Diffractive Geodesics of a Polygonal billiard

Luc Hillairet *

Abstract : we define the notion of diffractive geodesic for a polygonal billiard or more generally for an Euclidean surface with conical singularities. We study the local geometry of the set of such geodesics of given length and we relate it to a number that we call classical complexity. This classical complexity is then computed for any diffractive geodesic. As an application we describe the set of periodic diffractive geodesics as well as the symplectic aspects of the "diffracted flow".

AMS 2000 *Mathematics subject classification:* Primary 37D50

* Luc Hillairet, UMPA ENS-Lyon
46 allée d'Italie 69364 Lyon Cedex 7
Tel 04 72 72 84 18, e-mail: lhillair@umpa.ens-lyon.fr

1. Introduction

The propagation of singularities on a smooth Riemannian manifold states that the singularities of one solution of the wave equation propagate along the geodesics. This theorem has been proved in more general settings such as manifolds with smooth boundary, with conical singularities, polygonal domains (cf. [1, 2, 3]). In all these cases the propagation of singularities is true provided that one takes the suitable generalization of geodesics : i.e. broken (or reflective) geodesics in the boundary case, diffractive geodesics in the case of conical singularities. In order to understand more precisely the propagator of the wave equation, it is then very helpful to know how these "generalized" geodesics behave. One important issue is the description of the so-called geometric wave-front which consists in the endpoints of all the possible geodesics emanating from a given startpoint. The aim of this paper is to answer this question for the generalized (here diffractive) geodesics of an Euclidean surface with conical singularities (a setting which includes polygonal billiards). As it was previously said, this study is principally motivated by the description of the wave propagator on such surfaces. However, since the notion of diffractive geodesic is closely related to that of generalized diagonal (introduced by Katok in [4]), we also believe that this paper can provide some more understanding of the dynamical properties of polygonal billiards. To this purpose, we also remark here that the results we obtain are independent of whether the polygon is rational or not.

In the rest of the paper M will always be either an Euclidean surface with conical singularities (cf [5]) or a polygonal domain in $\mathbb{R}^2$. We will begin by defining the set of (possibly) diffractive geodesics of M and we will call $\Gamma_T(M)$ the set of all the geodesics of length T. Seeing an element of $\Gamma_T(M)$ as a mapping from $[0,T]$ in M, this set comes naturally equipped with a topology. Given $[p]$ any ordered sequence of conical points, we will also define $\Gamma_T^{[p]}(M)$ as the geodesics of length T that go through the conical points prescribed by $[p]$. The local geometry of $\Gamma_T(M)$ near a geodesic g is described by the number of strata $\Gamma_T^{[p]}$ to which g is adherent. This number, that we call *classical complexity* (see def 3) is the central object of this paper. The main result that we obtain is the following theorem (cf th. 3.8 page 13) that computes the classical complexity of a geodesic g from the sequence of its diffraction angles.

Theorem 1.1.
Let g be a geodesic of length T with n diffractive points and such that its sequence of diffraction angles is written

$$(\underbrace{\varepsilon_0\pi, \cdots, \varepsilon_0\pi}_{k_0}, \beta_{g,l_0}, \cdots, \beta_{g,l_1}, \underbrace{\varepsilon_1\pi, \cdots, \varepsilon_1\pi}_{k_1}),$$

$$\beta_{g,l_i} \neq \varepsilon_i\pi,$$

then we have :

1. *if $k_0 + k_1 < n$ then $c_c(g) = (k_0+1)(k_1+1)$,*
2. *if $k_0 + k_1 = n$ and $k_0k_1 \neq 0$ then $c_c(g) = (k_0+1)(k_1+1)$,*

3. *if* $k_0 + k_1 = n$ *and* $k_0 k_1 = 0$ *then* $c_c(g) = \frac{n(n+1)}{2} + 1$.

We also give two important applications when aiming at a trace formula for such surfaces : the description of the set of periodic orbits and the symplectic interpretation of the geometrical wave-front.

The paper is organized as follows. In the first section we will introduce the geometrical setting and the diffractive geodesics ; we will also establish the topological nature of $\Gamma_T(M)$. In the second section we will define the notion of classical complexity. We will then give some useful examples and finally, we will compute the classical complexity of any given diffractive geodesic. The last section will be devoted to applications.

2. Diffractive geodesics

(a) Geometrical Setting

The notion of Euclidean surface with conical singularities (E.s.c.s.) is defined in [5]. We recall that M is an E.s.c.s. if M can be partitioned in two sets $M = M_0 \cup P$, where M_0 is a (non-complete) Riemannian surface that is locally isometric to the Euclidean plane. The set P is discrete and, in the neighbourhood of each $p_i \in P$, M is locally modelled on the Euclidean cone of total angle α_i. In all the paper, the E.s.c.s. will always be assumed to be complete.

-Examples-

1. The Euclidean cone of angle α (denoted by $\mathcal{C}_\alpha$) is an E.s.c.s.. We denote by p the tip of the cone. The smooth part $\check{\mathcal{C}_\alpha}$ $(= \mathcal{C}_\alpha \backslash \{p\})$ is globally parametrized by $(r, x) \in (0, \infty) \times \mathbb{R}/\alpha\mathbb{Z}$ with the metric $dr^2 + r^2 dx^2$.
2. A simple way of constructing an E.s.c.s is by gluing Euclidean polygons along sides of same length. For instance, taking two copies of a polygon Q and gluing them along the sides gives an E.s.c.s. where each conical point corresponds to a vertex of the polygon. The angle α_i is then twice the corresponding angle of the polygon

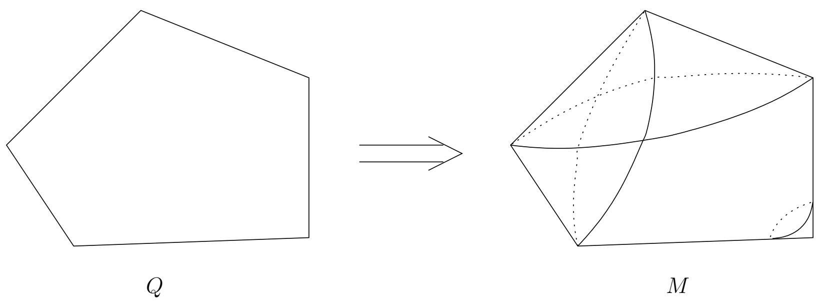

Figure 1. Doubling a polygon

This doubling method is standard in the study of billiards and was first used by Birkhoff.

3. A translation surface is a surface with conical singularities, and equipped with an atlas (outside the singularities) such that the transition functions are given by translations (cf [6] for a more precise definition). Such a translation surface is automatically an E.s.c.s., in particular the one that is associated with a rational polygon.

We now want to define the geodesics of such a surface. We begin by noting P_r the subset of P consisting of the conical points p_i such that $\alpha_i = \frac{2\pi}{k}$ (and we let P_d be the complement of P_r in P, the conical points in P_d will be called diffractive). Locally, near any point of $M_0 \cup P_r$, M is (up to a finite covering) isometric to $\mathbb{R}^2$. There is also no ambiguity in defining the geodesics as the projection of the geodesics of $\mathbb{R}^2$ (i.e. straight lines). This defines the *non-diffractive* geodesics of M. Any *non-diffractive* geodesic either can be prolongated to infinity or ends at a diffractive conical point in finite time. Since we need to prolongate such a geodesic, we give the following definition.

Definition 1.
A geodesic of an E.s.c.s. will be a mapping $g: \ [0, \infty) \to M$ such that

1. $g^{-1}(P_d)$ is discrete,
2. If $g(t) \in M_0 \cup P_r$ there exists ε such that the restriction of g to $(t-\varepsilon, t+\varepsilon)$ parametrizes *by arclength* a non-diffractive geodesic.

-Remarks-

1. We have found more convenient to define a geodesic as a mapping and not as a curve.
2. On the Euclidean cone of angle α ($\neq 2\pi/k$) this definition leads to two types of geodesics :
   - the *non diffractive* ones, that are straight lines avoiding the tip of the cone,
   - the diffractive ones, that are formed by the juxtaposition of an *incoming* and of an *outgoing* ray i.e. that are parametrized by :

$$\begin{cases} g^{\gamma}(t) = (T_1 - t, x_i) & t < T_1, \\ g^{\gamma}(T_1) = p \\ g^{\gamma}(t) = (t - T_1, x_o) & t > T_1. \end{cases} \tag{2.1}$$

   For such a geodesic, the angle $\beta = x_o - x_i$ is called the angle of diffraction, it belongs to $\mathbb{R}/\alpha\mathbb{Z}$. This angle of diffraction depends on the orientation of $\mathcal{C}_\alpha$.

3. The reason for this definition is the theorem of propagation of singularities on an Euclidean cone (cf [7]).

This definition implies that near a conical point, g parametrizes a geodesic of the corresponding cone, it allows us to define an angle of diffraction for each value of t such that

$g(t) \in P_d$.

-Remark-
These angles β depend on the orientation of M near the conical points. When M isn't orientable, there is no preferable choice. In this case, we choose an orientation near the beginning of the geodesic, we transport it along and we consider the angles relatively to this *compatible* orientation. Changing the orientation at the beginning will then multiply all the angles of diffraction by -1. This has no consequences for us since the information we need is invariant under this change (see condition (3.2) p. 12).

Notation : along a geodesic g we will denote by $p_{g,i}$ the $i-th$ diffractive conical point, $t_{g,i}$ the time at which the diffraction occurs, and $\beta_{g,i}$ the $i-th$ angle of diffraction. We will also denote by $[p]_g$ the sequence of diffractive points along g.

There is a globally defined distance d on M that is obtained by minimizing the length of curves. Locally, this distance coincide with that of the plane or of the corresponding Euclidean cone. We note here a diffractive geodesic g minimizes locally this distance near $g(t)$ if and only if

$$g(t) \in M_0 \text{ or } g(t) \in P_d, \text{ and } |\beta| \geq \pi.$$

This fact is a direct consequence of the explicit expression of the distance on the cone $\mathcal{C}_\alpha$ that is given by :

$$\begin{array}{ll} [r_1^2 + r_2^2 - 2r_1r_2\cos(|x_1,x_2|)]^{\frac{1}{2}} & \text{if } |x_1,x_2| \leq \pi, \\ r_1 + r_2 & \text{if } |x_1,x_2| \geq \pi, \end{array}$$

where $|x_1 - x_2|$ is the distance in $\mathbb{R}/\alpha\mathbb{Z}$.
Using this expression, for any geodesic g on the cone we have the following inequality

$$\forall t, \quad d(g(t),p) \geq |t - r_0|. \tag{2.2}$$

Since we aim at studying the set of all the geodesics on an E.s.c.s., it is helpful to first adress the topological nature of this set.

(i) *Topology of the set of geodesics*

By definition, a geodesic of length T is an element of $\mathcal{C}^0([0,T],M)$. The norm of uniform convergence gives us a topology on all the following sets.

Definition 2.

- For any subset N of M, we denote by $\Gamma_T(N)$ the set of all the geodesics of length T whose startpoint is in N. The notation Γ_T will be a shortcut for $\Gamma_T(M)$.
- Given any (ordered) finite sequence of diffractive conical points $[p] = [p_{i_1}, \cdots, p_{i_n}]$ we denote by $\Gamma_T^{[p]}$ the set of all the geodesics of length T having exactly n diffractive conical points such that $p_{g,j} = p_{i_j}$. The set corresponding to non-diffractive geodesics will be denoted by Γ_T^0.

-Remarks-

1. These sets may be empty. For instance, if the sequence $[p]$ has at least two elements, the sets $\Gamma_T^{[p]}$ are empty for small T.
2. Consider a geodesic g of $\Gamma_T^{[p]}$, any other geodesic g' in $\Gamma_T^{[p]}$ close to g is uniquely determined by its startpoint and its last diffraction angle. This parametrization shows that for any sequence $[p]$, and any time T, the set $\Gamma_T^{[p]}$ is a 3-dimensional manifold.

The following theorem establishes the topological nature of $\Gamma_T(N)$.

Theorem 2.1.
For any T, $\Gamma_T(N)$ is compact if and only if N is compact.

Proof : the mapping from $\Gamma_T(N)$ to M that associates to a geodesic its startpoint is continuous and onto N ; thus, if $\Gamma_T(N)$ is compact, then N is. Conversely, we first show that $\Gamma_T(N)$ is relatively compact. This is a consequence of Ascoli's theorem (cf [8]) since N is bounded, and any geodesic is 1-lipschitzian. Then, we show that $\Gamma_T(N)$ is closed when N is. Indeed, let g_n be a sequence of geodesics of $\Gamma_T(N)$ converging to a mapping g of $\mathcal{C}^0([0,T],M)$. If $g(t) \in M_0$, everything happens locally in the plane where a limit of straigth lines is again a straight line. It remains to show that $g^{-1}(P)$ is discrete. This is equivalent to proving that g can't stay at a conical point for a strictly positive amount of time, and this is ensured by the inequality (2.2). To finish the proof, the last thing to remark is that, since N is closed, $g(0) \in N$ and thus $g \in \Gamma_T(N)$. □

Corollary 2.2. *If M is compact, $\Gamma_T(M)$ is compact.*

Corollary 2.3. *The set $\Gamma_T(M)$ is always complete,*

Proof : since M is complete, $\mathcal{C}^0([0,T],M)$ is also complete, and in the proof of the theorem we have shown that $\Gamma_T(N)$ was closed whenever N was. Thus $\Gamma_T(M)$ is a closed set in a complete space. □

We now want to understand in a more precise manner the set Γ_T and in particular the stratification of it by the $\Gamma_T^{[p]}{}'s$. This is the goal of the following section. We will also completely forget about $\mathcal{C}^0$: from now on every topological statement is to be understood in $\Gamma_T(M)$ equipped with the topology of uniform convergence.

3. Classical complexity

As we already have pointed out, the geometry of one $\Gamma_T^{[p]}$ is rather simple so that the local geometry of Γ_T only comes frome the way these sets are close one to another. In order to understand this we introduce the following definition.

Definition 3.
The *classical complexity* of a geodesic g is the number of sequences of diffractive points $[p]$ such that

$$g \in Adh(\Gamma_T^{[p]}),$$

we denote it by $c_c(g)$.

-Remarks-

1. The number $c_c(g)$ answers the question : " how many sequences (possibly empty) of diffractive conical points $[p]$ are such that there exists a sequence $(g_n)_{n\in\mathbb{N}} \in (\Gamma_T^{[p]})^{\mathbb{N}}$ converging to g ?"
2. For fixed T and g, the number of possible sequences $[p]$ is bounded and thus we have the following equivalence :

$$c_c(g) = 1 \Leftrightarrow g \in Int(\Gamma_T^{[p]_g}).$$

(we recall that the Int is taken relatively to Γ_T.)

3. A priori, this definition has nothing to do with the complexity of an orbit in a polygonal billiard (cf [9] pp 63).

The case of non-diffractive geodesics is easily handled.

Lemma 3.1. *For all T, Γ_T^0 is open. Equivalently, if g is a non-diffractive geodesic then* $c_c(g) = 1$.

Proof : take $g \in \Gamma_T^0$, since g is continuous, $g([0,T])$ is compact and since P_d is discrete, there exists ε such that $\cup_t B_M(g(t), \varepsilon)$ doesn't intersect P_d. All the geodesics belonging to $B_{\Gamma_T}(g, \varepsilon)$ are then non-diffractive. □

Consider a sequence g_n converging to g. There exists ε and n_0 such that for all $n \geq n_0$ and for all i, the restriction of g_n to the interval $[t_{g,i} - \varepsilon, t_{g,i} + \varepsilon]$ can be identified with a geodesic on the corresponding cone. A consequence of the former lemma is that on the complement of these intervals, for large n, g_n can't have any diffraction. Thus the sequence $[p]$ such that $g \in Adh(\Gamma_T^{[p]})$ can only be obtained by deleting some of the diffractive points in $[p]_g$. Before adressing a general geodesic, it is instructive to study in detail some examples.

Example 1 : angles of $\pm\pi$ are special

In a plane wedge of angle $\alpha \neq \pi/k$. Consider an incoming ray γ hitting the tip of the sector, and consider the two families of parallel rays "above" and "under" γ. In each family, every ray will make the same reflections and eventually leave a neighbourhood of the vertex following the same direction. This gives two geodesics consisting of γ followed by the outgoing ray parallel to one of these directions. We denote these geodesics by $g^{\pm}$. Along these geodesics, the diffraction angle is $\pm\pi$ as it can be clearly seen by unfolding the orbit (see fig. 2). These two diffractive geodesics are by definition limits of non-diffractive ones. Conversely, consider a sequence of non-diffractive geodesic converging to γ on some small interval (a, b). This sequence can be decomposed into two subsequences depending on which side of the wedge the geodesic hits first. Since the sequence converges to γ on (a, b) one of these subsequences converges to g^+ and the other to g^-. If the sequence was known to converge, then only one of the two subsequences could be infinite and the limit is either g^+ or g^-.

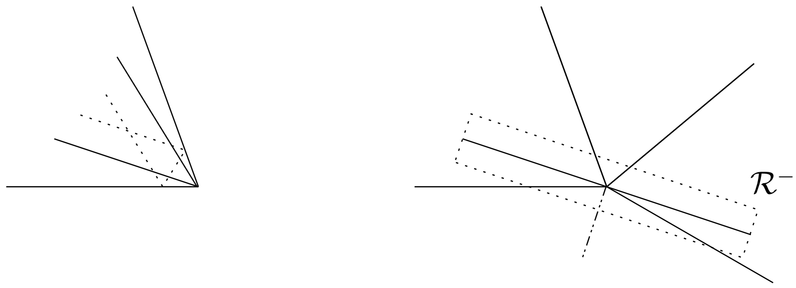

Figure 2. Definition of g^-

This example gives the complete classification on the cone by doubling the wedge.

Proposition 3.2. *Let g be a geodesic on the Euclidean cone of angle α ($\neq 2\pi/k$) then $c_c(g) = 2$ if and only if either g begins (or ends) at p, or g is diffractive in its interior with $\beta = \pm\pi$.*

-Remarks-

1. the notion of diffraction angle is not well-defined for an incoming (or outgoing) ray. However, such a geodesic is always a limit of non diffractive geodesics ; for instance the outgoing ray defined by $g(t) = (t, x_0)$ is the limit of the family $(g_\varepsilon)_{\varepsilon>0}$ defined by $g_\varepsilon(t) = (t + \varepsilon, x_0)$.

2. Since the diffraction angle is $\pm\pi$, there is a continuous choice of a normal vector $\vec{n}(t)$ along g^+ such that the mapping $j(t, s) = g^+(t) + s\vec{n}(t)$ is well-defined on $\mathbb{R}^2\backslash\{(t^+, s)\ s \geq 0\}$. Moreover j a local isometry into $\check{\mathcal{C}}_\alpha$.

3. This example shows that, on a general E.s.c.s., diffraction angles of $\pm\pi$ will play a special role. For instance we have the following lemma :

   Lemma 3.3. *A geodesic g such that all its angles of diffraction are different from $\pm\pi$ has classical complexity* 1.

   Proof : we can find ε such that, on each interval $[t_{g,i} - \varepsilon, t_{g,i} + \varepsilon]$, g_n is a geodesic of the corresponding cone converging to a diffractive geodesic whose angle of diffraction isn't $\pm\pi$. Necessarily, for large n g_n is thus diffractive at this conical point. □

Example 2 *: Rectangles with slits*

We want to generalize the first example by considering geodesics with several diffractions such that each diffraction angle is $\pm\pi$. Let g be such a geodesic. The first remark is that one can put around g a rectangle with slits. This is done by matching the local isometries j constructed for each diffraction (see Remark 2 above).

If the sequence of diffraction angle is $(\varepsilon_i \pi)$ we let

$$\mathcal{R} = [0,T] \times]-\delta, \delta[\ \setminus \ \cup S_i$$

where S_i is the segment (slit) $\{(t_{g,i}, s) \,|\, 0 \leq \varepsilon_i s < \delta\}$. There is a continuous choice of a normal vector $\vec{n}(t)$ along g such that, for δ small enough the mapping

$$\begin{aligned} j \ : \ \mathcal{R}^+ &\longrightarrow M \backslash P_d \\ (t,s) &\longmapsto g(t) + s\vec{n}(t). \end{aligned}$$

is a local isometry.

-Remark-

we have made the choice that a diffraction angle of $+\pi$ (resp. $-\pi$) corresponds to an upward (resp. downward) slit. See also the figure 3 for examples of such rectangles with slits.

These rectangles with slits allow us to compute simply the classical complexity of a geodesic such that all the diffraction angles are $\pm\pi$. For instance, if the sequence of diffraction angles is (π, π, π) we can construct approaching sequences of geodesics with the following diffractions : none, $[p_1]$, $[p_2]$, $[p_3]$, $[p_1, p_2]$, $[p_2, p_3]$, $[p_1, p_2, p_3]$ and thus $c_c(g) = 7$. If the sequence of diffraction angles is $(-\pi, \pi, -\pi)$ the possible diffraction sequence for an approaching sequence of geodesics is $[p_1, p_2]$, $[p_1, p_2, p_3]$, $[p_2]$, $[p_2, p_3]$ and thus $c_c(g) = 4$.

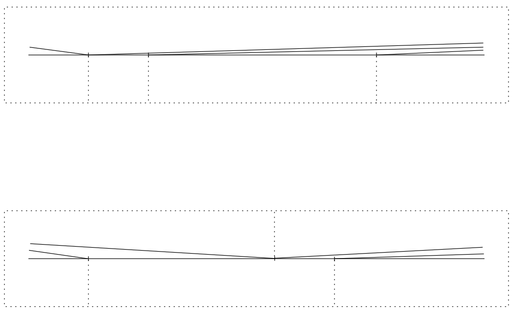

Figure 3. Rectangles with slits

Using these rectangles with slits, we can also find the diffractive geodesics that are limits of non-diffractive ones.

Lemma 3.4. *A geodesic g is a limit of non-diffractive ones if and only if the sequence of its diffraction angles can be written*

$$(\underbrace{\varepsilon_0\pi, \cdots, \varepsilon_0\pi}_{k_0}, \underbrace{-\varepsilon_0\pi, \cdots, -\varepsilon_0\pi}_{k_1}), \tag{3.1}$$

with $\varepsilon_0 = \pm 1$ *and* k_0, k_1 *possibly zero.*

Proof : we already know that all the diffraction angles must be $\pm\pi$. In any other case than those given in the lemma the position of the slits in the rectangle $\mathcal{R}$ forbids the existence of a sequence of non-diffractive geodesics approaching g. Conversely, if the sequence of diffraction angles is as in the lemma such a sequence is easily constructed. □

These examples show that not only the angles of $\pm\pi$ play a special role in computing the classical complexity but also the place they occupy in the sequence of diffraction angles. Among all geodesics, those having only π (or $-\pi$) deserve to be particularized. Their study is done in the following section.

(a) *Geodesics* $g^{\pm}$

We begin by constructing geodesics having only π (or $-\pi$) as diffraction angles. We start from a point m in the direction v, the geodesic is non diffractive for small times. If it reaches a conical point we define $t_1^+ = t_1^-$ to be the first time it does. We continue the geodesic in two ways, making angle $\pm\pi$. This gives two geodesics $g^{\pm}$. Each of these is defined until it reaches another conical point. If g^+ reaches a second diffractive point, we denote by t_2^+ the time it happens and continue g^+ making the angle of diffraction $+\pi$ and so on. We do the same with g^-. This construction gives, for any initial data (m, v) two infinite geodesics $g^{\pm}$ with sequences of diffraction times $(t_i^{\pm})$ and angles of diffraction $\beta_{g^{\pm},i} = \pm\pi$.

The existence of rectangles with slits along the geodesics $g^{\pm}$ gives the following proposition.

Proposition 3.5. *For all startpoint* m *and all time* T, *there is only a finite number of directions* v_i, *such that the geodesic emanating from* m *in the direction* v_i *hits a diffractive point before the time* T.

Proof : take such a direction v_i, it gives rise to two geodesics $g^{\pm}$. For any time T, we construct two rectangles $\mathcal{R}^{\pm}$ along $g^{\pm}(]0, T[)$ (see example 2 p.8) Take another geodesic emanating from m, for small times it can be lifted to a small segment in $\mathcal{R}^+$ and in $\mathcal{R}^-$. If the direction is close enough to v_i, this small segment can be prolongated to lenght T without leaving the rectangles $\mathcal{R}^{\pm}$. In one of these rectangles, it doesn't cross the slits so that the segment projects onto a non-diffractive geodesics. Since the set of directions is compact, there is only a finite number of directions that are diffractive before time T. □

Proposition 3.5 implies the following two technical lemmas that will reduce the computation of the classical complexity to a combinatorial problem. The first lemma will show that if a sequence of $\Gamma_T^{[p]}$ converges to g then $[p]$ is obtained from $[p]_g$ by deleting the first k_0 and the last k_1 conical points of $[p]_g$. The second lemma will then prove that the conical points deleted at the beginning must all have the same diffraction angle, which is moreover $\pm\pi$. A symmetrical statement is true for the conical points that are deleted at the end. (see fig. 4)

Lemma 3.6. *Let (g_n) be a sequence of geodesics of length T converging to g and such that for some $j_0 \leq j_1$, there exist two sequences (t_n^0) and (t_n^1) converging respectively to t_{g,j_0} and t_{g,j_1} such that*

$$\forall\, n,\ g_n(t_n^i) = p_{g,j_i}, \quad i = 0,\ 1.$$

Then the following holds :

$$\exists n_0,\ \forall n > n_0,\ g_n(t) = g(t - t_n^0 + t_{g,j_0}) \ \ on \ \ [t_n^0, t_n^1].$$

In particular, for $n > n_0$, g_n is also diffractive at $p_{g,j}$ for every $j_0 \leq j \leq j_1$.

Proof : on $[t_n^0, T]$, g_n is a geodesic emanating from p_{g,j_0} that is diffractive at some time t_n^1. Since g_n converges to g, proposition 3.5 shows that for large n, g_n and g follow the same outgoing ray at p_{g,j_0}. If $j_1 = j_0 + 1$ the conclusion then holds, and otherwise we can iterate the argument starting from p_{g,j_0+1}.
□

Lemma 3.7. *Let g be a geodesic emanating from m in the direction v that reaches a diffractive point before time T. Let (g_n) be a sequence of geodesics of length T emanating from m, non-diffractive on $]0,T]$ and such that :*

$$\exists\ 0 < a < b \mid g_{n|[a,b]} \rightarrow g_{|[a,b]}$$

then the geodesics $g^{\pm}$ emanating from (m, v) are the only accumulation points of the sequence (g_n).

we put rectangles $\mathcal{R}^{\pm}$ along $g^{\pm}$ respectively. For n large enough, each g_n corresponds to a segment in each rectangle but in only one it doesn't cross the slits. Since we are dealing with segments, convergence on $[a, b]$ implies convergence on $[0, T]$ and we are done. □

-Remarks-

1. A symmetrical statement is true for geodesics ending in m.

2. The assumption on the length of g_n can be relaxed if we know that g_n doesn't coincide with g. Indeed, using once again the rectangles $\mathcal{R}^{\pm}$, it can be shown that any geodesic that doesn't coincide with $g_{|[a,b]}$ but that is sufficiently close to $g_{|[a,b]}$ can be uniquely prolongated to a non diffractive geodesic defined on $[a, T]$.

3. We remind the reader that a limit of non-diffractive geodesics isn't necessarily a geodesic of type $g^{\pm}$ (See lemma 3.4 p. 9). The assumption that all the geodesics g_n emanate from m deals with this point.

These two lemmas lead to the computation of the classical complexity.

(b) Classical complexity : computation

Consider a geodesic g of length T with n diffractive points and assume that g is in $Adh(\Gamma_T^{[p]})$, the first diffractive point in $[p]$ is some p_{g,j_0}, and the last one is some p_{g,j_1} with $j_1 \geq j_0$. Lemma 3.6 implies that, since there exists a sequence of $\Gamma_T^{[p]}$ converging to g then, necessarily, $[p] = [p_{g,j_0}, p_{g,j_0+1} \cdots p_{g,j_1}]$. Then, using lemma 3.7, we show that

$$\exists\, \varepsilon_0, \varepsilon_1 \in \{+,-\} \mid \forall j < j_0,\ \beta_{g,j} = \varepsilon_0 \pi,\ \textit{and}\ \forall j > j_1,\ \beta_{g,j} = \varepsilon_1 \pi. \tag{3.2}$$

Conversely, suppose that j_0 and j_1 are such that $j_0 \leq j_1$ and satisfy condition (3.2) then we claim that there exists a sequence of geodesics in $\Gamma_T^{[p_{g,j_0},\cdots,p_{g,j_1}]}$ converging to g. Indeed, On $[t_{g,j_1}, T]$, the condition (3.2) implies that g is of type ε_1 and is thus a limit of non-diffractive rays emanating from p_{g,j_1}. The same is true on $[0, t_{g,j_0}]$ Since on this interval, g of type ε_0. The concatenation of a ray coming into p_{g,j_0} followed by g until p_{g,j_1} followed by a ray emanating from p_{g,j_1} forms a geodesic that can be as close as wanted to g.

Finally, computing the classical complexity amounts to enumerating the couples (j_0, j_1) satisfying (3.2) and adressing the possibility for g to be a limit of non-diffractive geodesics (which has been done in lemma 3.4).

It is always possible to write the sequence of diffraction angles in the following way :

$$(\underbrace{\varepsilon_0\pi, \cdots, \varepsilon_0\pi}_{k_0}, \beta_{g,l_0}, \cdots, \beta_{g,l_1}, \underbrace{\varepsilon_1\pi, \cdots, \varepsilon_1\pi}_{k_1}), \tag{3.3}$$

$$\beta_{g,l_i} \neq \varepsilon_i \pi,$$

where the subsequence $\beta_{g,l_0}, \cdots, \beta_{g,l_1}$ may be empty. This latter case corresponds to the geodesics that are limits of non-diffractive geodesics and we say that g is of empty type.

We then have the following discussion.

1. The geodesic g is not of empty type.
   Condition (3.2) is then equivalent to $j_0 \leq l_0$ and $j_1 \geq l_1$. Since the geodesic isn't a limit of non-diffractive ones we have

   $$c_c(g) = (k_0 + 1)(k_1 + 1).$$

2. The geodesic is of empty type and $k_0 k_1 \neq 0$
   The geodesic is a limit of non-diffractive ones and condition (3.2) is equivalent to

   $$j_0 \leq k_0 + 1,\quad j_1 \geq k_1 + 1,\quad j_1 \geq j_0.$$

   There are $(k_0 + 1)(k_1 + 1) - 1$ pairs satisfying this condition. Adding the non-diffractive geodesics, we find

   $$c_c(g) = (k_0 + 1)(k_1 + 1).$$

3. The geodesic is of empty type and $k_0k_1 = 0$
   The geodesic is then a limit of non-diffractive geodesics and condition (3.2) is equivalent to $j_0 \geq j_1$. This gives $\frac{n(n+1)}{2}$ pairs and the following complexity :
$$c_c(g) = \frac{n(n+1)}{2} + 1.$$

We resume these computations in the following theorem.

Theorem 3.8.
Let g be a geodesic of length T with n diffractive points and such that its sequence of diffraction angles is written in the form (3.3) then one of the following happens :

1. *g is not of empty type and $c_c(g) = (k_0+1)(k_1+1)$.*
2. *g is of empty type and $k_0k_1 \neq 0$ then $c_c(g) = (k_0+1)(k_1+1)$.*
3. *g is of empty type and $k_0k_1 = 0$ then $c_c(g) = \frac{n(n+1)}{2} + 1$.*

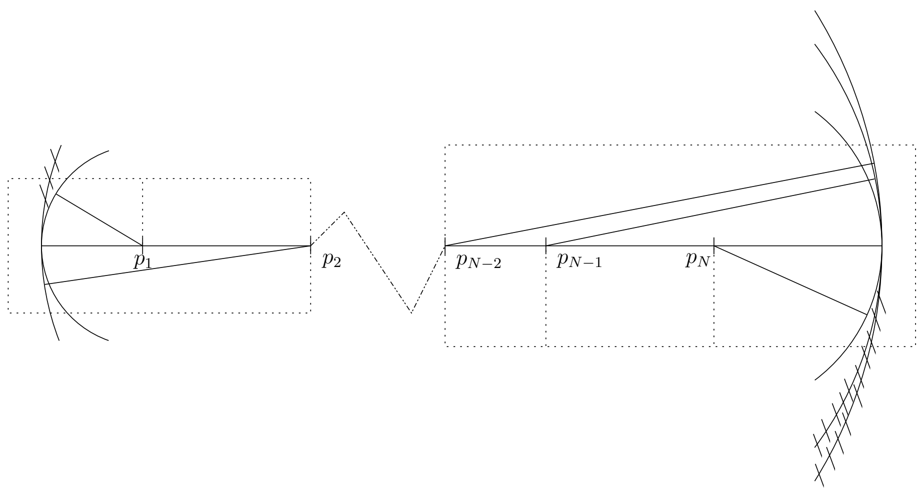

Figure 4. An example

4. Applications

We will give two straightforward applications of the previous discussion. The first one describes the periodic (eventually diffractive) geodesics of an E.s.c.s, and the second one describes geometrically what is to be understood as the canonical relation associated with the (diffractive) geodesic flow on an E.s.c.s.

(a) Periodic Geodesics

One question of interest (in particular when aiming at proving some kind of trace formula cf [10])) is to know whether a given periodic geodesic is isolated, or part of a family. We state the proposition in the case when M is oriented (see the remark after for the non-oriented case)

Proposition 4.1. *Let g be a periodic geodesic of length T of an oriented E.s.c.s. then one of the following occurs.*

1. *The geodesic g is non diffractive, it is then interior to a family of non-diffractive periodic geodesics of same length.*
2. *All the angles of diffraction are π (or $-\pi$), g is then the boundary of a family described in the first case.*
3. *In any other case, g is isolated in the set of periodic geodesics.*

Proof : in the first case everything happens in M_0 where the metric is Euclidean. Since M is oriented, the normal vector to g is well defined and the geodesics $g_\varepsilon(t) = g(t) + \varepsilon\vec{n}(t)$ are periodic and of same length. In the second case we use a rectangle of type $\pm$, and we can define geodesics parallel to g by using the same argument as in the non-diffractive case. The only difference is that, because of the slits, ε must be of chosen sign. In the third case, assume first that g has a diffraction angle $\beta_{g,j}$ different from $\pm\pi$. Consider an approaching sequence (g_n) of periodic geodesics. All these geodesics must go through $p_{g,j}$ at times $t_n + kT$. Lemma 3.6 implies that for large n, g_n and g coincide. If all the angles of g are $\pm\pi$ but not all of the same sign (which is adressed by case 2) then g (or its double) isn't of empty type. As a consequence there is also one conical point through which any approaching sequence of geodesics must go. Repeating the preceding argument gives the conclusion □

-Remarks-

1. If the surface isn't orientable, then in the first two cases, the geodesic can desorientate. The argument then breakdowns and g is isolated. In this case the former proposition remain true for the double of g.
2. The existence of a rectangle $\mathcal{R}^\pm$ along a geodesic g of the second type implies that the only periodic geodesics of period bounded by some T' close to g are the non-diffractive geodesics of the corresponding family. This is no more true if you allow the period to go to infinity. In fact, there are translation surfaces for which the geodesics emanating from a given point are periodic for a dense set of directions, in which case you can find a sequence g_n of periodic geodesics (of period $T_n \to \infty$) such that :
$$\forall T,\ \forall \varepsilon\ \exists n_0 \mid \sup_{[0,T]} d(g_n(t), g(t)) \leq \varepsilon.$$
3. We could define a notion of classical complexity for a periodic geodesic, by asking, for a given periodic geodesic g of period less than T how many different types of

periodic geodesics of period less than T can approach g. The proposition answers this question, but it tells more since it also proves that "most" diffractive periodic geodesics will be isolated.

One interesting question is, given a E.s.c.s., how complex can the classical complexity be ? The following proposition answers this question (at least partially).

Proposition 4.2. *Let M be an E.s.c.s. such that there exists a non-diffractive periodic orbit, then for any given N, there is a geodesic g such that $c_c(g) = N$.*

Proof : the existence of a non-diffractive periodic orbit implies the existence, at the boundary of the corresponding family, of a periodic diffractive geodesic g such that all its diffraction angles are π. We will construct a geodesic having a sequence of diffractions angles written in the form (3.3) with arbitrary k_0 and k_1. We pick a point on g and begin by following g for a sufficiently long time to have k_0 diffraction angles. We then leave g and go to another diffractive point, we follow then some diffractive geodesic that comes back to g and follow again g enough time to have k_1 diffractions. This gives the desired geodesic. □

The next application is concerned with symplectic aspects associated with Γ_T.

(*b*) *Symplectic aspects*

The set Γ_T gives a relation Λ_T from $T^*(M_0)$ to itself which is defined by

$$\begin{aligned}\Lambda_T \;=\; \{ \quad & (m_1, m_0, \mu_1, \mu_0) \in T^*(M_0) \times T^*(M_0) \mid \\ & \exists g \in \Gamma_T, \\ & |\mu_0| = |\mu_1|, \\ & g(0) = m_0 \quad \mu_0 = |\mu_0| \langle g'(0), \ . \rangle_{m_0} \\ & g(T) = m_1 \quad \mu_1 = |\mu_1| \langle g'(T), \ . \rangle_{m_1} \quad \},\end{aligned}$$

where $\langle ., . \rangle_m$ is the Euclidean scalar product in $T_m M_0$ and $|.|$ the associated norm.

-Remark-

The same definition on a smooth Riemannian manifold makes Λ_T the canonical relation associated with the geodesic flow at time T.

Given any subset $\mathcal{V}$ of Γ_T we can define $\Lambda_T^{\mathcal{V}}$ by asking, in the definition of Λ_T that the geodesic g belongs to $\mathcal{V}$. The classical complexity and the constructions made in the previous sections allow us to prove the following proposition.

Proposition 4.3.

Given any geodesic of length T starting and ending in M_0, there exists $c_c(g)$ lagrangian submanifolds $\Lambda_{g,i}$ of $T^*(M_0) \times T^*(M_0)$ such that for any sufficiently small neighbourhood $\mathcal{V}$ of g we have the following inclusion :

$$\Lambda_T^{\mathcal{V}} \subset \bigcup_{1}^{c_c(g)} \Lambda_{g,i}.$$

Furthermore each $\Lambda_{g,i}$ is determined by an explicit phase function and there exists Σ such that for $i \neq j$:

$$\Lambda_{g,i} \cap \Lambda_{g,j} = \Sigma,$$

the intersection being clean.

Proof : we take a geodesic g and $[p]$ such that $g \in Adh(\Gamma_T^{[p]})$ (by definition of $c_c(g)$ there are $c_c(g)$ choices for $[p]$). We will construct a lagrangian submanifold associated with $[p]$. We begin by $[p] \neq \emptyset$ there exists j_0 and j_1 such that $[p] = [p_{g,j_0}, \cdots p_{g,j_1}]$ and there exists ε_0 and ε_1 such that

$$\forall j < j_0 \;\; (resp.\ j > j_1), \;\; \beta_{g,j} = \varepsilon_0 \pi \;\; (resp.\ \varepsilon_1).$$

Each geodesic in $\Gamma_T^{[p]}$ consists of a ray coming in p_{g,j_0}, the portion of g between p_{g,j_0} and p_{g,j_1}, and a ray coming out p_{g,j_1}. There is a rectangle of type (ε_0) around the first diffractive points (if p_{g,j_0} is the first diffractive point, then the rectangle has no slits) and a local isometry j^{ε_0} such that $j^{\varepsilon_0}(t_{g,j_0}, 0) = p_{g,j_0}$. We can define $d_0(., p_{g,j_0})$ in a small neighbourhood of m_0 by

$$d_0(m, p_{g,j_0}) = d_{\mathbb{R}^2}((j^{\varepsilon_0})^{-1}(m), (t_{g,j_0}, 0)).$$

The same construction around the end of g gives $d_1(p_{g,j_1}, .)$ defined in a neighbourhood of m_1. The phase function

$$[d_0(m, p_{g,j_0}) + t_{g,j_1} - t_{g,j_0} + d_1(p_{g,j_1}, m') - T]\,\theta$$

defines a lagrangian submanifold that includes the part of Λ_T corresponding to the geodesics of $\Gamma_T^{[p]}$ close to g. For the non-diffractive geodesics close to g one should take as a phase function

$$\left[d_{\mathbb{R}^2}(j^{-1}(m), j^{-1}(m')) - T\right]\theta,$$

where j is the local isometry between the rectangle of type $(+)$, $(-)$ or $(\varepsilon, \cdots, \varepsilon, -\varepsilon, \cdots, -\varepsilon)$ and a neighbourhood of g. In the definition of these lagrangian submanifolds, we have not taken into account the slits, so that in fact the geodesics corresponding to Γ_T form a subset of the corresponding lagrangian submanifold. The set Σ corresponds to the geodesics obtained by "pushing g along itself". The fact that the intersections are clean is straightforward once good coordinates are chosen. □

On figure 4 the conormal sets to the circles correspond to some of the lagrangian submanifolds of the preceding proposition.

This proposition is really important since it describes what should be taken as the generalization of the geodesic flow (as long as propagation of singularities for the wave equation is considered). It also gives the geometric wave-front. In particular we would like to know whether the propagator for the wave equation is a Fourier Integral Operator and with which canonical transformation it is associated. This study implies that if $c_c(g) > 1$ then the propagator isn't in this class of operators.

References

1. V. Guillemin and R. Melrose. The Poisson summation formula for manifolds with boundary. *Adv. in Math.*, 32(3):204–232, 1979.
2. J. Wunsch. A Poisson relation for conic manifolds. *Math. Res. Lett.*, 9(5-6):813–828, 2002.
3. F.G. Friedlander. On the wave equation in plane regions with polygonal boundary. In *Advances in microlocal analysis (Lucca, 1985)*, pages 135–150. 1986.
4. A. Katok. The growth rate for the number of singular and periodic orbits for a polygonal billiard. *Comm. Math. Phys.*, 111(1):151–160, 1987.
5. M. Troyanov. *Les surfaces euclidiennes à singularités coniques.* l'Ens. Math. , 32:79–94, 1986.
6. P. Hubert and T. A. Schmidt. Invariants of translation surfaces. *Ann. Inst. Fourier (Grenoble)*, 51(2):461–495, 2001.
7. J. Cheeger and M. Taylor. On the diffraction of waves by conical singularities I et II. *Comm.Pure.Appl.Math.*, 35:275–331, 487–529, 1982.
8. J. Dixmier. *Topologie générale.* Presses Universitaires de France, Paris, 1981. Mathématiques. [Mathematics].
9. S. Tabachnikov. Billiards. *Panoramas et Synthèses*, (1), 1995.
10. J.J. Duistermaat and V.W. Guillemin. The spectrum of positive elliptic operators and periodic bicharacteristics. *Invent.Math.*, 29:39–79, 1975.