Notes about *Spectral radius of non-Hermitian random matrices*

(joint work with Charles Bordenave and Djalil Chafaï)

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1 Convergence and pushforward

The following two notions will be essential everywhere in this presentation.

Weak convergence

• For a topological space X we denote by $\mathcal{P}(X)$ the set of probability measures on X. We shall say that a sequence $(\mu_n)_n$ of elements of $\mathcal{P}(X)$ (weaky) converges to $\mu \in \mathcal{P}(X)$ if and only if

$$\lim_{n\to\infty} \int_X f \mathrm{d}\mu_n = \int_X f \mathrm{d}\mu$$

for every $f \in C_b(X)$ (i.e., bounded and continuous real function on X).

• The set $\mathcal{P}(X)$ will be endowed with the coarsest (smallest) topology such that $\mu \mapsto \int_X f \mathrm{d}\mu$ is continuous for every $f \in C_b(X)$.

Pushforward

Any measurable function $F : X \to Y$ between measurable spaces X and Y induces the (pushforward) map $F_* : \mathcal{P}(X) \to \mathcal{P}(Y)$ defined by

$$F_*(\mu)(C) = \mu(F^{-1}(C)).$$

2 Girko measure and absence of outliers

Fix a probability measure $\mu \in \mathcal{P}(\mathbb{C})$ such that $\int_{\mathbb{C}} |z|^2 \mathrm{d}\mu(z) = 1$ and $\int_{\mathbb{C}} z \mathrm{d}\mu(z) = 0$. The *Girko measure* $\mathbb{G}_n^\mu$ is the probability measure on the space of n by n matrices $M_n(\mathbb{C}) = \mathbb{C}^{\{1,\ldots,n\}^2}$,

$$\mathbb{G}_n^\mu = \mu^{\otimes \{1,\ldots,n\}^2}.$$

If we plot the eigenvalues of a matrix obtained by simulating the Girko measure $\mathbb{G}_n^\mu$ we would get something like Figure 1. We observe in these simulations that the eigenvalues seem to fill a disk centered at 0 and of radius $\sqrt{n}$. So, it would be convenient to consider the eigenvalues of the rescaled matrix $A/\sqrt{n}$. In this way the eigenvalues would seem to accumulate in the unit disk $\mathbb{D}$.

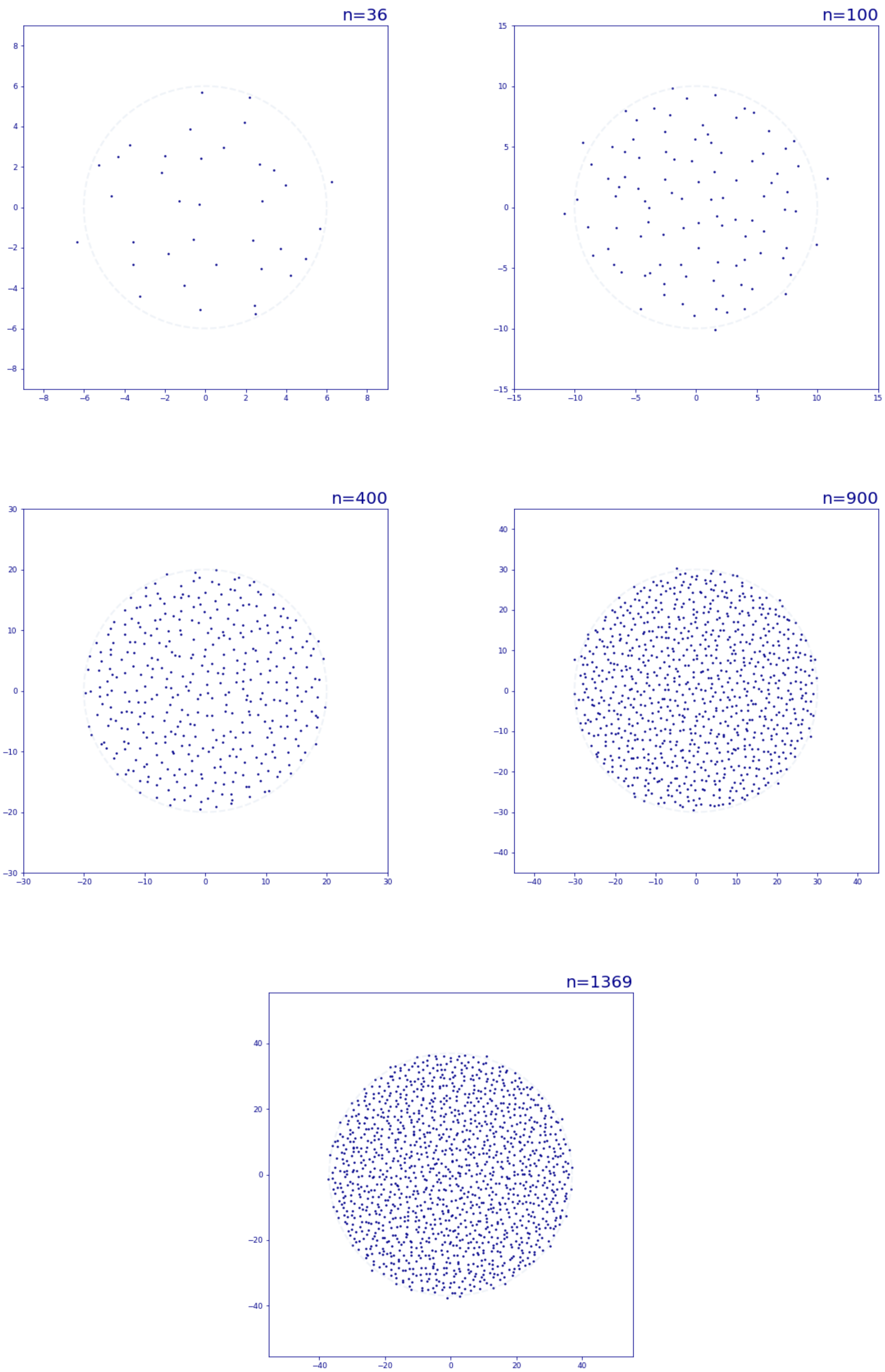

Figure 1: The eigenvalues of a Girko matrix where $\mu(\{1\}) = \mu(\{-1\}) = \mu(\{i\}) = \mu(\{-i\}) = 1/4$.

To state this more precisely, we consider the continuous map $\mathrm{ch} : M_n(\mathbb{C}) \to \mathbb{C}_n[z]$ given by

$$\mathrm{ch}(A) = \det\left(z - \frac{A}{\sqrt{n}}\right)$$

(here $\mathbb{C}_n[z]$ denotes the space of complex polynomials of degree n). To capture the information about the eigenvalues we define, for every continuous function $f : \mathbb{C} \to \mathbb{R}$, the continuous map $\mathrm{emp}_f : \mathbb{C}_n[z] \to \mathbb{R}$ given by

$$\mathrm{emp}_f(P) = \frac{1}{n} \sum_{P(z)=0} f(z).$$

The theorem of Tao and Vu (2010) implies that, for every $f \in C_b(\mathbb{C})$,

$$\lim_{n\to\infty} (\mathrm{emp}_f \circ \mathrm{ch})_* \mathbb{G}_n^\mu = \delta_{\frac{1}{\pi}\int_{\mathbb{D}} f \mathrm{d}\ell_{\mathbb{C}}},$$

where δ_x denotes the Dirac delta measure at x. Equivalently, this tells us that

$$\lim_{n\to\infty} \mathbb{G}_n^\mu\left(\left\{A \in M_n(\mathbb{C}) : \left|\frac{1}{n} \sum_{z \text{ eig. of } \frac{A}{\sqrt{n}}} f(z) - \frac{1}{\pi}\int_{\mathbb{D}} f \mathrm{d}\ell_{\mathbb{C}}\right| > \varepsilon\right\}\right) = 0$$

for every $\varepsilon > 0$. This explains the fact that the eigenvalues seem to fill uniformly the unit disk.

In particular, by bounding $1_{\{z\in\mathbb{C}:|z|>1+\delta\}}$ by a bounded continuous function supported outside $\mathbb{D}$ we can show that

$$\lim_{n\to\infty} \mathbb{G}_n^\mu\left(\left\{A \in M_n(\mathbb{C}) : \frac{\#\,\text{eigenvalues of } \frac{A}{\sqrt{n}} \text{ greater than } 1+\delta}{n} > \varepsilon\right\}\right) = 0$$

for every $\delta, \varepsilon > 0$, i.e., the proportion of eigenvalues at a finite distance from $\mathbb{D}$ goes to zero. Nevertheless, there may still be some eigenvalues outside $\mathbb{D}$ (sometimes called outliers) but it turns out there are not! This is one of the main results I wanted to tell you about.

Theorem 1 (Bordenave, Chafaï, G-Z, 2022)**.** *For every $\delta > 0$,*

$$\lim_{n\to\infty} \mathbb{G}_n^\mu\left(\left\{A \in M_n(\mathbb{C}) : \#\left[\textit{eigenvalues of } \frac{A}{\sqrt{n}} \textit{ greater than } 1+\delta\right] = 0\right\}\right) = 1.$$

3 Digression: Weyl polynomials and outliers

In this section, fix a probability measure $\mu \in \mathcal{P}(\mathbb{C})$ and suppose, for simplicity, that the conditions $\mu(\{0\}) = 0$, $\int_{\mathbb{C}} |z|^2 \mathrm{d}\mu(z) < \infty$ and $\int_{\mathbb{C}} z^2 \mathrm{d}\mu(z) = \int_{\mathbb{C}} z \mathrm{d}\mu(z) = 0$ are satisfied. We will think $\mathbb{C}_n[z]$ as $\mathbb{C}^{n-1} \times (\mathbb{C} \setminus \{0\})$ via the (not so standard) map $\mathrm{pol} : \mathbb{C}^{n-1} \times (\mathbb{C} \setminus \{0\}) \to \mathbb{C}_n[z]$ given by

$$\mathrm{pol}(a_0, \ldots, a_n) = \sum_{k=0}^{n} \frac{a_k (z\sqrt{n})^k}{\sqrt{k!}}.$$

The $\sqrt{n}$ term appears for a reason similar to the scaling for Girko matrices. We are interested in the *Weyl measure* $\mathbb{W}_n^\mu$ given by

$$\mathbb{W}_n^\mu = \mathrm{pol}_* \left(\mu^{\otimes n} \otimes \mu|_{\mathbb{C}\setminus\{0\}}\right).$$

If we plot the zeros of a polynomial obtained by simulating $\mathbb{W}_n^\mu$ we would get something like Figure 2. They seem to accumulate in $\mathbb{D}$. The theorem of Kabluchko and Zaporozhets (2014) confirms this[1] by implying that, for every $f \in C_b(\mathbb{C})$,

$$\lim_{n\to\infty} (\mathrm{emp}_f)_* \mathbb{W}_n^\mu = \delta_{\frac{1}{\pi}\int_{\mathbb{D}} f \mathrm{d}\ell_{\mathbb{C}}}.$$

Equivalently,

$$\lim_{n\to\infty} \mathbb{W}_n^\mu\left(\left\{P \in \mathbb{C}_n[z] : \left|\frac{1}{n}\sum_{P(z)=0} f(z) - \frac{1}{\pi}\int_{\mathbb{D}} f \mathrm{d}\ell_{\mathbb{C}}\right| > \varepsilon\right\}\right) = 0$$

for every $\varepsilon > 0$. Nevertheless, according to the simulations shown in Figure 2, there should be outliers in this case. This can be explained in the following way. Denote by $\mathcal{O}(\mathbb{D})$ the set of holomorphic functions on $\mathbb{D}$ endowed with the compact-open topology (the topology of uniform convergence on compact sets) and consider the continuous map $\widetilde{\mathrm{rec}} : \mathbb{C}_n[z] \to \mathcal{O}(\mathbb{D})$ given by

$$\widetilde{\mathrm{rec}}(P)(w) = \sqrt{\frac{n!}{n^n}}\, w^n P(1/w).$$

This is a normalized *reciprocal polynomial* and, up to the normalization, may be thought of as the polynomial seen from infinity (in a different coordinate system and different trivialization). So, the questions we may ask about $\mathbb{W}_n^\mu$ outside of $\mathbb{D}$ become questions about $\widetilde{\mathrm{rec}}_* \mathbb{W}_n^\mu$.

Theorem 2. *There exists $\nu \in \mathcal{P}(\mathcal{O}(\mathbb{D}))$ such that*

$$\lim_{n\to\infty} \widetilde{\mathrm{rec}}_* \mathbb{W}_n^\mu = \nu.$$

The measure ν depends on μ and it satisfies

$$\nu(\{g \in \mathcal{O}(\mathbb{D}) : g \textit{ has an infinite number of zeros}\}) = 1.$$

The measure ν is explicit and can be found, for instance, in [Butez, G-Z (2022)]. The last assertion can be obtained by relating this case to the case where μ is Gaussian and this is the reason the conditions on the first and second moments were needed.

Remarkably enough, a similar approach can be used for the Girko matrix case.

4 Reciprocal characteristic polynomial for the Girko measure

In this case, consider the continuous map $\mathrm{rec} : \mathbb{C}_n[z] \to \mathcal{O}(\mathbb{D})$ given by

$$\mathrm{rec}(P)(w) = w^n P(1/w).$$

The second main result I wanted to tell you about is the following.

Theorem 3 (Bordenave, Chafaï, G-Z, 2022)**.** *There exists $\nu \in \mathcal{P}(\mathcal{O}(\mathbb{D}))$ such that*

$$\lim_{n\to\infty} (\mathrm{rec} \circ \mathrm{ch})_* \mathbb{G}_n^\mu = \nu.$$

The measure ν depends only on $\int_{\mathbb{C}} z^2 \mathrm{d}\mu(z)$ and it satisfies

$$\nu(\{g \in \mathcal{O}(\mathbb{D}) : g \textit{ does not have zeros}\}) = 1.$$

[1] In fact, this holds for any non-deterministic μ such that $\int_{\mathbb{C}} \log(1+|z|)\mathrm{d}\mu(z) < \infty$. The first and second moment conditions asked are the simplest way I know of to ensure the existence of an infinite number of outliers.

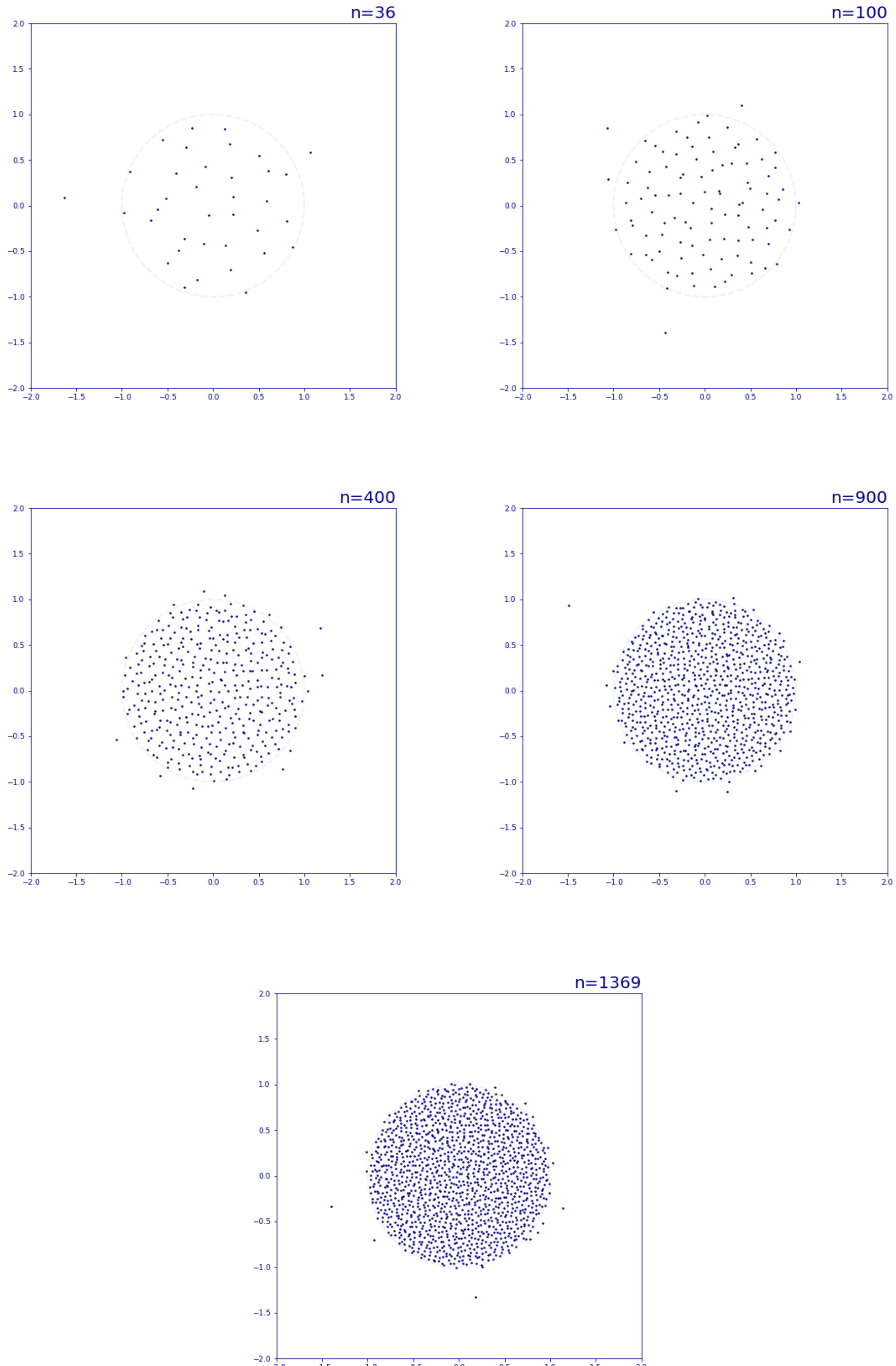

Figure 2: The zeros of a Weyl polynomial where $\mu(\{1\}) = \mu(\{-1\}) = \mu(\{i\}) = \mu(\{-i\}) = 1/4$.

This explains the absence of outliers. Notice that the polynomial we are interested in can be written as $(\mathrm{rec} \circ \mathrm{ch})(A) = \det\left(1 - w\frac{A}{\sqrt{n}}\right)$. The proof follows the following two steps:

1. Show that the sequence $((\mathrm{rec} \circ \mathrm{ch})_* \mathbb{G}_n^\mu)_n$ is precompact.
2. Given the map $\mathrm{coeff} : \mathcal{O}(\mathbb{D}) \to \mathbb{C}^{\mathbb{N}}$ defined by

$$\mathrm{coeff}\left(\sum_{k=0}^{\infty} a_k z^k\right) = (a_k)_k,$$

   show that $(\mathrm{coeff} \circ \mathrm{rec} \circ \mathrm{ch})_* \mathbb{G}_n^\mu$ converges.

For the first step, an argument inspired by Basak and Zeitouni (2020) allows us to obtain

$$\int_{M_n(\mathbb{C})} \left|\det\left(1 - w\frac{A}{\sqrt{n}}\right)\right|^2 \mathrm{d}\mathbb{G}_n^\mu(A) \leq \frac{1}{1 - |w|^2}.$$

Together with Montel's theorem, the subharmonicity of the norm square of a holomorphic function, Markov's inequality and Prokhorov's theorem we get the precompactness of the sequence.

For the second step, after a truncation argument inspired by Janson and Nowicki (1991), we may assume that μ is compactly supported. Then, to study $(\mathrm{coeff} \circ \mathrm{rec} \circ \mathrm{ch})_* \mathbb{G}_n^\mu$, we can use the identity $\det(e^B) = e^{\mathrm{Tr}(B)}$ which gives us

$$\det\left(1 - w\frac{A}{\sqrt{n}}\right) = \exp\left(-\sum_{k=0}^{\infty} \frac{w^k}{k} \frac{\mathrm{Tr}(A^k)}{\sqrt{n}^k}\right)$$

for w small enough. Then, the coefficients of the reciprocal characteristic polynomial are continuous functions (in fact, polynomials) of $\frac{\mathrm{Tr}(A^k)}{\sqrt{n}^k}$. It turns out that it is easier to consider the maps $T_k : M_n(\mathbb{C}) \to \mathbb{C}^k$ defined by

$$T_k(A) = \left(\frac{\mathrm{Tr}(A)}{\sqrt{n}}, \ldots, \frac{\mathrm{Tr}(A^k)}{\sqrt{n}^k}\right)$$

and to study the limit of $(T_k)_* \mathbb{G}_n^\mu$. For this, we begin by writing

$$\begin{aligned}\frac{\mathrm{Tr}(A^k)}{\sqrt{n}^k} &= \sum_{i_1,\ldots,i_k \in \{1,\ldots,n\}} a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_k i_1} \\ &= \frac{1}{\sqrt{n}^k} \sum_{\substack{i_1,\ldots,i_k \in \{1,\ldots,n\} \\ \text{the } i_m \text{repeat at least once}}} a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_k i_1} + \frac{1}{\sqrt{n}^k} \sum_{\substack{i_1,\ldots,i_k \in \{1,\ldots,n\} \\ \text{the } i_m \text{are all different}}} a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_k i_1}.\end{aligned}$$

The variance of the summand on the left converges to zero and it can be seen that its expected value converges. In fact, the only cycles $(i_1, \ldots, i_k, i_1)$ that will contribute to the limiting expected value are the double cycles, i.e., those of the form $(i_1, \ldots, i_{k/2}, i_1, \ldots, i_{k/2})$ but with no other repetitions. This gives us $(\int_{\mathbb{C}} z^2 \mathrm{d}\mu(z))^{k/2}$ if k is even and 0 if it is odd.

On the other hand, the summand on the right has zero expected value and its moments may be similarly understood by studying the closed paths $(i_1, \ldots, i_k, i_1)$ that contribute to the expected value of some product of terms.

At the end, this will imply the convergence of the reciprocal characteristic polynomial towards the exponential of a holomorphic function. This can be seen as the reason why the limit has no zeros and consequently a Girko matrix has no outliers (as $n \to \infty$).

References. The first two references are the ones that inspired some of the ideas.

- Janson and Nowicki. *The asymptotic distributions of generalized U-statistics with applications to random graphs.*
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