J Geom Anal
DOI 10.1007/s12220-012-9295-3

Krein Formula and S-Matrix for Euclidean Surfaces with Conical Singularities

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Received: 2 December 2010

Abstract Using the Krein formula for the difference of the resolvents of two self-adjoint extensions of a symmetric operator with finite deficiency indices, we establish a comparison formula for ζ-regularized determinants of two self-adjoint extensions of the Laplace operator on a Euclidean surface with conical singularities (E.s.c.s.). The ratio of two determinants is expressed through the value $S(0)$ of the S-matrix, $S(\lambda)$, of the surface. We study the asymptotic behavior of the S-matrix, give an explicit expression for $S(0)$ relating it to the Bergman projective connection on the underlying compact Riemann surface, and derive variational formulas for $S(\lambda)$ with respect to coordinates on the moduli space of E.s.c.s. with trivial holonomy.

Keywords Flat Laplacian · Determinant · Conical singularities · Complex structure

Mathematics Subject Classification (2000) 58J52 (30F30)

1 Introduction

Spectral geometry aims at understanding the relations between the spectrum of some Laplace operator in a given geometrical setting and geometric properties of the lat-

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Communicated by Michael Taylor.

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Published online: 23 February 2012

ter. Polygons and polyhedra are among the simplest shapes one can consider and one could hope in this setting for a better understanding. This leads naturally to studying the spectral geometry of Euclidean surfaces with conical singularities. Another motivation is the spectral theory of translation surfaces for which the geometric picture has many interesting developments (see [29], for instance).

One peculiarity of Laplacians on manifolds with conical points is that, due to the presence of conical points, a choice has to be made in order to get a self-adjoint operator. In this paper, we are interested in understanding how this choice affects several spectral quantities such as the resolvent and the zeta-regularized determinant. Depending on the self-adjoint extension, this zeta-regularization procedure is not as straightforward as usual because of unusual behavior of the zeta function, but it is still possible to define such a regularization (see [12, 17] and Sect. 5.3), and we will prove a comparison formula for these determinants.

Comparison formulas for regularized determinants for conical manifolds were first found in [22] using a surgery formula *à la* BFK (see [5]), and in [17] using a contour integral method based on a secular equation that defines the spectrum. One of our motivations was to understand how the comparison formulas for different self-adjoint extensions from [22] read in the case of Euclidean surfaces with conical singularities and whether it is possible to express the determinants of the non-Friedrichs self-adjoint extensions of the Laplacian on these surfaces through holomorphic invariants of the underlying Riemann surface (as it was done in [19] for the determinant of the Friedrichs extension). Indeed, Euclidean surfaces with conical singularities are our primary interest, and we will restrict to this setting although many statements still make sense for more general conical manifolds.

It turns out that the geometric interpretation of the formulas obtained in [22] and [17] is not that straightforward, and we have found it more convenient to establish the comparison formula for determinants using the Krein formula for the difference of resolvents of two self-adjoint extensions of a symmetric operator. We observe that the trace of the difference of two resolvents admits a nice representation through the so-called S-matrix of a Euclidean surface with conical singularities (E.s.c.s.) X. The latter matrix, or, more precisely, the meromorphic family of matrices $S(\lambda)$ is in some sense a characteristic feature of X. Indeed, we believe that some of the geometry of X (such as, for instance, the lengths of saddle-connections between conical points; see Remark 4.3) is encoded in $S(\lambda)$, although it seems quite difficult to retrieve this kind of information. We should also remark that this S-matrix allows to write down a secular equation that can then be treated using the approach of [17] so that what we propose here may be seen as a geometric interpretation for the latter method. The comparison with [22] is less straightforward; it relies on interpreting the S-matrix as some kind of limiting Dirichlet-to-Neumann operator on a circle around the conical point when the radius of that circle goes to 0. It can be noted here that, in contrast with [22], no extra condition is needed to obtain our formula.

We will thus prove the following theorem. The notion of *regular* self-adjoint extensions will be introduced in Definition 5.2, and, for these self-adjoint extensions, the expression $P + QS(0)$ makes sense (see Remark 5.5).

Theorem 1 *On a compact E.s.c.s. X, let $S(\lambda)$ be the S-matrix and Δ_F be the Friedrichs extension.*

Let P and Q be matrices that define a regular self-adjoint extension Δ_L, and define

$$D(\lambda) := \det\bigl(P + QS(\lambda)\bigr).$$

Let d be the dimension of $\ker(P + QS(0))$ *and let* $D^*(0) := \lim_{\lambda\to 0}(-\lambda)^{-d}D(\lambda)$.

There exist α_0 and Γ such that the asymptotic expansion of $D(-|\lambda|)$ as λ goes to ∞ is

$$\ln D\bigl(-|\lambda|\bigr) := \alpha_0 \ln\bigl(|\lambda|\bigr) + \Gamma + o(1).$$

The following identity then holds:

$$\det{}^*_\zeta(\Delta_L) = \exp(-\Gamma)D^*(0)\det{}^*_\zeta(\Delta_F),$$

in which $\det^*_\zeta$ *is the modified zeta-regularized determinant* (*see Definition* 5.14).

To fulfill our second aim, we then need to understand more explicitly what kind of geometric information is encoded in the family $S(\lambda)$. We focus on the limiting behavior when the spectral parameter goes to 0, since this is the regime that comes up in the comparison formula. We will prove that most of the matrix elements in this limit have an interpretation through values of the Bergman projective connection and the basic holomorphic differentials taken at the conical point in the corresponding distinguished holomorphic local parameter (see Sect. 4.3). Since we expect translation surfaces to have particular and interesting features, we will also say a word about the S-matrix on these special kinds of surfaces. Namely, we will derive variational formulas for the S-matrix when it is differentiated with respect to moduli parameters. These results answer most of the questions which motivated our study.

Organization of the Paper In Sect. 2, we will recall the basic facts about Euclidean surfaces with conical singularities. In particular, we will recall that these can be viewed as Riemann surfaces with flat conformal conical metric.

In Sect. 3, we recall some basic properties of the Friedrichs Laplace operator on E.s.c.s., and introduce the object of our primary interest—the S-matrix; we also derive here the standard formula for the derivative of the S-matrix with respect to λ.

In Sect. 4, we study the asymptotic behavior of $S(\lambda)$ as λ goes to $-\infty$, and find the geometric interpretation of $S(0)$. We also apply the variational formulas of [19] to obtain the variations of $S(\lambda)$ with respect to moduli parameters on translation surfaces.

In Sect. 5, we study various self-adjoint extensions of the Laplace operator on E.s.c.s. and prove the comparison formula for their ζ-regularized determinants.

2 Euclidean Surfaces with Conical Singularities

2.1 Euclidean Surfaces with Conical Singularities as Riemann Surfaces with Conformal Flat Conical Metrics

A Euclidean surface with conical singularities (E.s.c.s.) is a compact (orientable) surface glued from Euclidean triangles. One can take as an example of such a surface the boundary of a connected but not necessarily simply connected polyhedron in $\mathbb{R}^3$ [18].

When two triangles are glued together and after rotating one of the triangles around the common edge, we observe that the intrinsic geometry of the surface is locally that of the plane. There, the surface actually is smooth and equipped with a smooth Euclidean metric. At a vertex p where k triangles with angles $\vartheta_1, \dots, \vartheta_k$ are glued together, the surface is locally isometric to a neighborhood of the tip of the Euclidean cone of total angle $\theta_p = \vartheta_1 + \cdots + \vartheta_k$. The surface X is thus equipped with a Euclidean metric that is smooth except at the vertices p for which $\theta_p \neq 2\pi$.

It follows, for instance from [27], that X can be provided with a complex analytic structure becoming a compact Riemann surface $\tilde{X}$; moreover, the usual Euclidean metric on X gives rise to a flat *conformal* (i.e., defining the same complex structure) metric on $\tilde{X}$. Abusing notation slightly, from now on we won't make any difference between X and $\tilde{X}$.

On the other hand, consider a flat conformal metric m with conical singularities on a Riemann surface X. In a vicinity of a conical point p, m can be written as

$$m = \left|g(z)\right||z|^{2b}|dz|^2,$$

where z is a holomorphic local parameter near p, $z(p) = 0$, $b > -1$, and $g(z)$ is a holomorphic function of the local parameter such that $g(0) \neq 0$.

It is shown in [27] that one can choose a holomorphic change of variables $z = z(\zeta)$ such that

$$\left|g\big(z(\zeta)\big)\right|\left|z(\zeta)\right|^{2b}\left|z'(\zeta)\right|^2 = |\zeta|^{2b} \tag{2.1}$$

and, therefore,

$$m = |\zeta|^{2b}|d\zeta|^2 \tag{2.2}$$

in the local parameter ζ. This means that the Riemannian surface (X, m) near p is isometric to the standard Euclidean cone of angle $2\pi(b+1)$. Troyanov [27] showed that the Riemannian manifold (X, m) can be triangulated in such a way that all the conical points will be among the vertices of the triangulation, meaning thus that (X, m) is an E.s.c.s.

Definition 1 Let X be a compact Riemann surface with conformal flat conical metric (i.e., an E.s.c.s.) and let $p \in X$ be a conical point. Then any holomorphic local parameter ζ in which the metric takes the form (2.2) is called distinguished.

Notation We will denote by P the set of conical points and by $X_0 := X \setminus P$ the complement of P in X. We set $M := \mathrm{Card}(P)$ the number of conical points. At each $p \in P$, the total cone angle is denoted by θ_p.

2.1.1 Translation and Half-Translation Surfaces

A translation (resp., half-translation) surface is an E.s.c.s. that has trivial holonomy (resp., holonomy group $\mathbb{Z}_2$).

These are important examples of E.s.c.s. with very nice geometric properties (see [29] for a survey on these).

Translation surfaces are Riemann surfaces X that are equipped with a conformal flat conical metric given by the modulus square, $m = |\omega|^2$, of a holomorphic 1-form (an Abelian differential) ω. If P is a zero of ω of multiplicity k, then p is a conical point of the translation surface X with conical angle $2\pi(k+1)$. The moduli space H_g of pairs (X, ω) (where X is a compact Riemann surface of genus $g \geq 1$ and ω is a holomorphic 1-form on X) is stratified according to the multiplicities of the zeros of the 1-form ω. Denote by $H_g(k_1, \dots, k_M)$ the stratum consisting of pairs (X, ω), where ω has M zeros, $p_1, \dots, p_M$ of multiplicities $k_1, \dots, k_M$ (according to Riemann–Roch theorem, one has $k_1 + \dots + k_M = 2g - 2$). The stratum $H_g(k_1, \dots, k_M)$ is a complex orbifold of dimension $2g + M - 1$.

Let $(X, \omega) \in H_g(k_1, \dots, k_M)$. Choose a canonical basis of cycles $\{a_\alpha, b_\alpha\}$ on the Riemann surface X and take $M - 1$ contours γ_k, $k = 2, \dots, M$, on X connecting p_1 with $p_2, \dots, p_M$

The local coordinates on $H_g(k_1, \dots, k_M)$ (which are called Kontsevich–Zorich homological coordinates; see [20]) are given by the following integrals:

$$A_\alpha = \oint_{a_\alpha} \omega; \quad \alpha = 1, \dots, g,$$

$$B_\alpha = \oint_{b_\alpha} \omega; \quad \alpha = 1, \dots, g,$$

$$z_k = \int_{\gamma_k} \omega; \quad k = 2, \dots, M - 1.$$

A half-translation surface is a compact Riemann surface with flat conical metric $m = |q|$, where q is a meromorphic quadratic differential with at most simple poles.

Example 2.1 Consider the Riemann sphere $\mathbb{C}P^1$ with metric

$$\frac{|z|^2 |dz|^2}{\prod_{k=1}^{6} |z - z_k|},$$

where $z_k \in \mathbb{C}$, $z_k \neq 0$, and $z_i \neq z_k$ if $i \neq k$. This is a half-translation surface with 7 conical points $0, z_1, \dots, z_6$. The conical angle at 0 is 4π; the conical angles at each point z_k are equal to π.

Such a surface can be viewed by considering a Euclidean pair of pants (with one 4π singularity) and by sewing each leg and the waist with itself (thus creating the six π singularities).

3 The Friedrichs Laplacian and the S-Matrix

Let X be a compact E.s.c.s. In this section we will recall the definition of the Friedrichs Laplacian associated with the (singular) metric and define the so-called S-matrix. We will then collect several properties of this matrix.

We denote by Δ the minimal closed extension of the Euclidean Laplacian defined on $\mathcal{C}_0^\infty(X_0)$, and by Δ^* its adjoint with respect to the Euclidean L^2 scalar product

$$\langle u, v\rangle := \int_X u\overline{v}\, dx.$$

Near each conical point p, any $u \in \mathrm{dom}(\Delta^*)$ has the following asymptotic behavior in polar coordinates (r, θ) (see, e.g., [22–24], or [19]):

$$u(r,\theta) = \sqrt{2\theta_p}\big(a_0^+ + a_0^- \ln(r)\big) + \sum_\nu \sqrt{2|\nu|\theta_p}\big(a_\nu^+ r^{|\nu|} + a_\nu^- r^{-|\nu|}\big)\exp(i\nu\theta) + u_0, \tag{3.1}$$

where ν ranges over $N_p := \{\frac{2\pi}{\theta_p}\cdot k, |k \in \mathbb{Z}\backslash\{0\}, |k| < \frac{\theta_p}{2\pi}\}$, and $u_0 \in \mathrm{dom}(\Delta)$.

Notation We will denote by $N = \bigcup_{p\in P} N_p$, and we will abusively still denote by ν an element of N. Choosing an element ν of N thus amounts to choosing a conical point p and then some ν in N_p. Unless needed, we will omit the reference to p. The square root prefactors in (3.1) are just normalization constants. We will denote these constants by $C_0 := \sqrt{2\theta_p}$ and $C_\nu := \sqrt{2|\nu|\theta_p}$ (we recall that since ν implicitly depends on p, so does C_ν).

In the distinguished local parameter ζ near p we have, for $\nu = \frac{2\pi}{\theta_p}\cdot k$,

$$\zeta^k = r^\nu \exp(i\nu\theta) = \begin{cases} r^{|\nu|}\exp(i\nu\theta) & \text{if } \nu > 0,\\ r^{-|\nu|}\exp(i\nu\theta) & \text{if } \nu < 0.\end{cases} \tag{3.2}$$

$$\overline{\zeta}^{-k} = r^{-\nu}\exp(i\nu\theta) = \begin{cases} r^{|\nu|}\exp(i\nu\theta) & \text{if } \nu < 0,\\ r^{-|\nu|}\exp(i\nu\theta) & \text{if } \nu > 0.\end{cases} \tag{3.3}$$

Thus the asymptotic expansion (3.1) may also be written

$$\begin{aligned} u(\zeta,\bar\zeta) &= C_0\big(a_0^+ + a_0^- \ln(|\zeta|)\big)\\ &\quad + \sum_{k=1}^{\frac{\theta_p}{2\pi}-1} C_{k\frac{2\pi}{\theta_p}}\big(a_k^+\zeta^k + a_k^-\bar\zeta^{-k} + a_{-k}^+\bar\zeta^k + a_{-k}^-\zeta^{-k}\big) + u_0. \end{aligned} \tag{3.4}$$

A straightforward application of Green's formula (combined with the choice of the normalization constants C_0, C_ν) then implies that, for any u, v in $\mathrm{dom}(\Delta^*)$,

$$\big\langle \Delta^* u, v\big\rangle - \big\langle u, \Delta^* v\big\rangle = \sum_{p\in P}\Big[a_0^+\cdot\overline{b_0^-} - a_0^-\cdot\overline{b_0^+} + \sum_{\nu\in N_p}\big(a_\nu^+\cdot\overline{b_\nu^-} - a_\nu^-\cdot\overline{b_\nu^+}\big)\Big], \tag{3.5}$$

where the $a_\nu^\pm$ are the coefficients in the expansion of u and the $b_\nu^\pm$ those in the expansion of v.

Setting $\mathcal{G}(u, v) := \langle \Delta^* u, v\rangle - \langle u, \Delta^* v\rangle$, we define a Hermitian symplectic form on $\mathrm{dom}(\Delta^*)/\,\mathrm{dom}(\Delta)$ whose Lagrangian subspaces parameterize the self-adjoint extensions of Δ.

3.1 The Friedrichs Extension

For any $u \in \mathrm{dom}(\Delta)$, a straightforward integration by parts gives

$$\langle \Delta u, u\rangle = \int_X |\nabla u|^2 dx,$$

so that the Friedrichs procedure (see [4] Sect. 10.3 or [26] Theorem X.23) provides us with a self-adjoint extension that we denote by Δ_F. Since a function u in $\mathrm{dom}(\Delta_F)$ is characterized by $\nabla u \in L^2(X)$, we obtain the following lemma.

Lemma 3.1 *The Lagrangian subspace in* $\mathrm{dom}(\Delta^*)/\,\mathrm{dom}(\Delta)$ *that corresponds to the Friedrichs extension is*

$$\{a_\nu^- = 0\}.$$

Definition 2 We denote by $H^s := \mathrm{dom}(\Delta_F^{\frac{s}{2}})$ the scale of Sobolev spaces associated with it. In particular, we set $\mathrm{dom}(\Delta_F) := H^2$.

Remark 1 This definition of H^s is not completely standard. In particular, because of the conical singularities, for $m > 1$ the following inclusion is strict (see [11] for a much more detailed discussion about this fact):

$$\{u \in L^2 \mid \forall |\alpha| \le m, \partial^\alpha u \in L^2\} \subset H^m.$$

By standard spectral theory, the resolvent of Δ_F defines a continuous operator from H^s to H^{s+2}. We also recall that since X is compact, the Rellich-type injection theorem from [8] implies that Δ_F has compact resolvent, so that the spectrum is non-negative and discrete.

3.2 The S-Matrix

We will now define a matrix associated with the flat structure and with the choice of the Friedrichs extension.

First, for any ν, we fix $F_\nu = C_\nu r^{-|\nu|} \exp(i\nu\theta)\rho(r)$, where ρ is some fixed cut-off function that is identically 1 near the corresponding conical point p.

We define Λ_ν to be the linear functional on H^2 satisfying

$$\forall u \in H^2, \quad \Lambda_\nu(u) = \mathcal{G}(u, F_\nu). \tag{3.6}$$

We have the following lemma.

Lemma 3.2 *The linear functional Λ_ν is continuous on H^2 and*

$$\forall u \in H^2, \quad \Lambda_\nu(u) = a_\nu^+,$$

where a_ν^+ is the coefficient in the expansion (3.1) *of u near p.*

Proof The fact that $F_\nu \in \mathrm{dom}(\Delta^*)$ implies that Λ_ν is indeed continuous. The second statement follows from the respective asymptotic behaviors of F_ν and u near p. $\square$

Remark 3.3 The preceding lemma in particular implies that the linear functional Λ_ν doesn't depend on the choice of the cut-off function ρ.

For $\lambda \in \mathbb{C} \setminus [0, \infty)$, we set

$$G_\nu(\cdot; \lambda) := (\Delta_F - \lambda)^{-1}\Lambda_\nu.$$

Since Λ_ν is in H^{-2}, G_ν is in L^2, and for any $u \in H^2$, we have

$$\Lambda_\nu(u) = \big\langle(\Delta_F - \lambda)u, G_\nu(\cdot; \overline{\lambda})\big\rangle. \tag{3.7}$$

Since the resolvent is analytic in λ, $G_\nu(\cdot; \lambda)$ defines an analytic family of L^2 functions.

Observe that the latter equation is equivalent to

$$\big(\Delta^* - \lambda\big)G_\nu(\cdot; \lambda) = 0,$$

so that $G_\nu(\cdot; \lambda) \in \mathrm{dom}(\Delta^*)$. Moreover, by testing against an appropriate $u \in H^2$ we can compute the coefficients a_μ^- of G_ν. This yields $a_\mu^- = \delta_{\mu\nu}$ (where δ is the Kronecker symbol).

The following proposition gives a formula for G_ν.

Proposition 3.4 *For any $\lambda \in \mathbb{C} \setminus [0, \infty)$, set $f_\nu(\cdot; \lambda) := (\Delta^* - \lambda)F_\nu$ and $g_\nu(\cdot; \lambda) := -(\Delta_F - \lambda)^{-1} f_\nu(\cdot; \lambda)$. Then $g_\nu(\cdot; \lambda)$ is an analytic family in H^2 and*

$$G_\nu(\cdot\,; \lambda) = F_\nu(\cdot) + g_\nu(\cdot\,; \lambda).$$

Proof Computation shows that f_ν is in $L^2(X)$, which yields that g_ν is in H^2 since λ is in the resolvent set of Δ_F. Since f_ν and the resolvent depend analytically on λ, so does g_ν. By construction, $(\Delta^* - \lambda)(F_\nu + g_\nu) = 0$ and all the a_μ^- coefficients of $G_\nu - (F_\nu + g_\nu)$ vanish. This means that the latter function is in H^2 and thus is 0 since λ is in the resolvent set. $\square$

Example 3.5 Let us consider the complete cone $[0, \infty) \times \mathbb{R}/\alpha\mathbb{Z}$. Using separation of variables we have that $G_\nu(r, \theta; \lambda) = k(r)\exp(i\nu\theta)$. For $\nu \neq 0$, by definition k is the unique solution to

$$-k'' - \frac{1}{r}k' + \left(\frac{\nu^2}{r^2} - \lambda\right)k = 0,$$

which is in $L^2(rdr)$ and asymptotic to $C_\nu r^{-|\nu|}$ near 0. Thus k is proportional to $K_\nu(\sqrt{-\lambda}\, r)$, where K_ν is a Bessel–MacDonald function (see [25], for instance). For $\nu = 0$, the singular behavior is logarithmic, but $k(r)$ is still proportional to $K_0(\sqrt{-\lambda} r)$.

Definition 3.6 (The S-matrix) We define the S-matrix $S(\lambda)$ by

$$S_{\mu\nu}(\lambda) = \Lambda_\mu\big(g_\nu(\cdot;\lambda)\big). \tag{3.8}$$

Remark 3.7 Alternatively, $S_{\mu\nu}(\lambda)$ is the a_μ^+ coefficient of $g_\nu(\cdot;\lambda)$. It is also the a_μ^+ coefficient of $G_\nu(\cdot;\lambda)$. Observe that the entries of the S-matrix are numbered by non-integer numbers.

Using (3.7), we have the following alternative expression:

$$S_{\mu\nu}(\lambda) = \big\langle(\Delta_F - \lambda)g_\nu(\cdot;\lambda), G_\mu(\cdot;\overline{\lambda})\big\rangle = \big\langle f_\nu(\cdot;\lambda), G_\mu(\cdot;\overline{\lambda})\big\rangle.$$

It follows from the analyticity of g_ν that $S(\lambda)$ is analytic on $\mathbb{C}\setminus[0;\infty)$.

Example 3.8 We define $S_\alpha(\lambda)$ to be the S-matrix of the cone of angle α. According to Example 3.5, $S_\alpha(\lambda)$ is diagonal. Moreover, the asymptotic expansion of Bessel–MacDonald functions near 0 is

$$K_0(z) = -\ln(z) + \ln(2) - \gamma + o(1),$$
$$K_{|\nu|}(z) = \frac{\pi}{2\sin(|\nu|\pi)}\left[\frac{z^{-|\nu|}}{2^{-|\nu|}\Gamma(1-|\nu|)} - \frac{z^{|\nu|}}{2^{|\nu|}\Gamma(1+|\nu|)} + O\big(z^{2-|\nu|}\big)\right],$$

where Γ is the Euler gamma function and γ is Euler's constant (see, for instance, [25]). This yields

$$\big[S_\alpha(\lambda)\big]_{00} = \ln\big(\sqrt{-\lambda}\big) - \big(\ln(2) - \gamma\big),$$
$$\big[S_\alpha(\lambda)\big]_{\nu\nu} = -\frac{\Gamma(1-|\nu|)(-\lambda)^{|\nu|}}{2^{2|\nu|}\Gamma(1+|\nu|)}.$$

The interpretation of $S(\lambda)$ is given by the following lemma.

Lemma 3.9 *For any $\lambda \in \mathbb{C}\setminus[0,\infty)$ and any $F \in \ker(\Delta^* - \lambda)$. Denote by $A^\pm(F)$ the vector consisting of all the coefficients a_ν^- (resp., a_ν^+) of F. Then we have*

$$A^+ = S(\lambda)A^-.$$

Remark 3.10 Interpreting A^- as some kind of incoming data and A^+ as the outgoing data justifies the interpretation of the S-matrix as a scattering matrix.

Proof Set $\tilde{F} := \sum_\nu a_\nu^- G_\nu(\cdot;\lambda)$. Then $F - \tilde{F}$ is in dom(ker($\Delta^* - \lambda$)). Since all the a_ν^- vanish, $F - \tilde{F}$ actually is in dom(Δ_F). This implies $F = \tilde{F}$, since λ is in the resolvent set of Δ_F. Writing each $G_\nu = F_\nu + g_\nu$, we obtain:

$$a_\mu^+ = \Lambda_\mu\left(\sum_\nu a_\nu^- g_\nu\right) = \sum_\nu S(\lambda)_{\mu\nu} a_\nu^- .$$

□

Remark 3.11 Until now we haven't used the fact that the underlying metric actually is Euclidean with conical singularities. The preceding construction is fairly general and can be made on any manifold with conical singularities. Actually, it can be done in an abstract manner for any symmetric operator with (equal) finite deficiency indices (compare with Sect. 13.4 of [14]).

Before coming to the main aim of this paper, which is to understand how much geometric information is contained in the S-matrix, we first derive two basic properties of $S_{\mu\nu}(\lambda)$.

3.3 Derivative of the S-Matrix

In this subsection, a dot will mean differentiation with respect to λ, and we prove the following lemma.

Lemma 3.12 *On* $\mathbb{C} \setminus [0, \infty)$, *we have*

$$\dot{S}_{\mu\nu} = \left\langle G_\nu(\cdot;\lambda), G_\mu(\cdot;\overline{\lambda})\right\rangle. \tag{3.9}$$

Proof We start from the relation

$$(\Delta_F - \lambda) g_\nu(\cdot;\lambda) = -\Delta^* F_\nu(\cdot) + \lambda F_\nu(\cdot),$$

which we differentiate with respect to λ. Since F_ν doesn't depend on λ, and g_ν is analytic in H^2, we obtain

$$(\Delta_F - \lambda)\dot{g}_\nu(\cdot;\lambda) = F_\nu(\cdot) + g_\nu(\cdot;\lambda) = G_\nu(\cdot;\lambda).$$

This gives

$$\begin{aligned} \dot{S}(\lambda)_{\mu\nu} &= \Lambda_\mu\left(\dot{g}_\nu(\cdot;\lambda)\right) \\ &= \Lambda_\mu\left((\Delta_F - \lambda)^{-1} G_\nu(\cdot;\lambda)\right) \\ &= \left\langle G_\nu(\cdot;\lambda), G_\mu(\cdot;\overline{\lambda})\right\rangle, \end{aligned}$$

where we have used (3.7) for the last identity. □

3.4 Relation with the Resolvent Kernel

Denote by $R(x, x'; \lambda)$ the resolvent kernel of the Friedrichs extension Δ_F.

Fix $x' \in X_0$. As a function of the first argument, $R(\cdot, x'; \lambda)$ is locally in H^2 near each conical point p. Thus, according to (3.1), there exists a collection $a_\nu^+(x'; \lambda)$ such that in the neighborhood of p we have the following asymptotic expansion:

$$R\big(r\exp(i\theta), x'; \lambda\big) = \sum_{\nu\in N_p} C_\nu a_\nu^+\big(x'; \lambda\big) r^{|\nu|}\exp(i\nu\theta) + r_0, \qquad (3.10)$$

with $r_0 \in \overline{\mathcal{C}_0^\infty(X_0)}^{H^2}$.

Using (3.6), we see that $a_\nu^+(x'; \lambda) = \mathcal{G}(R(\cdot, x'; \lambda), F_\nu)$, and thus the former expansion may be differentiated with respect to x' in any compact set of X_0.

The following proposition makes the relation between $a_\nu^+(x'; \lambda)$ and $G_\nu(x'; \lambda)$ more explicit.

Proposition 3.13 *For any $x' \in X_0$, we have*

$$G_\nu\big(x'; \lambda\big) = a_\nu^+\big(x'; \lambda\big), \qquad (3.11)$$

where $a_\nu^+(x'; \lambda)$ is the previously described coefficient in the asymptotic expansion of $R(\cdot, x'; \lambda)$ near p.

In other words, $G_\nu(x'; \lambda)$ is obtained by selecting in the resolvent kernel $R(x, x'; \lambda)$ some particular term in the asymptotic behavior $x \to p$. Using $\overline{R(x', x; \overline{\lambda})} = R(x, x'; \lambda)$, there are similar statements when we fix x and let x' tend to p.

Proof Denote by Δ_1 the Euclidean Laplace operator on $\mathcal{C}_0^\infty(X \setminus (P \cup \{x'\}))$. This operator fits in the general theory described in Sect. 3 by considering that x' actually is the vertex of a cone of angle 2π. In particular, Green's formula (3.1) is still valid, provided we take into account log singularities at x'. The resolvent kernel $R(\cdot, x'; \lambda)$ and $G_\nu(\cdot; \lambda)$ both belong to $\mathrm{dom}(\Delta_1^*)$. The singularities of R are described by the functions a_ν^+ near the conical points, and R has a log singularity near x', whereas G_ν is smooth near x' and its singular behavior near the conical points G_ν is prescribed by (3.10). Green's formula thus yields:

$$\begin{aligned}&\big\langle\big(\Delta_1^* - \lambda\big)R\big(\cdot, x'; \lambda\big), G_\nu(\cdot\overline{\lambda})\big\rangle - \big\langle R\big(\cdot, x'; \lambda\big), \big(\Delta_1^* - \overline{\lambda}\big)G_\nu\big(x'; \overline{\lambda}\big)\big\rangle \\ &\quad = \overline{G_\nu\big(x'; \overline{\lambda}\big)} - a_\nu^+\big(x'; \lambda\big).\end{aligned}$$

Since $(\Delta_1^* - \lambda)R(\cdot, x'; \lambda) = 0 = (\Delta_1^* - \overline{\lambda})G_\nu(x'; \overline{\lambda})$, we obtain

$$\overline{G_\nu\big(x'; \overline{\lambda}\big)} = a_\nu^+\big(x'; \lambda\big).$$

We now use the fact that $G_\nu(x; \lambda)$ is analytic for $\lambda \in \mathbb{C}\backslash[0, \infty)$ and real for real (and negative) λ. Thus, by analytic continuation,

$$\overline{G_\nu\big(x'; \overline{\lambda}\big)} = G_\nu\big(x', \lambda\big).$$

□

4 The S-Matrix of E.s.c.s.

In this section we try to understand what kind of geometric information is encoded in the S-matrix of a Euclidean surface with conical singularities. We begin by studying the asymptotic behavior of $S(\lambda)$ as λ goes to $-\infty$.

4.1 $S(-|\lambda|)$ for Large λ

It is a general fact that the behavior of the resolvent kernel when λ goes to $-\infty$ is a local quantity.

This is confirmed by the following lemma.

Lemma 4.1 *When λ goes to ∞ then*

$$\big[S(-|\lambda|)\big]_{\mu\nu} = O\big(|\lambda|^{-\infty}\big),$$

if μ and ν do not correspond to the same conical point.

When μ and ν correspond to the same conical point p of angle α, then we have

$$\big[S(-|\lambda|)\big]_{\mu\nu} = \big[S_\alpha(-|\lambda|)\big]_{\mu\nu} + O\big(|\lambda|^{-\infty}\big),$$

where S_α denotes the S-matrix on the infinite cone of total angle α.

Moreover, both identities may be differentiated with respect to λ.

Proof We use the representation of the resolvent kernel using the heat kernel (which we denote here by $\mathcal{P}(t,x,x')$):

$$R\big(x,x';-|\lambda|\big) = \int_0^\infty \exp\big(-t|\lambda|\big)\mathcal{P}\big(t,x,x'\big)\,dt. \tag{4.1}$$

We now use a standard construction of a parametrix for the heat kernel (see [7], for instance). We first enumerate the set of conical points writing $P := \{p_i, 1 \le p_i \le M\}$. Then, for each p_i we choose $\tilde{\chi}_i$ and χ_i, two smooth cut-off functions such that $\mathrm{supp}(\chi_i) \subset \{\tilde{\chi}_i = 1\}$; χ_i is identically 1 near p and X is isometric to a neighborhood of the tip of the cone of angle θ_{p_i} on the support of $\tilde{\chi}_i$. We complete the collections $(\chi_i)_{i\le M}$ and $(\tilde{\chi}_i)_{i\le M}$ to $(\chi_i)_{i\le \tilde{M}}, (\tilde{\chi}_i)_{i\le \tilde{M}}$ in such a way that $(\chi_i)_{i\le \tilde{M}}$ is a partition of unity, $\tilde{\chi}_i$ is identically 1 on the support of χ_i and, for $M < i \le \tilde{M}$, X is isometric to a neighborhood of the origin in $\mathbb{R}^2$ on the support of $\tilde{\chi}_i$. We also set $\mathcal{P}_i$ to be the heat kernel on the cone corresponding to p_i if $i \le M$ and on the plane otherwise and define

$$\tilde{\mathcal{P}}\big(t,x,x'\big) = \sum_{i=1}^{\tilde{M}} \tilde{\chi}_i(x)\mathcal{P}_i\big(t,x,x'\big)\chi_i(x).$$

Using Duhamel's principle and the fact that $\mathcal{P}_i$ quickly decays away from the diagonal (see (1.1) of [7]) yields that $\tilde{\mathcal{P}}(t) - \mathcal{P}(t)$ maps L^2 into H^s for any s, and

$$\big\|\tilde{\mathcal{P}}(t) - \mathcal{P}(t)\big\|_{L^2\to H^s} = O\big(t^\infty\big)$$

when t goes to 0, so that $\tilde{\mathcal{P}}$ is a parametrix for the heat kernel.

Inserting into (4.1) and integrating against f_ν we obtain

$$g_\nu\big(x;-|\lambda|\big)=\tilde{\chi}_i(x)\int_0^\infty\int_X\mathcal{P}_i\big(t,x,x'\big)f_\nu\big(x';-|\lambda|\big)dS\big(x'\big)dt+r_\lambda(x),$$

where the remainder $r_\lambda\in H^2$ and $\|r_\lambda\|_{H^2}=O(|\lambda|^{-\infty})$ and the index i corresponds to the conical point corresponding to ν. The first statement follows. The second one also follows by remarking that F_ν, f_ν, and Λ_ν can also be seen as living on the cone, and that the latter equation is also valid on the complete cone. Differentiating with respect to λ amounts to replacing $\mathcal{P}$ by $\Delta_F\mathcal{P}$, and we can use the same argument. □

Using Example 3.8, we obtain the following proposition as a corollary.

Proposition 4.2 *When λ goes to ∞ we have*

$$\big[S\big(-|\lambda|\big)\big]_{\mu\nu}=O\big(|\lambda|^{-\infty}\big)\quad \textit{if } \mu\neq\nu,$$

$$\big[S\big(-|\lambda|\big)\big]_{\nu\nu}=-\frac{\Gamma(1-|\nu|)}{2^{2|\nu|}\Gamma(1+|\nu|)}\cdot|\lambda|^{|\nu|}+O\big(|\lambda|^{-\infty}\big),\quad \textit{if } \nu\neq 0,$$

$$\big[S\big(-|\lambda|\big)\big]_{00}=\frac{1}{2}\ln\big(|\lambda|\big)-\big(\ln(2)-\gamma\big)+O\big(|\lambda|^{-\infty}\big).$$

Remark 4.3 It would be interesting to study the asymptotic behavior of $S(\pm i|\lambda|)$. It is then expected to see contributions of periodic diffractive orbits (compare with [16]).

4.2 Explicit Formulas for $S(0)$

In this subsection we will show that for $\nu\neq 0$ the coefficient $S_{\mu\nu}(\lambda)$ is continuous at $\lambda=0$ and may be expressed using standard objects of the Riemannian surface X.

Recall that, in the distinguished local parameter ζ near some conical point P the asymptotic expansion was given in (3.4). It follows that we have

$$\begin{cases} F_\nu(\zeta,\overline{\zeta})\sim C_\nu\zeta^{-k} & k>0,\\ F_\nu(\zeta,\overline{\zeta})\sim C_\nu\overline{\zeta}^{k} & k<0,\end{cases}$$

where, as usual, ν and k are related by the relation $\nu=\frac{2\pi}{\theta_p}\cdot k$.

We first prove the following lemma.

Lemma 4.4 *If $\nu\neq 0$ then $G_\nu(\cdot;\lambda)$ is continuous at $\lambda=0$ and $G_\nu(\cdot;0)$ is a harmonic L^2 function on X such that*

$$\begin{cases} G_\nu(\zeta,\overline{\zeta};0)=\zeta^{-k}+O(1) & k>0,\\ G_\nu(\zeta,\overline{\zeta};0)=\overline{\zeta}^{k}+O(1) & k<0.\end{cases}$$

Proof Recall that we have set $G_\nu(\cdot;\lambda) = F_\nu + g_\nu(\cdot;\lambda)$, where $g_\nu(\cdot;\lambda)$ is the unique solution to

$$(\Delta_F - \lambda)\, g_\nu(\cdot;\lambda) = -\big(\Delta^* - \lambda\big) F_\nu.$$

Since $\int_X (\Delta^* - \lambda) F_\nu \, dx = 0$, the continuity at 0 follows from the fact that the $\ker(\Delta_F)$ consists only of the constant function. By continuity we obtain that $G_\nu(\cdot;0)$ is a solution to $\Delta^* G_\nu(\cdot;0) = 0$ and, therefore, $G_\nu(\cdot;0)$ is harmonic on X_0. □

Remark 4.5 Let ζ denote the distinguished local parameter near a fixed $p \in P$. The problems

$$\begin{cases} \Delta U_k = 0 & \text{on } X \setminus P \\ U_k \sim \zeta^{-k} + O(1), & \text{as } \zeta \to 0 \end{cases} \tag{4.2}$$

for $0 < k < \frac{\theta_p}{2\pi}$ and

$$\begin{cases} \Delta U_k = 0 & \text{on } X \setminus P \\ U_k \sim \overline{\zeta}^{k} + O(1), & \text{as } \zeta \to 0 \end{cases} \tag{4.3}$$

for $-\frac{\theta_p}{2\pi} < k < 0$ have solutions only up to an additive constant. On the other hand, the problem

$$\begin{cases} \Delta u = 0 & \text{on } X \setminus P \\ u \sim \log r + O(1), & \text{as } \zeta \to 0 \end{cases}$$

doesn't have a solution. Thus the behavior of the coefficients $S_{0\nu}(\lambda)$ and $S_{\mu 0}(\lambda)$ may not even be properly defined for $\lambda = 0$. When writing $S(0)$ we will implicitly assume that only the coefficients $S_{\mu\nu}$ with nonzero μ and ν are considered (see also Remark 5.5).

In the next subsection we construct solutions to the problems (4.2), (4.3) since they give the functions $G_\nu(\cdot;0)$ from which the coefficients $S_{\mu\nu}$ can be computed (for nonzero μ and ν).

4.3 Special Solutions and an Explicit Expression for $S(0)$

Choose a canonical basis of cycles $\{a_\alpha, b_\alpha\}$ on the Riemann surface X, and let $\{v_\alpha\}_{\alpha=1,\dots,g}$ be the corresponding basis of holomorphic normalized differentials. Let $\mathbb{B}$ be the matrix of b-periods of X.

We have the following proposition.

Proposition 4.6 *Fixing P a conical point and $k \in \mathbb{N}$, there exist Ω_k and Σ_k such that*

(1) *Ω_k and Σ_k are meromorphic differentials of the second kind on X with only one pole of order $k+1$ at P.*

(2) *In the distinguished local parameter near P, they satisfy*

$$\begin{cases} \Omega_k(\zeta) = -\frac{k}{\zeta^{k+1}} d\zeta + O(1) \\ \\ \Sigma_k(\zeta) = -\frac{ik}{\zeta^{k+1}} d\zeta + O(1). \end{cases} \tag{4.4}$$

(3) *All the a-periods and b-periods of $\Omega_k(P, \cdot)$ and $\Sigma_k(P, \cdot)$ are purely imaginary.*

Proof Let $\omega(\cdot, \cdot)$ be the canonical meromorphic bidifferential on the Riemann surface X (see [9], p. 3) for which the following asymptotic expansion holds:

$$\frac{\omega(\zeta(Q_1), \zeta(Q_2))}{d\zeta(Q_1) d\zeta(Q_2)} = \frac{1}{(\zeta(Q_1) - \zeta(Q_2))^2} + \frac{1}{6} S_B\big(\zeta(Q_2)\big) + o(1)$$

as $Q_1 \to Q_2$, where S_B is the Bergman projective connection. Moreover, ω is normalized in such a way that

$$\begin{cases} \displaystyle\oint_{a_\alpha} \frac{\omega(\cdot, \zeta)}{d\zeta}\bigg|_{\zeta=0} = 0 \\ \displaystyle\oint_{b_\alpha} \frac{\omega(\cdot, \zeta)}{d\zeta}\bigg|_{\zeta=0} = 2\pi i \frac{v_\alpha(\zeta)}{d\zeta}\bigg|_{\zeta=0}, \end{cases} \tag{4.5}$$

for $\alpha = 1, \ldots, g$.

Let $(c_\alpha)_{\alpha=1,\ldots,g}$ be coefficients to be chosen later, and consider the meromorphic differential

$$-\frac{\omega(\cdot, \zeta)}{d\zeta}\bigg|_{\zeta=0} + \sum_{\alpha=1}^{g} c_\alpha v_\alpha. \tag{4.6}$$

We want to choose c_α in (4.6) so that all the a-periods and b-periods of this differential are purely imaginary. The vanishing of the real parts of all a-periods implies that all the constants c_α are purely imaginary. The vanishing of the real part of the period over the cycle b_β then gives:

$$\mathrm{Re}\left(\oint_{b_\beta} \sum c_\alpha v_\alpha\right) = \mathrm{Re}\left(\oint_{b_\beta} \frac{\omega(\cdot, \zeta)}{d\zeta}\bigg|_{\zeta=0}\right).$$

Using the fact that the c_α are known to be purely imaginary and the normalization of ω recalled in (4.5), we obtain the following system of equations:

$$\sum_{\alpha=1}^{g} [\mathrm{Im}\mathbb{B}]_{\beta\alpha} c_\alpha = 2\pi i \,\mathrm{Im}\left(\frac{v_\beta}{d\zeta}\bigg|_{\zeta=0}\right). \tag{4.7}$$

Since $\mathrm{Im}\,(\mathbb{B})$ is invertible, this uniquely determines c_α.

In order to get Σ_1 we apply the same method of searching coefficients c_α so that the meromorphic differential

$$-i\frac{\omega(\cdot,\zeta)}{d\zeta}+\sum_{\alpha=1}^{g}c_\alpha v_\alpha$$

has purely imaginary periods. The system of equations we obtain is similar to (4.7) except that $\mathrm{Im}(\frac{v_\beta}{d\zeta}|_{\zeta=0})$ is replaced by $\mathrm{Re}(\frac{v_\beta}{d\zeta}|_{\zeta=0})$. It still has a solution using the same invertibility of $\mathrm{Im}(\mathbb{B})$.

To get Ω_k and Σ_k with an arbitrary $k\geq 1$, we repeat the same construction, taking the first term in (4.6) to be

$$\frac{(-1)^k}{(k-1)!}\left[\frac{d}{d\zeta}\right]^{k-1}\frac{\omega(\cdot,\zeta)}{d\zeta}\bigg|_{\zeta=0}.$$

We will obtain an equation similar to (4.7) so that eventually, the existence result thus follows from the existence of ω and the fact that the matrix $\mathrm{Im}\,(\mathbb{B})$ is invertible. □

This proposition gives the following corollary.

Corollary 4.7 *Let Ω_k and Σ_k be defined by the preceding proposition. Then the following formula defines a function f_k which is harmonic in $X\setminus\{P\}$*:

$$f_k(Q)=\mathrm{Re}\left\{\int_{P_0}^{Q}\Omega_k\right\}-i\,\mathrm{Re}\left\{\int_{P_0}^{Q}\Sigma_k\right\}. \tag{4.8}$$

Moreover, in the distinguished local parameter near P, f_k admits the following asymptotic behavior:

$$f_k(\zeta)=\frac{1}{\zeta^k}+O(1).$$

Proof Since all the a-periods and b-periods of Ω and Σ are purely imaginary, f_k is indeed well defined on X. The remaining statements follow from the construction. □

By considering $C_\nu f_k$ or $C_\nu\overline{f_k}$ we obtain the functions $G_\nu(\cdot;0)$ up to an additive constant. This additive constant is harmless when computing the matrix elements $S_{\mu\nu}(0)$.

4.3.1 Examples

(1) *A conical point of angle $2\pi<\beta\leq 4\pi$ on a Euclidean surface of genus* ≥ 1. In this case one has $n=1$.

Proposition 4.6 combined with the asymptotics of ω yield

$$\int_{P_0}^{\zeta}\Omega_1(P,\cdot)=\frac{1}{\zeta}+c_0+\Bigg[-\frac{1}{6}S_B(\zeta)\Big|_{\zeta=0} + 2\pi i\sum_{\alpha=1,\beta=1}^{g}\big((\operatorname{Im}\mathbb{B})^{-1}\big)_{\alpha\beta}\operatorname{Im}\left\{\frac{v_\beta(\zeta)}{d\zeta}\Big|_{\zeta=0}\right\}\frac{v_\alpha(\zeta)}{d\zeta}\Big|_{\zeta=0}\Bigg]\zeta + O\big(\zeta^2\big) \tag{4.9}$$

with some constant c_0, and

$$\int_{P_0}^{\zeta}\Sigma_1(P,\cdot)=\frac{i}{\zeta}+d_0+\Bigg[-\frac{i}{6}S_B(\zeta)\Big|_{\zeta=0} + 2\pi i\sum_{\alpha=1,\beta=1}^{g}\big((\operatorname{Im}\mathbb{B})^{-1}\big)_{\alpha\beta}\operatorname{Re}\left\{\frac{v_\beta(\zeta)}{d\zeta}\Big|_{\zeta=0}\right\}\frac{v_\alpha(\zeta)}{d\zeta}\Big|_{\zeta=0}\Bigg]\zeta + O\big(\zeta^2\big) \tag{4.10}$$

with some constant d_0.

Denoting the expressions in square brackets in (4.9) and (4.10) by A and B respectively, one gets the asymptotics

$$f_1(\zeta,\overline{\zeta})=\frac{1}{\zeta}+\text{const}+\frac{A-iB}{2}\zeta+\frac{\overline{A}-i\overline{B}}{2}\overline{\zeta}+O\big(|\zeta|^2\big)$$

and, therefore,

$$S_p(0)=\begin{pmatrix} * & * & * \\ * & \frac{A-iB}{2} & \frac{\overline{A}-i\overline{B}}{2} \\ * & \frac{A+iB}{2} & \frac{\overline{A}+i\overline{B}}{2}\end{pmatrix}, \tag{4.11}$$

where the index p means that we have written down only the coefficients of $S(0)$ that correspond to indices $\nu \in N_p$

(2) *A Euclidean sphere with one* 4π *singularity and six* π *singularities*. Consider the surface of Example 2.1, i.e., the Riemann sphere with metric

$$\frac{|z|^2|dz|^2}{\prod_{k=1}^{6}|z-z_k|}.$$

We consider the part of the S-matrix with non-zero indices μ and ν. We thus only have to consider the asymptotic behavior near 0 and compute the coefficients $S_{\frac{1}{2}\frac{1}{2}}$, $S_{-\frac{1}{2}-\frac{1}{2}}$, $S_{-\frac{1}{2}\frac{1}{2}}$, and $S_{\frac{1}{2}-\frac{1}{2}}$.

The distinguished local parameter ζ in a vicinity of the conical point $z=0$ is given by

$$\zeta(z)=\left(\int_0^z\frac{w\,dw}{\sqrt{\prod_{k=1}^{6}(w-z_k)}}\right)^{1/2}.$$

The special solution f_1 is now not only harmonic, but even holomorphic in $\mathbb{C}P^1 \setminus 0$, and is nothing but the function A/z with some constant A.

One has

$$\frac{A}{z} = \frac{1}{\zeta} + \text{const} + S_{\frac{1}{2}\frac{1}{2}}(0)\zeta + O\big(\zeta^2\big).$$

Therefore, $A = \frac{dz}{d\zeta}|_{\zeta=0}$ and a simple calculation shows that

$$S_{\frac{1}{2}\frac{1}{2}}(0) = -\frac{1}{6}\frac{z'''(\zeta)z'(\zeta) - \frac{3}{2}(z''(\zeta))^2}{(z'(\zeta))^2}\bigg|_{\zeta=0} = -\frac{1}{6}\{z,\zeta\}|_{\zeta=0},$$

where $\{z,\zeta\}$ is the Schwarzian derivative. One also has $S_{-\frac{1}{2}-\frac{1}{2}}(0) = \overline{S_{\frac{1}{2}\frac{1}{2}}(0)}$ and $S_{\frac{1}{2}-\frac{1}{2}}(0) = S_{-\frac{1}{2}\frac{1}{2}}(0) = 0$.

In the very symmetric case where the z_k form a regular hexagon, the computation yields that $z = c\cdot\zeta(1 + O(\zeta^6))$ so that $S_{\frac{1}{2}\frac{1}{2}}$ and $S_{-\frac{1}{2}-\frac{1}{2}}$ also vanish.

4.4 S-Matrix as a Function on the Moduli Space of Holomorphic Differentials: Variational Formulas

Let $(X,\omega) \in H_g(k_1,\dots,k_M)$ and let $S(\lambda)$ be the S-matrix corresponding to a conical point of the translation surface $(X, |\omega|^2)$ (i.e., one of the zeros of the holomorphic one-form ω). Here we derive the variational formulas for $S(\lambda)$ with respect to Kontsevich–Zorich homological coordinates on $H_g(k_1,\dots,k_M)$.

Proposition 4.8 *Let $z(p) = \int^p \omega$. Introduce the following (closed) (1-1)-form on X_0:*

$$\Theta_{\mu\nu} = \big[G_\mu(z;\lambda)\big]_{z\bar z} G_\nu(z;\lambda)d\bar z + \big[G_\mu(z;\lambda)\big]_z\big[G_\nu(z;\lambda)\big]_z dz.$$

Then the variational formulas hold:

$$\frac{\partial S_{\mu\nu}(\lambda)}{\partial A_\alpha} = 2i\oint_{b_\alpha}\Theta_{\mu\nu};\quad \alpha = 1,\dots,g, \tag{4.12}$$

$$\frac{\partial S_{\mu\nu}(\lambda)}{\partial B_\alpha} = -2i\oint_{a_\alpha}\Theta_{\mu\nu};\quad \alpha = 1,\dots,g, \tag{4.13}$$

$$\frac{\partial S_{\mu\nu}(\lambda)}{\partial z_k} = 2i\oint_{p_k}\Theta_{\mu\nu};\quad k = 2,\dots,M, \tag{4.14}$$

where the integrals in (4.14) are taken over some small contours encircling conical points p_k.

Proof The method of proof closely follows [19]. We will prove only the variational formulas with respect to coordinates A_α, since the other formulas can be established similarly.

According to [19] (Proposition 2, p. 84) one has

$$\partial_{A_\alpha} R(x,y;\lambda) = 2i\oint_{b_\alpha} R(x,z;\lambda)R_{z\bar z}(y,z;\lambda)d\bar z + R_z(x,z;\lambda)R_z(y,z;\lambda)\,dz. \tag{4.15}$$

(Here $R(x,y;\lambda)$ stands for the resolvent kernel of the Friedrichs extension; one has $R_{z\bar{z}}(x,z;\lambda)=\frac{\lambda}{4}R(x,z;\lambda)$.) On the other hand, by the definition of g_ν we have

$$g_\nu(x;\lambda)=-\iint_X \big[R(x,y;\lambda)(\Delta-\lambda)F_\nu(y)\big]dy, \tag{4.16}$$

and Lemma 7 on p. 88 of [19] reads as

$$\partial_{A_\alpha}\iint_X \Phi(x,\overline{x};\text{moduli})dx=\iint_X \partial_{A_\alpha}\Phi(x,\overline{x},\text{moduli})dx + \frac{i}{2}\oint_{b_\alpha}\Phi(x,\overline{x},\text{moduli})d\overline{x}. \tag{4.17}$$

The cycle b_α does not intersect the support of F_ν, and the terms F_ν and $(\Delta-\lambda)F_\nu$ are moduli independent; therefore,

$$\partial_{A_\alpha}G_\nu(x;\lambda)=\partial_{A_\alpha}(F_\nu+g_\nu)=\partial_{A_\alpha}g_\nu(x;\lambda).$$

Using (4.16) and (4.17), we obtain

$$\begin{aligned}\partial_{A_\alpha}G_\nu(x;\lambda)&=2i\iint_X dy\big[(\Delta-\lambda)F_\nu(y)\big]\oint_{b_\alpha}\big\{R(x,z;\lambda)R_{z\bar{z}}(y,z;\lambda)d\bar{z}\\&\quad+R_z(x,z;\lambda)R_z(y,z;\lambda)dz\big\}\\&=2i\oint_{b_\alpha}R(x,z;\lambda)\left[\iint_X\frac{\lambda}{4}R(y,z;\lambda)(\Delta-\lambda)F_\nu(y)dy\right]d\bar{z}\\&\quad+2i\oint_{b_\alpha}R_z(x,z;\lambda)\left[\iint_X R_z(y,z;\lambda)(\Delta-\lambda)F_\nu(y)dy\right]dz\\&=2i\oint_{b_\alpha}R_{z\bar{z}}(x,z;\lambda)g_\nu(z;\lambda)d\bar{z}+R_z(x,z;\lambda)\big[g_\nu(z;\lambda)\big]_z dz\\&=2i\oint_{b_\alpha}R_{z\bar{z}}(x,z;\lambda)G_\nu(z;\lambda)d\bar{z}+R_z(x,z;\lambda)\big[G_\nu(z;\lambda)\big]_z dz.\end{aligned}$$

We finally obtain

$$\partial_{A_\alpha}g_\nu(\lambda,x)=2i\oint_{b_\alpha}R_{z\bar{z}}(x,z;\lambda)G_\nu(z;\lambda)d\bar{z}+R_z(x,z;\lambda)\big[G_\nu(z;\lambda)\big]_z dz.$$

Using Proposition 3.13 and (3.10) to identify the behavior near the conical points of the different terms, we obtain

$$\partial_{A_\alpha}S_{\mu\nu}=2i\oint_{b_\alpha}\big[a_\mu^+(z;\lambda)\big]_{z\bar{z}}G_\nu(z;\lambda)d\bar{z}+\big[a_\mu^+(z;\lambda)\big]_z\big[G_\nu(z;\lambda)\big]_z dz.$$

Using Proposition 3.13, this gives the result.

5 Krein's Formula and Relative Determinants

There are several ways of defining determinants of operators acting on an infinite-dimensional space. We recall the following two basic constructions: first, a *perturbative determinant* when the operator is a trace-class perturbation of the identity; and second, *zeta-regularization*, which is used in particular for Laplacian-like operators.

Both these approaches can also be used to define relative determinants when comparing two operators H_0 and H_1 in which one is thought to be a perturbation of the other. Krein's formula is a classical tool in this setting and usually applies when the difference $f(H_1) - f(H_0)$ is trace-class for some simple function f. In that case it is possible to define a relative perturbative determinant (see [28]). This approach applies well to the case when H_0 and H_1 are different self-adjoint extensions of a symmetric operator that has finite deficiency indices. Indeed, in that case the difference of the resolvents is a finite-rank operator and, moreover, the perturbative determinant is actually the determinant of a finite-dimensional matrix.

We will thus adapt these techniques to our setting. The method is clearly identified in the literature (see [28] and also [5]), and the main task here consists of identifying the perturbative determinant in terms of the boundary condition and the S-matrix.

Once this is done, we will use this determinant to define a zeta-regularization and compare the determinants that are obtained this way.

Remark 5.1 We insist here that we will actually use the perturbative determinant to show that zeta-regularization is possible and then to compare the two definitions of determinants. In particular, all the issues that are relative to zeta-regularization may be expressed using the perturbative determinant (when the latter can be defined).

5.1 Krein's Formula and Perturbative Determinant

One convenient way of parameterizing the self-adjoint extensions of Δ is by using two matrices P and Q in the following way (see [21]).

We first construct two vectors $A^{\pm}$ that collect the coefficients $a_\nu^{\pm}$. We organize these coefficients so that the first n_{p_1} entries correspond to the first conical point p_1, then we put the data corresponding to the second conical points, and so on.

A Lagrangian subspace L in $\mathrm{dom}(\Delta^*)/\,\mathrm{dom}(\Delta)$ can be parameterized by a system of linear equations of the following form:

$$PA^- + QA^+ = 0,$$

where P and Q are square matrices satisfying $\mathrm{rank}(P, Q)$ is maximal and P^*Q is self-adjoint. We fix two such matrices and denote by Δ_L the corresponding self-adjoint extensions.

It is possible to find a basis in which the $n \times 2n$ matrix $(P\ Q)$ has the following block-decomposition ([21]):

$$\begin{pmatrix} P_2 & P_3 & Q_1 & 0 \\ 0 & P_1 & 0 & 0 \end{pmatrix}, \tag{5.1}$$

in which P_1 and Q_1 are invertible and $L := Q_1^{-1} P_2$ is self-adjoint.

Definition 5.2 We will call an extension Δ_L *regular* if functions in $\mathrm{dom}(\Delta_L)$ are not allowed to have logarithmic singularities. Equivalently, Δ_L is regular if and only if for any $u \in \mathrm{dom}(\Delta_L)$, and any conical point p, the coefficient $a^-_{p,0}$ of u vanishes.

The following observation (based on the classical Krein formula) is the key technical result of the present paper.

Proposition 5.3 *For any* $\lambda \in \mathbb{C} \setminus (\mathrm{spec}(\Delta_F) \cup \mathrm{spec}(\Delta_L))$ *the following identity holds*:

$$\mathrm{Tr}\big((\Delta_L - \lambda)^{-1} - (\Delta_F - \lambda)^{-1}\big) = -\mathrm{Tr}\big(\big(P + QS(\lambda)\big)^{-1} Q\dot{S}(\lambda)\big),$$

where the dot indicates derivation with respect to λ.

Proof Let λ be in the union of the resolvent sets of Δ_F and Δ_L, and let f be in L^2. We search a matrix $X = [x_{\mu\nu}]$ such that we have the following Krein formula (see, e.g., [4] or [1], Theorem A.3)

$$(\Delta_L - \lambda)^{-1} f = (\Delta_F - \lambda)^{-1}(f) + \sum_{\mu,\nu} x_{\mu\nu} G_\mu(\cdot;\lambda)\Lambda_\nu\big[(\Delta_F - \lambda)^{-1}(f)\big]. \quad (5.2)$$

We denote by $u = (\Delta_F - \lambda)^{-1}(f)$ and we compute the vectors $A^\pm$ of the right-hand side

$$a^-_\mu = \sum_\nu x_{\mu\nu}\Lambda_\nu(u),$$

$$a^+_{\mu'} = \Lambda_{\mu'}(u) + \sum_{\mu,\nu} x_{\mu\nu}\big[S(\lambda)\big]_{\mu'\mu}\Lambda_\nu(u).$$

Denoting by Λ the vector $\Lambda_\nu(u)$ we thus have

$$A^- = X\Lambda, \qquad A^+ = \big(I + S(\lambda)X\big)\Lambda.$$

Plugging into the self-adjoint condition, we obtain that the following relation is satisfied.

$$\big[PX + Q\big(I + S(\lambda)X\big)\big]\cdot\Lambda = 0.$$

Using the block decomposition (5.1), we see that

$$P + QS(\lambda) = \begin{pmatrix} P_2 + Q_1 S(\lambda) & * \\ 0 & P_1 \end{pmatrix}.$$

Since λ is in both resolvent sets, Λ is arbitrary and the preceding system always has a solution. We obtain that $(P_2 + Q_1 S(\lambda))$ must be invertible and hence $P + QS(\lambda)$. Finally, we obtain

$$X = -\big(P + QS(\lambda)\big)^{-1} Q.$$

Denoting by $\Pi_{\mu\nu}(\lambda)$ the (rank one) operator defined from H^2 into L^2 by

$$\Pi_{\mu\nu}(\lambda)(u) = G_\mu(\cdot;\lambda)\Lambda_\nu(u),$$

(5.2) may be rewritten:

$$(\Delta_L - \lambda)^{-1} - (\Delta_F - \lambda)^{-1} = \sum_{\mu,\nu} x_{\mu\nu}\Pi_{\mu\nu}(\lambda)\circ(\Delta_F - \lambda)^{-1}.$$

Observe that the right-hand side is finite rank, so that we can trace this equation and obtain

$$\operatorname{Tr}\big((\Delta_L - \lambda)^{-1} - (\Delta_F - \lambda)^{-1}\big) = \sum_{\mu,\nu} x_{\mu\nu}\operatorname{Tr}\big(\Pi_{\mu\nu}(\lambda)\circ(\Delta_F - \lambda)^{-1}\big).$$

Using Lemma 5.4 below and Lemma 3.12, we obtain

$$\begin{aligned}\operatorname{Tr}\big((\Delta_L - \lambda)^{-1} - (\Delta_F - \lambda)^{-1}\big) &= \sum_{\mu,\nu} x_{\mu\nu}\big\langle G_\mu(\cdot;\lambda), G_\nu(\cdot;\overline{\lambda})\big\rangle \\ &= \sum_{\mu\nu} x_{\mu\nu}\big[\dot{S}(\lambda)\big]_{\nu\mu} \\ &= -\operatorname{Tr}\big(\big(P + QS(\lambda)\big)^{-1} Q\dot{S}(\lambda)\big). \qquad \square\end{aligned}$$

It remains to prove the following lemma.

Lemma 5.4 *The trace of the rank one operator* $\Pi_{\mu\nu}(\lambda)\circ(\Delta_F - \lambda)^{-1}$ *is given by*

$$\operatorname{Tr}\big(\Pi_{\mu\nu}(\lambda)\circ(\Delta_F - \lambda)^{-1}\big) = \big\langle G_\mu(\cdot;\lambda), G_\nu(\cdot;\overline{\lambda})\big\rangle.$$

Proof Let e_n be an orthonormal basis of L^2. Then

$$\begin{aligned}\big\langle \Pi_{\mu\nu}(\lambda)\circ(\Delta_F - \lambda)^{-1}e_n, e_n\big\rangle &= \big\langle G_\mu(\cdot;\lambda), e_n\big\rangle \cdot \Lambda_\nu\big((\Delta_F - \lambda)^{-1}e_n\big) \\ &= \big\langle G_\mu(\cdot;\lambda), e_n\big\rangle \cdot \big\langle e_n, G_\nu(\cdot;\overline{\lambda})\big\rangle.\end{aligned}$$

Summing over n and using Parseval's identity gives the lemma. □

We may now define D on the union of the resolvent sets of Δ_F and Δ_L by

$$D(\lambda) = \det\big(P + QS(\lambda)\big). \tag{5.3}$$

Remark 5.5 When the extension is regular, the matrix $P + QS(\lambda)$ doesn't involve the coefficients $S_{\mu\nu}$ whenever μ or ν is 0 (because these are multiplied by a zero entry of Q). Hence the matrix $P + QS(0)$ makes perfect sense and can be computed using the results of Sect. 4.2.

The preceding proposition gives

$$\mathrm{Tr}\big((\Delta_L-\lambda)^{-1}-(\Delta_F-\lambda)^{-1}\big)=-\frac{D'(\lambda)}{D(\lambda)}. \tag{5.4}$$

This implies that $\frac{D'}{D}$ extends to a meromorphic function with poles that correspond to eigenvalues of Δ_L and Δ_F and with residues $\dim(\ker(\Delta_L-\lambda))-\dim(\ker(\Delta_F-\lambda))$.

Since $\frac{D'}{D}$ is the logarithmic derivative of D, it is convenient to give a name to $\ln(D)$. We thus denote by $\Omega\subset\mathbb{C}$ the set obtained by removing a downward vertical cut starting at each eigenvalue of Δ_F and Δ_L, i.e.,

$$\Omega=\mathbb{C}\setminus\big\{\lambda-it,\lambda\in\mathrm{spec}(\Delta_F)\cup\mathrm{spec}(\Delta_L),t\in(-\infty,0]\big\},$$

and, on Ω, we define the function $\tilde{\xi}$ by $\tilde{\xi}(\lambda):=-\frac{1}{2i\pi}\ln(\det(P+QS(\lambda)))$.

Observe that on Ω we have, by definition,

$$D(\lambda)=\exp\big(-2i\pi\tilde{\xi}(\lambda)\big). \tag{5.5}$$

The function $\tilde{\xi}$ is intimately related to the spectral shift function ξ (see [13, 28]). Although the latter is usually used in settings with continuous spectrum, it is possible to define it even when H_0 and H_1 have pure point spectrum. In the latter case, it follows from the definitions that ξ is the step-function: $\xi(t):=N_1(t)-N_0(t)$, where N_i is the counting function associated with H_i.

It follows from our definition of $\tilde{\xi}$ that the function ξ defined on $\mathbb{R}$ by

$$\xi(t):=-\frac{1}{2\pi}\mathrm{Arg}\big(D(t)\big)=\mathrm{Re}\,\tilde{\xi}(t)$$

is a step function with jumps located at the eigenvalues of Δ_F and Δ_L. Moreover, the jumps are exactly the differences $\dim(\ker(\Delta_L-\lambda))-\dim(\ker(\Delta_F-\lambda))$. We thus obtain the spectral shift function of Δ_F and Δ_L (compare with [28] Theorem 1 p. 272).

In our setting, the Birman–Krein formula would be (5.5) (compare with [28] p. 272) and would follow, in our case, from our definitions. In the next subsection we will prove that, using D, one may define a determinant of Δ_L via zeta-regularization and then establish the relation

$$\det_\zeta(\Delta_L-\lambda)=C_0\cdot D(\lambda)\det_\zeta(\Delta_F-\lambda), \tag{5.6}$$

in which C_0 is some constant that we will also determine.

In particular, we will now prove that D allows us to recover the "exotic" features of the zeta function associated with Δ_L. This unusual behavior has been extensively studied in [17] in a setting very close to ours, and in [12] in greater generality. Our main contribution here is the interpretation of D using S-matrix that, in some sense, gives a geometrical interpretation to the "secular equation" method of [17].

5.2 Comparing Determinants

The procedure here is not as straightforward as usual because of unusual behavior of the zeta function near $s = 0$. In particular, $\zeta(s, \Delta_L)$ will admit an analytic continuation that is regular at 0 only if L is regular (though with possible unusual poles). This unusual behavior has been extensively studied in the literature (see [12, 17, 22]); from our point of view, it is linked with the asymptotic behavior of $D(\lambda)$ for large negative λ. We thus begin by deriving this asymptotic expansion.

5.2.1 $D(\lambda)$ for Large Negative λ.

The analysis that follows is closely related to the one performed in [17]. This is not surprising, since the asymptotic regime as λ goes to $-\infty$ is local. In particular, the function $D(-|\lambda|) := \det(P + QS(-|\lambda|))$ on a cone has to be compared to the function $F(i\sqrt{|\lambda|})$ in [17].

We first use Proposition 4.2 and consider all possible sums of the exponents ν_i that appear in this proposition. This gives us a collection of numbers that we order and denote by $\alpha_0 > \alpha_1 > \cdots > \alpha_k > \cdots$. Expanding now the determinant, and ordering the terms, we get

$$D(-|\lambda|) = \sum_{\text{finite}} a_{kl} |\lambda|^{\alpha_k} (\ln|\lambda|)^l + O\big(|\lambda|^{-\infty}\big).$$

By definition, there are no logarithms in the expansion corresponding to a regular self-adjoint extension; therefore, in that case, the expansion reads:

$$D(-|\lambda|) = \sum_{\text{finite}} a_k |\lambda|^{\alpha_k} + O\big(|\lambda|^{-\infty}\big).$$

We set l_k to be the largest integer l such that $|\lambda|^{\alpha_k}(\ln|\lambda|)^l$ appears in the expansion, and we set $\beta_k = \alpha_0 - \alpha_k$. We have

$$D(-|\lambda|) = a_{0l_0} |\lambda|^{\alpha_0} (\ln|\lambda|)^{l_0} \Bigg[1 + \sum_{l \geq 1} a_{0l} (\ln|\lambda|)^{-l} + \sum_{\beta_k > 0} \sum_{-l_k}^{l_0} a_{kl} |\lambda|^{-\beta_k} (\ln|\lambda|)^l + O\big(|\lambda|^{-\infty}\big) \Bigg].$$

We denote by $F(\lambda) = [1 + \sum_{l\geq 1} a_{0l}(\ln|\lambda|)^{-l} + \sum_{\beta_k>0} \sum_{-l_k}^{l_k} a_{kl} |\lambda|^{-\beta_k}(\ln|\lambda|)^l + O(|\lambda|^{-\infty})]$.

Taking the logarithmic derivative, we obtain

$$-\frac{D'(-|\lambda|)}{D(-|\lambda|)} = 2i\pi \tilde{\xi}'\big(-|\lambda|\big) = \frac{\alpha_0}{|\lambda|} + \frac{l_0}{|\lambda| \ln|\lambda|} + \frac{F'(\lambda)}{F(\lambda)}.$$

By inspection we find

$$\frac{F'(\lambda)}{F(\lambda)} = \begin{cases} O(|\lambda|^{-\beta_1 - 1}) & \text{regular case,} \\ O(|\lambda|^{-1}(\ln|\lambda|)^{-2}) & \text{otherwise.} \end{cases}$$

Lemma 5.6

(1) *In the regular case, there exist three positive numbers* α_0, β_1, *and* M *such that the estimate*

$$\left|2i\pi\tilde{\xi}'(-|\lambda|) - \frac{\alpha_0}{|\lambda|}\right| \le M|\lambda|^{-\beta_1-1} \tag{5.7}$$

holds for λ *large enough.*

(2) *In the other cases, there exist two positive real numbers* α_0 *and* β_1, *a non-negative integer* l_0, *and a constant* M *such that the estimate*

$$\left|2i\pi\tilde{\xi}'(-|\lambda|) - \frac{\alpha_0}{|\lambda|} - \frac{l_0}{|\lambda|\ln|\lambda|}\right| \le M\cdot|\lambda|^{-1}\big(\ln|\lambda|\big)^{-2} \tag{5.8}$$

holds for $|\lambda|$ *large enough.*

In the regular case, for any $C > 0$, define $h_C(s)$ for $\mathrm{Re}(s)$ large enough by

$$h_C(s) = 2i\pi\int_C^\infty \lambda^{-s}\tilde{\xi}'\big(-|\lambda|\big)\,d\lambda - \frac{\alpha_0}{s}\exp\big(-s\ln(C)\big). \tag{5.9}$$

The estimates of the previous lemma imply the following corollary. We restrict to the regular case, although similar statements are valid in the non-regular case (with extra logarithmic singularities; see [17]).

Proposition 5.7 *For a regular extension, the function* h_C *extends to a holomorphic function on* $\{\mathrm{Re}(s) \ge -\beta_1\}$.

Proof We have

$$\int_C^\infty \lambda^{-s}2i\pi\tilde{\xi}'(-|\lambda|)d\lambda = \int_C^\infty \lambda^{-s}\left[2i\pi\tilde{\xi}'(-|\lambda|) - \frac{\alpha_0}{\lambda}\right]d\lambda + \int_C^\infty \lambda^{-s}\frac{\alpha_0}{\lambda}\,d\lambda.$$

The second integral on the right-hand side is computed directly:

$$\int_C^\infty \lambda^{-s}\frac{\alpha_0}{\lambda}\,d\lambda = \frac{\alpha_0}{s}\exp(-s\ln C),$$

so that h_C actually represents the first integral. Lemma 5.6 then gives that h_C extends to a holomorphic function on $\mathrm{Re}\, s > -\beta_1$. □

5.3 Zeta-Regularization

For any A and any C that is large enough, for any $\tilde{\lambda} \in \Omega$ such that $\mathrm{Re}(\tilde{\lambda}) > A$ we choose a cut $c_{\tilde{\lambda}} \subset \Omega$ that starts from $-\infty + i0$ and ends at $\tilde{\lambda}$. We may choose it in such a way that it begins with the interval $(-\infty, -C]$.

For any $\tilde{\lambda}$ and any $s \in \mathbb{C}$, the function $\lambda \mapsto (\lambda - \tilde{\lambda})^{-s}$, which is well defined when $\lambda - \tilde{\lambda}$ is a positive real number, extends to a holomorphic function on the complement of the cut $c_{\tilde{\lambda}}$. Moreover, when λ goes to the cut $c_{\tilde{\lambda}}$ from above or from below, we have the following jump condition:

$$\lim_{\lambda \downarrow c_{\tilde{\lambda}}} \exp(-i\pi s)(\lambda - \tilde{\lambda})^{-s} = \lim_{\lambda \uparrow c_{\tilde{\lambda}}} \exp(i\pi s)(\lambda - \tilde{\lambda})^{-s}.$$

For λ on $c_{\tilde{\lambda}}$, we define $(\lambda - \tilde{\lambda})_0^{-s}$ to be this common limit.

Let A^+ be any number greater than A that is neither an eigenvalue of Δ_F nor of Δ_L. Define a contour γ that avoids $c_{\tilde{\lambda}}$ and that consists of one part that encloses the half-line $\{x \geq A^+\}$ and then small circles that enclose the eigenvalues of Δ_L and Δ_F that are smaller than A^+.

For $\mathrm{Re}(s) > 1$ we have

$$\begin{aligned}\zeta(s, \Delta_L - \tilde{\lambda}) &= \frac{1}{2i\pi}\mathrm{Tr}\left(\int_\gamma (\lambda - \tilde{\lambda})^{-s}(\Delta_L - \lambda)^{-1} d\lambda\right), \\ &= \frac{1}{2i\pi}\mathrm{Tr}\left(\int_{c_{\tilde{\lambda},\varepsilon}} (\lambda - \tilde{\lambda})^{-s}(\Delta_L - \lambda)^{-1} d\lambda\right),\end{aligned}$$

in which $c_{\tilde{\lambda},\varepsilon}$ denotes the contour obtained by following $c_{\tilde{\lambda}}$ at a distance ε. The second identity comes from the Cauchy integral formula since, when $\mathrm{Re}(s) > 1$, the contribution of a large circle centered at $\tilde{\lambda}$ tends to zero when the radius grows to infinity.

The same formulas are true for Δ_F, and using the fact that $(\Delta_L - \lambda)^{-s}$ and $(\Delta_F - \lambda)^{-s}$ are trace class for $\mathrm{Re}\, s > 1$, we can exchange the contour integration and the trace operation to obtain

$$\zeta(s, \Delta_L - \tilde{\lambda}) - \zeta(s, \Delta_F - \tilde{\lambda}) = \frac{1}{2i\pi}\int_{c_{\tilde{\lambda},\varepsilon}} (\lambda - \tilde{\lambda})^{-s}\mathrm{Tr}\big((\Delta_L - \lambda)^{-1} - (\Delta_F - \lambda)^{-1}\big) d\lambda.$$

Using Proposition 5.3 and the definition of $\tilde{\xi}$, we obtain

$$\zeta(s, \Delta_L - \tilde{\lambda}) - \zeta(s, \Delta_F - \tilde{\lambda}) = \int_{c_{\lambda,\varepsilon}} (\lambda - \tilde{\lambda})^{-s}\tilde{\xi}'(\lambda) d\lambda.$$

We rewrite the right-hand side in the following form: $\zeta_1(s) + \zeta_2(s)$ where ζ_1 corresponds to the part of the contour $c_{\lambda,\varepsilon}$ that is in the half-plane $\{\mathrm{Re}\,\lambda \leq -C\}$, and ζ_2 is the remaining part of that contour.

The function ζ_2 extends to an entire function of s, and for $\mathrm{Re}(s) < 1$ we may let ε go to 0, giving

$$\forall s,\ \mathrm{Re}(s) < 1,\ \zeta_2(s) = 2i\sin(\pi s)\int_{-C}^{\tilde{\lambda}} (\lambda - \tilde{\lambda})_0^{-s}\tilde{\xi}'(\lambda) d\lambda,$$

where the integral is along the part of the cut $c_{\tilde{\lambda}}$ that belongs to the half-plane $\{\mathrm{Re}(\lambda) > -C\}$.

For ζ_1 we may first let ε go to 0 and obtain:

$$\zeta_1(s) = 2i\sin(\pi s)\int_{-\infty}^{-C} (\lambda - \tilde{\lambda})_0^{-s}\tilde{\xi}'(\lambda)d\lambda.$$

We make a further reduction by using the following technical lemma.

Lemma 5.8 *On $\mathbb{C}\times\{|z|<1\}$, we define $\rho(s,z) = (1-z)^{-s} - 1$. For any $r<1$, and any $R>0$, the following holds for any $|z|\le r$, and any $|s|\le R$:*

$$\left|\rho(s,z)\right| \le \frac{\exp(\frac{Rr}{1-r})}{1-r}\cdot |s|\cdot|z|. \tag{5.10}$$

Proof We start from

$$\rho(s,z) = \sum_{k\ge 1}\frac{(-s)^k[\ln(1-z)]^k}{k!}.$$

By integration we have $|\ln(1-z)|\le \frac{1}{1-r}|z|$ so that

$$|\rho(s,z)| \le \exp\left(\frac{|s||z|}{1-r}\right) - 1 = \int_0^{\frac{|s||z|}{1-r}}\exp(v)\,dv.$$

The claim then follows. □

For $\mathrm{Re}(\lambda)\le -C$, there exists some $r<1$ such that $|\frac{\tilde{\lambda}}{\lambda}|\le r$. We can thus write

$$(\lambda-\tilde{\lambda})^{-s} = \lambda^{-s}\left(1+\rho\left(s,\frac{\tilde{\lambda}}{\lambda}\right)\right)$$

for any λ such that $\mathrm{Re}(\lambda)\le -C$ and $\lambda\notin(-\infty,-C)$.

Fix some R. For s such that $\mathrm{Re}(s)>0$ and $|s|\le R$, using the bound in Lemma 5.8 we may let ε go to zero and write

$$\zeta_1(s) = 2i\sin(\pi s)\int_{-\infty}^{-C}|\lambda|^{-s}\tilde{\xi}'(\lambda)\,d\lambda + 2i\sin(\pi s)\tilde{R}_C(s,\tilde{\lambda})$$

where

$$\tilde{R}_C(s,\tilde{\lambda}) = \int_{-\infty}^{-C}|\lambda|^{-s}\tilde{\xi}'(\lambda)\rho\left(s,\frac{\tilde{\lambda}}{\lambda}\right)d\lambda.$$

Using Lemma 5.8 and Lemma 5.6 we find that, for any extension (regular or not) $\tilde{R}_C(\cdot,\lambda)$ can be analytically continued to $\mathrm{Re}(s)>-1$, and that $\tilde{R}_C(0)=0$.

Adding up ζ_1 and ζ_2 we obtain the following proposition.

Proposition 5.9 *For any extension, the function $R_C(s,\tilde{\lambda})$ which is defined for s large by*

$$R_C(s,\tilde{\lambda}) = \zeta(s,\Delta_L-\tilde{\lambda}) - \zeta(s,\Delta_F-\tilde{\lambda}) - 2i\sin(\pi s)\int_{-\infty}^{-C}|\lambda|^{-s}\tilde{\xi}'(\lambda)\,d\lambda - \zeta_2(s)$$

can be analytically continued to $\mathrm{Re}(s) > -1$, *and* $R_C(s,\tilde{\lambda})$ *vanishes at least at second order at* $s=0$.

Proof By inspection and using the definitions of the different functions that appear in the expression of R_C, we find that

$$R_C(s,\tilde{\lambda}) = 2i\sin(\pi s)\tilde{R}_C(s,\tilde{\lambda}).$$

Using the bounds given by Lemmas 5.6 and 5.8, we find a constant $\tilde{C}$ such that

$$\forall\lambda < -C, \left| |\lambda|^{-s}\tilde{\xi}'(\lambda)\rho\left(s,\frac{\tilde{\lambda}}{\lambda}\right)\right| \leq \tilde{C}|s|\cdot|\lambda|^{-\mathrm{Re}(s)-2},$$

where $\tilde{C}$ depends on $C,\tilde{\lambda}$ and is uniform for $|s|\leq R$. The claim follows. □

In particular, in the regular case, we obtain the following corollary (compare with [23]).

Corollary 5.10 *If* L *defines a regular extension, then* $(s-1)\zeta(s,\Delta_L-\tilde{\lambda})$ *extends to a holomorphic function on* $\mathrm{Re}(s) > -\beta_1$.

Proof The zeta-regularization of the Friedrichs extension is well known and well studied, starting from the small-time asymptotics of the heat kernel (obtained, for instance, from [7]). The function $(s-1)\zeta(\Delta_F-\tilde{\lambda})$ is thus known to extend holomorphically to $\mathbb{C}$ (see [2, 3, 17, 19]). Moreover, the preceding proposition yields that

$$(s-1)\zeta(s,\Delta_L-\tilde{\lambda}) = (s-1)\cdot\left[\zeta(s,\Delta_F-\tilde{\lambda}) + \frac{\sin(\pi s)}{\pi}\left(h_C(s) + \frac{\alpha_0}{s}\exp(-s\ln C)\right) + \zeta_2(s) + R_C(s,\tilde{\lambda})\right].$$

The statement thus follows by examining the analytic continuation of each individual term. □

Remark 5.11 By evaluating everything at $s=0$ we obtain

$$\zeta(0,\Delta_L-\tilde{\lambda}) = \zeta(0,\Delta_F-\tilde{\lambda}) + \alpha_0.$$

In the regular case, we can thus define the regularized zeta determinant by the usual formula

$$\det_\zeta(\Delta_L-\tilde{\lambda}) = \exp\bigl(-\zeta'(0,\Delta_L-\tilde{\lambda})\bigr),$$

and we obtain the following theorem.

Theorem 2 *Let* L *define a regular extension and set* Γ *to be*

$$\Gamma = \lim_{\lambda\to\infty}\ln\bigl(D(-|\lambda|)\bigr) - \alpha_0\ln\bigl(-|\lambda|\bigr). \tag{5.11}$$

Then, for any $\tilde{\lambda} \in \Omega$ *we have*

$$\det_\zeta(\Delta_L - \tilde{\lambda}) = e^{-\Gamma} \cdot D(\tilde{\lambda}) \cdot \det_\zeta(\Delta_F - \tilde{\lambda}). \tag{5.12}$$

Proof According to the preceding proposition, we have

$$\zeta'(0, \Delta_L - \tilde{\lambda}) - \zeta'(0, \Delta_F - \tilde{\lambda}) = \zeta_2'(0) + h_C(0) - \alpha_0 \ln(C).$$

We compute

$$\zeta_2'(0) = 2i\pi\big[\tilde{\xi}(\tilde{\lambda}) - \tilde{\xi}(-C)\big].$$

Combining the two, we find

$$\begin{aligned}\zeta'(0, \Delta_L - \tilde{\lambda}) - \zeta'(0, \Delta_F - \tilde{\lambda}) &= 2i\pi\tilde{\xi}(\tilde{\lambda}) - 2i\pi\tilde{\xi}(-C) + h_C(0) - \alpha_0 \ln(C) \\ &= 2i\pi\tilde{\xi}(\tilde{\lambda}) + \ln\big(D(-C)\big) - \alpha_0 \ln(C) + h_C(0).\end{aligned}$$

This implies the result with Γ replaced by the quantity $\ln(D(-C)) - \alpha_0 \ln(C) + h_C(0)$ (which proves in particular that the latter doesn't depend on C large enough). When we let C go to infinity, on the one hand $\ln(D(-C)) - \alpha_0 \ln(C)$ goes to Γ, and on the other hand, since

$$h_C(0) = \int_C^\infty \left(2i\pi\tilde{\xi}'(-|\lambda|) - \frac{\alpha_0}{\lambda}\right) d\lambda$$

and the function inside the integral is L^1, $h_C(0)$ goes to 0. This finishes proving the theorem. □

Remark 5.12 As soon as h_C allows the definition of the relative zeta determinant of $\Delta_L - \tilde{\lambda}$ and $\Delta_F - \tilde{\lambda}$, then, using Theorem 2 and differentiating with respect to λ, we recover a well-known fact of this theory:

$$\partial_{\tilde{\lambda}}\big(\ln\det(\Delta_L - \tilde{\lambda}) - \ln\det(\Delta_F - \tilde{\lambda})\big) = 2i\pi\tilde{\xi}'(\tilde{\lambda})$$

(compare with [6, 10, 15]).

Remark 5.13 For non-regular extensions, it is still possible to analytically continue ζ to $\operatorname{Re} s > 0$ and to define a zeta-regularized determinant by picking some coefficient in the asymptotic expansion of $\zeta(s, \Delta_L - \tilde{\lambda})$ near 0 (see [17]). Note, however, that the limit $\tilde{\lambda} \to 0$ will be problematic.

5.4 Proof of Theorem 1

In order to get the theorem of the Introduction, we now let $\tilde{\lambda}$ go to 0. We thus modify the zeta-regularized determinant in order to exclude the eigenvalue 0. Define by δ_L (resp., δ_F) the dimensions of $\ker(\Delta_L)$ (resp., $\ker(\Delta_F)$). Equation (5.4) implies that 0 is a pole of $\frac{D'}{D}$ with residue $d := \delta_L - \delta_F$ so that we can define $D^*(0)$:

$$D^*(0) := \lim_{\lambda \to 0} D(\lambda)(-\lambda)^{-(\delta_L - \delta_F)}.$$

On the other hand, we define the modified zeta function by

$$\zeta^*(s, \Delta_F - \tilde{\lambda}) = \zeta(s, \Delta_F - \tilde{\lambda}) - \delta_F(-\tilde{\lambda})^{-s}$$

and the corresponding modified determinant. □

Definition 5.14 Let L define a regular extension (or $L = F$); the modified zeta determinant of Δ_L is defined by

$$\overset{*}{\underset{\zeta}{\det}}(\Delta_L) = \lim_{\tilde{\lambda}\to 0} (-\tilde{\lambda})^{-\delta_L} \underset{\zeta}{\det}(\Delta_L - \tilde{\lambda}).$$

Using this definition for Δ_L and Δ_F, and plugging into (5.12), the powers of $-\tilde{\lambda}$ cancel out and we may let $\tilde{\lambda}$ go to zero. We thus obtain the theorem in the Introduction (Theorem 1).

When $d = 0$, the prefactor $D^*(0)$ may be computed using the method of Sect. 4.3. When $d > 0$, then this method has to be refined to compute more terms in the Taylor expansion of $S(\lambda)$ at $\lambda = 0$. In the following example, we will pay special attention to addressing the question of the kernel of $P + QS(0)$.

5.5 On the Euclidean Sphere with One 4π and Six π Singularities

We consider the Euclidean sphere with six π singularities and one 4π conical point. We define

$$A^\pm = \begin{pmatrix} a_0^\pm \\ a_{-\frac{1}{2}}^\pm \\ a_{+\frac{1}{2}}^\pm \\ \tilde{A}^\pm \end{pmatrix},$$

where $a_i^\pm$, $i = -\frac{1}{2}, 0, \frac{1}{2}$ correspond to the 4π singularity and $\tilde{A}^\pm$ are the coefficients corresponding to the remaining six π singularities. Recall that for each of the latter there are only two coefficients $a_0^\pm$.

A regular extension thus relates only the coefficients $a_{\pm\frac{1}{2}}^\pm$.

We define P_θ and Q_θ by

$$P_\theta := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta I_2 & 0 \\ 0 & 0 & I \end{pmatrix}, \qquad Q_\theta := \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sin\theta I_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This choice defines a regular self-adjoint extension (which is, moreover, invariant under complex conjugation). We have

$$D(\lambda) = \det\bigl(P + QS(\lambda)\bigr) = \det\bigl(\cos\theta I_2 + \sin\theta \tilde{S}(\lambda)\bigr),$$

where $\tilde{S}$ is the 2×2 matrix obtained from S by erasing the first row and column (which correspond to $a_0^\pm$) and all the rows and columns corresponding to $\tilde{A}^\pm$.

According to Proposition 4.2, when $\theta \neq 0, \pi$, the asymptotic expansion of D is given by

$$\ln D\big(-|\lambda|\big) = 2|\nu| \ln\big(|\lambda|\big) + \ln\left(\left[\frac{\Gamma(1-|\nu|)}{2^{2|\nu|}\Gamma(1+|\nu|)} \sin\theta\right]^2\right) + O(1),$$
$$= \ln\big(|\lambda|\big) + \ln[\sin\theta]^2 + O(1),$$

since $|\nu| = \frac{1}{2}$.

Finally, we obtain that, for any $\theta \neq 0, \pi$ such that $-\text{cotan}(\theta)$ isn't an eigenvalue of $\tilde{S}(0)$, the following holds:

$$\overset{*}{\underset{\zeta}{\det}}(\Delta_L) = \frac{\det(\cos\theta I_2 + \sin\theta \tilde{S}(0))}{\sin^2\theta} \cdot \det{}_\zeta^*(\Delta_F).$$

Acknowledgements The research of LH was partly supported by the ANR programs *NONaa* and *Teichmüller*.

The research of AK was supported by NSERC. AK thanks Hausdorff Research Institute for Mathematics (Bonn) and Laboratoire de Mathématiques Jean Leray (Nantes) for hospitality. AK also thanks the MATPYL program for supporting his travel and stay in Nantes, where this research began.

We acknowledge useful conversations with G. Carron and with D. Korotkin, whose advice in particular helped us simplify some constructions from Sect. 4.2.

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