Introduction to Macdonald measures

I Symmetric functions

II Macdonald measures (Borodin-Corwin 2011)

III The Whittaker measure (O'Connell 2009)

Schur functions

$$s_\lambda(x) = \frac{\det\left(x_i^{\lambda_j+n-j}\right)}{\det\left(x_i^{n-j}\right)} \qquad \begin{array}{l} x=(x_1 \dots x_n) \\ \lambda=(\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0) \end{array}$$

Cauchy identity

$$\sum_\lambda s_\lambda(x)\, s_\lambda(y) = \prod_{i,j} \frac{1}{1-x_i y_j} =: \Pi(x,y)$$

defines a probability measure $\mathbb{P}(\lambda) = \frac{1}{\Pi(x,y)} s_\lambda(x)\, s_\lambda(y)$

→ random permutations

→ last passage percolation

→ TASEP (interacting particle system)

→ Random matrices, free fermions, etc...

$S_\lambda(x_1 \dots x_n)$ is a symmetric polynomial of n variables. We extend their definition to any partition:
$\ell(\lambda) = \#\{i,\ \lambda_i > 0\}$ If $\ell(\lambda) > n$ $s_\lambda(x_1 \dots x_n) = 0$

Now, the def is such that:

$$S_\lambda(x_1, \dots, x_n, 0) = S_\lambda(x_1 \dots x_n) \qquad \text{(stability)}$$

We may regard S_λ as a symmetric function in infinitely many variables.

The power sum $p_k = \sum_{i=1}^{n} x_i^k$ form an algebra basis of symmetric polynomials in n variables.

We may let $p_k = \sum_{i=1}^{+\infty} x_i^k$ and define

$$\text{Sym} = \overset{\text{algebra}}{\text{ring}} \text{ of symmetric functions} = \mathbb{C}[p_1, p_2, p_3, \dots]$$

It is endowed with the Hall inner product:

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_{\ell(\lambda)}} \quad \langle p_\lambda, p_\mu \rangle = z_\lambda \mathbb{1}_{\lambda=\mu}$$

where $z_\lambda = \prod_{i \geq 1} i^{m_i} m_i!$ $\qquad \lambda = (1^{m_1} 2^{m_2} \dots)$

Using the Cauchy identity + $\sum_\lambda \frac{1}{z_\lambda} p_\lambda(x)\, p_\lambda(y) = \Pi(x,y)$

one gets $\langle s_\lambda, s_\mu \rangle = \mathbb{1}_{\lambda=\mu}$. (this is an equivalence)

<u>Macdonald functions</u>

$(q,t) \in (0,1)$

same as before

$$z_\lambda(q,t) := z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}$$

$$\langle P_\lambda, P_\mu \rangle_{q,t} = z_\lambda(q,t)\, \mathbb{1}_{\mu=\lambda}$$

<u>Rq</u> : $q=t$ recovers the Hall inner product

<u>Theorem</u> (Macdonald 1987)

For each partition λ, $\exists!\ P_\lambda(\cdot\,; q,t) \in Sym$

such that $P_\lambda(x\,;q,t) = \sum_{\mu \le \lambda} u_{\lambda\mu}\, m_\lambda(x)$

monomial sym. poly.

$(\lambda \geq \mu \Leftrightarrow \forall i \ \ \lambda_1 + \dots + \lambda_i \geq \mu_1 + \dots + \mu_i)$ $\quad \sum x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \dots$

where $\begin{cases} u_{\lambda\mu} \in \mathbb{Q}(q,t) \\ u_{\lambda\lambda} = 1 \end{cases}$

and $\langle P_\lambda, P_\mu \rangle_{q,t} = 0$ if $\lambda \neq \mu$.

Define $\bullet\ Q_\lambda = b_\lambda P_\lambda$ where $b_\lambda = \langle P_\lambda, P_\lambda \rangle_{q,t}$

so that $\langle P_\lambda, Q_\mu \rangle_{q,t} = \mathbb{1}_{\lambda=\mu}$.

$$\bullet\ \Pi(x,y\,;q,t) = \prod_{i,j} \frac{(t x_i y_j\,;q)_\infty}{(x_i y_j\,;q)_\infty}$$

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\ldots$$

q-Pochhammer symbol

A direct computation shows that

$$\sum_\lambda \frac{1}{z_\lambda(q,t)} p_\lambda(x)\, p_\lambda(y) = \Pi(x,y\,;q,t)$$

$$\Rightarrow \boxed{\sum_\lambda P_\lambda(x\,;q,t)\, Q_\lambda(y\,;q,t) = \Pi(x,y\,;q,t)}$$

Cauchy identity for Macdonald functions.

We may develop the theory, define skew Macdonald functions, prove branching rules, Pieri identities, etc...

$$P_\lambda(x) = \sum_{\substack{\text{tableaux } T \\ \text{of shape } \lambda}} \varphi(T)\, x^T \qquad x^T = x_1^{\#1\text{'s in } T}\, x_2^{\#2\text{'s in } T} \cdots$$

$\varphi(T) \in \mathbb{Q}(q,t)$ explicit

Rq: One way to construct Macdonald functions is to impose $P_\lambda = \sum_{\mu \leq \lambda} u_{\lambda\mu} m_\mu$ are eigenfunctions of an operator $\tilde{D}$ such that $\tilde{D} P_\lambda = d_\lambda P_\lambda$, d_λ distinct, and $\tilde{D}$ is self adjoint w.r.t $\langle \cdot, \cdot \rangle_{q,t}$

Def
$$D^{(n)} = \sum_{i=1}^{n} \prod_{j \neq i} \frac{t x_i - x_j}{x_i - x_j} T_{q,i}$$

$$T_{q,i} f(x_1 \dots x_n) = f(x_1 \dots q x_i \dots x_n)$$

Prop
$$D^{(n)} P_\lambda(x_1 \dots x_n) = \underbrace{\left(q^{\lambda_1} t^{n-1} + \dots + q^{\lambda_n} t^0\right)}_{e_1(q^\lambda t^\delta)} P_\lambda(x_1 \dots x_n)$$

Rq: One constructs higher order D_r such that
$$D_r^{(n)} P_\lambda = e_r(q^\lambda t^\delta) P_\lambda$$

Rq: When $q=t$, $\boxed{P_\lambda = s_\lambda = Q_\lambda}$

Rq: P_λ may be thought of as a Fourier basis
$$P_\lambda \leftrightarrow e^{ix\lambda}$$
$$D_r^{(n)} \leftrightarrow (\partial_x)^r$$
One knows explicit inversion formulas...

Macdonald measure

$$\mathbb{P}^{q,t}(\lambda) = \frac{1}{\Pi(x,y;q,t)} P_\lambda(x) Q_\lambda(y)$$

Discuss positivity $\quad x_i, y_j \in (0,1)$.

How to compute observables: act on the Cauchy identity with an operator diagonalised by Macdonald fct.

Let A: Sym $\to$ Sym st $AP_\lambda = d_\lambda P_\lambda$.

$$\sum_\lambda P_\lambda(x) Q_\lambda(y) = \Pi(x,y)$$

$$\frac{1}{\Pi(x,y)} \sum_\lambda d_\lambda P_\lambda(x) Q_\lambda(y) = \frac{A\Pi(x;y)}{\Pi(x,y)}$$

$$\mathbb{E}^{q,t}[d_\lambda].$$

<u>Exemple</u> Acting with $(D_1)^k$, we get

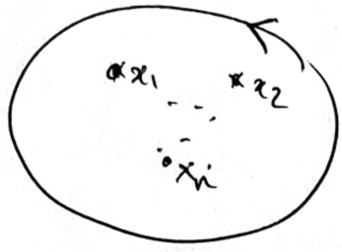
$$\mathbb{E}\left[\left(q^{\lambda_1}t^{n-1} + \cdots + q^{\lambda_n}\right)^k\right]$$

from that, in certain cases we may extract the distribution of marginals such as λ_1.

If $G(x) = g(x_1)\cdots g(x_n)$

holomorphic inside contour and non zero.

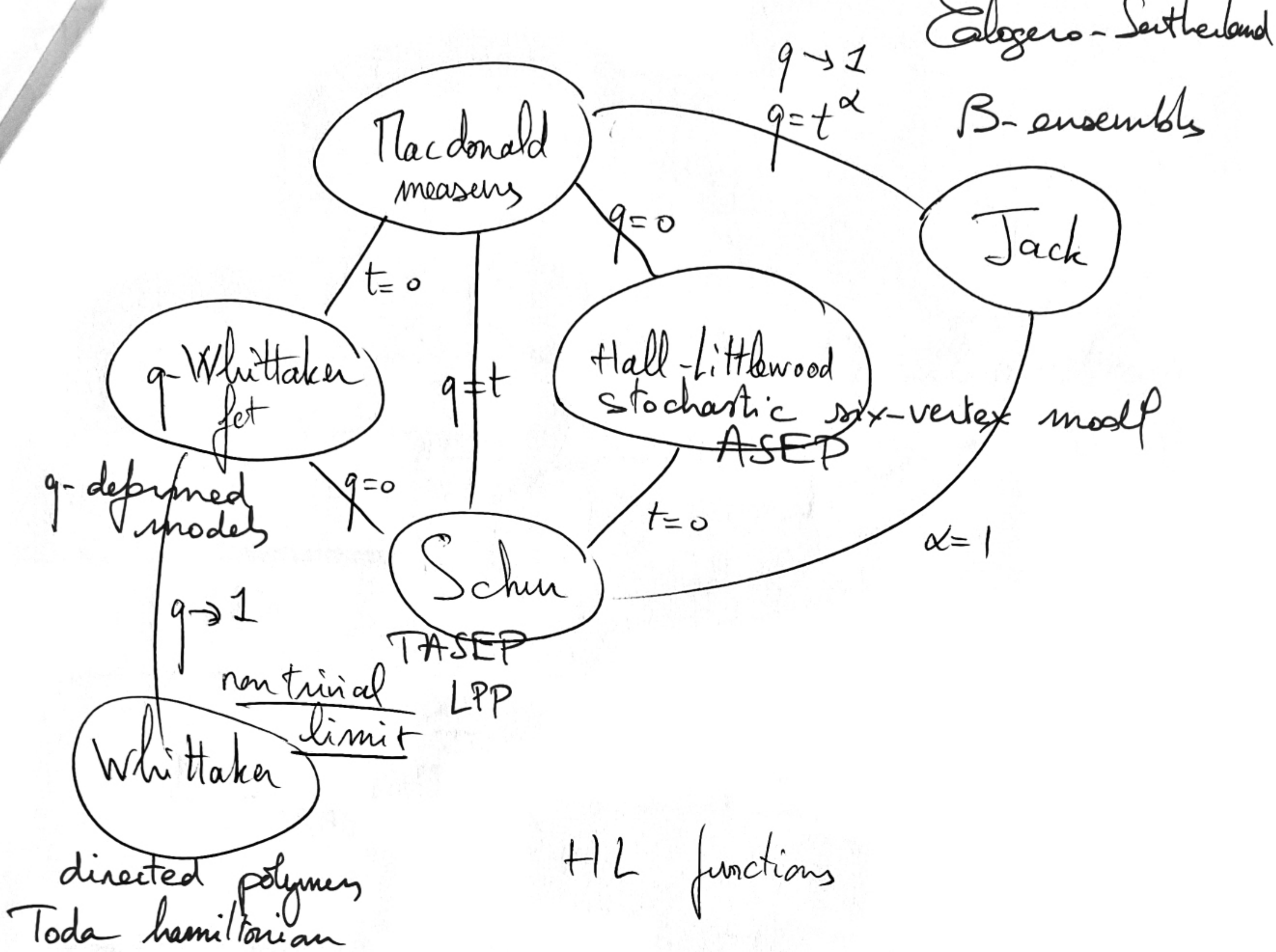
$$\frac{D_1 G(x)}{G(x)} = \frac{1}{2i\pi}\oint \frac{dz}{qz-z}\prod_{m=1}^{n}\frac{tz-x_m}{z-x_m}\times\frac{g(qz)}{g(z)}.$$



more generally one gets integral formulas for

$$\frac{D_1^k\,\Pi(x,y)}{\Pi(x,y)} \quad \text{or} \quad \frac{D_r\,\Pi(x,y)}{\Pi(x,y)}$$

Rq: Instead of just one partition one can define "natural" measures on sequence of partitions, compute more observables, ...



HL functions

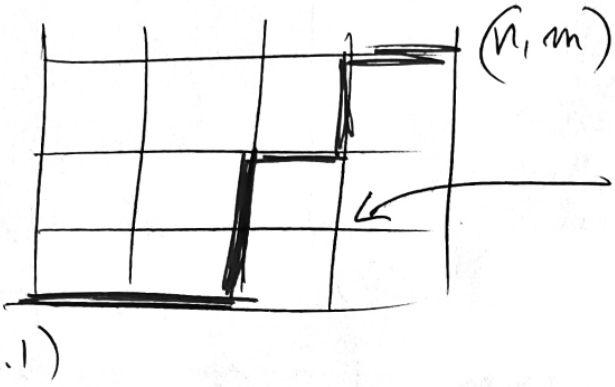
$$c(\lambda) = \prod_{i \geq 0} \frac{(1-t)^{m_i}}{(t;t)_{m_i}}$$

$$P_\lambda = c(\lambda) \sum_{\sigma \in S_n} \sigma \left(\prod_{i<j} \frac{x_i - t x_j}{x_i - x_j} \prod_{i=1}^{n} x_i^{\lambda_i} \right)$$

<u>Question</u> What happens to RSK ?

III The Whittaker measure

Directed polymers



Typically random $\omega_{ij} > 0$

$$Z(n,m) = \sum_{\substack{\text{paths} \\ (1.1) \to (n,m)}} \prod_{i,j \in \text{path}} \omega_{ij}$$

One is generally interested in the probability measure $\mathbb{P}(\pi) = \dfrac{\prod_{i,j\in\pi} \omega_{ij}}{Z(n,m)}$. As usual in statistical mechanics, most information about the large scale behaviour is contained in the asymptotics of the partition function.

Whittaker functions ~~Whittaker~~ introduced by Whittaker 1927 and more generally Jacquet 1967

- Fourier coefficients of automorphic forms $GL_2(\mathbb{R})$
- Integrals of Whittaker fcts arise in number theory [Stade]
- Geometry
- Eigenbasis of the quantum Toda lattice

$gl_n(\mathbb{R})$ (archimedean) Whittaker fct

$\alpha \in \mathbb{C}^n \qquad x \in \mathbb{R}_+^n$

$$\begin{tikzcd}[column sep=small]
 & & & z_{11} \arrow[dr] & & & \\
 & & z_{22} \arrow[ur] \arrow[dr] & & z_{21} \arrow[dr] & & \\
 & z_{33} \arrow[ur] \arrow[dr] & & z_{32} \arrow[ur] \arrow[dr] & & z_{31} \arrow[dr] & \\
z_{44} \arrow[ur] \arrow[d, equal] & & z_{43} \arrow[ur] \arrow[d, equal] & & z_{42} \arrow[ur] \arrow[d, equal] & & z_{41} \arrow[d, equal] \\
x_4 & & x_3 & & x_2 & & x_1
\end{tikzcd}$$

$$\Psi_\alpha(x) = \int \prod_i \left(\frac{\prod_j z_{ij}}{\prod_j z_{i-1,j}} \right)^{\alpha_i} \exp\left(- \sum_{a \to b} \frac{a}{b} \right) \prod_{ij} \frac{dz_{ij}}{z_{ij}}$$

"favors $a \le b$"

Cauchy identity (Bump – Stade)

If $\mathrm{Re}[\alpha_i + \beta_j] > 0 \quad \forall i,j$

$$\int_{\mathbb{R}_+^n} \Psi_{-\alpha}(x)\, \tilde{\Psi}_{-\beta}(x) \frac{dx}{x} = \prod_{ij} \Gamma(\alpha_i + \beta_j)$$

$$\int \prod(\ )^{\alpha_i} \prod(\ )^{\beta_i} \exp\Big(\sum_{a \to b} \cdots\Big) \qquad \prod_{ij} \int w_{ij}^{-(\alpha_i+\beta_j)} e^{-1/w_{ij}} \frac{dw_{ij}}{w_{ij}}$$

The geometric RSK correspondance is a birational map (volume preserving)

$$(\omega_{ij}) \in \mathbb{R}_+^{n\times m} \longleftrightarrow (z_{ij}) \in \mathbb{R}_+^{n\times m}$$

It is such that $z_{n,m} = Z(n,m)$.

$u>0$

$$\mathbb{E}\left[e^{-u\, Z(n,m)}\right] = \int e^{-u x_1}\, \psi_{-\alpha}(x)\, \tilde{\psi}_{-\beta}(x)\, \frac{dx}{x}$$

Here $\omega_{ij} \sim \text{Gamma}^{-1}(\alpha_i+\beta_j) \quad \times \prod_{i,j} \frac{1}{\Gamma(\alpha_i+\beta_j)}$

~~Analogy with the~~

leads to $\quad (\alpha_i=\alpha \,,\, \beta_i=\beta)$

$$\mathbb{P}\left(\frac{\log Z(n,m) - cn}{c'\, n^{1/3}} \leq x\right) \longrightarrow F_{TW}(x)$$

Analogy with the Schur case

$$\sum S_\lambda(x)\, S_\lambda(y) = \prod \frac{1}{1-x_i y_j}$$

$$\sum_{P,Q} x^P y^Q \qquad\qquad \sum_M \prod (x_i y_j)^{m_{ij}}$$

Young tableaux of the same shape that can be encoded by an array

$$\begin{array}{ccccc} & & z_{11} & & \\ & z_{22} & & z_{21} & \\ z_{33} & & z_{32} & & z_{31} \end{array}$$

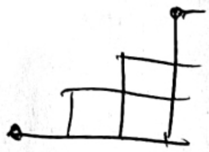
integers

RSK : bijection $(P,Q) \longleftrightarrow M \in \mathbb{N}^{n\times m}$

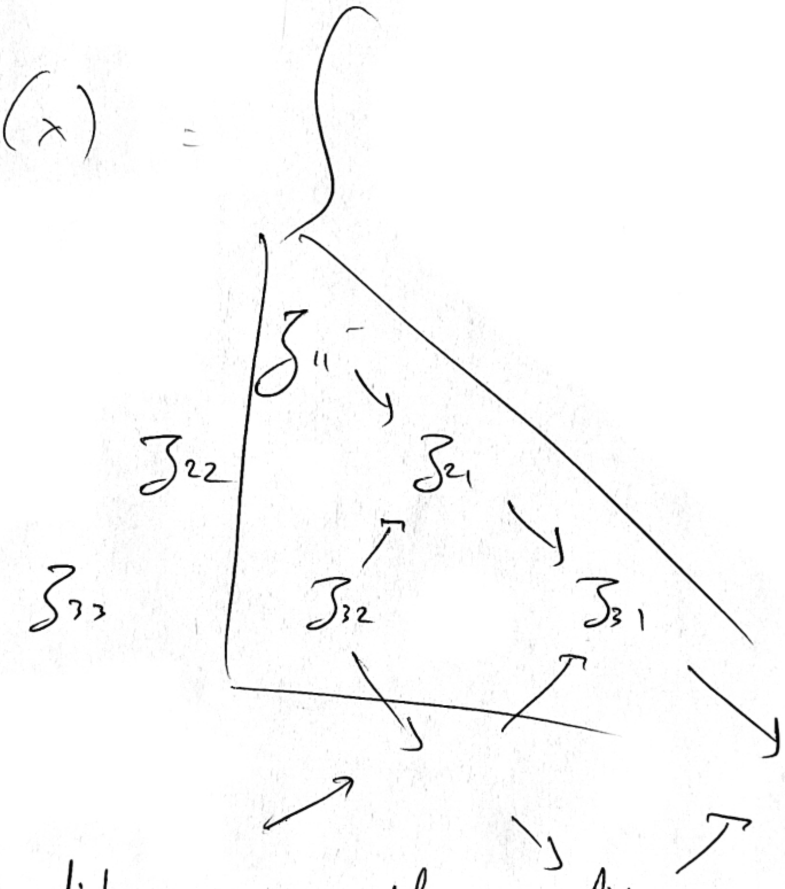
which matches the monomials in the Cauchy identity.

Further directions:

- Other types and Littlewood identities are related to polymers on other domains



$$\Psi^{SO_{2n+1}}(x) =$$



- Cauchy identity and orthogonality relations are satisfied by other families of functions, for instance partition functions of vertex models....

cf Amol