Positively multiplicative graphs, representation theory and alcove walks

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Consider a finite oriented graph Γ

- rooted at $\mathbf{v}_0$ with vertices $\mathbf{v}_0, v_1, \ldots, v_{n-1}$
- whose edges are weighted by monomials cR^β in $R_1, \ldots, R_k$ with $c \geq 0$.

Its adjacency matrix A belongs to $M_n(\mathbb{A})$ with $\mathbb{A} = \mathbb{R}[R_1, \ldots, R_k]$

Example : Γ_2 with

$$M_2 = \begin{pmatrix} 0 & R_1 + R_2 \\ 1 & 0 \end{pmatrix}$$

Γ is positively multiplicative if there exists an algebra $\mathcal{A}$ s.t.

$$\mathbb{A}[A] \subset \mathcal{A} \subset M_n(\mathbb{A})$$

with a distinguished basis $\mathfrak{B} = \{b_0, b_1, \dots, b_{n-1}\}$ s.t.

1. $b_0 = 1$,
2. for all $j = 0, \dots, n-1$
$$Ab_j = \sum_{i=0}^{n-1} a_{i,j} b_i$$
3. the structure constants $c_{i,j}^k$ of the products $b_i b_j$ belong to $\mathbb{A}_+$ the set of polynomials in $R_1, \dots, R_k$ with nonnegative coefficients.

Rq : by considering the cone $\mathcal{C} = \bigoplus_{i=0}^{n-1} \mathbb{A}_+ b_i$, this is equivalent to

$$b_0 = 1, \quad A \in \mathcal{C} \quad \text{and} \quad \mathcal{C}^2 = \mathcal{C}$$

In the previous example, we have

$$\mathbb{A}[A] = \mathbb{A} I_2 \oplus \mathbb{A} A$$

and Γ_2 rooted at v_0 is PM with $\mathfrak{B} = \{b_0 = I_2, b_1 = A\}$ since

$$A^2 = (R_1 + R_2) I_2, \quad A \times I_2 = I_2 \times A = A.$$

We can expand Γ_2 has the infinite graph Γ_2^e with set of vertices

$$\{(v_0, R_1^a R_2^b), (v_1, R_1^a R_2^b) \mid (a, b) \in \mathbb{N}^2\}$$

and arrows

$$(v_0, R_1^a R_2^b) \to (v_1, R_1^a R_2^b)$$

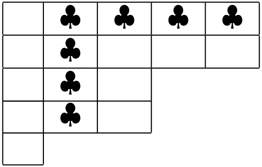
$$(v_1, R_1^a R_2^b) \begin{array}{l} \nearrow \ (v_0, R_1^{a+1} R_2^b) \\ \searrow \ (v_0, R_1^a R_2^{b+1}) \end{array}$$

II. Combinatorics

A partition of rank l is a sequence $\lambda = (\lambda_1 \geq \cdots \geq \lambda_m) \in \mathbb{Z}_{\geq 0}$ s.t. $\lambda_1 + \cdots + \lambda_m = l$.
λ is encoded by its Young diagram.
Each box c in λ has a hook length $h(c)$

Example :

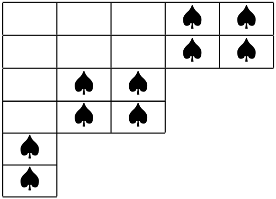
$\lambda = (5, 5, 3, 3, 1) \leftrightarrows$



with $h(1, 2) = 7$.

Let $k \geq 1$ be an integer.
A $(k+1)$-core is a partition λ with no hook length equal to $k+1$.
Write $|\lambda|_k$ for nb of boxes with hook length less or equal to k.

Example : The partition

$\lambda =$ 

is a 4-core with $|\kappa|_3 = 10$ (but not a 3-core).

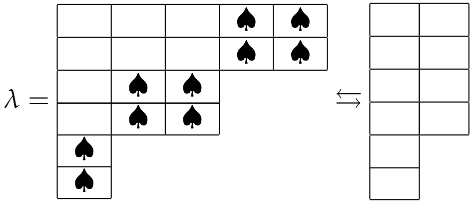
Rq : λ is a $(k+1)$-core i.f.f. its transposed $\mathrm{tr}(\lambda)$ is.

A partition is k-bounded when its parts are at most k.
There is a bijection

$$\{\lambda \mid k+1\text{-core s.t. } |\lambda|_k = l\} \underset{\mathfrak{c}^{-1}}{\overset{\mathfrak{c}}{\leftrightarrows}} \{\mu \mid k\text{-bounded of rank } l\}$$

obtained by deleting the boxes with hook lengths greater than k and next left align.

Example : For the 4-core



The map $\iota = \mathfrak{c}^{-1} \circ \mathrm{tr} \circ \mathfrak{c}$ is an involution on the k-bounded partitions.

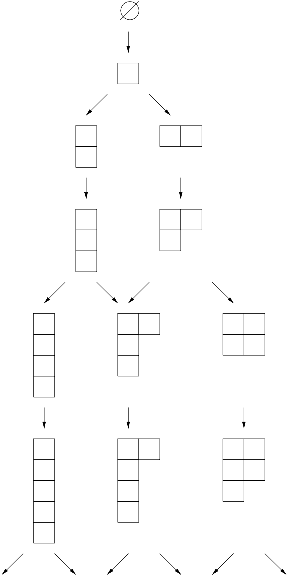
Let $\mathcal{Y}^k$ the graph with vertices the k-bounded partitions with oriented edges $\lambda \rightarrow \mu$ s.t.

- μ is obtained by adding one box to λ
- $\iota(\mu)$ is obtained by adding one box to $\iota(\lambda)$.

$\mathcal{Y}^k$ can also be interpreted as

- the poset of $(k+1)$-core (by using c^{-1})
- the orbit of the basic weight Λ_0 of $\widehat{\mathfrak{sl}}_{k+1}$ under the action of the affine Weyl group $\widehat{\mathfrak{S}}_{k+1}$ of type $A_k^{(1)}$.
- the poset of alcoves in the dominant Weyl chamber in type A_k.

Observe that $\lim_{k \rightarrow +\infty} \mathcal{Y}^k = \mathcal{Y}$ is the Young lattice of ordinary partitions.



The graph $\mathcal{Y}^2$

III. Harmonic functions

A function $f : \mathcal{Y}^k \to \mathbb{R}_{\geq 0}$ is harmonic when $f(\emptyset) = 1$ and for any $\lambda \in \mathcal{Y}^k$

$$f(\lambda) = \sum_{\lambda \to \mu} f(\mu).$$

The positive harmonic functions parametrize the central Markov chains on $\mathcal{Y}^k$: the transition matrix associated to f is

$$\Pi(\lambda, \mu) = \frac{f(\mu)}{f(\lambda)} 1_{\lambda \to \mu}$$

and

$$\Pi(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(l)}) = \frac{f(\lambda^{(l)})}{f(\lambda^{(1)})}$$

only depends on the ends of the trajectory $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(l)}$.

If f and g are positive harmonic, for any $t \in [0,1]$

$$tf + (1-t)g$$

also is. The set of such functions is a convex cone.

Problem : Find the extremal harmonic functions on $\mathcal{Y}^k$?

The graph $\mathcal{Y}^k$ is positively multiplicative : there exists a $\mathbb{R}$-algebra $\mathcal{A}$ with a distinguished basis $\mathbb{B} = \{s_\lambda^{(k)} \mid \lambda \in \mathcal{Y}^k\}$ s.t.

- $s_\emptyset^{(k)} = 1$
- $s_\lambda^{(k)} s_1^{(k)} = \sum_{\lambda \to \mu} s_\mu^{(k)}$
- $s_\lambda^{(k)} s_\mu^{(k)}$ decomposes on $\mathbb{B}$ with nonnegative coefficients.

The algebra $\mathcal{A}$ is the cohomology ring of $\widehat{\mathfrak{S}}_{k+1}/\mathfrak{S}_{k+1}$ the affine Grassmannian and the $s_\lambda^{(k)}$ are the affine Schubert classes (Lam 2008).

Theorem (Kerov-Vershik 1989) : The nonnegative extremal harmonic functions of $\mathcal{Y}^k$ are given by the morphisms $\theta : \mathcal{A} \rightarrow \mathbb{R}$ s.t. $\theta(s_1^{(k)}) = 1$ and $\theta(s_\lambda^{(k)}) \geq 0$ for any $\lambda \in \mathcal{Y}^k$ by setting

$$f(\lambda) = \theta(s_\lambda^{(k)})$$

IV. The k-Schur functions

Let $\Lambda = Sym_{\mathbb{R}}(x_1, \ldots, x_n, \ldots)$ be the algebra of symmetric functions.
The functions

$$h_a = \sum_{1 \leq i_1 \leq \cdots \leq i_a} x_{i_1} \cdots x_{i_a} \text{ with } a \geq 1$$

generate Λ

By results of T. Lam

- $\mathcal{A} = \langle h_1, \ldots, h_k \rangle$
- the $s_\lambda^{(k)}$ coincide with the k-Schur functions (Lascoux and al. (2003)).

The $s_\lambda^{(k)}$ can be computed by "Pieri rules" $s_\lambda^{(k)} \times h_a$ encoded in $\mathcal{Y}^{(k)}$.
In particular

$$s_1^{(k)} s_\lambda^{(k)} = \sum_{\lambda \rightarrow \mu \text{ in } \mathcal{Y}_k} s_\mu^{(k)}.$$

Observe that $\lim_{k \rightarrow +\infty} s_\lambda^{(k)} = s_\lambda$ is the usual Schur function associated to λ.

When $\mathrm{h}(\lambda) \leq k$, we get $s_\lambda^{(k)} = s_\lambda$.

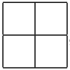
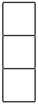
$\lambda \in \mathcal{Y}^k$ is k-irreducible when λ does not contain any rectangle $R_a = (k-a+1) \times a$ for $a = 1, \ldots, k$.

Theorem : For all $\lambda \in \mathcal{P}^{(k)}$, their exists a unique decomposition

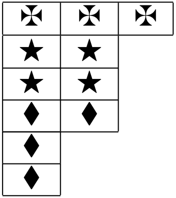
$$s_\lambda^{(k)} = s_{R_1}^{p_1} \cdots s_{R_k}^{p_k} s_\kappa^{(k)}$$

with $\kappa \in \mathcal{P}_{\mathrm{irr}}^{(k)}$.

Example : For $k = 3$ we have

$R_1 =$ , $R_2 =$ , $R_3 =$ 

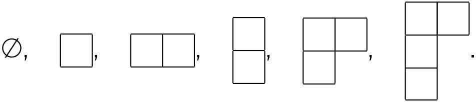
and for

$\lambda = (3, 2, 2, 2, 1, 1) =$ 

$$s_{\lambda}^{(3)} = s_{(3)} s_{(2,2)} s_{(2,1,1)}^{(3)}.$$

We have $\mathrm{card}(\mathcal{P}_{\mathrm{irr}}^{(k)}) = k!$

Example :
For $k = 2$, there are 2 irreducible partitions $\emptyset$, and $\square$.
For $k = 3$, there are 6 irreducible partitions



The positive morphisms θ are such that.

$$\begin{cases} \theta(s_{\emptyset}) = 1 \\ \theta(s_{R_a}) \geq 0 \text{ for } a = 1, \ldots, k \\ \theta(s_{\kappa}^{(k)}) \geq 0 \text{ for } \kappa \in \mathcal{P}_{\mathrm{irr}}^{(k)}. \end{cases}$$

By using the rectangle factorization, one can write for $\kappa \in \mathcal{P}_{\mathrm{irr}}^{(k)}$

$$s_{\kappa}^{(k)} \cdot s_{(1)} = \sum_{\kappa \to \mu} s_{\mu}^{(k)} = \sum_{\kappa' \in \mathcal{P}_{\mathrm{irr}}^{(k)}} m_{\kappa,\kappa'}(s_{R_1}, \ldots, s_{R_k}) s_{\kappa'}^{(k)}$$

where $m_{\kappa,\kappa'}(s_{R_1}, \ldots, s_{R_k}) \in \mathbb{Z}_{\geq 0}[s_{R_1}, \ldots, s_{R_k}]$ defines a matrix $k! \times k!$ written $M_k(s_{R_1}, \ldots, s_{R_k})$.

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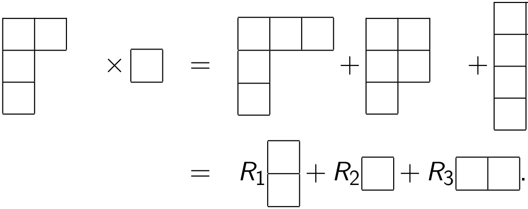
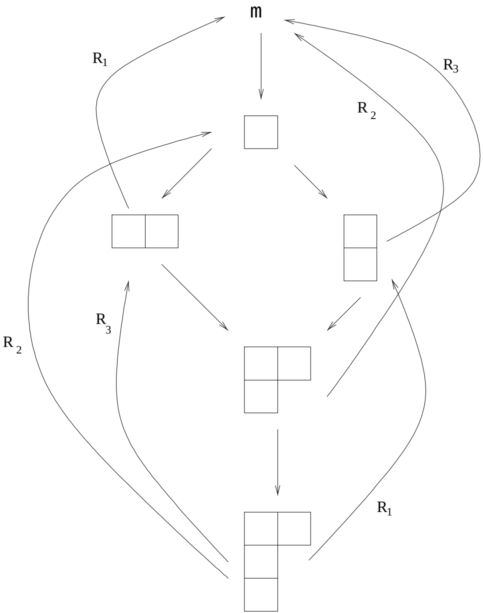


The matrix $M_k(R_1, \ldots, R_k)$ is the matrix of a simply connected graph $\Gamma_{A_k^{(1)}}$.

Example : For $k = 3$, we get

$$M_3 = \begin{pmatrix} 0 & 0 & R_1 & R_3 & R_2 & 0 \\ 1 & 0 & 0 & 0 & 0 & R_2 \\ 0 & 1 & 0 & 0 & 0 & R_3 \\ 0 & 1 & 0 & 0 & 0 & R_1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

which is the matrix of the simply connected graph $\Gamma_{A_3^{(1)}}$.


m
R1
R2
R3
R3
R2
R1

For $i = 1, \dots, k-1$, set $\alpha_i = e_i - e_{i+1}$ in $\mathbb{R}^k$ and $\alpha_0 = -(\alpha_1 + \cdots + \alpha_k)$.

Consider the tessellation of $\mathbb{R}^k$ by alcoves defined from the hyperplanes

$$H_{i,m} = \{v \in \mathbb{R}^k \mid (v, \alpha_i) = m\}$$

with $i = 0, \dots, k-1$ et $m \in \mathbb{Z}$.

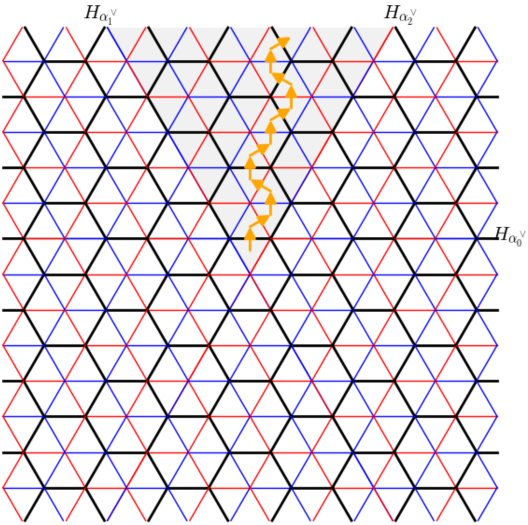


FIG.: A reduce walk on the alcoves for $k = 2$

The dominant alcoves are those whose points satisfy $(v, \alpha_i) \geq 0$ for all $i = 1, \ldots, k$.

They are in bijection with the $(k+1)$-cores or the k-bounded partitions.

A path in $\mathcal{Y}^k$ gives a reduced expression in $\widehat{\mathfrak{S}}_{k+1}/\mathfrak{S}_{k+1}$: an hyperplane can only be crossed once.

Set $\mathcal{S}_k = \{(u_1, \dots, u_k) \in \mathbb{R}^k_{\geq 0} \mid u_1 + \dots + u_k = 1\}$.

Theorem (Tarrago-L 2018) :

1. To each $\vec{r} \in \mathcal{S}_k$ corresponds a unique morphism $\theta : \mathcal{A} \to \mathbb{R}$ with $\theta(s_1) = 1$, positive on the k-Schur functions and s.t. $\theta(s_{R_a}) = r_a$ for all $a = 1, \dots, k$.
2. The correspondence is explicit : given $\vec{r}$, the $\theta(s^{(k)}_\kappa)$'s are the coordinates of the Perron Frobenius vector in M_k specialized in $\vec{r}$.
3. To each $\vec{r} \in \mathcal{S}_k$ corresponds a central random walk $(v_n)_{n \geq 0}$ on dominant alcoves satisfying a law of large numbers.

VI General result on positively multiplicative graphs

Remind : Γ is a finite oriented graph Γ

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Warning : the property of being PM depends on the chosen root !

Examples

- The graphs $\Gamma_{A_k^{(1)}}$ and their analogues for all the affine root systems.
- Every graph constructed from a fusion algebra $\mathcal{F}$ with basis $\mathfrak{B} = \{b_0, b_1, \ldots, b_{n-1}\}$ and an element

$$a = \sum_{i=0}^{n-1} x_i b_i \text{ with } x_i \geq 0 \text{ for } i = 0, \ldots, n-1.$$

- In particular for $\mathcal{F}$ a character ring or a group algebra.

Fewer results

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4. Simple algorithm to compute the coefficients $c_{i,j}^{k}$ when $\dim \mathbb{A}[A] = n$.
5. Characterization of the finite PM which can be rooted at any vertex in terms of Cayley graphs.

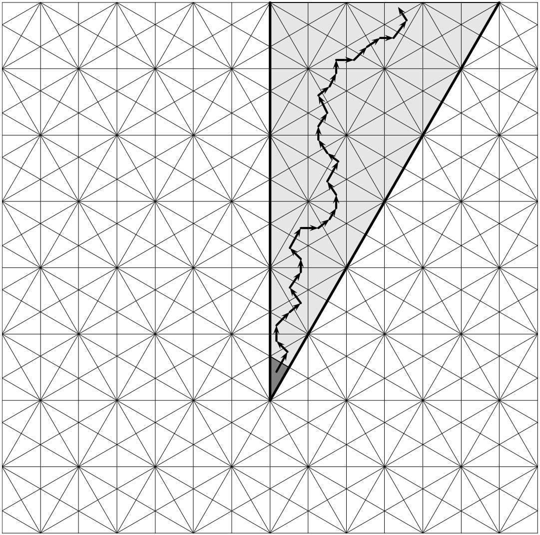

FIG.: Alcove walk of type G_2

Hamermesh graphs

Let χ_λ the character associated to $\lambda \vdash n$ for the symmetric group $\mathfrak{S}_n$. We have

$$\chi_\lambda \times \chi_{(n-1,1)} = (l_\lambda - 1)\chi_\lambda + \sum_{\mu \neq \lambda} \chi_\mu$$

where μ is obtained by moving one box of λ and l_λ is the number of distinct parts of λ.

The associated Hamermesh graph is PM with $\mathfrak{B} = \{\chi_\lambda \mid \lambda \dashv n\}$.

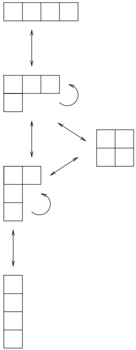

Fig. : Graphe de Hamermesh pour $\mathfrak{S}$

Affine orbit of a dominant weight

R is an affine root system of rang k and $W_a = \langle s_i, i = 0, \ldots, k \rangle$ its affine Weyl group.
For each weight $\lambda \in P$, let W^λ be the set of minimal length representatives in W_a / W_λ.

W^λ has a graph structure Γ_λ given by the weak Bruhat order on $W = \langle s_i, i = 1, \ldots, k \rangle$

$$\forall i \in \{1, \ldots, k\} \quad w \longrightarrow ws_i \text{ if } \ell(ws_i) = \ell(w) + 1.$$

We have

$$W_a = W \ltimes Q$$

where Q is the root lattice of R.

For each $\lambda \in P_+$, the graph $\Gamma_{\lambda,a}$ is obtained by adding to Γ_λ the edges

$$w \longrightarrow w' \text{ if } s_0 w = w' t_\beta \text{ with } \beta \in Q.$$

Theorem (GLP) : For all $\lambda \in P_+$ the graph $\Gamma_{\lambda,a}$ rooted in $w_0 W_\lambda$ is positively multiplicative.