APPROXIMATING SYSTEMS OF DIFFERENTIAL EQUATIONS WITH RANDOM INPUTS OR BOUNDARY CONDITIONS

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ABSTRACT

It is not very easy to get reasonable simulation schemes with high order of convergence for reflected stochastic differential equations, because of the intricate behavior of the diffusion near the boundary. But a simple Milstein-type scheme can be performed if diffusion and reflection act into separate directions; this is the case in many practical situations, some of which are presented.

1 INTRODUCTION

There has been in the last fifteen years a growing interest in drawing up numerical methods for solving stochastic differential equations. The big book by Kloeden and Platen [8] is devoted to them, other recent books ([2],[5],[16]) mention them while numerous publications (among them [6],[7],[12],[13]) have made significant contributions to the development of both strong and weak approximations. In the following study we shall restrict to mean square approximation.

Approximation and simulation of reflected stochastic differential equations have been treated by several authors ([10],[15]); in this case there are some difficulties to work out a practical Euler-type scheme, and it seems hopeless to achieve a higher order of magnitude. However, if diffusion and constraint work into separate directions, a convenient Milstein scheme may be proposed as we shall see.

Section 2 sets out a Skorohod-type equation which is a particular case of a generalized Skorohod-type equation displayed in a recent work by Cépa [3]. With a function having continuous derivative is associated a new function which reflects at the boundary of an interval and has discontinuous derivative at some points inside this interval. In Section 3 we introduce the stochastic differential equation we are interested in; existence and uniqueness are proved under the usual growth and Lipschitz conditions. In Section 4 a Milstein scheme approximating the solution is presented and the order of convergence turns out to be one as in the ordinary stochastic differential equations. Section 5 is devoted to applications. We first explain why the above equations may be fitted to model some evolutions in economics. Then we give two examples related to the dynamics of engineering systems; the first one deals with the travel of an airplane in turbulent system and is borrowed from Poirion and Angélini [14]; the second one explores the hysteretic response of structures when exposed to severe earthquake loading and is borrowed from Bouc [1], Krée [9], Suzuki and Minai [17].

2 A DETERMINISTIC REFLECTION PROCESS

If n is a nonnegative integer, we consider two finite sets of numbers

$$-\infty \leq \alpha = \gamma_0 < \gamma_1 < \cdots < \gamma_n < \gamma_{n+1} = \beta \leq +\infty$$

and

$$+\infty = e_0 > e_1 > \cdots > e_{n+1} > e_{n+2} = -\infty \ .$$

Let $T > 0$ and let $\dot{g}$ be a continuous function on $[0, T]$. For x a real number in $[\alpha, \beta]$, we put

$$g(t) = x + \int_0^t \dot{g}(s)\, ds \qquad 0 \leq t \leq T \ .$$

We consider a new function y on $[0, T]$ having $y(0) = x$ and derivative

$$\begin{aligned} \frac{dy}{dt}(t) &= \textstyle\sum_{k=0}^{n} (\dot{g}(t) + e_{k+1}) 1_{\{\gamma_k < y(t) < \gamma_{k+1}\}} \\ &\quad + \textstyle\sum_{k=0}^{n+1} [\max(0, \dot{g}(t) + e_{k+1}) + \min(0, \dot{g}(t) + e_k)] 1_{\{y(t) = \gamma_k\}} \ . \end{aligned}$$

Let $C^{(\alpha,\beta)}$ be the class of C^1-functions g on $[0,T]$ having $\alpha \leq g(0) \leq \beta$, and on this class let Γ be the mapping $g \mapsto y$. We define

$$\|g\|_T = \sup_{0 \leq t \leq T} |g(t)| \; .$$

Proposition 1 *The mapping Γ is Lipschitz continuous from $C^{(\alpha,\beta)}$ endowed with the uniform topology into $C([0,T])$ and*

$$\|\Gamma(g) - \Gamma(h)\|_T \leq 2\|g - h\|_T$$

for any g, h in $C^{(\alpha,\beta)}$.

Proof.- Write $y = \Gamma(g)$ and $z = \Gamma(h)$. Let t in $[0,T]$ and assume that $y(t) > z(t)$. Let

$$\begin{aligned} \tau &= \sup\{s \in [0,t] : y(s) = z(s)\} && \text{if } \{\;\} \neq \emptyset \\ &= 0 && \text{if } \{\;\} = \emptyset \; . \end{aligned}$$

For any s in $[\tau, t]$, we have $y(s) \geq z(s)$ and thus

$$\frac{dy}{dt}(s) - \frac{dz}{dt}(s) \leq \dot{g}(s) - \dot{h}(s) \; .$$

Therefore

$$y(t) - z(t) \leq \int_\tau^t (\dot{g}(s) - \dot{h}(s))ds \, + \, y(\tau) - z(\tau) \; .$$

If $\tau > 0$, we obtain

$$y(t) - z(t) \leq (g(t) - h(t)) - (g(\tau) - h(\tau))$$

and if $\tau = 0$, we obtain

$$y(t) - z(t) \leq g(t) - h(t) \; .$$

In both cases, $y(t) - z(t) \leq 2\|g - h\|_T$ and we are done. $\square$

It is worth noting that the mapping Γ is local in the following sense: if the restrictions of g and h to $[0,U] \subset [0,T]$ are identical, then the restrictions of $\Gamma(g)$ and $\Gamma(h)$ to $[0,U]$ are as well identical.

3 SOME REFLECTED STOCHASTIC DIFFERENTIAL EQUATIONS

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with a right-continuous filtration $(\mathcal{F}_t, 0 \le t \le T)$ and let $(W_t, 0 \le t \le T)$ be an adapted d-dimensional Wiener process. Let D be a closed set in some Euclidean space; a Borel function from $[0,T] \times D$ to $\mathbb{R}$ is said to satisfy a uniform growth condition on D if there exists a constant $C < \infty$ such that for all x in D and t in $[0,T]$ we have

$$|f(t,x)| \le C\,(1 + |x|)$$

and it is said to satisfy a uniform Lipschitz condition if for all x, y in D and t in $[0,T]$ we have

$$|f(t,x) - f(t,y)| \le C\,|x - y|\ .$$

Let $I = \{1, \ldots, m\}$, $J = \{m+1, \ldots, m+n\}$, $K = \{1, \ldots, d\}$. For any $i \in J$, let be given a nonnegative integer n_i and two finite sets of numbers

$$\begin{array}{l} -\infty \le \alpha^i = \gamma_0^i < \gamma_1^i < \cdots < \gamma_{n_i}^i < \gamma_{n_i+1}^i = \beta^i \le +\infty \\ +\infty = e_0^i > e_1^i > \cdots > e_{n_i+1}^i > e_{n_i+2}^i = -\infty\ . \end{array}$$

Let Γ^i be the mapping associated with these numbers as detailed in Section 2. We put

$$D = \mathbb{R}^m \times (\mathbb{R}^n \cap (\prod_{i \in J} [\alpha^i, \beta^i]))$$

and let $(b^i, i \in I \cup J)$ and $(\sigma_k^i, i \in I, k \in K)$ be continuous functions satisfying uniform growth and Lipschitz conditions on D. Let finally $\eta \in D$. We consider the following stochastic differential system on $[0,T]$

$$\begin{array}{lcll} X_t^i & = & \eta^i + \int_0^t b^i(s, X_s) ds + \sum_{k \in K} \int_0^t \sigma_k^i(s, X_s)\, dW_s^k & i \in I \\ X_t^i & = & \Gamma^i(\eta^i + \int_0^\cdot b^i(s, X_s) ds)(t) & i \in J \end{array} \qquad (1)$$

where $X = (X^1, \ldots, X^{m+n})$ is a D-valued continuous adapted process. If for i in J we define

$$\begin{array}{rcl} \tilde{b}^i(t,x) & = & \sum_{k=0}^{n_i} (b^i(t,x) + e_{k+1}^i) 1_{\{\gamma_k^i < x^i < \gamma_{k+1}^i\}} \\ & & + \sum_{k=0}^{n_i+1} [\max(0, b^i(t,x) + e_{k+1}^i) + \min(0, b^i(t,x) + e_k^i)] 1_{\{x^i = \gamma_k\}}\ , \end{array}$$

an alternative way to write the second set of equations in (1) is to set

$$X_t^i = \eta^i + \int_0^t \tilde{b}^i(s, X_s)ds \quad i \in J \ .$$

In this way the system (1) turns to be an ordinary stochastic differential system with discontinuous drift.

Theorem 1 *Under the above assumptions on η, $b = (b^i, i \in I \cup J)$, $\sigma = (\sigma_k^i, i \in I, k \in K)$, the system (1) has a pathwise unique strong solution and there exists $C_1 < \infty$ such that for all s, t in $[0, T]$,*

$$\mathbb{E}[|X_s - X_t|^2] \leq C_1 |s - t| \ . \tag{2}$$

Proof.- a) It is easily checked that any solution X satisfies

$$\mathbb{E}[\sup_{0 \leq t \leq T} |X_t|^2] < \infty \ . \tag{3}$$

In fact, this is obtained by stopping X when leaving a bounded domain in D, using the Doob inequality, the uniform growth condition on the coefficients, the Lipschitz property of the Γ^i's, the Gronwall lemma and the monotone convergence theorem which allows to relax the stopping procedure.

b) Let X be an adapted continuous process which satisfies (3). The continuous process Y defined by

$$\begin{array}{lcll} Y_t^i & = & \eta^i + \int_0^t b^i(s, X_s)ds + \sum_{k \in K} \int_0^t \sigma_k^i(s, X_s)\, dW_s^k & i \in I \\ Y_t^i & = & \Gamma^i(\eta^i + \int_0^{\cdot} b^i(s, X_s)ds)(t) & i \in J \end{array}$$

satisfies as well

$$\mathbb{E}[\sup_{0 \leq t \leq T} |Y_t|^2] < \infty \ .$$

The proof is standard for $i \in I$ and for $i \in J$ we use the estimate

$$\mathbb{E}[\sup_{0 \leq t \leq T} (Y_t^i)^2] \leq 2\|\Gamma^i(\eta^i)\|_T^2 + 8\mathbb{E}[(\int_0^T |b^i(s, X_s)|ds)^2] \ .$$

Considering now the norm

$$\|X\|^* = \sup_{0 \leq t \leq T} e^{-ct}(\mathbb{E}[\sup_{0 \leq s \leq t} |X_s|^2])^{1/2} \ ,$$

it is easily checked that for $c > 0$ large enough, the mapping $X \mapsto Y$ is contracting; this proves existence and uniqueness.

c) For i in I, it is well known ([2], Proposition 5.A.4.2 or [8], Theorem 3.8) that

$$\mathbb{E}[(X_t^i - X_s^i)^2] \le C_2\,|t-s|$$

for some $C_2 < \infty$. For i in J and $s < t$,

$$\begin{aligned}
\mathbb{E}[(X_t^i - X_s^i)^2] &\le 4\mathbb{E}[\sup_{0\le r\le t}(\textstyle\int_0^r b^i(u,X_u)du - \int_0^{r\wedge s} b^i(u,X_u)du)^2] \\
&\le 4(t-s)\mathbb{E}[\textstyle\int_s^t (b^i(u,X_u))^2 du] \\
&\le C_3\,(t-s)^2
\end{aligned}$$

for some $C_3 < \infty$. $\square$

4 A MILSTEIN SCHEME

Now we address the issue of approximating the solution of the system (1) through a simple numerical scheme allowing an easy Monte-Carlo simulation. It is well known that if b and σ satisfy uniform growth and Lipschitz conditions in the second variable and are Hölder continuous of order 1/2 in the first variable, the solution of the stochastic differential system

$$X_t = \eta + \int_0^t b(s,X_s)ds + \int_0^t \sigma(s,X_s)dW_s$$

can be approximated by

$$\bar{X}_t = \eta + \int_0^t b(\tau_s,\bar{X}_{\tau_s})ds + \int_0^t \sigma(\tau_s,\bar{X}_{\tau_s})dW_s\ ,$$

where N is a positive integer and

$$\tau_s = \frac{T}{N}\left[\frac{Ns}{T}\right]$$

(here $[x]$ denotes the integral part of x). This Euler-Maruyama scheme has order 1/2, that is

$$\|\sup_{0\le t\le T}|X_t - \bar{X}_t\|_2 \le C\,N^{-1/2}\ .$$

Things are more complicated in case of stochastic differential equations with normal reflection at the boundary of a convex closed set F. If p denotes the orthogonal projection onto F, a simple Euler scheme is given by

$$\bar{X}_t = p(\bar{X}_{\tau_t} + b(\tau_t, \bar{X}_{\tau_t})(t-\tau_t) + \sigma(\tau_t, \bar{X}_{\tau_t})(W_t - W_{\tau_t}))\ .$$

When F is a convex polyhedron, Slomiński [15] has proved that this scheme has order $1/2$-ε. This order can be brought up to $1/2$ by using the more sophisticated Euler-Peano scheme; unfortunately this last scheme is not easily implementable unless the polyhedron is a rectangular parallelepiped [10].

Reverting to ordinary stochastic differential equations we can improve the order up to 1 with the so-called Milstein scheme [11] which involves some Ito-Taylor development of $b(t, X_t)$ and $\sigma(t, X_t)$. We shall see below that with the special system (1) in Section 3 can also be associated a Milstein scheme of order 1 that is easily implementable.

Henceforth X denotes the solution of system (1) in Section 3. We consider the Milstein scheme

$$\begin{array}{rcll} \bar{X}^i_t & = & \eta^i + \int_0^t b^i(\tau_s, \bar{X}_{\tau_s})ds + \sum_{k\in K} \int_0^t \sigma^i_k(\tau_s, \bar{X}_{\tau_s})dW^k_s & \\ & & + \sum_{\alpha\in I, k,l\in K} \int_0^t \sigma^\alpha_l(\tau_s, \bar{X}_{\tau_s}) \frac{\partial \sigma^i_k}{\partial x^\alpha}(\tau_s, \bar{X}_{\tau_s})(W^l_s - W^l_{\tau_s})dW^k_s & i\in I \\ \bar{X}^i_t & = & \Gamma^i(\eta^i + \int_0^\cdot b^i(\tau_s, \bar{X}_{\tau_s})ds)(t) & i \in J\ . \end{array}$$

It is well known that we need more regularity on the coefficients for using a Milstein scheme than an Euler one: see the set of conditions stated in Theorem 10.3.5 of [8]. We introduce the operators

$$\begin{array}{rcl} L^0 & = & \frac{\partial}{\partial t} + \sum_{\alpha\in I} b^\alpha \frac{\partial}{\partial x^\alpha} + \sum_{\alpha\in J} \tilde{b}^\alpha \frac{\partial}{\partial x^\alpha} + \frac{1}{2}\sum_{\alpha,\beta\in I, q\in K} \sigma^\alpha_q \sigma^\beta_q \frac{\partial^2}{\partial x^\alpha \partial x^\beta} \\ L^l & = & \sum_{\alpha\in I} \sigma^\alpha_l \frac{\partial}{\partial x^\alpha}\ ,\ \ l\in K\ , \end{array}$$

and state the following set of assumptions:

$$|b^i(t,x) - b^i(t,y)| \ \le\ C_1\,|x-y| \qquad (4)$$

$$|\sigma^j_k(t,x) - \sigma^j_k(t,y)| \ \le\ C_1\,|x-y| \qquad (5)$$

$$
\begin{aligned}
|L^l\sigma_k^j(t,x) - L^l\sigma_k^j(t,y)| &\leq C_1\,|x-y| && (6)\\
|b^i(t,x)| &\leq C_2\,(1+|x|) && (7)\\
|\sigma_k^j(t,x)| &\leq C_2\,(1+|x|) && (8)\\
|L^0 b^i(t,x) &\leq C_2\,(1+|x|) && (9)\\
|L^0\sigma_k^j(t,x)| &\leq C_2\,(1+|x|) && (10)\\
|L^l b^i(t,x)| &\leq C_2\,(1+|x|) && (11)\\
|L^l\sigma_k^j(s,x) - L^l\sigma_k^j(t,x)| &\leq C_3\,(1+|x|)|s-t|^{1/2} && (12)
\end{aligned}
$$

for all $s,t\in[0,T]$, $x,y\in D$, $i\in I\cup J$, $j\in I$, $k,l\in K$, where the constants C_1,C_2,C_3 do not depend on $(t,x,y,)$.

Theorem 2 *Under the assumptions (4) to (12), there exists $C<\infty$ such that*

$$
\|\sup_{0\leq t\leq T}|X_t-\bar{X}_t|\|_2 \leq C\,N^{-1}\,.
$$

Proof.- We put $\delta = T/N$. Every positive constant independent of N will be denoted by C. For $i\in I$, we have

$$
X_t^i - \bar{X}_t^i = \sum_{j=1}^{7} I_t^j
$$

where

$$
\begin{aligned}
I_t^1 &= \textstyle\int_0^t (b^i(s,X_s) - b^i(\tau_s,X_{\tau_s}))\,ds\\
I_t^2 &= \textstyle\int_0^t (b^i(\tau_s,X_{\tau_s}) - b^i(\tau_s,\bar{X}_{\tau_s}))\,ds\\
I_t^3 &= \textstyle\sum_{k\in K}\int_0^t[\sigma_k^i(s,X_s) - \sigma_k^i(\tau_s,X_{\tau_s}) - \sum_{l\in K}\int_{\tau_s}^s L^l\sigma_k^i(u,X_u)dW_u^l]\,dW_s^k\\
I_t^4 &= \textstyle\sum_{k\in K}\int_0^t(\sigma_k^i(\tau_s,X_{\tau_s}) - \sigma_k^i(\tau_s,\bar{X}_{\tau_s}))\,dW_s^k\\
I_t^5 &= \textstyle\sum_{k,l\in K}\int_0^t[\int_{\tau_s}^s(L^l\sigma_k^i(u,X_u) - L^l\sigma_k^i(u,X_{\tau_s}))dW_u^l]\,dW_s^k\\
I_t^6 &= \textstyle\sum_{k,l\in K}\int_0^t[\int_{\tau_s}^s(L^l\sigma_k^i(u,X_{\tau_s}) - L^l\sigma_k^i(\tau_s,X_{\tau_s}))dW_u^l]\,dW_s^k\\
I_t^7 &= \textstyle\sum_{k,l\in K}\int_0^t[\int_{\tau_s}^s(L^l\sigma_k^i(\tau_s,X_{\tau_s}) - L^l\sigma_k^i(\tau_s,\bar{X}_{\tau_s}))dW_u^l]\,dW_s^k\,.
\end{aligned}
$$

From the respective assumptions (4), (5), (6), and (12), we have the estimates

$$
\begin{aligned}
\mathbb{E}[\sup_{0\leq s\leq t}(I_s^j)^2] &\leq C\,\textstyle\int_0^t\mathbb{E}[\sup_{0\leq u\leq s}|X_u-\bar{X}_u|^2]\,ds \quad j=2,4,7,\\
\mathbb{E}[\sup_{0\leq t\leq T}(I_t^6)^2] &\leq C\,\delta^2\,.
\end{aligned}
$$

This last estimate is also obtained for I^5 by using again assumption (6) and estimate (2). To estimate I^1 and I^3 we need to apply the Ito formula to $b^i(s, X_s)$ and $\sigma^i_k(s, X_s)$ respectively. Assumption (10) yields easily

$$\mathbb{E}[\sup_{0 \le t \le T} (I^3_t)^2] \le C\,\delta^2 \,.$$

The ordinary integral in the Ito development of $b^i(s, X_s)$ is easily estimated from assumption (9); as for the stochastic integral we use assumption (11) and the technique described in Lemma 10.8.1 of [8]. Both estimates yield

$$\mathbb{E}[\sup_{0 \le t \le T} (I^1_t)^2] \le C\,\delta^2 \,.$$

As regards $i \in J$, the problem is reduced through Proposition 1 to the same estimations as for I^1 and I^2. An application of the Gronwall lemma allows to conclude. $\square$

Let us discuss for a while the practicality of the associated algorithm. It is well known that the simulation of each stochastic integral

$$\int_{r\delta}^{(r+1)\delta} (W^l_s - W^l_{r\delta})\, dW^k_s$$

is difficult for $k \neq l$: see [8], p.347 and [4]. Nevertheless, if the commutativity conditions

$$L^l \sigma^i_k = L^k \sigma^i_l \qquad i \in I, k, l \in K$$

hold, we only handle independent Gaussian variables.

The behavior of the components $\bar{X}^i, i \in J$ turns out to be quite simple. During a time interval $[r\delta, (r+1)\delta]$, if for instance $\gamma^i_k \le \bar{X}^i_{r\delta} < \gamma^i_{k+1}$ and $b^i(r\delta, \bar{X}_{r\delta}) + e_{k+1} > 0$, then $\bar{X}^i_t$ increases with successive velocities $b^i(r\delta, \bar{X}_{r\delta}) + e_{j+1}, j = k, k+1, \ldots$, until either time is over or this expression becomes negative, in which case it stops.

5 APPLICATIONS

We select three applications of the model in Section 3.

1. *Economic Control.*

   The price of economic goods is often governed by differential equations in which the coefficients depend on the prices of other goods which are submitted to random fluctuations. Some institution may want to prevent this price from leaving a fixed interval $[\alpha, \beta]$ and acts with appropriate means to modify the governing equations in order to slow down any undesirable evolution. Very often this intervention is gradual but stepwise as described in Section 2: some thresholds are chosen between α and β, and between two neighbouring levels the action has constant intensity.

2. *Aircraft Automatic Control* and *Wind Turbulence.*

   Aerospace structures are subjected to wind loads which are induced by the atmospheric turbulence. In [14], Poirion and Angélini have proposed to write the vertical turbulence equation in the form

   $$dx_t = -x_t\, dt + \sigma\, dw_t \qquad \sigma > 0\ .$$

   A simplified model for the velocity of a control surface of an airplane is given by the differential equation

   $$dy_t = \alpha\, x_t\, dt - \beta\, y_t\, dt \qquad \alpha, \beta > 0$$

   as long as $|y_t|$ is less than some threshold ξ; then it gets stuck until the dynamics allows the absolute value of the velocity to decrease. A stochastic simulation has been carried out in [14]; the result of Section 4 proves that the order of convergence is one.

3. *Hysteretic Response to Earthquake Excitation.*

   When exposed to stress, some structures may have first an elastic behavior and next a nonperfect plastic behavior when the stress becomes larger. Suppose that an earthquake generates an acceleration $-\dot{W}$ in some horizontal direction, where $\dot{W}$ is a Gaussian white noise. The motion of the structure is governed by the equation

$$\ddot{x} + 2h\dot{x} + c = \dot{W} \ ,$$

where $h > 0$ characterizes the structural damping and $-c$ is the restoring force caused by the deformation x. This restoring force has hysteretic properties and is often written ([1],[9],[17]) in the form

$$c = rx + (1-r)z$$

where r is the ratio of post-yielding stiffness to pre-yielding stiffness and z is a memory variable. The function z is subjected to the constraint $|z| \leq 1$ and has derivative $\dot{z}$ given by

$$\dot{z} = \begin{cases} \dot{x}1_{\{\dot{x}<0\}} & \text{if } z = 1 \\ \dot{x} & \text{if } -1 < z < 1 \\ \dot{x}1_{\{\dot{x}>0\}} & \text{if } z = -1 \ . \end{cases}$$

Writing $\dot{x} = y$, we get the following system

$$\begin{aligned} dx_t &= y_t\,dt \\ dy_t &= -(rx_t + 2hy_t + (1-r)z_t)\,dt + dW_t \\ dz_t &= y_t(1 - 1_{(\{z_t=1\}\cap\{y_t>0\})\cup(\{z_t=-1\}\cap\{y_t<0\})})\,dt \ . \end{aligned}$$

This system is similar to those in Section 3 and, since the diffusion coefficient is constant, the above Milstein scheme is in fact an Euler-type scheme with order one as proved in Section 4.

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