

Eigenvalue Variations and Semiclassical Concentration

Luc Hillairet

1. Introduction

In this paper, we aim at explicating some relations between the behavior of analytic eigenvalue branches of a spectral problem and concentration properties of its eigenfunctions. In order to illustrate this we consider a semiclassical Schrödinger operator $-h^2\Delta + V$ on \mathbb{R}^d (in which Δ is the nonpositive Euclidean Laplace operator). Viewing h as an analytic parameter, this problem enters the usual perturbation theory and the eigenvalues are organized into analytic eigenbranches $E_j(h)$. It is then natural to address the behavior of these eigenbranches when h goes to 0. This point of view is fundamentally different from the usual semiclassical perspective. Indeed, any analytic eigenbranch is expected to cross a typical energy window $[E_1, E_2]$ so that any semiclassical expansion in this window corresponds actually to different eigenbranches. Our point is that knowing how the eigenfunctions concentrate in the semiclassical regime yields some result on the behavior of eigenbranches. For instance, we have the following theorem (see Section 3).

Theorem 1. *Let $E_j(h)$ be an analytic eigenbranch of the family of operators $-h^2\Delta + V$ on $L^2(\mathbb{R}^d)$. The following then holds.*

- (1) *The eigenbranch $E_j(h)$ converges to some limit E_0 when h tends to 0.*
- (2) *The limit E_0 is necessarily a critical value of the potential V .*

We will also use the same kind of ideas to prove an integrated remainder estimate for an analytic variation of metrics on a smooth compact manifold X . (see Theorem 10 (in Section 4).

As we have already pointed out, the study of analytic eigenbranches and semiclassical analysis are two different worlds so that such a relation between them is a priori quite surprising. However, as it will become clear later, the formula that expresses the derivative of any analytic eigenbranch is actually a semiclassical quantity. This is the fact we will be exploiting here. This idea is also used in the two recent papers [8] and [5]. In [8] we deal with the limiting behavior of analytic eigenbranches in a singular setting. The main ingredient of the proof consists in controlling certain quadratic forms evaluated on the eigenfunctions of the spectral problem under consideration. We claim here that such a control may actually be

2010 *Mathematics Subject Classification.* Primary 35P20; Secondary 58J37.
This is the final form of the paper.

interpreted as a condition on the associated semiclassical measures. Such an interpretation is not necessarily helpful since rather little is known in general about semiclassical measures and the overall strategy of [8] relies more on direct estimates for solutions of one-dimensional ODEs. In [5], a part of the proof relies on the fact that an assumption on semiclassical measures allows to prove an integrated remainder estimate of the form we will consider here. Section 4 can actually be seen as a formalization of this argument of [5].

Acknowledgments. The ideas in this paper grew out from several discussions with Chris Judge and Andrew Hassell, I would like to thank both of them for their stimulating influence. I would also like to thank the CRM in Montréal and the organizers of the workshop “Spectrum and Dynamics” in April, 2008 and the MSRI in Berkeley where part of this work was done. Finally, I would like to thank Gilles Carron, Yves Colin de Verdière, Patrick Gérard and Andrew Hassell who made several comments on the first version of this paper that helped me improve the results and the exposition.

2. Quantization and semiclassical measures

In this section we will briefly recall some known facts about quantization and semiclassical measures. We will use both the classical and the semiclassical theory of pseudodifferential operators (See [4, 10] for background on these notions)

In the case of Schrödinger operators in \mathbb{R}^d , the principal symbol of a (h -) pseudodifferential operator is a function in $T^*(\mathbb{R}^d)$. When working on a smooth compact Riemannian manifold, we will denote by S^*X the quotient of $T^*X \setminus \{0\}$ by $(x, \xi) \sim (x, \lambda\xi)$, $\lambda > 0$. The subsets of S^*X are the conical subsets of $T^*X \setminus \{0\}$. The symbol of a pseudodifferential operator A is defined by testing A against fast oscillatory functions. It is then a function on $T^*M \setminus \{0\}$. The symbol of a classical pseudodifferential operator is conically invariant and thus defines a function on S^*X .

Remark 2. The principal symbol of a pseudodifferential operator may also be read off from the Schwartz kernel of A . Observe, however, that when performing this identification, one has to pay careful attention to the density that is used (especially when dealing with varying Riemannian metrics).

A crucial point is the ability of quantizing any reasonable function on S^*X . This is done via a so-called quantization procedure. There are several possible choices, but it is useful to choose a quantization that respects positivity (see [3, 7])¹. Thus, for any function a on S^*X there exists a pseudodifferential operator $\text{Op}^+(a)$ of order 0 whose principal symbol is a . If A is any other pseudodifferential operator with principal symbol a then $\text{Op}^+(a) - A$ is a pseudodifferential operator of order -1 . Moreover if, a is nonnegative, then $\text{Op}^+(a)$ is also nonnegative, as an operator.

In the semiclassical setting, the homogeneity condition on a is dropped and we may quantize any smooth function on $T^*(X)$ with compact support.

Using this quantization, we define the so-called semiclassical measures. We present the construction in the semiclassical setting since in the classical one we will use a nonconcentration estimate rather than semiclassical measures.

¹It is actually critical that such a quantization exists.

We start with a Schrödinger operator on \mathbb{R}^d , and we consider a sequence of normalized eigenfunctions u_h of the equation

$$-h^2\Delta u_h + V(x)u_h = E_h u_h,$$

such that the corresponding eigenvalues E_h converge to some noncritical energy E_0 . To any compactly supported function a the map $a \mapsto \langle \text{Op}^+(a)u_h, u_h \rangle$ defines a measure that converges weakly (up to extracting a subsequence). Any such limit is called a semiclassical measure.

Remark 3. One could in fact associate semiclassical measures to any bounded sequence in L^2 . In the sequel we will consider only semiclassical measures that are associated with eigenfunctions of the Laplace operator.

Among the basic facts satisfied by any semiclassical measure, we mention only the two following properties (see [3, 7]).

- (1) Any semiclassical measure is a probability measure supported on the energy surface $\Sigma_{E_0} := \{|\xi|^2 + V(x) = E_0\}$.
- (2) Any semiclassical measure is invariant under the Hamiltonian flow associated with $|\xi|^2 + V(x)$ on Σ_{E_0} .

Remark 4. Under suitable hypotheses, we may extend the definition to smooth functions that are not necessarily compactly supported in $T^*\mathbb{R}^d$. In particular, for a sequence of eigenfunctions with energy E_h converging to E_0 , the quantity $\int \|\nabla u_h\|^2 dx$ is uniformly bounded. This allows to prove that $\int |\xi|^2 d\mu < +\infty$ and then that

$$\int |\xi|^2 d\mu = \lim_{h \rightarrow 0} \int \|\nabla u_h\|^2 dx.$$

In the classical setting we can also define semiclassical measures. The main difference is that instead of a small parameter going to zero, we have to consider a sequence of eigenfunctions with eigenvalues growing to infinity and we recover a probability measure on S^*X . It can be pointed out that the estimate in Corollary 14 that we use in Section 4 is a kind of nonconcentration estimate. Indeed, such an estimate would give some information on any semiclassical measure when passing to the high energy limit. The assumptions on semiclassical measure in Section 3 and on geometric control in Section 4 are thus very similar in nature.

3. Schrödinger operators from the analytic viewpoint

In this section we consider the semiclassical Schrödinger equation in $X = \mathbb{R}^d$. The eigenvalue problem consists in looking for eigenfunctions u_h and eigenvalues E_h such that

$$(1) \quad -h^2\Delta u_h + V(x)u_h = E_h u_h,$$

in which Δ is the (nonpositive) Laplace operator associated with the Euclidean metric. Let us assume that V is smooth and confining (i.e., $\lim_{|x| \rightarrow \infty} V(x) = +\infty$) so that there is a complete orthonormal set of eigenfunctions and the corresponding eigenvalues grow to infinity.

This eigenvalue problem may be put into a generalized eigenvalue problem in the sense of Kato (cf. [9]). We change notations by letting $h = t$ thus emphasizing

that we will now consider an analytic perturbation problem. An integration by parts leads to the following equivalent problem

$$(2) \quad q_t(u_t, v) = E_t \cdot n(u, v),$$

In which q_t is the quadratic form

$$q_t(u) = t^2 \int |\nabla u(x)|^2 dx + \int V(x)|u(x)|^2 dx$$

defined on $\mathcal{D} = H^1(X) \cap L^2((1 + V(x)^2)^{1/2} dx)$ and n is the standard Riemannian quadratic form on $L^2(X)$. Observe that although q_t is real-analytic for $t \in \mathbb{R}$, \mathcal{D} is the form domain of q_t only for $t \neq 0$. Consequently, usual analytic perturbation theory tells us that the eigenvalues organize into real-analytic eigenbranches for $t \in (0, 1]$. A standard question in this setting is to address the limiting behavior of these eigenbranches when t goes down to 0.

Theorem 5. *Let E_t be an eigenvalue branch of (2), then E_t converges to a limit E_0 when t goes to 0. Moreover, this limit E_0 is a critical value of V .*

PROOF. Let E_t be an eigenvalue branch and u_t the corresponding normalized eigenvector branch. The derivative of E_t may be computed by differentiating (2) (see [9]), this yields

$$(3) \quad \dot{E}_t = 2t \int_X |\nabla u_t|^2(x) dx.$$

This expression is obviously nonnegative so that E_t is a nondecreasing function of t . Since, for any t we have $E_t \geq \min_X V(x)$, E_t has a limit E_0 when t goes to 0.

The condition on concentration of semiclassical measures then enters in the following lemma.

Lemma 6. *Assume that for any semiclassical measure μ at the energy E_0 ,*

$$(C) \quad \int_{\Sigma_{E_0}} |\xi|^2 d\mu(x) > 0.$$

Then, for any eigenbranch (E_t, u_t) such that $\lim E_t = E_0$,

$$(4) \quad \exists c > 0, \quad \liminf_{t \rightarrow 0} t^2 \cdot \int_X |\nabla u_t(x)|^2 dx \geq c.$$

Assuming this lemma, we finish the proof of the theorem. If we suppose, for a contradiction, that E_0 satisfies condition (C), then using (3), and (4) we get

$$\dot{E}_t \geq \frac{2c}{t},$$

and this is a contradiction since E_t has to converge when t goes down to 0. So condition (C) is not fulfilled and there exists a semiclassical measure at energy E_0 such that

$$\int_{\Sigma_{E_0}} |\xi|^2 d\mu = 0.$$

This implies that the support of μ is contained in the level set $\{\xi = 0\} \subset \Sigma_{E_0}$. Since $d\mu$ has to be invariant by the Hamiltonian flow

$$\begin{cases} \dot{x} = 2\xi \\ \dot{\xi} = -\nabla V(x) \end{cases}$$

this forces E_0 to be a critical value of V . □

It remains to prove Lemma 6.

PROOF OF LEMMA 6. As it is standard in this kind of settings (see [6] or [2]), we actually work in the reverse direction, assuming that (4) is not true. We may thus find a sequence t_n going to 0 and corresponding eigenfunctions and eigenvalues E_{t_n} and u_{t_n} , such that

$$t_n^2 \int_X |\nabla u_{t_n}(x)|^2 dx \leq \frac{1}{n}.$$

We now change notations again and let $t_n = h$ since the rest of the argument is of semiclassical nature (and, as usual in this setting h actually stands for h_n). We thus have

$$(5) \quad 0 \leq h^2 \int_X |\nabla u_h(x)|^2 dx \leq \varepsilon(h).$$

where $\varepsilon(h)$ is some function going to 0 with h . The sequence u_h is L^2 normalized so that we may extract subsequences and find associated semiclassical measures at energy E_0 since $\lim E_h = E_0$. Let $d\mu$ be one of these semiclassical measures, by definition (see Remark 4 above)

$$(6) \quad \lim_{h \rightarrow 0} h^2 \cdot \int_X |\nabla u_h(x)|^2 dx = \int_{\Sigma_{E_0}} |\xi^2| d\mu,$$

where the limit is understood along the subsequence defining μ . Putting (5) and (6) together yields

$$\int_{\Sigma_{E_0}} |\xi^2| d\mu = 0,$$

thus finishing the proof of the lemma. \square

Remark 7. Since the energy surface is defined by $|\xi|^2 + V(x) = E_0$, we can replace everywhere

$$\int_{\Sigma_{E_0}} |\xi^2| d\mu$$

by

$$\int_{\Sigma_{E_0}} (E_0 - V(x)) d\mu.$$

In dimension 1 this result can be refined using that the spectrum of a one-dimensional Schrödinger operator is known to be simple. If we assume that the potential V has only one nondegenerate minimum, then the spectrum near the bottom of the energy is known (see [4]) and there is an infinite number of eigenvalues close to the minimum. Since the eigenvalue branches cannot cross, the nondegenerate minimum is then only possible limit.²

²I would like to thank San Vũ Ngọc and Frédéric Faure for pointing out that, in the one-dimensional case, the simplicity of the spectrum allows to improve the result.

4. Integrated remainder estimates

In this section, we consider a smooth compact manifold X of dimension $d \geq 2$ and a real-analytic family of Riemannian metrics $(g_\tau)_{\tau \in [-1,1]}$. We denote by Δ_τ the Laplace operator associated with g_τ and by $E_j(\tau)$ the associated analytic branches of eigenvalues. We will also denote by $\langle \cdot, \cdot \rangle_\tau$ the scalar product on $L^2(X, g_\tau)$. We now define the usual following counting functions:

$$N(\tau, E) = \#\{j \mid E_j(\tau) \leq E\},$$

$$R_M(\tau, E) = \#\{j \mid E_j(\tau) \in (E - M, E + M]\} = N(\tau, E + M) - N(\tau, E - M).$$

Remark 8. Observe that since (g_τ) is analytic in τ , the scalar products $\langle \cdot, \cdot \rangle_\tau$ define equivalent norms. The same is true at the level of H^1 : the norms $\|\cdot\|_{H^1(X, g_\tau)}$ defined by:

$$\|u\|_{H^1(X, g_\tau)}^2 = \int_M g_\tau(\nabla_\tau u, \nabla_\tau u) \, \text{dvol}_{g_\tau} + \|u\|_\tau^2,$$

are all equivalent.

The following lemma expresses the variation of E_t using a family of pseudodifferential operators. It also serves as a definition of the family A_τ that will be used in the rest of the paper.

Lemma 9. *There exists a real-analytic family of symbols a_τ , and, for any τ_0 , there exists a constant C such that, for any normalized eigenbranch $(E_n(\tau), u_n(\tau))$ we have*

$$\forall |\tau| \leq \tau_0, \quad \left| \frac{\dot{E}_n(\tau)}{E_n(\tau)} - \langle \text{Op}^+(a_\tau)u_n(\tau), u_n(\tau) \rangle_\tau \right| \leq CE_n(\tau)^{-1/2}.$$

We will denote by $A_\tau = \text{Op}^+(a_\tau)$.

PROOF. First, by differentiating the eigenvalue equation

$$E_n(\tau) = \langle \Delta_\tau u_n(\tau), u_n(\tau) \rangle_\tau,$$

we find a second-order differential operator $\dot{\Delta}_\tau$ such that

$$\dot{E}_n(\tau) = \langle \dot{\Delta}_\tau u_n(\tau), u_n(\tau) \rangle_\tau.$$

(Observe that since the branch is normalized and using the eigenvalue equation we have that $\langle \Delta \dot{u}, u \rangle = E \langle \dot{u}, u \rangle = 0$). Denote by a_τ the principal symbol of $\dot{\Delta}_\tau(\Delta_\tau + 1)^{-1}$, and let $A_\tau = \text{Op}^+(a_\tau)$. By definition $R_\tau = \dot{\Delta}_\tau(\Delta_\tau + 1)^{-1} - A_\tau$ is an analytic family of pseudodifferential operators of order -1 . In particular, R_τ is uniformly bounded from L^2 into H^1 . This yields the bound

$$\left| \frac{\dot{E}_n(\tau)}{E_n(\tau) + 1} - \langle \text{Op}^+(a_\tau)u_n(\tau), u_n(\tau) \rangle_\tau \right| \leq CE_n(\tau)^{-1/2}.$$

We thus get

$$\left| \frac{\dot{E}_n(\tau)}{E_n(\tau)} - \langle (1 + E_n^{-1}) \text{Op}^+(a_\tau)u_n(\tau), u_n(\tau) \rangle_\tau \right| \leq CE_n(\tau)^{-1/2}(1 + E_n^{-1}),$$

from which the claimed bound follows since $|\langle \text{Op}^+(a_\tau)u_n(\tau), u_n(\tau) \rangle_\tau|$ is uniformly bounded. \square

The main result of this section is then the following theorem (the notion of geometric control is defined in Definition 11 in the following section).

Theorem 10. *Suppose that a_0 is nonnegative and that there exists $\varepsilon > 0$ such that the subset $\{(x, \xi) \in S^*X, a_0(x, \xi) > \varepsilon\}$ geometrically controls (S^*X, g_0) , then for any fixed M , there exists τ_0, E_0 and K such that*

$$(7) \quad \forall E > E_0, \quad \int_{-\tau_0}^{\tau_0} R_M(t, E) dt \leq KE^{d/2-1}.$$

The universal remainder in Weyl's law gives that for a fixed t , $R_M(t, E) = O(E^{(d-1)/2})$ as E goes to infinity. The theorem thus says that averaging with respect to the perturbation greatly improves this estimate.

The rest of the section is devoted to the proof of this theorem. The central part of the argument is actually interesting by itself and consists in proving a somewhat explicit uniform control estimate.

4.1. A uniform control estimate. We begin by recalling the notion of geometric control observing that since the velocity is constant along a geodesic, the geodesic flow of a Riemannian metric is well-defined on S^*X .

Definition 11. We say that an open subset \mathcal{U} of S^*X *geometrically controls* (S^*X, g) if for any $(x, \xi) \in S^*X$, there exists $T \in \mathbb{R}$ and $(x_0, \xi_0) \in \mathcal{U}$ such that $(x, \xi) = \Phi^T(x_0, \xi_0)$ where $\Phi(\cdot, \cdot)$ denotes the geodesic flow of g .

It is well-known that geometric control implies a nonconcentration estimate for eigenfunctions. More precisely, if U is an open subset of X such that $\mathcal{U} = S^*U$ geometrically controls (S^*X, g) then, there exists some positive c and an energy E_0 such that

$$(8) \quad \int_U |u_n|^2 d\text{vol}_g \geq c \int_X |u_n|^2 d\text{vol}_g,$$

for any eigenfunction u_n of the Laplace operator associated with g such that $E_n \geq E_0$. (Actually the restriction on E for this estimate may be released using the principle of unique continuation for solutions of second order elliptic PDEs see Remark 13 below).

Such a control estimate can be proved by a contradiction argument relying on known properties of semiclassical measures ([2, 6] for the scheme of such an argument). Such an approach can probably be adapted to get a control estimate that is uniform in τ . We propose here a slightly different proof that also allows us to get some uniform control on c .

Proposition 12. *Let \mathcal{U} be an open subset of S^*X that geometrically controls (S^*X, g_0) . There exists $\tau_0 \in \mathbb{R}$, $E_0 \in \mathbb{R}$, a zeroth-order pseudodifferential operator Π on X and $c > 0$ such that:*

- (1) *The symbol π of Π is supported in \mathcal{U} and, $0 \leq \pi(x, \xi) \leq 1$ on S^*X .*
- (2) *For any $|\tau| \leq \tau_0$ and any u eigenfunction of Δ_τ with energy greater than E_0 then the following nonconcentration estimate holds:*

$$(9) \quad \|\Pi u\|_\tau \geq c \|u\|_\tau.$$

Remark 13. Estimate (8) may be extended to any eigenfunction, without any restriction on the energy. Such an extension requires that one knows that an eigenfunction of a Laplace operator cannot vanish on an open subset (thus proving the control estimate for any finite number of eigenfunctions). The principle of unique continuation gives this property for eigenfunctions of the Laplace operator.

It is not clear to the author if this nonvanishing property holds in the cotangent bundle. It can also be observed that, strictly speaking, the formulation with the microlocal cutoff only has a meaning in the high energy limit since the microlocal cutoff is defined modulo smoothing operators.

PROOF. Let $(x, \xi) \in S^*X$, the geometric control assumption implies that there exists some $T (= T(x, \xi))$ and $(x_0, \xi_0) \in \mathcal{U}$ such that $\Phi_0^T(x_0, \xi_0) = (x, \xi)$. Using the continuity properties of solutions of an O.D.E. both with respect to initial conditions and with respect to parameters, we can find

- some $\tau_{\max}(x, \xi)$,
- two conical neighborhoods of (x, ξ) : $\mathcal{V} = \mathcal{V}_{(x, \xi)}$ and $\mathcal{W} = \mathcal{W}_{(x, \xi)}$,
- a conical neighborhood $\tilde{\mathcal{U}} = \tilde{\mathcal{U}}_{(x, \xi)}$ of (x_0, ξ_0)

such that \mathcal{V} is compactly included in \mathcal{W} and $\Phi_\tau^{-T}(\mathcal{W}) \subset \tilde{\mathcal{U}} \Subset \mathcal{U}$ for any $|\tau| \leq \tau_{\max}(x, \xi)$. Since S^*X is compact, we can find a finite collection $((x_i, \xi_i))_{1 \leq i \leq N}$ such that $S^*X = \bigcup_{i=1}^N \mathcal{V}_{(x_i, \xi_i)}$. We now use the index i to denote any quantity formerly indexed by (or attached to) (x_i, ξ_i) . Denote by τ_0 the minimum of all $\tau_{\max}(x_i, \xi_i)$. We can then find a smooth partition of unity π_i such that each π_i is identically 1 in \mathcal{V}_i and identically 0 outside \mathcal{W}_i . Note that, by construction,

$$\forall |\tau| \leq \tau_0, \quad \bigcup_{i=1}^N \Phi_\tau^{-T_i}(\mathcal{W}_i) \subset \bigcup_{i=1}^N \tilde{\mathcal{U}}_i \Subset \mathcal{U}.$$

We denote by π_0 a smooth function which is identically 1 in $\bigcup_{i=1}^N \tilde{\mathcal{U}}_i$ and 0 outside \mathcal{U} . Denote by $\Pi_i = \text{Op}^+(\pi_i)$, any by $U_\tau(T) = \exp iT\sqrt{\Delta_\tau}$, we claim that for any $i > 0$, and for any $|\tau| \leq \tau_0$, the operator $R_i = \Pi_i U_\tau(T_i)(\text{Id} - \Pi_0)$ is smoothing. Indeed, its wave-front set is included

$$\{(y, \eta, x, \xi) \mid (y, \eta) \in \text{supp}(\pi_i), (x, \xi) \in \text{supp}(1 - \pi_0), (y, \eta) = \Phi_\tau^{T_i}(x, \xi)\},$$

which is empty by construction. In particular, each $R_{i, \tau}$ is bounded from $H^{-1}(X)$ into $L^2(X)$. Moreover, a careful analysis of the construction of the Hadamard parametrix (as it is for instance presented in [1]) shows that there exists a uniform constant C such that:

$$(10) \quad \forall |\tau| \leq \tau_0, \forall w \in H^{-1}(X), \quad \|R_i w\|_{L^2(X, g_0)} \leq C \|w\|_{H^{-1}(X, g_0)},$$

(see the appendix for a brief sketch of proof).

Since the $(\pi_i)_{i>1}$ are a partition of unity, $\sum_{i>1} \Pi_i = \text{Id}$. We thus have

$$(11) \quad \|u\|_0^2 = \sum_{i=1}^N \langle \Pi_i u, u \rangle_0.$$

Using that u is an eigenfunction of Δ_τ , we have

$$|\langle \Pi_i u, u \rangle_0| = |\langle \Pi_i U_\tau(T_i) u, u \rangle_0| \leq |\langle \Pi_i U_\tau(T_i) \Pi_0 u, u \rangle_0| + |\langle R_i u, u \rangle_0|.$$

We use Cauchy–Schwarz inequality on both terms and the following facts on the norm of the different operators:

- Π_i is bounded from $L^2(X)$ to itself,
- $U_\tau(T)$ is an isometry from $L^2(X)$,
- and (10).

This yields:

$$|\langle \Pi_i u, u \rangle_0| \leq (c_i \|\Pi_0 u\|_0 + C \|u\|_{H^{-1}(X, g_0)}) \cdot \|u\|_0.$$

Using that the $H^{-1}(X, g_\tau)$ norms are all equivalent and that $\|u\|_{H^{-1}(X, g_\tau)}^2 = (1 + E)^{-1} \|u\|_\tau^2$ when u is an eigenfunction of Δ_τ , we get

$$|\langle \Pi_i u, u \rangle_0| \leq (c_i \|\Pi_0 u\| + C(1 + E)^{-1/2} \|u\|_0) \cdot \|u\|_0,$$

for a constant C independent of $\tau \in [-\tau_0, \tau_0]$.

Summing all these inequalities and denoting by \tilde{c} the maximum of the c_i we obtain:

$$\|u\|_0 \leq (N\tilde{c}) \|\Pi_0 u\| + CN(1 + E)^{-1/2} \|u\|_0.$$

This yields the estimate in the proposition with $\Pi = \Pi_0$ and $c^{-1} = 2\tilde{c}N$ as soon as $CN(1 + E)^{-1/2}$ is less than $\frac{1}{2}$. \square

This proposition has the following useful corollary.

Corollary 14. *Let a_τ be an analytically dependent family of nonnegative symbols of order 0 such that $\{a_0(x, \xi) > \varepsilon\}$ geometrically controls (S^*X, g_0) then there exist positive κ, τ_0 and E_0 such that, for any $|\tau| \leq \tau_0$ and any u eigenfunction of Δ_τ with energy greater than E_0 then*

$$\langle A_\tau u, u \rangle_0 \geq \kappa \|u\|_\tau^2,$$

where A_τ denotes $\text{Op}^+(a_\tau)$.

Remark 15. In the following proof, we will use several times the fact that, for any symbols of order 0, $\text{Op}^+(bc) - \text{Op}^+(b) \circ \text{Op}^+(c)$ is a pseudodifferential operator of order -1 . We will denote by R any operator arising from this kind of operation. Observe that if b and c depend analytically on τ , then R also depends analytically on τ .

PROOF. Let τ_0, \tilde{E}_0 and Π be given by applying Proposition 12 to $\mathcal{U} = \{a_0(x, \xi) > \varepsilon\}$. If needed, we restrict τ_0 so that, for any $|\tau| \leq \tau_0$, a_τ is bounded below by $\varepsilon/2$ on \mathcal{U} . Observe that

$$\begin{aligned} \langle A_\tau u, u \rangle_0 &= \langle A_\tau \Pi^2 u, u \rangle_0 + \langle A_\tau (\text{Id} - \Pi^2) u, u \rangle_0 \\ &= \langle \text{Op}^+(a_\tau \pi^2) u, u \rangle_0 + \langle \text{Op}^+(a_\tau (1 - \pi^2)) u, u \rangle_0 + \langle R_\tau u, u \rangle_0 \\ &\geq \langle \text{Op}^+(a_\tau \pi^2) u, u \rangle_0 + \langle R_\tau u, u \rangle_0, \end{aligned}$$

where the last inequality holds because $\pi(x, \xi) \leq 1$ and we have chosen a positive quantization. Since the support of π is included in \mathcal{U} we have

$$\forall (x, \xi) \in S^*X, a_\tau(x, \xi) \pi^2(x, \xi) \geq \frac{\varepsilon}{2} \pi^2(x, \xi),$$

which is quantized into

$$\langle \text{Op}^+(a_\tau \pi^2) u, u \rangle_0 \geq \frac{\varepsilon}{2} \langle \text{Op}^+(\pi^2) u, u \rangle_0 \geq \frac{\varepsilon}{2} \|\Pi u\|_0^2 + \langle R_\tau u, u \rangle_0,$$

with another remainder operator R_τ . Putting these two inequalities together, we obtain

$$(12) \quad \langle Au, u \rangle_0 \geq \frac{\varepsilon}{2} \|\Pi u\|_0^2 + \langle R_\tau u, u \rangle_0.$$

Since R is of order -1 and the norms on H^{-1} associated with $(g_\tau)_{|\tau| \leq \tau_0}$ are uniformly equivalent, there exists some uniform C such that

$$|\langle R_\tau u, u \rangle_0| \leq CE^{-1/2} \|u\|_\tau.$$

Using Proposition 12, for $|\tau| \leq \tau_0$ and $E \geq E_0$, we have $\|\Pi u\|_0 \geq c_1 \|\Pi u\|_\tau \geq c_2 \|u\|_\tau$ so that we get

$$(13) \quad \langle A_\tau u, u \rangle \geq \left(c \frac{\varepsilon}{2} - CE^{-1/2} \right) \cdot \|u\|_\tau^2.$$

The corollary then holds with $\kappa = c\varepsilon/4$ as soon as E is large enough so that $CE^{-1/2} \leq c\varepsilon/4$. \square

4.2. Proof of Theorem 10. We consider normalized eigenbranches and we start from the formula for the derivative of the eigenfunctions: (in the following equation, everything except the constant C depends on τ)

$$\frac{d}{d\tau} \ln(E) \geq \langle Au, u \rangle - CE^{-1/2}.$$

Under the assumptions of the theorem, we may use Proposition 12 and its corollary. This provides us with $\kappa > 0$, τ_0 and E_0 such that

$$(14) \quad \forall |\tau| \leq \tau_0, \forall E(\tau) \geq E_0, \quad \frac{d}{d\tau} \ln(E) \geq \kappa.$$

The rest of the argument is merely reproduced from [5].

Denote by $R_M(E) = \int_{-\tau_0}^{\tau_0} R_M(t, E) dt$. A straightforward manipulation of the integral yields:

$$(15) \quad R_M(E) = \sum_j \text{Leb}\{\tau \mid E_j(\tau) \in (E - M, E + M)\},$$

where Leb denotes the Lebesgue measure. We consider now E such that $E - M \geq E_0$. According to (14), any eigenbranch that crosses the energy $E - M$ has to do it in an increasing manner. This implies that any eigenbranch for which the set $I_j := \{t \mid E_j(\tau) \in (E - M, E + M)\}$ is not empty satisfies $E_j(-\tau_0) \leq E + M$ and $E_j(\tau_0) \geq E - M$. In particular, the sum in (15) has only a finite number of positive terms, and moreover, this number is bounded by $N_{-\tau_0}(E + M)$ which is bounded by $CE^{d/2}$ according to Weyl's law. Estimate (14) also implies that I_j is an interval whose extremities we denote by τ_j^- and τ_j^+ .³ Integrating (14) we obtain

$$\kappa(\tau_j^+ - \tau_j^-) \leq \ln\left(\frac{E_j(\tau_j^+)}{E_j(\tau_j^-)}\right) \leq \ln\left(\frac{E + M}{E - M}\right) \leq \frac{C}{E},$$

as soon as E is large enough (independently of j). This bounds each term of the sum in (15), since the number of terms is also bounded explicitly using Weyl's law, we obtain

$$R_M(E) \leq C \cdot E^{d/2-1}.$$

Remark 16. The fact that a_0 is nonnegative implies in particular that the derivative of $\text{vol}_{g_\tau} X$ is increasing. In particular, the variation cannot be volume-preserving.

³We do not specify whether I_j is open or closed at each of its extremity since this fact depends on whether τ_j^\pm coincide or not with $\pm\tau_0$, and it is irrelevant when computing the Lebesgue measure.

Remark 17. Corollary 14 implies that for any semiclassical measure associated with Δ_τ , the following lower bound holds:

$$\int a \, d\mu \geq \kappa.$$

If we replace the control assumption in Theorem 10 by the following (weaker) assumption:

$$\forall \tau, \forall \mu \text{ semiclassical measure associated with } \Delta_\tau, \quad \int a \, d\mu > 0,$$

then, following the same lines of proof, we would get a 'almost everywhere' result in the spirit of [5].

Remark 18. This result can probably be extended to manifolds with boundary with the precaution that, in this case we would have to control the possible escape of the mass to the boundary.

Examples. (1) A natural variation to look at is to simply take $g_\tau = \exp(\tau)g_0$. In this case the control assumption is trivially satisfied so that we get a uniform integrated remainder. However, this is not a new result since looking at a fixed energy interval and dilating the volume is equivalent to fixing the volume and dilating the energy interval so that the integrated remainder is actually a regularized remainder in Weyl's law. And it is known that integrating the remainder in Weyl's law with respect to the energy yields a much better estimate than the universal remainder term.

(2) For a more general conformal variation on a surface ($d = 2$) we have $g_\tau := (1 + \tau f(x))^2 g_0$, the eigenvalue problem can be put in the form

$$\int_X \nabla_0 u \nabla_0 \phi \, d\text{vol}_{g_0} = E \int_X u \phi (1 + \tau f(x))^2 \, d\text{vol}_{g_0}.$$

This gives $a_0(x, \xi) = 2f(x)$. Choosing f nonnegative and supported in a control region yields examples where the theorem applies.

(3) On the contrary, the variation considered in [5] on a Donnelly surface doesn't satisfy the control estimate because of the cylinder of periodic orbits and of periodic geodesic in the hyperbolic wings.

Appendix A. Uniform Hadamard parametrix

The aim of this appendix is to give a (rather sketchy) justification of the uniform estimate (10).

Let X be a smooth manifold without boundary and g_τ a real-analytic family of riemannian metrics.

Denote by $U_\tau(s)$ the propagator of the half-wave equation associated with Δ_τ :

$$U_\tau(s) = \exp(is\sqrt{\Delta_\tau}),$$

and denote by $k_\tau(s, x, y)$ its Schwartz kernel. This kernel $k_\tau(s, x, y)$ is obtained by restricting the Schwartz kernel of $\cos(s\sqrt{\Delta_\tau})$ to the region $\sigma > 0$ of the cotangent bundle $T^*(\mathbb{R} \times X \times Y)$

We start from the Hadamard parametrix construction as explained for instance in [1]. Concretely, for s less than some small ε and x, y such that $d_\tau(x, y) \leq 2\varepsilon$, we

make the ansatz

$$\tilde{k}_N(\tau, s, x, y) = \sum_1^N u_k(\tau, x, y) T_{\alpha_d+k}(s^2 - d_\tau^2(x, y)),$$

in which the u_k are smooth functions of x, y and $T_{\alpha_d+k}(z)$ are oscillatory distributions of \mathbb{R} with wave-front in $\{(z = 0, \zeta > 0)\}$. The coefficients u_k satisfy some transport equations from which it is straightforward that they also depend smoothly on τ . We fix a smooth cutoff function $\rho := \rho(d_\tau(x, y))$ that cuts off in $d_\tau(x, y) \leq 2\varepsilon$ and we let $k_N = \tilde{k}_N \times \rho$. A computation then yields that k_N solves the half-wave equation up to a remainder term $r_N(\tau, s, x, y)$ which is $\mathcal{C}^{l(N)}$ where $l(N)$ tends to infinity with N . Applying Duhamel's principle, we get

$$k(\tau, s, x, y) - k_N(\tau, s, x, y) = \int_0^s k(\tau, s - s', x, w) r_N(\tau, s', w, y) ds' dw.$$

Here again, a detailed analysis shows that, for $s \leq \varepsilon$ and for any N , we finally have

$$k(\tau, s, x, y) = k_N(\tau, s, x, y) + R_N(\tau, s, x, y),$$

in which everything is smooth in τ , and R_N is $\mathcal{C}^{l(N)}$. Since k_N has a fixed order as a distribution of (s, x, y) , applying it to a C^l function gives a $C^{l-\beta}$ with a loss β that is independent of N .

We now fix some T , and we use the semigroup properties of U to write

$$U_\tau(T) = U_\tau(\delta) \circ U_\tau(\delta) \circ \dots \circ U_\tau(\delta),$$

with a fixed small δ . We now plug into this expression the decomposition $U_\tau = K_N + R_N$. Expanding everything gives a large sum in which each term is a composition of K_N factors with R_N factors. We address the regularity of each term. Each term has regularity $N - p\beta$ where p is the number of K_N factors. Thus any term except the one with only K_N factors may be made as regular as wanted by choosing N large enough. Moreover, everything is regular in τ .

It remains to address the term with only K_N factors. Each K_N is explicit and this composition is addressed using stationary phase. It is somewhat tedious in the general case because for instance of conjugate points. However it is much simpler when we have an initial and final microlocal cutoff that destroys everything in the wave-front set. Indeed a uniform nonstationary phase then applies that yields a uniform C^m bound for any m .

References

1. P. H. Bérard, *On the wave equation on a compact Riemannian manifold without conjugate points*, Math. Z. **155** (1977), no. 3, 249–276.
2. Nicolas Burq and Maciej Zworski, *Geometric control in the presence of a black box*, J. Amer. Math. Soc. **17** (2004), no. 2, 443–471 (electronic).
3. Y. Colin de Verdière, *Ergodicité et fonctions propres du laplacien*, Comm. Math. Phys. **102** (1985), no. 3, 497–502.
4. M. Dimassi and J. Sjöstrand, *Spectral asymptotics in the semi-classical limit*, London Math. Soc. Lecture Note Ser., vol. 268, Cambridge Univ. Press, Cambridge, 1999.
5. A. Hassell, *Ergodic billiards that are not quantum unique ergodic*, with an appendix by A. Hassell and L. Hillairet, Ann. of Math. (2) **171** (2010), no. 1, 605–618.
6. A. Hassell, L. Hillairet, and J. Marzuola, *Eigenfunction concentration for polygonal billiards*, Comm. Partial Differential Equations **34** (2009), no. 4-6, 475–485.
7. B. Helffer, A. Martinez, and D. Robert, *Ergodicité et limite semi-classique*, Comm. Math. Phys. **109** (1987), no. 2, 313–326.

8. L. Hillairet and C. Judge, *The eigenvalues of a domains with small slits*, Trans. Amer. Math. Soc., to appear.
9. T. Kato, *Perturbation theory for linear operators*, Classics Math., Springer, Berlin, 1995. Reprint of the 1980 edition.
10. M. E. Taylor, *Pseudodifferential operators*, Princeton Math. Ser., vol. 34, Princeton Univ. Press, Princeton, NJ, 1981.

LABORATOIRE DE MATHÉMATIQUES JEAN LERAY UMR CNRS 6629, UNIVERSITÉ DE NANTES,
2 RUE DE LA HOUSSINIÈRE, BP 92208, 44322 NANTES CEDEX 3, FRANCE
E-mail address: `Luc.Hillairet@math.univ-nantes.fr`