\frac{b_0 b_1 (R_0 R_1)^{\frac{1}{2}}}{l^{\frac{1}{2}}}\right]\Bigg|_{\substack{R_1=\phi_1(z_1)\\ R_0=l-\phi_1(z_1)\\ x_0=x_{0d}\\ x_1=x_{1i}.}} \theta^{\alpha_0+\alpha_1-\frac{1}{2}}.$$

The proof is a straightforward application of the method of stationary phase applied to the composition of FIOs (cf. [8, Theorem 2.4.1, p. 38]).

This lemma will allow us to describe the microlocalized propagator U_g. Before doing so, we have to recall and simplify some notations associated with g. Since we are dealing with one fixed geodesic g we can drop the index g in the lists $p_{g,j},\ \beta_{g,j}\cdots$ So that along g we have n diffractive points $p_1\cdots p_n$, the corresponding angle are α_i

and the diffraction angle of g in p_i is β_i (we recall that since the geodesic is regular, $\beta_i \neq \pm\pi \,\mathrm{mod}(\alpha_i)$). The length l_j will be that of g between p_j and p_{j+1}. We take (R_0, x_0) the polar coordinates centred at p_1 in a neighbourhood of m_0, and (R_1, x_1) the polar coordinates centred at p_n in the neighbourhood m_1. We fix the origin of angles so that $x_0 = 0$ corresponds to the incoming g and $x_1 = 0$ to the outgoing g. We also define the following functions:

$$d_g(m_0, m_1) = d_{\alpha_1}(\beta_1 - x_0) d_{\alpha_2}(\beta_2) \cdots d_{\alpha_n}(x_1),$$

$$l_g(m_0, m_1) = R_0 \times l_1 \times l_2 \cdots \times l_{n-1} \times R_1.$$

The following theorem describes the microlocalized propagator along a regular diffractive geodesic.

Theorem 5. *Let g be a regular diffractive geodesic with n diffractions. With the former notations and in the neighbourhood of (T_0, m_0, m_1), the operator U_g (defined by (2)) is a FIO associated with the lagrangian manifold Λ_g. Microlocally, its kernel can be written as*

$$U_g(t, m_0, m_1) = \int_{\theta>0} e^{i\theta\left[t-(R_0+\sum_{j=1}^{n-1} l_j+R_1))\right]} k_g(t, m_0, m_1, \theta)\, |d\theta|,$$

where the leading term of k_g is

$$k_g(t, m_0, m_1) \sim (2\pi)^{\frac{n-3}{2}} e^{-\frac{(n-1)i\pi}{4}} \frac{d_g(m_0, m_1)}{(l_g(m_0, m_1))^{\frac{1}{2}}}\, \theta^{-\frac{n-1}{2}}.$$

The proof is by induction on n.

For $n = 2$, we apply Lemma 4 with $B_1(t, m_1, m) = U_{\alpha_2}(t - t_0)\Pi_1$ and $B_0(m, m_0) = U_{\alpha_1}(t_0)$. Using Theorem 4, we have

$$b_0(m, m_0, \theta_0) \sim \frac{1}{2\pi} \frac{d_{\alpha_1}(x_0(m) - x_0)}{(R_0(m)R_0)^{\frac{1}{2}}} p_0(m, m_0, \theta_0)$$

and

$$b_1(t, m_1, m, \theta_1) \sim \frac{1}{2\pi} \frac{d_{\alpha_2}(x_1 - x_1(m))}{(R_1(m)R_1)^{\frac{1}{2}}} p_1(m_1, m, \theta_1)$$

and the composition takes place in $\mathbb{R}^2$ around a segment of length l_1. The functions p_0 and p_1 are homogenous (near infinity) of degree 0 in θ_i and take the cut-offs Π_i

into account. Using Lemma 4, we find that U_g can be written with the phase function $[t - R_0 - l_1 - R_1]\theta$ and with a symbol k_g whose leading term is

$$k_g \sim (2\pi)^{\frac{3}{2}} \frac{1}{4\pi^2} e^{-i\frac{\pi}{4}} \frac{d_{\alpha_1}(\beta_1 - x_0) d_{\alpha_2}(x_1)}{(R_0(m) R_0 R_1(m) R_1)^{\frac{1}{2}}} \frac{(R_0(m) R_1(m))^{\frac{1}{2}}}{l_1^{\frac{1}{2}}} \theta^{-\frac{1}{2}} p(m_1, m_0, \theta).$$

This can be simplified to give the desired result (remark that p is identically 1 in a microlocal neighbourhood of (m_1, m_0, θ)). We get the expression for $n+1$ diffractions from that for n by applying Lemma 4 once again. □

We collect some features of the microlocalized propagator along g.

- Each diffraction gains $\frac{1}{2}$ order of regularity.
- Each diffraction shifts the phase of $\frac{3\pi}{4}$ or of $-\frac{\pi}{4}$ depending on the sign of $d_\alpha(\beta)$.

The fact that U_g is a FIO will allow us to compute the contribution of a regular diffractive periodic orbit by applying exactly the same techniques as in the smooth case (see Section 4.1). Before getting to the trace we first derive an expression for the propagator near the optical boundary.

3.3. Optical boundary

Lemma 2 tells us that the propagator near the optical boundary will be obtained by applying an explicit FIO to the distribution G_α localized near the $(y = 1, x = \pm\pi)$. The structure of this distribution is given by the following lemma.

Lemma 5. *There exists two lagrangian distributions $R_{\pm,\alpha}$ in $\mathcal{D}'(\mathbb{R} \times \mathbb{R}/\alpha\mathbb{Z})$ associated with the lagrangian manifold $N^*\{y = 1\}$ such that, in the neighbourhood of $(y = 1, x = \pi)$ (resp. $(y = 1, x = -\pi)$), the following descriptions hold*:

$$G^1_\alpha(y, x) = H_\alpha(y, x) - \frac{1}{\pi} \arctan\left(\frac{\mathrm{ch}^{-1} y}{\pi - x}\right) + R_{+,\alpha}(y, x),$$

resp.

$$G^1_\alpha(y, x) = H_\alpha(y, x) - \frac{1}{\pi} \arctan\left(\frac{\mathrm{ch}^{-1} y}{\pi + x}\right) + R_{-,\alpha}(y, x).$$

The remainders $R_{\pm,\alpha}$ can be represented as oscillatory integrals:

$$R_{\pm,\alpha} = \int \exp\left[i(y-1)\theta\right] r_{\pm,\alpha}(x, \theta)\, |d\theta|,$$

where the leading term of the symbol $r_\pm$ is $O(|\theta|^{-\frac{3}{2}})$.

Proof. We take a cut-off ρ localizing near $(y = 1, z = \pi)$. The distribution G^1_α is the sum of $G^1\rho$ and of the periodization of $(1 - \rho)G^1$. The latter is obtained exactly as G^1_α away from $z = \pm\pi$. □

Remarks.

1. We should recall that $\alpha \neq 2\pi/k\ (k \in \mathbb{N})$. Thus, the cut-off ρ can be chosen so that only π (or only $-\pi$) is in the union of the supports of $\rho(. + k\alpha)$.
2. The most singular term does not depend on α.
3. There exists some classes of distribution that are associated to the intersection of lagrangian manifolds (see [16,20,23]). It is not clear if G belongs to one of these classes. To be slightly more precise, the distributions constructed in [23] can be thought of as distributions admitting the following kind of expansion:

$$u \sim \sum_k \sum_j x_+^{\alpha_0 - j} y_+^{\beta+k+j},$$

whereas in our case, we would get some (formal) expansion reading:

$$u \sim \sum_k \sum_j x_+^{\alpha_0 - 2j} y_+^{\beta+k+j}$$

and it is not completely clear what sense has to be given to such an expansion (although the more general construction of [20] probably allows such kind of expansions).

After applying the FIO A_0 we get the following description of U_α near the optical boundary:

Proposition 4 (*Near* Σ). *Near the optical boundary* Σ, *the propagator* $U_{\alpha,N}$ *can be written as the sum of three terms*:

$$U_{\alpha,N} = U^{\mathrm{ft}}_{\alpha,N} + U^{\mathrm{ds}}_{\alpha,N} + U^{\mathrm{dr}}_{\alpha,N}.$$

Each of these has the following description:

1. *The operator* $U^{\mathrm{ft}}_{\alpha,N}$ *corresponds to the free propagation localized to the "classical region"*:

$$U^{\mathrm{ft}}_{\alpha,N}(t, m_1, m_0) = U_{0,N}(t, m_0, m_1)H(\pi - (x_1 - x_0)),$$

2. *the operator* $U^{\mathrm{ds}}_{\alpha,N}$ *contributes to the diffracted front its kernel can be written as*

$$U^{\mathrm{ds}}_{\alpha,N}(t, R_1, x_1, R_0, x_0) = \int_{\theta>0, w>0} e^{i\theta[f(t,R_1,R_0)-\mathrm{ch}\, w]} a_N(t, R_1, R_0, \theta) i\theta^{-1} \times \frac{\pi-(x_1-x_0)}{w^2+(\pi-(x_1-x_0))^2}\rho_1(\pi-(x_1-x_0))\rho_2(\mathrm{ch}\, w)\, |dw\, d\sigma|,$$

3. *the operator* $U^{\mathrm{dr}}_{\alpha,N}$ *contributes regularly to the diffracted front. It is a FIO associated to* $N^*\{t-R_0-R_1=0\}$. *If one takes* $\sigma(t-R_0-R_1)$ *as the generating phase function, the symbol is* $0(|\sigma|^{-N})$.

Proof. We plug into the expression of $U_{\alpha,N}$ given in Lemma 2 the decomposition provided by Lemma 5. We get three terms.

1. The first one is given by

$$A_{N,\alpha}[H_\alpha(y, x)]$$

and is identified as the free propagator with cut-off.
2. The second one is given by applying $A_{N,\alpha}$ to $\arctan\left(\frac{\mathrm{ch}^{-1}(y)}{\pi-x}\right)\rho(y, x)$, which gives $U^{\mathrm{ds}}_{\alpha,N}$ after making the change of variables $y = \mathrm{ch}\, w$ and one integration by parts in w. (The integration by parts also give some integral involving the derivative of ρ with respect to y, but this is identically 0 near $y = 1$ and for wave-front reasons the corresponding operator is smoothing.)
3. The third one is obtained by applying $A_{\alpha,N}$ to $R_{+,\alpha}$. Sine $R_{+,\alpha}$ is a lagrangian distribution, we are led to exactly the same computations than for U_α away from Σ. This gives the result. Concerning the order it can be easily found by remarking that $A_{N,\alpha}$ is N times smoother than A_0 and that $A_{0,\alpha}$ applied to a symbol of order $O(|\sigma|^{-\frac{3}{2}})$ gives a symbol of order 0. □

Remarks.

1. The superscript ft stand for free and truncated, dr for diffractive and regular, and ds for diffractive and singular.
2. This decomposition is "essentially" unique. Indeed, it only depends of the choice of the cut-off function ρ used in the proof of Lemma 5. More precisely, if we have two decompositions then $U^{\mathrm{ds}} - \tilde{U}^{\mathrm{ds}}$ will be a FIO with the same wave front-relation and same order as U^{dr}.
3. Getting away from $x = \pi$, the singularities of $G_{1,\alpha}$ split into those living on $y + \cos x = 0$ and those living on $y = 1$. On the cone, we obtain the part corresponding to the primary front (that matches with U^{ft}_α) and the part corresponding to the diffractive front away from the optical boundary (that matches the sum of the two other terms).

4. It is of some interest to look at the wave front of each operator we have written. The wave-front of $U^{\mathrm{dr}}_{\alpha,N}$ is contained in the diffractive front. That of $U^{\mathrm{ft}}_{\alpha,N}$ is a subset of $\Lambda_0 \cup \Lambda_t$ where $\Lambda_t = N^*\{x_1 - x_0 = \pi\}$ corresponds to the cut-off. That of $U^{\mathrm{ds}}_{\alpha,N}$ is a subset of $\Lambda_d \cup \Lambda_t$. One peculiar feature of the description we have given is that we have apparently created some singularities on Λ_t. There we are assured that $U^{\mathrm{ft}}_{\alpha,N}$ and $U^{\mathrm{ds}}_{\alpha,N}$ have the same order of magnitude since they must exactly compensate. This artificial singularity is not so disturbing since when we will take the trace, it will disappear due to wave-front reasons.
5. Near $\Lambda_d \backslash \Sigma$, the operator $K^{\mathrm{ds}}_{\alpha,N}$ is a FIO matching with the propagator away from the optical boundary.

An expansion for a general microlocalized propagator could be obtained the following way. Each time a diffraction angle of $\pm\pi$ occurs we replace it by the latter sum, and each time we have a sequence of consecutive regular diffractive angles we replace it by the corresponding FIO. Since we do not know how to simply compose the operators occurring in the description near the optical boundary, such an expansion is, at this stage, useless. We will write it for one angle of diffraction of angle $\pm\pi$ when we will compute the trace of a regular family of periodic orbits. This will be done after we have addressed the case of a regular diffractive periodic orbit.

4. Leading contribution to the trace formula

In this section, we will compute the leading part created by a periodic orbit of length L. We begin by microlocalizing along g, this means that for any point m of the periodic orbit, we take U_g a microlocalized propagator along g (we recall that it implies that the cut-offs are identically 1 in a microlocal neighbourhood of g). We next take some test-function $h(t)$ that localizes near 0 and we form the following quantity:

$$I(s) = \left\langle \mathrm{Tr}\left[U(t)U_g(L)\right], h(t)\exp(-ist)\right\rangle,$$

of which we study the asymptotic behaviour when s goes to ∞.

4.1. Regular periodic diffractive orbit

We consider here a regular diffractive periodic geodesic of length L. We begin by addressing $I(s)$ when m is a regular point of g. Since we have localized near $m \in M_0$ at the beginning the computation takes place over M_0 and since the geodesic is regular, $U_g(t)$ is a FIO. The computation of the trace will thus run exactly as in the smooth case. We introduce some notations before stating the proposition.

For a regular diffractive geodesic g with n diffractive points, we define

$$D_g = \prod d_{\alpha_i}(\beta_i),$$

and

$$P_g = \prod l_i .$$

Lemma 6. *With the preceding notations, the behaviour of $I(s)$ for s going to infinity is given by*

$$I(s) \sim s^{-\frac{n}{2}}(2\pi)^{\frac{n}{2}} e^{-\frac{ni\pi}{4}} \frac{D_g}{P_g^{\frac{1}{2}}} h(L) e^{-isL} \int \rho(g(u)) \, |du|.$$

We will denote by c_g the constant $(2\pi)^{\frac{n}{2}} e^{-\frac{ni\pi}{4}} \frac{D_g}{P_g^{\frac{1}{2}}}$.

Proof. There is a cut-off ρ identically 0 near the conical points such that the trace is obtained (up to a smooth remainder) by applying the operator $\pi_* \tilde{\Delta}^*$ to $U(t)U_g(L)\rho$ (where $\tilde{\Delta}^*$ is the restriction to the diagonal and π_* the integration on M_0, cf. [8]). Furthermore, with this cut-off, and for t small enough we have (up to a smooth remainder) $U(t)U_g(L) = U_g(t+L)$. Since U_g is a known FIO, the action of $\pi_* \tilde{\Delta}^*$ followed by testing against $h(t)\exp(-ist)$ is given by the following oscillatory integral:

$$I(s) = \int e^{-ist} e^{i\theta[t - R_0(m) - R_1(m) - \sum l_j]} k_g(t, m, m, \theta) f(t) \rho(m) \, |dt \, dm \, d\theta|.$$

The distances $R_0(m)$ and $R_1(m)$ are the radial components of the polar coordinates of m centred at p_1 and p_n, respectively. Using the homogeneity of the phase, we are led to evaluate the following:

$$I(s) = s \int e^{is[-t + (t - R - d_1(R,x) - \sum l_j)\theta]} \tilde{k}_g(t, R, x, s\theta) \rho(R, x) R \, |dt \, dR \, dx \, d\theta|,$$

where $d_1(R, x) = R_1(m)$. This is done by performing a stationary phase in (t, x, θ), uniform with respect to R.

The critical points and the hessian matrix are given by

$$\begin{cases} -1 + \theta & = 0 \\ -\theta \partial_x d_1 & = 0 \\ t - R - d_1 - \sum l_j & = 0 \end{cases} ; \quad |H| = \begin{vmatrix} 0 & 0 & 1 \\ 0 & -\dfrac{\partial^2}{\partial x^2} d_1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = d_1^{-1} R l_n .$$

We finally get the following equivalent:

$$I(s) \sim s\left(\frac{2\pi}{s}\right)^{\frac{3}{2}} e^{-i\frac{\pi}{4}} e^{-isL} f(L) \int k_g(L, R, 0, s) \frac{d_1^{\frac{1}{2}}}{(Rl_n)^{\frac{1}{2}}} \rho(R) R\, |dR|.$$

Plugging into this formula the principal part of k_g (cf. Theorem 5) gives the result. □

We have now to compute the contribution given by the geodesic near the conical point. We recall that by definition, a microlocalized propagator such that the initial point is conical can be written $U_{\tilde{g}}(t-t_0)U_\alpha(t_0)$ for t_0 small where the initial point of $U_{\tilde{g}}$ is $g(t_0)$. The first thing to say is that we can shift the operator so that the trace is computed above M_0. This is explained in detail in [17]. We recall here the main lines of the proof. Using the cyclicity of the trace and the fact that the trace of a regularizing operator is smooth, we get (up to a smooth remainder)

$$\mathrm{Tr}(U(t)U_g(L)) = \mathrm{Tr}\left[U_{\tilde{g}}(t+L-2t_0)\chi U_\alpha(t_0)\rho U_\alpha(t_0)\chi\right],$$

where the initial point of $\tilde{g}$ is $g(t_0)$, χ localizes near $g(t_0)$ and ρ near p.

We now write $\rho = 1-(1-\rho)$ so that we the operator of which we take the trace can be written as

$$\left[U_{\tilde{g}}(t+L-2t_0)\chi U_\alpha(2t_0)\chi\right] - \left[U_{\tilde{g}}(t+L-2t_0)\chi U_\alpha(t_0)(1-\rho)U_\alpha(t_0)\chi\right].$$

Due to the wave-front relations, we can insert new microlocal cut-offs in U_α microlocalizing away from the optical boundary so that all the operators occurring in the latter expression are FIOs. In the FIO class the cut-offs act by multiplication on the principal symbol. Thus, the operator $U_g(T-2t_0)\chi U_\rho(2t_0)\chi$ (where $U_\rho(2t_0) = U_\alpha(2t_0) - U_\alpha(t_0)(1-\rho)U_\alpha(t_0))$ is a FIO associated with the same lagrangian manifold as $\chi U_g(T)\chi$. Furthermore the principal symbol is simply multiplied by $\rho(|t_0 - R_0|)$.

To evaluate $I_\rho(s) = \langle \mathrm{Tr} U_{\tilde{g}}(t-2t_0)\chi U_\rho(2t_0)\chi, h(t)e^{-its}\rangle$, we are thus led to exactly the same computations as for Lemma 6 up to multiplication by the cut-off functions.

Lemma 7. *When s goes to infinity, $I_\rho(s)$ has the following leading term*:

$$I_\rho(s) \sim c_g s^{-\frac{n}{2}} h(L) e^{-isL} \int \rho(|R|)\, |dR|,$$

where c_g is given in Lemma 6.

Proof. We use the proof of Lemma 6 and the fact that the cut-offs act by multiplication on the principal symbol. We get as leading term

$$I(s) \sim c_g \int_{R_0>0} \chi(g(R_0+t_0))\rho(|t_0-R_0|)\,|dR_0|.$$

We let then $R = t_0 - R_0$ and remark that, by construction χ is identically 1 when ρ is non-zero. □

We get the contribution of a regular diffractive periodic orbit by adding these local contributions.

Theorem 6. *The contribution of a regular periodic diffractive orbit g of period L, and of primitive length L_0 is given at leading order by*

$$I_{\mathrm{tot}}(s) \sim s^{-\frac{n}{2}} c_g h(L) e^{-isL} L_0,$$

with $c_g = (2\pi)^{\frac{n}{2}} e^{-\frac{ni\pi}{4}} \frac{D_g}{P_g^{\frac{1}{2}}}$, *where n is the number of diffractions, and D_g and P_g depend on the angles of diffraction and of the length of the different geodesic segments of g and are given in Definition* 4.1.

Proof. The contribution of g is given by the sum

$$I_{\mathrm{tot}}(s) = \sum \langle \mathrm{Tr}(U_g \rho_{m_i}), e^{-ist} f(t)\rangle,$$

where the m_i are points on g and $\sum \rho_{m_i}$ is identically 1 in a neighbourhood of $g_{|[0,L_0]}$. Each term of this sum has already been computed and we get

$$I_{\mathrm{tot}}(s) \sim s^{-\frac{n}{2}} c_g f(L) e^{-isL} \sum \int \rho_{m_i}(g(t))\,|dt|,$$

which is exactly the conclusion of the theorem. □

Several comments have to be made on this expression.

1. This contribution is valid for g and for all its multiples.
2. The leading order depends on the number of diffractions. In the trace there cannot be any cancellations between the contributions of two geodesics with different numbers of diffractions. Although we expect this fact to be true for any type of geodesics we have proved it only for regular diffractive ones.
3. Each diffraction gain $\frac{1}{2}$ order of regularity. The most singular contribution is given by periodic orbits with one diffraction and it is already $\frac{1}{2}$ smoother than the contribution of an isolated periodic geodesic on a smooth manifold (cf. [8]).

4.1.1. Application: triangles are spectrally determined

Corollary 1. *Let T be a euclidean triangle and $S_{\mathrm{D}}(T)$ (resp. $S_{\mathrm{N}}(T)$) be the spectrum of the euclidean laplacian in T with Dirichlet boundary condition (resp. Neumann). The equality $S_{\mathrm{D}}(T) = S_{\mathrm{D}}(T')$ (or $S_{\mathrm{N}}(T) = S_{\mathrm{N}}(T')$) holds if and only if T and T' are isometric.*

The proof consists in two steps: first we show that the length of an altitude is spectrally determined and then we recover the triangle from the knowledge of the altitude, the area and the perimeter. This result was first proved by Durso in [9] using this method.

We consider a triangle ABC with sides a, b, c (respectively, opposite to A, B and C) and angles $\hat{A}$, $\hat{B}$, and $\hat{C}$. We also denote by α the angle at A (i.e. $\alpha = \hat{A}$) and by β the angle between the altitude emanating from A and b. We consider M the surface obtained by doubling the triangle. Each altitude contained in the triangle corresponds to a geodesic of M uniquely diffractive (whose length is twice that of the altitude). We choose the altitude emanating from A and we assume that α is not $\frac{\pi}{k}$. If the corresponding geodesic on M is regular and since its angle of diffraction is 2β (the angle at the diffractive point is 2α), we get the following contribution

$$I(s) \sim \sqrt{\pi} e^{-\frac{i\pi}{4}} \frac{d_{2\alpha}(2\beta)}{l^{\frac{1}{2}}} h(2l) e^{-isl} s^{-\frac{1}{2}}, \tag{11}$$

where l is the length of the altitude.

We have now to know when an altitude is regular or not. Denote by s_a, s_b, s_c the linear part of the orthogonal symmetry across sides $a, b,$ and c. Unfolding the triangle, we see that the altitude emanating from A is not regular if and only if there exists N such that either $(s_c s_b)^N = -\mathrm{Id}$ or $s_c(s_b s_c)^N = s_a$ or $s_b(s_c s_b)^N = s_a$. The first case cannot happen since it implies $\alpha = \pi/2N$. The second (resp. third) case implies $\hat{A} = \hat{B}/N$ (resp. $\hat{A} = \hat{C}/N$). In particular, this shows that the smallest altitude is always regular and thus brings the contribution (11) in the trace. Since we have chosen the smallest altitude, there cannot be any cancellation with another periodic geodesic and thus, we have exactly described the leading order of the singularity of the trace at $2l$. It shows that there is indeed a singularity there and thus that the length of the smallest altitude is spectrally determined. Actually, because of the doubling procedure we have shown that this length is determined by $S_{\mathrm{N}} \cup S_{\mathrm{D}}$. We can get the same result with the spectrum of Dirichlet (or Neumann) by symmetrisation (this gives the contribution computed in [9]).

Using Weyl's law, the area and the perimeter of the triangle are also determined by S_{D}. Knowing these three parameters, we can recover the triangle in the following way. First the area and the altitude gives us one side of the triangle (say $[AB]$). Using the perimeter, C has to be on one particular ellipse of foci A and B, but C also has to be at a given distance of the line (AB) (using once again the altitude). This gives four possible points but the four corresponding triangles are isometric.

We now address the contribution of a regular family of periodic orbit.

4.2. Regular family

We recall that a regular family is a family of periodic orbits such that on each boundary only one diffraction occurs. It implies that the orbits of the family are necessarily primitive. There will be three types of operators of which we will have to take the trace. The simple ones are those corresponding to a microlocalized propagator in the interior of the family. The two other kinds will be at the boundary and respectively near and away from the conical point.

The contribution of the orbits in the interior of the family are easily handled since the microlocalized propagator only involves the free propagation, and taking the trace involves exactly the same computation as for a non-degenerate family of periodic orbits in the smooth case (cf. [8]). It is even simpler as in [8] since the metric is euclidean.

We recall the notations: ρ localizes near an interior point m_0 of the family, and is chosen so that its support is included in the support of the family. We also have

$$I(s) = \langle \mathrm{Tr}(U_g(t+L)\rho), h(t)e^{-ist} \rangle.$$

Lemma 8. *For a geodesic g in the interior of the family (g), the contribution I has for leading term*

$$I(s) \sim \frac{e^{i\frac{\pi}{4}}}{\sqrt{2\pi}} s^{\frac{1}{2}} \frac{1}{\sqrt{L}} h(L) e^{-isL} \int \rho(m_0)\,|dm_0|.$$

Proof. The microlocalized propagator is given by free propagation. We have to address

$$I(s) = \int e^{-ist} e^{i\theta\left[(t^2 - D^2(m_0,m_0))\right]} k_0(t, m_0, m_0, \theta)\rho(m_0)\,|dt\,dm_0\,d\theta|,$$

where

$$U_0(t, m_0, m_1) = \int e^{i\theta\left[(t^2 - D^2(m_0,m_1))\right]} k_0(t, m_0, m_1, \theta)\,|d\theta|$$

is the free propagator $e^{it\sqrt{\Delta_0}}$ in $\mathbb{R}^2$, for which we have the explicit expression:

$$k_0(t, m_0, m_1, \theta) \sim \frac{e^{i\frac{\pi}{4}}}{\pi\sqrt{\pi}} t\theta^{\frac{1}{2}}$$

(cf. [1] for example).

Here, we have $D(m_0, m_0) = L$, and the lemma follows from the application of a stationary phase argument. More precisely, there is a non-degenerate submanifold of critical points and we can apply the results of [8]. From a technical point of view, we have to perform the stationary phase with respect to (t, θ) uniformly in m_0. $\square$

We have now to address the contribution of the boundary orbits $g^{\pm}$. Both lead to exactly the same computation and we only write down the details for one. We thus consider the contribution of an uniquely diffractive orbit such that the diffraction angle is π. As it was already pointed out, the propagator is no more a FIO, so that we have to justify the fact that the trace is obtained by restricting to the diagonal and then integrating over M. This procedure is valid as soon as the considered operator acts continuously from L^2 into some H^N, with N large enough. Thus, the trace of $U_{g,N}$ can be handled this way. Using the functional calculus, we have the following identity:

$$I(s) = \left\langle \mathrm{Tr}(U_{g,N}(t+L)\rho),\, \frac{d^N}{dt^N}\left[e^{-ist}h(t)\right]\right\rangle.$$

Since we are dealing with a regular family of periodic orbits, there is only one diffraction so that $U_{g,N}$ is in fact $U_{\alpha,N}$. When ρ localizes near a regular point of g_+ we can compute $I(s)$ by putting $m_0 = m_1 = m$ and integrating over M. When ρ localizes near the conical point we want to use the cyclicity trick to get back to the same expression. However, since we are no more in the FIO class, it is not clear how the new cut-off functions will act at leading order. We thus have to write down what happens at the conical point and see the kind of operator we have to deal with.

By the cyclicity trick, we have to take the trace of the operator

$$U_{\tilde{g},N_1}(t+L-2t_0)\chi U_{\alpha,N_0}(t_0)\rho U_{\alpha,N_2}(t_0)\chi,$$

where $\tilde{g}$ starts at $g(t_0)$ and ends at $g(L-t_0)$. This portion is non-diffractive so that $U_{\tilde{g},N_1}$ can be replaced by U_{0,N_1}. As usual we write $\rho = 1-[1-\rho]$ and get on the one hand the operator

$$U_{\tilde{g},N_1}(t-2t_0)\chi U_{\alpha,N_0+N_2}(2t_0)\chi$$

and on the other hand the operator

$$U_{\tilde{g},N_1}(t-2t_0)\chi U_{\alpha,N_0}(t_0)[1-\rho]U_{\alpha,N_2}(t_0)\chi. \tag{12}$$

This latter operator contains two factors U_{α,N_i} but the cut-off functions and a wave-front argument imply that both cannot be diffractive at the same time, so that one can be replaced by U_{0,N_i}. This gives two operators which are written as (12) with one α replaced by 0. We can compose this factor U_0 with the other one (after using once again the cyclicity when needed).

Eventually all the operators we have to take the trace can be written as

$$\tilde{U}_{0,N_1'}(t+L-t')U_{\alpha,N_0'}(t')\chi',$$

where the following properties hold:

- χ cuts off away from the conical points,
- $\tilde{U}$ is a FIO associated with the free propagation, it differs from U_0 at leading order by multiplication by some cut-off function ρ,
- $U_{\alpha,N_1'}$ is U_α cut-off near the diffraction angle $|x_1 - x_0| = \pi$,
- t' is either t_0 or $2t_0$
- N_0' and N_1' are either one of the $N_i's$ or the sum of two of them.

In the following, we will drop the $'$ and return to the notation $t' = t_0$.

Remark. The contribution of the propagator near a regular point can also be written as the trace of an operator satisfying the preceding properties. We only have to write $U_g(t+L) = \tilde{U}_0(t+L-t')U_\alpha(t')$ (up to a smoothing operator).

Taking the trace against a test function $\frac{d^N}{dt^N}\left[e^{-ist}h(t)\right]$, we have

$$\tilde{I}_s = \int \tilde{U}_{0,N_1}(t-t_0,m_0,m_1)U_{\alpha,N_0}(t_0,m_1,m_0)\chi(m_0)\frac{d^N}{dt^N}\left[e^{-ist}h(t)\right]|dt\,dm_1\,dm_0|.$$

Writing down explicitly the derivative of the test function, $\tilde{I}(s)$ has for leading order $(is)^N I(s)$, where $I(s)$ is

$$I(s) = \int \tilde{U}_{0,N_1}(t+L-t_0,m_0,m_1)U_{\alpha,i,N_0}(t_0,m_1,m_0)\chi(m_0)e^{-ist}h(t)\,|dt\,dm_1\,dm_0|.$$

We now use the decomposition proved in Lemma 4 and get three terms I^{ft}, I^{ds}, I^{dr} corresponding respectively to each term in

$$U^{\mathrm{ft}}_{\alpha,N_0} + U^{\mathrm{ds}}_{\alpha,N_0} + U^{\mathrm{dr}}_{\alpha,N_0}.$$

4.2.1. Contribution of I^{ft}

The integral giving I^{ft} can be written as

$$I^{\mathrm{ft}}(s) = \int \tilde{U}_{0,N_1}(t+L-t_0,m_0,m_1)U_{0,N_0}(t_0,m_1,m_0) \times H(\pi-x_1-x_0)\rho(m_0)h(t)e^{-ist}\,|dt\,dm_0\,dm_1|.$$

We replace $\tilde{U}_{0,N_1}$ and U_{0,N_0} by their expression in oscillatory integral and perform the stationary phase. The only difference with the contribution of the interior of the family is the cut-off function $H(\pi-x_0-x_1)$, so that the stationary phase involves an integral on a domain with boundary. One has to be a little more careful but since the set of critical points intersects the boundary transversally, the leading order is the same as the one computed in Lemma 8. We get:

Lemma 9. *At leading order, the integral* $I^{\mathrm{ft}}(s)$ *is given by*

$$I^{\mathrm{ft}} \sim (is)^{-N} \frac{e^{i\frac{\pi}{4}}}{\sqrt{2\pi}} s^{\frac{1}{2}} \frac{1}{\sqrt{L}} \int_{\mathcal{A}_g} \rho(m_0)\, |dm_0|.$$

In this formula, $\mathcal{A}_g$ *is the domain swept out by the family* (g)*, and* ρ *takes into account the cut-off* χ *and the one in* $\tilde{U}$*. The remainder has for order* $s^{-N-\frac{1}{2}}$*.*

This contribution is exactly of the same kind as that from the interior. Both will match to give the leading contribution involving the area $\mathcal{A}_g$.

4.2.2. Contribution of I^{dr}

This one is the simplest since U^{dr} is a FIO and we are led to the same computations as in the regular diffractive case. That gives the lemma.

Lemma 10. *The contribution* $I^{\mathrm{dr}}(s)$ *is of order* $s^{-N-\frac{1}{2}}$*.*

We do not look for a more precise result since this contribution is already 1 time smoother than I^{ft}.

Remark. We have already said that the contribution of a regular diffractive is $\frac{1}{2}$ times smoother than the contribution of a non-diffractive isolated orbit. The contribution I^{ft} is $\frac{1}{2}$ more singular than the latter so that we eventually get a shift of 1 between I^{ft} and I^{dr}.

4.2.3. Contribution of I^{ds}

We will once again apply a stationary phase argument to evaluate I^{ds}. We will here only sketch this evaluation. The technical material can be found in Appendix B.

We have to compute:

$$\begin{aligned} I^{\mathrm{ds}}(s) = \int & e^{i\theta_1\left[[(t-t_0)^2 - D_1^2(m_0,m_1)]\right]} \rho_1 \tilde{\rho}_1 k_{0,N_1}(t-t_0, m_0, m_1, \theta_1) \\ & \times e^{i\sigma[f(t,R_1,R_0) - \mathrm{ch}\, w]} a_{N_0}(t, R_1, R_0, \sigma) \sigma^{-1} \frac{\pi - (x_1 - x_0)}{w^2 + (\pi - (x_1 - x_0))^2} \\ & \times \rho_1(\pi - (x_1 - x_0)) \rho_2(\mathrm{ch}\ w) \rho_0 \tilde{\rho}_0\, |dw\, d\sigma\, dm_0\, dm_1\, d\theta\, d\sigma|. \end{aligned} \tag{13}$$

We use the homogeneity in θ and σ and let $z = \pi - (x_1 - x_0)$ (keeping x_1). We then perform a stationary phase in $(t, R_1, x_1, \theta, \sigma)$ uniformly with respect to (R, w, z).

This results in a complete expansion:

$$I^{\mathrm{ds}}(s) = s^{-N+\frac{1}{2}} \sum s^{-k} I_k,$$

in which each I^k can be written as

$$I_k(s) = \int e^{i[-s\psi(R,w,z)]} a_k(R,w,z) \frac{z}{z^2+w^2} \, |dR\, dw\, dz|,$$

where the coefficients a_k are $\mathcal{C}^\infty$ and compactly supported in R, w, z.

The complete proof is in the appendix. Here we will only stress the important features of the resulting integral

- $\psi(R,w,x)$ is obtained by writing the critical points of the phase and evaluating t there (as a function of the parameters).
- The functions a_k are expressed by the stationary phase expansion in terms of the complete symbol in I_{ds} (and its derivatives) evaluated at the critical point.
- The principal order can be found as follows: the use of the homogeneity gives the power $\frac{1}{2} - N_0 + \frac{3}{2} - N_0 - 1 + 2$, i.e. $-N+3$ in front. The stationary phase involves 5 oscillatory variables, that gives: $-N+3-\frac{5}{2} = -N+\frac{1}{2}$.

Furthermore, $a_0(R,0,0) = t_0^{-1}$ is also prescribed. The new phase $\psi(R,w,z)$ has for Taylor expansion

$$\psi(R,w,z) = \psi_0(R) + Q(R,w,z) + \psi_2(R,w,z),$$

where $Q(R,w,z)$ is the quadratic form (w.r.t. w,z)

$$Q_{|(R,0,0)} = \begin{pmatrix} \dfrac{R(t_0-R)}{t_0} & 0 \\ 0 & \dfrac{R(t_0-R)}{t_0} \end{pmatrix}.$$

With these properties of ψ we evaluate I_k by stationary phase as the following lemma shows.

Lemma 11. *Let the phase $\psi(R,w,z)$ be $\mathcal{C}^\infty$ and have the following behaviour near $(w=0, z=0)$:*

$$\psi(R,w,z) = \psi_0(R) + Q(R,w,z) + \psi_2(R,w,z),$$

with $\psi_0,\ Q,\ \psi_2 \in \mathcal{C}^\infty$ and $Q(R,w,z)$ is a definite quadratic form in (w,z). Assume the remainder can be written

$$\psi_2(R, r\cos\theta, r\sin\theta) = r^3 g(R,r,\theta), \quad g \in \mathcal{C}^\infty.$$

Then, the integral I_k has a complete expansion in powers of s and the leading term is

$$I_k(s) \sim s^{-\frac{1}{2}} \int_{R,\theta} e^{is[\psi_0(R)]} a(R,0,0)\sin(\theta)|Q(R,\cos\theta,\sin\theta)|^{-\frac{1}{2}}\,|dR\,d\theta|.$$

Proof of Lemma 11. We use polar coordinates for (w, z)

$$I(s) = \int e^{is[\psi(R,r\cos\theta,r\sin\theta)]}\tilde{a}(R,r,\theta)\,|dr\,dR\,d\theta|,$$

with $\tilde{a}(R,r,\theta) = a(R, r\cos\theta, r\sin\theta)\sin\theta$. The assumptions permit us to do a stationary phase w.r.t. r uniformly in (R,θ) and this gives the result. □

If we come back to I^{ds}, the principal term is of order s^{-N} and we get it by writing the coefficient I^0 given by Lemma 11. That is

$$\int e^{is[\psi_0(R)]}|R_0(t_0 - R_0)|^{-\frac{1}{2}}\sin(\theta)\,|d\theta\,dR_0|,$$

which is eventually 0. Thus we have proved The contribution of the boundary is of negligible order with respect to the contribution of the interior. □

More precisely, the contribution of I^{ds} is comparable to the first remainder term of I^{ft}.

Summing everything up we get the theorem.

Theorem 7. *The contribution of a regular family of periodic orbits is*

$$\frac{e^{i\frac{\pi}{4}}}{\sqrt{2\pi}} s^{\frac{1}{2}} \frac{1}{\sqrt{L}} h(L) e^{-isL} |\mathcal{A}_g|,$$

where $|A_g|$ is the area swept out by the family and L is the (*necessarily*) *primitive length of the family.*

We sum up the contributions that we have computed. The contribution of the interior matches with I^{ft}. This matching, as in the case of regular diffractive orbits is made by the cut-off functions ρ_m.

This contribution is exactly the same as the contribution of a family of periodic orbit in the smooth case. The only difference is that the area swept is that of a surface with boundary. The boundary only brings correcting terms (this agrees with [2]). This computation only holds for primitive family since the iterates of a regular family will never be regular.

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Appendix A. periodization of G

The goal of this appendix is to periodize G and to show that away from the optical boundary G is a lagrangian distribution that admits the expansion

$$G(y,z) = \sum a_k(z)(y-1)_+^{\frac{1}{2}+k}.$$

We denote by I_ε the interval $]1-\varepsilon, 1+\varepsilon[$, and $J_{\varepsilon'} = \mathbb{R}\backslash\{|z \pm \pi| < \varepsilon'\}$. Using a wave-front argument, the behaviour of the propagator near the diffracted wave-front and away from the optical boundary depends only of G in $I_\varepsilon \times J_{\varepsilon'}$, once we have chosen ε and ε' small enough. In the following everything (in particular the constants) depends on these two numbers ε and ε'. In order to simplify the notations, we will not explicit this dependence and denote simply by $I \times J$ any set $I_\varepsilon \times J_{\varepsilon'}$.

We recall the definition of G:

$$G(y,z) = H(y+\cos z)H(\pi - |z|) - \frac{H(y-1)}{\pi}\left[\arctan\left(\frac{\mathrm{ch}^{-1} y}{\pi - z}\right) + \arctan\left(\frac{\mathrm{ch}^{-1} y}{\pi + z}\right)\right].$$

On $I \times J$, $H(y+\cos z)H(\pi-|z|)$ is identically 1 or identically 0 it can thus be periodized and the periodization results in a function of $I \times (\mathbb{R}/\alpha\mathbb{Z})$ smooth away from the optical boundary.

To study the remaining term in G we first derive with respect to y. We let

$$G_\pm(y,z) = \frac{\pi \pm z}{(\mathrm{ch}^{-1}(y))^2 + (\pi \pm z)^2},$$

so that we have

$$G(y,z) = R(y,z) - \frac{1}{\pi}\int_1^y \left[G_+(u,z) + G_-(u,z)\right](u^2-1)_+^{-\frac{1}{2}}\,|du| \tag{14}$$

on $I \times J$, and R is smooth and can be periodized in a smooth function.

Since $(u^2-1)_+^{-\frac{1}{2}}$ is a lagrangian distribution that is independent of z, all we have to show is that

$$\frac{\pi \pm z}{(\mathrm{ch}^{-1}(u))^2 + (\pi \pm z)^2}$$

is smooth on $I \times J$ and can be periodized in a function of $I \times (\mathbb{R}/\alpha\mathbb{Z})$ smooth away from the optical boundary.

We first remark that since $\mathrm{ch}^{-1}(u)$ is a primitive of $(y-1)_+^{-\frac{1}{2}}$, there exists $c(u)$ smooth on $]0,2[$ such that

$$\mathrm{ch}^{-1}(u) = c(u)(u-1)_+^{\frac{1}{2}}.$$

Taking squares, this ensures that the function $\left[\mathrm{ch}^{-1}(u)\right]^2$ can be extended to $]0,2[$ in a smooth, positive function $h(u)$. Moreover we have $h(u) \sim 2(u-1)$ in the neighbourhood of 1.

The functions $G_\pm$ can also be written as

$$G_\pm(u,z) = \frac{\pi \pm z}{h(u) + (\pi \pm z)^2}$$

and can thus be extended to smooth functions on $I \times J$.

Performing integration (14), it is then clear that on $I \times J$, G is a lagrangian distribution associated with $N^*\{y=1\}$. We get its asymptotic expansion by writing the Taylor expansion for $G_\pm$ in the neighbourhood of $u=1$, followed by integration (14). Hence, we have

$$G(y,z) \sim \sum_k g_k(z)(y-1)_+^{\frac{1}{2}+k}. \tag{15}$$

We now want to periodize term by term this expansion. The following lemma gives the needed estimates on the partial derivatives of $G_\pm$ to do that.

Lemma A.1. *For all $k, l \in \mathbb{N}$ such that $k+l \geqslant 1$, there exist two functions $P_{\pm,k,l}(y,z)$ that are polynomial w.r.t. z. such that*

$$\partial_y^k \partial_z^l G_\pm(y,z) = \frac{P_{\pm,k,l}(y,z)}{\left[h(y) + (\pi \pm z)^2\right]^{1+k+l}}.$$

Moreover, the degree of $P_{\pm,k,l}$ is $2(k-1)+l+1$ if $k \geqslant 1$ and $l+1$ if $k=0$.

The proof is an induction on both k and l.

A.1. First step

We derive once $G_\pm$ with respect to u or z:

$$\partial_u G_\pm(u,z) = \frac{-h'(u)(\pi \pm z)}{\left[h(u) + (\pi \pm z)^2\right]^2}$$

$$\partial_z G_\pm(u,z) = \frac{-2z(\pi \pm z) \pm \left[h(u) + (\pi \pm z)^2\right]}{\left[h(u) + (\pi \pm z)^2\right]^2}.$$

We check the lemma by inspection.

A.2. Induction

We derive the expression given in the lemma with respect to u and z:

$$\partial_u^{k+1}\partial_z^l G_\pm(u,z) = \frac{-(k+l+1)h'(u)P_{\pm,k,l} + \left[h(u) + (\pi \pm z)^2\right]\partial_u P_{\pm,k,l}}{\left[h(u) + (\pi \pm z)^2\right]^{k+l+2}},$$

$$\partial_u^{k}\partial_z^{l+1} G_\pm(u,z) = \frac{-2(k+l+1)P_{\pm,k,l} + \left[h(u) + (\pi \pm z)^2\right]\partial_z P_{\pm,k,l}}{\left[h(u) + (\pi \pm z)^2\right]^{k+l+2}}.$$

The claimed lemma follows.

Inspecting the respective degrees of the numerator and of the denominator we get the

Corollary A.1. *For all k, l such that $k + l \geqslant 1$, the following estimate holds uniformly on $I \times J$*

$$|\partial_u^k \partial_z^l G_\pm(u,z)| \leqslant C(|z|+1)^{-2}.$$

It remains to study the case $k = l = 0$. In this case, we cannot address $G_\pm$ separately, but summing them we have

$$G_+(u,z) + G_-(u,z) = \frac{2\pi h(u) + 2\pi(\pi^2 - z^2)}{(h(u) + (\pi - z)^2)(h(u) + (\pi + z)^2)}$$

and thus the estimate:

$$|G_+(u,z) + G_-(u,z)| \leqslant C(|z|+1)^2,$$

uniformly on $I \times J$.

With these estimates, the asymptotic expansion (15) can be term by term periodized. This gives a lagrangian distribution G_α on $I \times (\mathbb{R}/\alpha\mathbb{Z})$ away from the optical boundary. This distribution is associated with the lagrangian $N^*\{y = 1\}$ and the leading term is (according to the expansion)

$$G_\alpha(y,x) = C\left(\sum_k \frac{1}{\pi^2 - (x + k\alpha)^2}\right)(y-1)_+^{\frac{1}{2}}.$$

Appendix B. stationary phase

The main aim of this appendix is to give some details on the stationary phase arguments we have used in the paper to evaluate I^{ds}. More precisely, we are concerned with the stationary phase argument before the integration in x, w. Since we will not need the exact expression of the principal symbol, we will be rather careless with it and focus mainly on the order and on the dependence with respect to the parameters.

We start with expression (13)

$$I^{\mathrm{ds}}(s) = \int e^{i\theta_1[[(t-t_0)^2 - D_1^2(m_0,m_1)]]}\rho_1\tilde{\rho}_1 k_{0,N_1}(t-t_0, m_0, m_1, \theta_1)$$
$$\times e^{i\sigma[f(t,R_1,R_0)-\mathrm{ch}\, w]} a_{N_0}(t, R_1, R_0, \sigma)\frac{\pi - (x_1 - x_0)}{w^2 + (\pi - (x_1 - x_0))^2}$$
$$\times \rho_1(\pi - (x_1 - x_0))\rho_2(\mathrm{ch}\, w)\rho_0\tilde{\rho}_0\, |dw\, d\sigma\, dm_0\, dm_1\, d\theta\, d\sigma|.$$

We replace all the cut-off functions in m_0, m_1 by one that we denote by ρ.

We also replace k_{0,N_1} and a_{N_0} by their leading part; that of a_{N_0} can be simply deduced from that of a_0 by using the pseudo-differential calculus.

We first dilate the phase variables θ and σ. Actually, we first have to insert some cut-off (depending on s) ensuring that after the dilation we get an integral with compact support in the new variables. This is a standard procedure, that is resumed in the following steps: we look at the stationary points, it gives a compact set in which $s\theta$, and $s\sigma$ have to live, we then insert the corresponding cut-off. Using integration by parts with respect to all the variables except σ and θ, the left-over is shown to be $O(s^{-\infty})$.

We also let $x = \pi - x_1 + x_0$. At leading order we have

$$I^{\mathrm{ds}}(s) \sim Cs^3(is)^{-N}\int_{w>0} e^{is[\Phi]}[2i(t-t_0)]^{1-N_1}\theta^{\frac{1}{2}-N_1}\left(\frac{t_0}{R_0R_1}\right)^{1-N_0}\sigma^{\frac{1}{2}-N_0}$$
$$\times\frac{x}{w^2+x^2}\rho f(t)(R_0R_1)^{\frac{1}{2}}\, |dw\, dR_0\, dR_1\, dx_1\, dx\, d\theta\, d\sigma\, dt|.$$

To simplify the computation we can make the change $\sigma \leftarrow \frac{\sigma}{2R_0R_1}$ and we are led to evaluate the integral

$$J(s) = \int_{w>0} e^{is[\Phi]}[2i(t-t_0)]^{1-N_1}\theta^{\frac{1}{2}-N_1}t_0^{1-N_0}\theta^{\frac{1}{2}-N_0}\frac{x}{w^2+x^2}\rho$$
$$\times f(t)\, |dw\, dR_0\, dR_1\, dx_1\, dx\, d\theta\, d\sigma\, dt|,$$

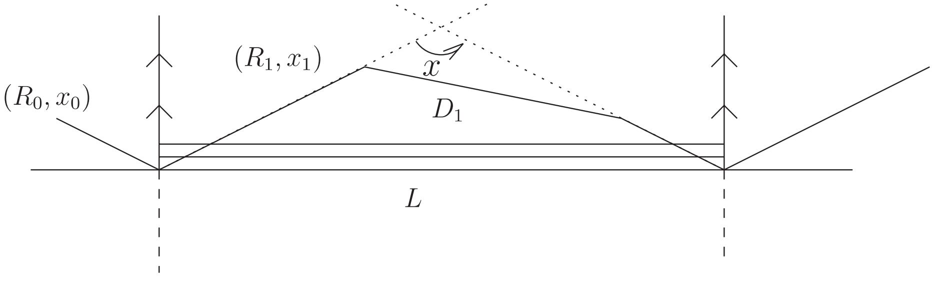
in which the phase is

$$\Phi(R_0, x, w, t, R_1, x_1, \theta, \sigma) = -t + \theta[(t-t_0)^2 - D_1^2] + \sigma[t_0^2 - R_0^2 - R_1^2 - 2R_0R_1\mathrm{ch}\, w]$$

and D_1^2 is given by

$$D_1^2 = (R_1 \cos x_1 + R_0 \cos(x + x_1) - L)^2 + (R_1 \sin x_1 + R_0 \sin(x + x_1))^2.$$

The following picture summarizes the notations.



The critical points of the phase (w.r.t. $t, R_1, x_1, \theta, \sigma$ are given by the (non-linear) system:

$$\begin{aligned}
\partial_t \Phi &= -1 + 2(t - t_0)\theta = 0,\\
\partial_\theta \Phi &= (t - t_0)^2 - D_1^2 = 0,\\
\partial_\sigma \Phi &= t_0^2 - R_0^2 - R_1^2 - 2R_0 R_1 \mathrm{ch}\, w = 0,\\
\partial_{R_1} \Phi &= -\theta \partial_{R_1}(D_1^2) - 2\sigma(R_1 + R_0 \mathrm{ch}\, w) = 0,\\
\partial_{x_1} \Phi &= -\theta \partial_{x_1}(D_1^2) = 0.
\end{aligned}$$

Remarking that the last line implies that the segment joining m_0 to m_1 has to be parallel to the limiting periodic geodesic, we have the following system for the critical points. It determines (t, R_1, x_1) as a function of the parameters (R_0, x).

$$(\Sigma) \begin{cases} t_0^2 - R_0^2 - R_1^2 - 2R_0R_1 \mathrm{ch} w & = 0,\\ R_1 \sin(x_1) + R_0 \sin(x + x_1) & = 0,\\ L(L + t_0 - t)\sin(x_1) - R_0 \sin(x) & = 0. \end{cases}$$

We will denote by ψ the function of (R_0, w, x) that gives the t component of the solution of the system (Σ).

At the critical point determined by (R_0, w, x), the hessian matrix is

$$H = \begin{vmatrix} 2\theta & 2(t-t_0) & 0 & 0 & 0 \\ 2(t-t_0) & 0 & 0 & 0 & -\partial_{R_1}(D_1^2) \\ 0 & 0 & 0 & -2(R_1 + R_0 \mathrm{ch}(w)) & 0 \\ 0 & 0 & 0 & -\theta\partial^2_{R_1^2}(D_1^2) & -\theta \sin x_1 \\ 0 & -\partial_{R_1}(D_1^2) & -2(R_1 + R_0\mathrm{ch}(w)) & -\theta \sin x_1 & -\theta\partial^2_{x_1^2}(D_1^2) \end{vmatrix}.$$

Since the critical point corresponding to the parameters $(R_0, x = 0, w = 0)$ is given by $t = L, x_1 = \pi, R_1 = t_0 - R_0, \theta = \frac{1}{2(L-t_0)}, \sigma = \frac{1}{t_0}$, we can rewrite the hessian matrix there.

$$H_{(R_0,0,0)} = \begin{vmatrix} \dfrac{1}{2(L-t_0)} & 2(L-t_0) & 0 & 0 & 0 \\ 2(L-t_0) & 0 & 0 & 0 & -(L-t_0) \\ 0 & 0 & 0 & -2t_0 & 0 \\ 0 & 0 & -2(t_0) & \dfrac{t_0}{L-t_0} & 0 \\ 0 & -(L-t_0) & 0 & 0 & -\dfrac{t_0}{(L-t_0)} \end{vmatrix}.$$

We thus have $|H_{(R_0,0,0)}| = 16t_0^3(L-t_0)$, and the signature is -1.

We can thus apply the stationary phase uniformly with respect to the parameters. We get

$$J(s) \sim \left(\frac{2\pi}{s}\right)^{\frac{5}{2}} e^{-i\frac{\pi}{4}} \frac{f(L)}{\sqrt{L}} \int_{w>0,R_0,x} e^{i[-st(R,w,x)]}\tilde{a}(R,w,x)\rho(R,w,x) \\ \times \frac{x}{w^2+x^2}\,|dR\,dw\,dx|,$$

where $\tilde{a}$ is smooth function of (R, w, x) such that

$$\tilde{a}(R,0,0) = t_0^{-1}.$$

Indeed, we get $\tilde{a}$ by evaluating the amplitude of J at the critical point divided by $|H|^{\frac{1}{2}}$. Everything has been computed when $(w, x) = (0, 0)$ and it gives the result.

We thus have proved the lemma

Lemma B.1. *The leading term of I^{ds} is given by*

$$I^{\mathrm{ds}}(s) \sim C(is)^{-N} s^{\frac{1}{2}} \int_{w>0,R_0,x} e^{i[-s\psi(R,w,x)]}\tilde{a}(R,w,x)\rho(R,w,x)\frac{x}{w^2+x^2}\,|dR\,dw\,dx|,$$

in which the constant depends in particular of h and L. The phase ψ is the component t of the solution of system (Σ). The amplitude $\tilde{a}$ is smooth and ρ is a compactly supported function.

The last thing we have to check is that $\psi(R_0, w, x)$ fulfills the hypothesis of Lemma 11. To do that we have to compute the partial hessian of ψ, w.r.t. (x, w). Using (Σ), we find: $(\partial_w \psi)_{|(R,0,0)} = 0$, $(\partial_x \psi)_{|(R,0,0)} = 0$ and

$$D^2_{(w,x)}\psi_{|(0,0)} = \begin{pmatrix} \dfrac{R_0(t_0 - R_0)}{t_0} & 0 \\ 0 & \dfrac{R_0(t_0 - R_0)}{t_0} \end{pmatrix}.$$

We check by inspection that it is positive definite and that justifies the last stationary phase argument in Lemma 11.

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