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Spectral decomposition of square-tiled surfaces

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Abstract We consider N copies of a square S_0 and define selfadjoint extensions of the Euclidean Laplacian acting on $\left(\mathcal{C}_0^\infty(S_0)\right)^N$ by choosing some boundary conditions that are parametrized by two unitary matrices H and V acting on $\mathbb{C}^N$. Denoting by $\mathrm{Sp}(\Delta_N, H, V)$ the spectrum of such an operator we derive conditions on H and V so that the following spectral decomposition holds:

$$\mathrm{Sp}(\Delta_N, H, V) = \bigcup_{1\le i\le k} \mathrm{Sp}(\Delta_{N_i}, H_i, V_i) \quad \text{with} \sum_1^k N_i = N.$$

If H and V are permutation matrices this gives a spectral decomposition of the spectrum of the square-tiled surface defined by the corresponding permutations. We apply this to derive examples related to isospectrality and to high multiplicity.

0 Introduction

Square-tiled surfaces can be endowed with a singular Euclidean metric that has many interesting geometrical and dynamical properties (see [1]). In this paper, we address the Laplacian associated with this metric. It is indeed natural to expect that the particular geometrical features of these surfaces have influence on the spectrum of their Laplacian.

We will show that the Laplacian on any square-tiled surface belongs to a class $\mathcal{O}_N$ of operators in which we can easily implement the transplantation theory of Bérard and Buser ([2,3]). This will result in a general spectral decomposition theorem (see Theorem 3). This theorem may be applied to various settings and we we will use it to prove the two following theorems.

Theorem 1 *Let M_6, M_3, T_1, T_2 be the square-tiled surfaces as defined in Fig. 1. The following identity holds*:

$$Sp(M_6) + 2\,Sp(T_1) \;=\; 2\,Sp(M_3) + Sp(T_2).$$

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This theorem may be interpreted as isospectrality between two non-connected surfaces. It also provides new examples of non-trivial linear combinations between the spectra of branched coverings of two-dimensional tori (see [4,5]).

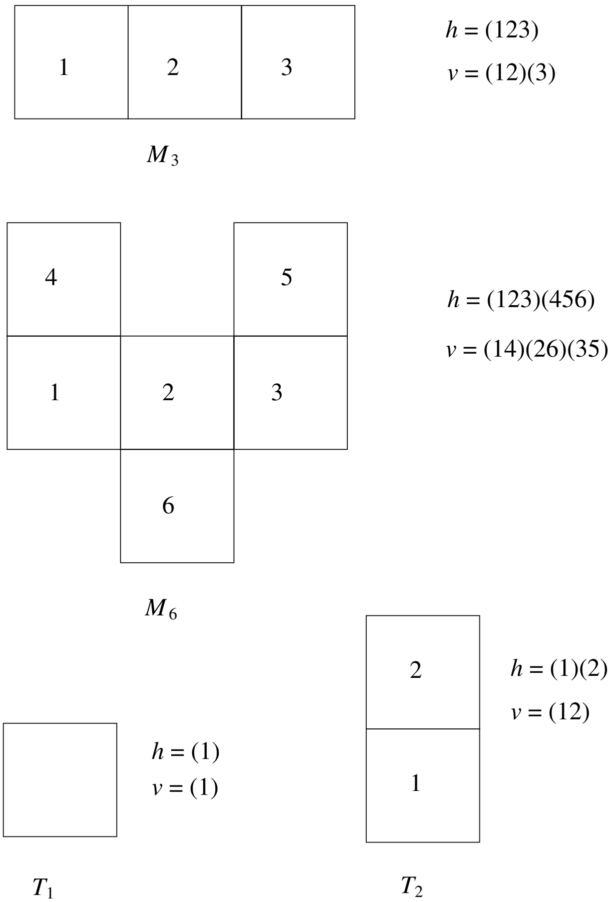
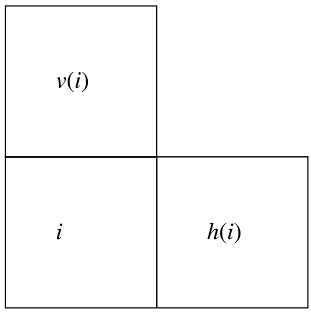


Fig. 1 Isospectral configuration

Fig. 2 Gluing pattern



Theorem 2 *There exists a sequence of square-tiled surfaces M_n such that the genus g_n goes to infinity and such that, for any ε, the following inequality holds for n large enough*:

$$s\text{-}m(M_n) \geq g_n^{\frac{1}{2}-\varepsilon}$$

where s-m denotes the structural multiplicity.

The notion of structural multiplicity is defined in Sect. 6. The bound in $\sqrt{g}$ is interesting with respect to the more general study of multiplicity in Riemannian geometry (see [6]).

Since any square-tiled surface may be seen as a branched covering of the torus, the spectral decomposition theorem may also be understood as a Peter–Weyl type theorem in the theory of Riemannian coverings, or more generally in the theory of operators commuting with some group action. However, it should be noted that, at first sight, there is no obvious group action commuting with the Laplacian of a square-tiled surface.

1 Square-tiled surfaces

In this section, we will review the definition of a square-tiled surface and give some basic facts concerning its geometry. We refer to [1] and the references therein for a much more complete presentation of these objects.

We denote by S_0 the square $[0, 1]_x \times [0, 1]_y$. To contruct a square-tiled surface, we take N squares $(S_i)_{1\leq i\leq N}$ each one isometric to S_0 and we identify pairwise the horizontal (resp: vertical) sides of the tiles S_i. We will denote by (x_i, y_i) the coordinates on S_i.

Definition 1 (Square-tiled surface) Consider N isometric squares $(S_i)_{1\leq i\leq N}$, and two permutations h and v of $\mathfrak{S}_N$. The square-tiled surface associated with (h, v) is defined by

$$M = \bigcup_{i=1}^{N} S_i / \sim,$$

with the following identification:

$$\begin{cases} (x_i, 1) \sim (x_{v(i)}, 0) \\ (1, y_i) \sim (0, y_{h(i)}) \end{cases}$$

We will denote by G the subgroup of $\mathfrak{S}_N$ generated by h and v.

With this definition, for any i, $h(i)$ denotes the tile located on the right of the ith tile and $v(i)$ the tile above. The permutations h and v thus describe the gluing pattern in the horizontal (resp. vertical) direction. (see Fig. 2)

Remarks 1. Denote by $p_1 \dots p_k$ the points on M corresponding to the vertices of the tiles, and let $M_0 = M\backslash\{p_1, \dots, p_k\}$. Using this cell-decomposition to compute the Euler characteristic of M gives us the genus of the surface

$$g = \frac{N-k+2}{2}. \tag{1}$$

2. Actually, the fact that S_0 is a square is irrelevent for everything we will say. Everything works for parallelogram-tiled surfaces.

Most of the geometric properties of the square-tiled surface M can be directly deduced from the knowledge of h and v. For instance, M is connected if and only if G acts transitively on $\{1 \dots N\}$.

The Euclidean metric on each square defines an Euclidean metric on M_0. In the neighbourhood of p_i, the surface M is obtained by gluing together a certain number of tiles so that the metric is locally that of an Euclidean cone. The following lemma gives the way of recovering the angles of the conical singularities from h and v.

Lemma 1 *Let c be the commutator of h and v ($c = h^{-1}v^{-1}hv$). In the decomposition of c as a product of cycles, each cycle of length l corresponds to one conical singularities of angle $2l\pi$.*

Proof Starting from tile i, the tile $c(i)$ is obtained by going up, then to the right, then down, then to the left. If i is in the support of a l-cycle, then l sequences *up-right-down-left* must be done before getting back to the ith tile. □

Each point p_i in $M\backslash M_0$ corresponds to one cycle in c. In particular the fixed point of c correspond to 'fake' singularities (for which the cone angle is 2π).

The vector fields ∂_{x_i} and ∂_{y_i} defined on S_i can be smoothly prolongated to vector fields ∂_x and ∂_y on M_0. We define the quadratic form Q on $\mathcal{C}_0^\infty(M_0)$ by

$$Q(u) = \int_M |\partial_x u|^2 + |\partial_y u|^2 \, dxdy,$$

where $dxdy$ is the Euclidean area. This quadratic form is positive and we denote by Δ the self-adjoint operator given by the Friedrichs procedure (see [7] p. 176). By construction, the domain of Δ is characterized by

$$\mathrm{dom}(\Delta) = \{u \in L^2(M) \,|\, \Delta^* u \in L^2(M) \quad \text{and} \quad Q(u) < \infty\},$$

where Δ^* denotes the Euclidean Laplacian taken in the distributional sense on $\mathcal{C}_0^\infty(M_0)$. The only result we need to know about the spectral theory of this operator is that the spectrum of Δ is still purely ponctual (this derives for instance from the Rellich-type theorem proved in [8]). The eigenvalues will be denoted by $0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_n \leq \cdots$.

For any $g \in G$ the mapping $(x_i, y_i) \mapsto (x_{g(i)}, y_{g(i)})$ defines a piecewise isometry of M. Note however that this mapping is not continuous unless G is abelian (which implies that M is a torus). In particular, $g.u$ may have jump discontinuities even when u is smooth. The domain of Δ is thus not invariant by this action of G implying that this action and Δ do not commute. We can however do the following construction. Let ψ_n be the nth eigenfunction of the torus T obtained by identifying the parallel sides of S_0, and let ϕ be defined on M by copying ψ_n in each S_i. A straightforward computation based on Green's formula implies that ϕ belongs to dom(Δ) so that $\Delta\phi = \mu_n\phi$ where μ_n is the nth eigenvalue of the torus. Eventually, we have the following inclusion:

$$\mathrm{Sp}(T) \subset \mathrm{Sp}(M).$$

Remark If we denote by $\mathcal{N}_M(E)$ (resp. $\mathcal{N}_T(E)$) the counting function for the eigenvalues of M (resp. for T), then Weyl's law states that

$$\mathcal{N}_M(E) \sim cNE \quad \text{and} \quad \mathcal{N}_T(E) \sim cE,$$

so that, heuristically, if the complement of Sp(T) in Sp(M) is the spectrum of some self-ajoint operator, this operator should be defined on a surface tiled by $N - 1$ squares.

2 Self-adjoint extensions of the Laplacian on N squares

We denote by $H^1(S_0)$ the usual Sobolev space, i.e the set of $u's$ in $L^2(S_0)$ such that

$$\int_{S_0} |\partial_x u|^2 + |\partial_y u|^2 \, dxdy < \infty.$$

The traces on the edges are well defined and continuous from $H^1(S_0)$ in $H^{\frac{1}{2}}(0, 1)$ (see [9] for example). We denote them by $\gamma_{0l}, \gamma_{0r}, \gamma_{0u}, \gamma_{0d}$.[1]

Consider H and V two elements of $\mathbb{U}_N(\mathbb{C})$ the set of unitary matrices of size N. For convenience, we will ask that the group generated by H and V in $\mathbb{U}_N(\mathbb{C})$ is finite. We denote by $D(H, V)$ the set of all $\vec{v}$ in $\left[H^1(S_0)\right]^N$ satisfying the following boundary condition:

$$\overrightarrow{\gamma_0 v_l} = H\overrightarrow{\gamma_0 v_r}, \quad \overrightarrow{\gamma_0 v_d} = V\overrightarrow{\gamma_0 v_u} \tag{2}$$

On $\left[H^1(S_0)\right]^N$ we also define the quadratic form Q by

$$Q_N(\vec{u}) = \sum_1^N \int_{S_0} |\partial_x u_i|^2 + |\partial_y u_i|^2 \, dxdy.$$

The following definition provides us with a class $\mathcal{O}_N$ of operators.

Definition 2 (Class $\mathcal{O}_N$) Choose H and V in $\mathbb{U}_N(\mathbb{C})$ such that the group generated by H and V is finite. The quadratic form Q_N considered on $D(H, V)$ defines a unique self-adjoint operator that is denoted by (Δ_N, H, V). We denote by $\mathcal{O}_N$ the set of all the operators that can be obtained this way.

Remark There are other self-adjoint extensions of the Laplacian defined on N squares. We could make the gluing conditions depend on the point or take non-unitary H and V.

Example If $N = 1$, H (resp. V) is the multiplication by $e^{i\alpha}$ (resp. $e^{i\beta}$) with α and β in $\pi\mathbb{Q}$. Studying the corresponding operator on the square amounts to study two Floquet operators on $(0, 1)$. These are exactly the operators that come up when computing the spectrum of a torus that covers a smaller one.

The symmetric group $\mathfrak{S}_N$ can be represented in $\mathbb{U}_N(\mathbb{C})$ by asking that an orthonormal basis $(\vec{e}_i)$ is transformed according to $\sigma.\vec{e}_i = \vec{e}_{\sigma(i)}$. Let h and v be two elements of $\mathfrak{S}_N$, we denote by H and V their images by this representation, the following proposition then holds.

Proposition 1 *Let h and v be two permutations of $\mathfrak{S}_N$ and H and V the corresponding permutation matrices. The associated element of $\mathcal{O}_N$ is the Euclidean Laplacian on the square-tiled surface M defined by h and v.*

Proof Let Δ_1 be the element of $\mathcal{O}_N$ defined by H and V. By definition, Δ is the unique self-adjoint extension of the Euclidean Laplacian defined on $\mathcal{C}_0^\infty(M_0)$ whose domain contains $H^1(M)$. Using Green's formula and the definition of $D(H, V)$, Δ_1 is shown to satisfy both properties. □

In the sequel, we will focus on some particular subclasses of $\mathcal{O}_n$ that we describe now.

[1] Subscripts l and r are for left and right, u and d for up and down.

2.1 Subclasses of $\mathcal{O}_n$

2.1.1 Special boundary conditions

We specify two types of boundary conditions.

- We will denote by $\mathcal{S}_N$ the subset of $\mathcal{O}_N$ consisting of the operators for which H and V are permutations matrices.
- We will denote by $\mathcal{F}_N$ the operators of $\mathcal{O}_N$ such that H and V satisfy the following condition

$$\exists h, v \in \mathfrak{S}_n, \quad \exists(\omega_i), (\zeta_i) \in \mathbb{U}^N \mid$$
$$H\vec{e}_i = \omega_i \vec{e}_{h(i)}, \quad V\vec{e}_i = \zeta_i \vec{e}_{v(i)}.$$

  These operators will be called of generalized Floquet type.

Remark In the definition of the Floquet-type operator, we have taken ω_i and ζ_i of modulus 1. Actually, the fact that the group generated by H and V is finite imposes that the ω_i's and ζ_i's are roots of the unity.

Proposition 1 tells us that $\mathcal{S}_N$ is the subclass of $\mathcal{O}_N$ that contains exactly the Laplacians on any square-tiled surface. The class $\mathcal{F}_N$ contains $\mathcal{S}_N$ by letting $\omega_i = \zeta_i = 1$.

2.1.2 Operators associated with Cayley graphs

We can construct an operator in $\mathcal{O}_N$ by mimicking the construction of the Cayley graph of a group. We start from a group G generated by two elements h and v and we define the square-tiled surface M_G in the following way: for each element g in G we take a square that we label by g. We now glue these squares so that the square on the right of g is hg and that above g is vg. The corresponding permutation is the left-multiplication by h (resp. by v) in the horizontal (resp. vertical) direction.[2] For these surfaces, the number of squares is exactly the cardinal of the group generated by h, v.

We denote by $\mathcal{C}_N$ the set of all operators obtained by this procedure. Since $\mathcal{C}_N$ consists of Laplacians on square-tiled surfaces we have $\mathcal{C}_N \subset \mathcal{S}_N$. This inclusion is strict. Indeed, any choice of h and v in $\mathfrak{S}_N$ gives a square-tiled surface with N squares but N is not necessarily the cardinal of the group generated by h, v.

To get all the translation surfaces, we must consider the more general following construction.

2.1.3 Operators associated with Schreier graphs

Let G be a group generated by two elements h and v and consider a subgroup G_0. We construct a square-tiled surface in the following way: we take one square for each coset gG_0 and define the permutations in the horizontal (resp. vertical) direction as the action of h (resp. v) by left-multiplication on G/G_0 (i.e. $h(gG_0) = (hg)G_0$).

Any square-tiled surface is obtained this way by considering G the group generated by h and v and G_0 the stabilizer of a square. The graph underlying this construction is the Schreier graph associated with G, G_0 and the set of generators (h, v) (see [11]).

[2] This construction is already explained in [10].

Finally, we have the following chain of inclusions:

$$\mathcal{C}_N \subset \mathcal{S}_N \subset \mathcal{F}_N \subset \mathcal{O}_N,$$

where $\mathcal{C}_N$ corresponds to Laplacians on square-tiled surfaces associated with Cayley graphs, $\mathcal{S}_N$ corresponds to all the square-tiled surfaces (or equivalently the operators associated with Schreier graphs).

2.2 A group-theoretic dictionary

We now give a more group-theoretic flavour to the class $\mathcal{O}_N$. We will use [12] and [13] as constant references for representation theory.

Let G be a finite group generated by two elements h and v, and let ρ be a unitary representation of G in $\mathbb{C}^N$. We define Δ_ρ to be the element of $\mathcal{O}_N$ such that $H = \rho(h)$ and $V = \rho(v)$.

Remarks 1. The notation is slightly ambiguous since to define Δ_ρ we need not only ρ but also the generators h and v.
2. Any operator in $\mathcal{O}_N$ can be written Δ_ρ. Indeed, we just have to let G be the group generated by H and V and, since H and V are in $\mathbb{U}_N(\mathbb{C})$ we have a unitary representation of G.

We will keep on using both notations Δ_ρ and (Δ_N, H, V) since the first one stresses the underlying representation whereas the second notation makes the boundary condition clearer. The Laplacian of a square-tiled surface can thus be written Δ_ρ where ρ is the representation of G in $\mathbb{U}^N(\mathbb{C})$ by permutation matrices.

The classes of operators described in the preceding section all correspond to special classes of representations. We have the following dictionary between special types of representations and the corresponding set of operators.

Any representation of dimension N	$\mathcal{O}_N$
Regular representation of G	$\mathcal{C}_N$
Quasi-regular representation associated with $G_0 \subset G$	$\mathcal{S}_N$
Representation induced by a one-dimensional representation	$\mathcal{F}_N$

3 Spectral decomposition in $\mathcal{O}_N$

The main aim of this section is to find conditions on H and V so that the spectrum of (Δ, H, V) may be decomposed. We first give a definition of what we mean by spectral decomposition.

Definition 3 (Spectral Decomposition) Let $(A_i)_{0\leq i\leq n}$ be a collection of operators with pure point spectrum and $(a_i)_{1\leq i\leq n}$ a collection of integers, we will write

$$\mathrm{Sp}(A_0) = \sum_{i=1}^{n} a_i\, \mathrm{Sp}(A_i), \tag{3}$$

if the following holds

$$\forall \mu \in \mathbb{C}, \quad m_0(\mu) = \sum_{1}^{n} a_i\, m_i(\mu),$$

where $m_i(\mu)$ denotes the multiplicity of μ in the spectrum of A_i.

Remark This definition allows us to do algebraic sums of spectra. We remark however that the equality $\mathrm{Sp}(A) = \mathrm{Sp}(B) - \mathrm{Sp}(C)$ can hold only if $\mathrm{Sp}(C) \subset \mathrm{Sp}(B)$.

If the representation ρ decomposes as $\rho = \rho_1 + \rho_2$ then the corresponding matrices H_1, H_2, V_1, V_2 are block-diagonal and $\left[H^1(S_0)\right]^N$ can be decomposed accordingly. We get

$$\mathrm{Sp}(\Delta_\rho) = \mathrm{Sp}(\Delta_{\rho_1}) + \mathrm{Sp}(\Delta_{\rho_2}). \tag{4}$$

We will now prove the following general decomposition theorem.

Theorem 3 *Let G be a finite group generated by two elements h and v. Let $(\rho_i)_{0\leq i\leq n}$ be unitary representations of G and $(\chi_i)_{0\leq i\leq n}$ their corresponding characters. If $\chi_0 = \sum_{i=1}^n a_i\chi_i$ for some collection (a_i) of integers then the following spectral decomposition holds*:

$$Sp(\Delta_{\rho_0}) = \sum_{i=1}^n a_i Sp(\Delta_{\rho_i}). \tag{5}$$

The proof we will give relies on the transplantation theory that was introduced by Bérard and Buser [2,3] to explain the original result of Sunada [14]. We will also sketch a second proof showing that we can actually use the general theory of Riemannian coverings.

3.1 The transplantation method

We recall here the general philosophy of transplantations (see [3] or [2]). Consider two domains that are built of N isometric elementary bricks. Transplanting a function of the first domain into the second one consists in restricting the function to the elementary bricks and then in recombining the parts thus obtained using some matrix A in $\mathrm{GL}_N(\mathbb{C})$. This operation commutes formally with any differential operator. The transplantation method becomes fruitful in settings where this commutation is not only formal. It is the difficult part of the usual theory to find such settings and it requires quite sophisticated group-theoretic arguments (see [3,10]).

From this general point of view it is natural to use transplantations to study the class $\mathcal{O}_N$. Actually, since the operators in $\mathcal{O}_n$ are defined using boundary conditions, the commutation property will be even easier to establish.

Proposition 2 *Consider H_1, H_2, V_1, V_2 elements of $\mathbb{U}_N(\mathbb{C})$ and denote by Δ_i the corresponding operators in $\mathcal{O}_N$. Let A be in $GL_N(\mathbb{C})$, such that*

$$AH_1 = H_2A, \quad \text{and} \quad AV_1 = V_2A, \tag{6}$$

then we have:

$$A\Delta_1 = \Delta_2 A. \tag{7}$$

Proof Since Δ^* commutes with A, the only problem in proving (7) is to check that the domains are consistent. We recall that

$$\vec{v} \in \mathrm{dom}(\Delta_i) \Leftrightarrow \begin{cases} \vec{v} \in D(H_i, V_i), \\ \exists M,\ \forall \vec{w} \in D(H_i, V_i),\ |\langle \nabla\vec{v}|\nabla\vec{w}\rangle| \leq M\|\vec{w}\|, \end{cases}. \tag{8}$$

Equation (6) implies that $AD(H_1, V_1) = D(H_2, V_2)$. Moreover, when A satisfies (6), so does A^{*-1} so that we also have $A^{*-1}D(H_1, V_1) = D(H_2, V_2)$. Let $\vec{v}_1$ be in $\mathrm{dom}(\Delta_1)$ then

$\vec{v}_2 = A\vec{v}_1$ is in $D(H_2, V_2)$ and for any $\vec{w}_2$ in dom(Δ_2) there exists $\vec{w}_1$ in $D(H_1, V_1)$ such that $\vec{w}_2 = A^{*-1}\vec{w}_1$. The following computation then holds:

$$\langle \nabla \vec{v}_2 | \nabla \vec{w}_2 \rangle = \langle \nabla A\vec{v}_1 | \nabla A^{*-1}\vec{w}_1 \rangle$$
$$= \langle \nabla \vec{v}_1 | \nabla \vec{w}_1 \rangle.$$

Since $\vec{v}_1 \in$ dom(Δ_1), this last expression is bounded by $M\|\vec{w}_1\|$ so that we have eventually proved:

$$\forall \vec{w}_2 \in D(H_2, V_2), \quad |\langle \nabla \vec{v}_2 | \nabla \vec{w}_2 \rangle| \leq \tilde{M}\|\vec{w}_2\|,$$

with $\tilde{M} = M\|A^*\|$. This ends the proof. □

As usual in transplantation theory, this proposition is used to prove the following corollary.

Corollary 1 *Let Δ_1 and Δ_2 be two operators of $\mathcal{O}_N$ such that there exists A satisfying condition* (6) *then the following holds*:

$$Sp(\Delta_1) = Sp(\Delta_2),$$

(*counting multiplicities*). *In particular, if ρ_1 and ρ_2 are isomorphic representations of G, then*

$$Sp(\Delta_{\rho_1}) = Sp(\Delta_{\rho_2}).$$

Proof The former proposition implies that A conjugates Δ_1 and Δ_2. □

Remarks 1. The transplantation A actually gives an explicit relation between eigenfunctions of Δ_1 and Δ_2.
2. As we have already remarked, the preceding proposition is really simple because we consider the whole class $\mathcal{O}_N$; if we want to study isospectrality for square-tiled surfaces (i.e. in $\mathcal{S}_n$) the work is not finished yet, since we have to know when $H,\ V,\ AHA^{-1},\ AVA^{-1}$ can be simultaneously permutations. However, there are examples where this condition is satisfied (see [3,4]).

Proof of theorem 3 We consider first the case when the a_i's are positive. We construct a finite-dimensional representation ρ of G by letting $\rho(h)$ (resp. $\rho(v)$) be the block diagonal matrix such that there are exactly a_i blocks $\rho_i(h)$ (resp. $\rho_i(v)$) for $1 \leq i \leq n$. It is obvious that the characters of ρ and ρ_0 are equal so that the representations ρ and ρ_0 are equivalent. The conclusion of Theorem 3 then follows from the preceding corollary and Eq. (4) When the a_i's are not necessarily positive, we begin by ordering them so that the first k are negative and the last $n-k$ are positive and we rewrite the condition $\chi_0 = \sum a_i \chi_i$ as $\chi_0 + \sum_{i=1}^{k} a_i \chi_i = \sum_{k+1}^{n} a_i \chi_i$. The conclusion then follows. □

Before presenting several applications of Theorem 3, we say a word about an alternative proof that uses more from the general theory of Riemannian coverings.

3.2 The common cover method

First, we consider Δ in the class $\mathcal{C}_N$. It corresponds to the Laplacian on a surface M constructed using the Cayley graph of G (see Sect. 2.1). By construction we can label the squares using the elements of G. There is a natural action of G defined on $L^2(M)$ by

$$S_g\left(\sum_{g'} u_{g'}\vec{e}_{g'}\right) = \sum_{g'} u_{g'}\vec{e}_{g'g^{-1}}. \tag{9}$$

This time, the domain of Δ is invariant under S_g so that S_g commutes with Δ. We are thus led to the general theory of an operator commuting with some group action. In this setting we can define an operator $\tilde{\Delta}_\rho$ for each representation of G (see [14]). This $\tilde{\Delta}_\rho$ can then be identified with Δ_ρ and Theorem 3 then follows from the general Peter–Weyl theorem of this theory (see [14]).

For a general square-tiled surface M, we consider the corresponding group G. There is a surface M_G whose Laplacian is in $\mathcal{C}_{|G|}$. This allows us to define operators Δ_ρ associated with the representations of G. The Laplacian on M is then shown to correspond to the quasiregular representation associated with the stabilizer of a square and we get the decomposition of the spectrum of Laplacian using the decomposition of the quasiregular representation into irreducible ones.

Remark As a branched covering of the torus, a square-tiled surface is not necessarily *normal*. More precisely, the normal coverings correspond exactly to the surfaces whose Laplacian is in $\mathcal{C}_N$. However a non-normal covering can always be covered by a normal one (see [15]). This is the construction we have sketched above.

3.3 First applications

We first give three direct applications of Theorem 3. This will illustrate the fact that results from representation theory may be interpreted in $\mathcal{O}_N$ (or in one of its subclasses that we have pointed out).

1. First, we use the fact that the representation ρ of a subgroup G of $\mathfrak{S}_N$ by permutation matrices is never irreducible since the vector $\vec{Z} = \sum_{i=1}^{N} \vec{e}_i$ is $G-$invariant. Application of Theorem 3 then gives the following.

   Proposition 3 *Let M be a square-tiled surface with N squares, and T the torus with one square. There exists A in $\mathcal{O}_{N-1}$ such that*:

   $$Sp(\Delta_M) = Sp(\Delta_T) + Sp(A). \tag{10}$$

   Proof the Laplacian on M is some Δ_ρ where ρ is the representation of G by permutation matrices. Since $\vec{Z}$ is invariant, we have $\rho = t + \rho_1$ where t is the trivial representation and ρ_1 is ρ restricted to $F_1 = \{\vec{Z}\}^\perp$. Identifying Δ_t as the Laplacian on a torus gives the result. □

   In Sect. 5, we will construct examples where Δ_{ρ_1} can itself be decomposed.
2. Let now be M a surface such that its Laplacian lies in $\mathcal{C}_N$. Then it can be written Δ_ρ where ρ is the regular representation of G. Applying Theorem 3 then gives the following proposition (compare with [14] Prop. 3)

   Proposition 4 *Let $\Delta \in \mathcal{C}_N$. Let G be the associated finite group and ρ_λ be the set of irreducible representations of G. let d_λ, χ_λ be the dimension and character of ρ_λ then*:

   $$Sp(\Delta) = \sum_{irr.\ rep.} d_\lambda \, Sp(\Delta_{\rho_\lambda}).$$

   Proof The decomposition of the regular representation in irreducible ones is known to be:

   $$\chi_G = \sum_{irr.\ rep.} d_\lambda \, \chi_\lambda.$$

   The proposition then follows from Theorem 3. □

3. Combining Brauer's theorem (see [12] Theorem 23 p. 29) and Theorem 3 gives the following proposition.

Proposition 5 *Let* (Δ, H, V) *be an operator in* $\mathcal{O}_N$ *then the following spectral decomposition holds*:

$$Sp(\Delta_N, H, V) = \sum a_i\, Sp(\Delta_{N_i}, H_i, V_i),$$

where each (Δ_{N_i}, H_i, V_i) *is in* $\mathcal{F}_{N_i}$.

This proposition is even more interesting if we give another interpretation of the operators in $\mathcal{F}_N$. We recall that an operator A in $\mathcal{F}_N$ is associated with h, v and two collections of roots of the unity (see p. 398) . It is very likely that any operator in $\mathcal{F}_N$ can be realized as a Laplacian on the square-tiled surface defined by h and v with magnetic flux lines going through its conical points (see [16] for a definition of these operators in the plane).

Remark The class $\mathcal{O}_N$ provides us with operators associated with representations of groups that are generated by two elements, i.e. to quotients of F_2, the free group generated by two elements. Putting things into this perspective rises new questions such as adapting results from the general theory of Riemannian coverings ([20] for instance) when $SL(2, \mathbb{R})$ is replaced by F_2. This kind of questions also has to be related with the work of Brooks on regular graphs ([21]) with the main difference that more geometry is involved in our setting.

4 Self-transplantations

The transplantation method allows us to construct pairs of isospectral square-tiled surfaces by looking for permutation matrices H_1, V_1, H_2, V_2 and for A such that

$$AH_1 = H_2 A \quad AV_1 = V_2 A.$$

(see [4] for such an example). Furthermore, from the spectral point of view, it is quite interesting to have operators commuting with the Laplacian. This leads to the following definition.

Definition 4 (Self-transplantations) Let (Δ_N, H, V) be an operator in $\mathcal{O}_N$. Any element A of $\mathrm{GL}_N(\mathbb{C})$ such that

$$AH = HA, \quad AV = VA,$$

will be called a *self-transplantation of* (Δ_N, H, V).

Let ρ be the representation such that $(\Delta_N, H, V) = \Delta_\rho$, the set of self-transplantations coincide with the commutant set of the representation

$$\mathrm{Com}(\rho) = \{ A \in \mathrm{GL}_N(\mathbb{C}) \,|\, \forall g \in G,\ A\rho(g) = \rho(g)A \}.$$

Since we consider only unitary representation, $\mathrm{Com}(\rho)$ is a linear subspace that is invariant under adjoint operation and its dimension can be computed using the following proposition.

Proposition 6 *Let* G *be a finite group generated by* h, v. *Let* ρ *be some representation of* G *in* $\mathbb{U}_N(\mathbb{C})$. *For any irreducible representation* ρ_λ, *we denote by* m_λ *the corresponding multiplicity in* ρ. *The dimension of the vector space of self-transplantations of* Δ_ρ *is then* $k = \sum m_\lambda^2$.

When Δ_ρ *belongs to the class* $\mathcal{S}_N$, *then* G *is isomorphic to a subgroup of* $\mathfrak{S}_N$ *and the dimension* k *is also the number of orbits of the diagonal action of* G *on* $\{1 \ldots N\}^2$.

Proof The first statement is a direct application of Schur's lemma (see [12] p. 12). The second statement is a reformulation of exercise 2 p. 30 of [12]. We sketch a proof here. Let B be a matrix in $\mathrm{Com}(\rho)$ then, for any $g \in G \subset \mathfrak{S}_N$, we have $b_{g(i)g(j)} = b_{ij}$ so that the entries of B are constant along any orbit of the diagonal action of G on $\{1, \dots, N\}^2$. For each orbit $\mathcal{O}_k$ of this action, we define $B^{(k)}$ by $b^{(k)}_{ij} = 1$ if and only if (i, j) belongs to $\mathcal{O}_k$. Since the orbits form a partition of $\{1, \dots, N\}^2$, the $B^{(k)}$ $'s$ will be linearly independent, yielding the result. □

Examples 1. Let M be a connected square-tiled, then the corresponding group $G \subset \mathfrak{S}_n$ acts transitively on $\{1, \dots, N\}$. The diagonal is an orbit of the action we consider and we can take $B^{(1)} = I$, the identity matrix.
2. Let ρ be such that Δ_ρ is the Laplacian on a connected square-tiled surface. The dimension of $\mathrm{Com}(\rho)$ is 2 if and only if G acts transitively on the off-diagonal terms. This also means that the decomposition (10) cannot be further decomposed and that the representation ρ_1 (defined in the proof of Prop. 3) is irreducible. In this case, a basis of $\mathrm{Com}(\rho)$ is given by I, J where I is the identity and J has all its entries equal to 1.
3. On the contrary, when ρ is the regular representation of some group G (meaning that Δ_ρ is in some $\mathcal{C}_N$,) we get

$$\dim(\mathrm{Com}(\rho)) = \sum d_\lambda^2 = |G|.$$

In this case, the elements of $\mathrm{Com}(\rho)$ are linear combination of the S_g (defined in Sect. 3.2).

We will now construct other examples of surfaces where the dimension of the set of self-transplantations is greater than 2. This will lead us to finer spectral decompositions than Eq. (10) and to examples related to isospectrality.

5 An interesting class of square-tiled surfaces

We start by considering, for p prime, the group $\mathrm{GA}(\mathbb{F}_p)$ of invertible affine transformations acting on $\mathbb{F}_p$ i.e. we have

$$\mathrm{GA}(\mathbb{F}_p) = \left\{ \begin{array}{rl} f : \mathbb{F}_p & \to \mathbb{F}_p \\ x & \mapsto ax + b \end{array}, \ | \ a \in \mathbb{F}_p^*, \ b \in \mathbb{F}_p \right\}.$$

The group $\mathbb{F}_p^*$ is cyclic with $(p-1)$ elements so that for each divisor d of $(p-1)$ there is a group of index d in $\mathbb{F}_p^*$ that we denote by $\mathbb{G}_d$. This group is generated by an element a_d. We now let

$$G_d = \{f \in \mathrm{GA}(\mathbb{F}_p) \mid a \in \mathbb{G}_d\},$$

This group can be seen as a subgroup of $\mathfrak{S}_p$ generated by two elements h and v that correspond to:

$$f_h : \begin{array}{rl} \mathbb{F}_p & \to \mathbb{F}_p \\ x & \mapsto x+1, \end{array} \qquad f_v : \begin{array}{rl} \mathbb{F}_p & \to \mathbb{F}_p \\ x & \mapsto a_d x. \end{array}$$

Remark The group corresponding to $d = \frac{p-1}{2}$ is actually the dihedral group of order $2p$. More generally, if we denote $q = \frac{p-1}{d}$ then G_d is isomorphic to $\mathbb{F}_p \rtimes \mathbb{Z}/q\mathbb{Z}$.

We construct the square-tiled surfaces with p squares defined by h and v. Using proposition 6, we can compute the dimension of the set of self-transplantations on this surface.

Lemma 2 *The diagonal action of G_d on $\mathbb{F}_p^2$ has exactly $d+1$ orbits.*

Proof The group $\mathbb{G}_d$ acts transitively on $\mathbb{F}_p$ so that the diagonal of $\mathbb{F}_p^2$ is one orbit of the diagonal action. We pick now two off-diagonal points (x, y) and (x', y') that are in the same orbit. There exist $a \in \mathbb{G}_d$ and $b \in \mathbb{F}_p$ such that

$$x' = ax + b, \quad y' = ay + b.$$

This implies $x' - y' = a(x - y)$ so that $x' - y'$ and $x - y$ are in the same class modulo $\mathbb{G}_d$. The converse is immediate so that the off-diagonal orbits are in bijection with the number of classes modulo $\mathbb{G}_d$. □

This gives us examples of surfaces with non-trivial spectral decomposition. In order to find the reduced operators we let

$$\vec{f_k} = \frac{1}{\sqrt{p}} \sum_{l=1}^{p} \exp\left(\frac{2ikl\pi}{p}\right) \vec{e_l}.$$

The family $(\vec{f_k})_{1 \le k \le p}$ is an orthonormal basis of $\mathbb{C}^p$. A straightforward computation shows that $\vec{f_k}$ is an eigenvector of H with eigenvalue $\exp(-\frac{2ik\pi}{p})$ and that the action of V induces a cyclic permutation on the $\vec{f_k}$ such that k belongs to the same orbit modulo $\mathbb{G}_d$. This implies that the spectrum of (Δ_p, H, V) can be decomposed into the spectrum of the torus and the spectrum of d operators on $\frac{p-1}{d}$ squares. For each of these, denoting by H_r, V_r the boundary condition, we can find a basis such that V_r is the permutation matrix associated with a $\frac{p-1}{d}$−cycle (we just have to reorder the $\vec{f_j'}s$), and H_r is a diagonal matrix whose diagonal entries are pth roots of unity.

Example For $p = 5$, $d = q = 2$, we get the following. We identify $\mathbb{F}_p$ with $\{1, \dots, 5\}$. The only element of order 2 in $\mathbb{F}_p^*$ is 4 so that we have

$$h = (12345), \quad v = (14)(23)(5).$$

The surface M is of genus 3 and has one singularity of angle 10π.

The invariant subspaces are generated by $\vec{f_5}$, $(\vec{f_1}, \vec{f_4})$, $(\vec{f_2}, \vec{f_3})$. The corresponding matrices are:

$$V_1 = V_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad H_1 = \begin{pmatrix} \omega & 0 \\ 0 & \omega^4 \end{pmatrix}, \quad H_2 = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^3 \end{pmatrix},$$

where $\omega = \exp(\frac{2i\pi}{5})$.

We then have the following decomposition:

$$\mathrm{Sp}(M) = \mathrm{Sp}(T) + \mathrm{Sp}(\Delta_{N_1}, H_1, V_1) + \mathrm{Sp}(\Delta_{N_2}, H_2, V_2),$$

with $N_1 = N_2 = 2$.

5.1 Isospectrality

We refer to [17] for an account on the long history of isospectrality; we also refer to [2,3,10, 14] for some results about it and to [18] for recent results close to what we are dealing with (Fig. 1).

In the preceding section, we have found a group $G = \mathbb{F}_p \rtimes \mathbb{Z}/q\mathbb{Z}$ and a surface M_p associated with it. We will denote by M_{pq} the surface in $\mathcal{C}_{pq}$ associated with the Cayley graph of G relatively to generators h and v. We will also denote by T_1 the torus with 1 square and by T_q the torus with q squares in one column. We then have the following.

Theorem 4 *The following spectral identity holds*

$$Sp(M_{pq}) + q\, Sp(T_1) = q\, Sp(M_p) + Sp(T_q), \tag{11}$$

where M_{pq}, M_p, T_1, T_q are defined above

Proof From the preceding section, we already have

$$\mathrm{Sp}(M_p) = \mathrm{Sp}(T_1) + \sum_1^d \mathrm{Sp}(\rho_i),$$

where each ρ_i is of dimension q. First note that two ρ_i are not isomorphic since $\rho_i(h)$ is diagonal and the eigenvalues are different for different i. We now claim that these ρ_i are irreducible. Indeed, we have proved that the dimension of the set of self-transplantations was $d+1$. The alternative expression $\sum m_\lambda^2$ (see Prop. 6) implies that each ρ_i is irreducible. The group G also has q one-dimensional representations corresponding to the factor $\mathbb{Z}/q\mathbb{Z}$. These one-dimensional representations all arise in the spectrum of the torus T_q with q squares in one column (defined by $H = Id$ and V a q-cycle). Recall that $qd = p-1$, since $p\frac{p-1}{d} = q + d(\frac{p-1}{d})^2$, the classical formula $|G| = \sum d_\lambda^2$ (see [12] p. 18) tells us that we have found all the irreducible representations of G. It remains to compare the decomposition into irreducible representations of the spectrum of M_p, the spectrum of T_q and the spectrum of M_{pq} (which is given by Prop. 4) to get the formula. □

Theorem 1 is the specification of this theorem to the case $p = 3$, $q = 2$. In this case, the group G is $\mathfrak{S}_3$.

6 High multiplicity surfaces

A spectral decomposition in the form of Eq. (5) implies that $\mathrm{Sp}(\Delta_\rho)$ has some multiplicity as soon as some $a_i > 1$. For instance, if $a_1 = 2$ then each eigenvalue of Δ_{ρ_1} will have multiplicity 2 as an eigenvalue of Δ_ρ. Since the counting functions for Δ_ρ and Δ_{ρ_1} have comparable growth, this implies that *many* eigenvalues of Δ_ρ have multiplicity at least 2.

This is precised by the following definition. Let Δ be an operator whose spectrum consists only in eigenvalues. We define the set $\mathrm{Sp}(\Delta, m) = \{\lambda_i^{(m)}\}$, where

$$\forall i,\ \exists j(i) \mid \lambda_i^{(m)} = \lambda_{j(i)},$$
$$\lambda_0^{(m)} = \min\{\lambda_j,\ m(\lambda_j) \geq m\}$$
$$\lambda_{i+1}^{(m)} = \min\{\lambda_j,\ m(\lambda_j) \geq m,\ \lambda_j \geq \lambda_{j(i)+m}\}.$$

We also introduce $\mathcal{N}(E, m)$ the counting function for $\mathrm{Sp}(\Delta, m)$.

Definition 5 (Structural multiplicity) The structural multiplicity of Δ is the greatest m (possibly ∞) for which the following holds

$$\exists c > 0,\ E_0,\ \mid \forall E > E_0,\quad \frac{\mathcal{N}(E,m)}{\mathcal{N}(E)} > c. \tag{12}$$

It is denoted by s-m(Δ).

The multiplicities a_i of the irreducible representations in ρ accounts for some multiplicity in the spectrum of Δ_ρ. In particular s-m$(\Delta_\rho) \geq \max(a_i)$. However, even when ρ_λ is irreducible, its spectrum may have some multiplicity and the spectrum of two Δ_{ρ_λ} may also intersect. Still, we expect that these latter phenomena occur only exceptionally.

Since for a surface M whose Laplacian is in $\mathcal{C}_N$, the multiplicity of each irreducible representation is its dimension, we get easily examples of surfaces with arbitrarily high structural multiplicity. Comparing this number to the genus of the surface M gives the following result of which theorem 2 is a weakening.

Proposition 7 *There exists a sequence of square-tiled surfaces M_n such that the genus g_n goes to infinity and such that the following inequality holds*:

$$C_0 \exp(-c_0\gamma(3g_n - 1))\sqrt{g_n} \leq s\text{-}m(\Delta_{M_n}),$$

where γ is the reciprocical function of the usual Γ function.

Proof We choose $h = (1 \cdots n-1)(n)$ and $v = (1)(2)\cdots(n-2)(n-1\ n)$. The group G generated by h and v is $\mathfrak{S}_n$ and we construct M_n the surface associated with the corresponding Cayley graph. The commutator c of h and v is a 3-cycle; to get the genus of M_n we address the orbits of $\mathfrak{S}_n$ under multiplication by c. Each orbit has 3 elements (g, cg, c^2g) and thus there are $\frac{n!}{3}$ 6π-conical points on M_n. Using formula (1), the genus of M_n is thus

$$g_n = \frac{n!}{3} + 1.$$

We now take advantage of the fact that the maximal dimension among the irreducible repesentations of $\mathfrak{S}_n$ is (almost) known (see [19]). More precisely we have the following inequality:

$$\exp(-c_0 n)\sqrt{n!} \leq d_n \leq \exp(-c_1 n)\sqrt{n!}.$$

This results in the proposition. □

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