On logarithmic Sobolev inequalities on a bounded interval

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Introduction

With regard to the generators of diffusion semigroups, a nice tool to prove logarithmic Sobolev inequalities (LSI) with best or at least very good constants is the so-called curvature-dimension condition of Bakry and Emery [2]. Their proof relies on a very strong assumption: existence of a dense algebra in the domain of the generator A which is stable under the actions of A, of the associated semigroup and by composition with C^∞-functions. This is often unrealistic and in many cases some additional work has to be completed to carry out several integrations by parts. The following study is concerned with Kolmogorov semigroups associated with diffusion processes taking values in a bounded open interval: in some cases, every thing goes well.

1 Langevin equation on a bounded interval

Let $-\infty < a < b < \infty$ and $I = (a, b)$. Let ϕ be a real convex C^∞ function on I with the property

$$(H1): \quad \int_{a+} e^{2\phi} dx = \int^{b-} e^{2\phi} dx = \infty .$$

We set

$$\rho = C^{-1} e^{-2\phi}$$

where

$$C = \int_a^b e^{-2\phi} dx .$$

The probability measure μ with density ρ in I is called the *Boltzmann measure* associated with ϕ.

If B is a linear Brownian motion and X_0 is an independent random variable with values in I, the stochastic differential equation

$$X_t = X_0 + B_t - \int_0^t \phi'(X_s) ds = X_0 + B_t + \int_0^t \frac{\rho'}{2\rho}(X_s) ds \tag{1}$$

has a unique strong solution with values in I: using hypothesis $(H1)$, Feller's test for explosions ensures that the process (X_t) never hits the boundary. We write (X_t^x) for the solution with initial value $X_0 = x \in I$.

For any bounded Borel function on I and $t \geq 0$, we set

$$P_t f(x) = \mathbb{E}[f(X_t^x)] .$$

Cépa and Jacquot have proved in [3] that the measure μ is reversible with respect to the Markov kernel (P_t): for any bounded Borel functions f and g and $t \geq 0$,

$$\int_a^b P_t f(x)\, g(x)\, d\mu(x) = \int_a^b f(x)\, P_t g(x)\, d\mu(x) \; .$$

They also proved that for any $x \in I$, the distribution of (X_t^x) converges to μ as t goes to infinity in the variation metric with exponential speed. Let us finally note that hypoelliptic properties entail that for any bounded Borel function on I, $P_t f(x)$ is C^∞ on $(0, \infty) \times I$.

2 Kolmogorov semigroups

The kernels (P_t) extend to a Markov continuous semigroup of selfadjoint operators on $L^2(\mu)$, also denoted by (P_t). We define the infinitesimal generator A of (P_t) in $L^2(\mu)$

$$Af = L^2 - \lim_{t \to 0} \frac{P_t f - f}{t}$$

with domain $D(A)$ the subspace of $L^2(\mu)$ where such a limit exists. The operator A is negative selfadjoint and $D(A)$ is dense in $L^2(\mu)$. If S is the symmetric linear operator defined on $D(S) = C_c^\infty(I)$ by

$$Sf = \frac{1}{2\rho}(\rho f')' = \frac{1}{2} f'' + \frac{\rho'}{2\rho} f' \; ,$$

then A is a selfadjoint extension of S. If S^* is the adjoint of S,

$$D(A) \subset D(S^*) = \{ f \in L^2(\mu) \cap C^1(I) : f' \text{ is a.c. and } \frac{1}{2\rho}(\rho f')' \in L^2(\mu) \} \; .$$

It is not an easy task to completely know $D(A)$ and one has generally to content with subspaces of $D(A)$. Let $C_b^1(I)$ be the set of functions with continuous bounded derivative.

Lemma 1 *Let $f \in C_b^1(I) \cap C^2(I)$. Then,*

$$\frac{1}{2\rho}(\rho f')' \in L^2(\mu) \Longleftrightarrow f \in D(A)$$

and in this case

$$Af = \frac{1}{2\rho}(\rho f')' = \frac{1}{2} f'' + \frac{\rho'}{2\rho} f' \; .$$

Proof. Use Ito's formula as in the proof of Theorem 2.2.25 in [6]. □

We will make a systematic use of the integration by parts formula:

Lemma 2 *Assume $f \in C_b^1(I), g \in C_b^1(I) \cap C^2(I) \cap D(A)$. Then*

$$\int_a^b f\, Ag\, d\mu = -\frac{1}{2} \int_a^b f' g'\, d\mu \; .$$

Proof. $(H1)$ entails $\rho(a+) = \rho(a-) = 0$. An usual integration by parts gives the result.
$\square$

We will need the following result.

Lemma 3 *If $f \in C_b^1(I)$, then, for any $t \geq 0$, $P_t f \in C_b^1(I)$.*

Proof. Let x and $x+h$ be in I. Then,

$$\begin{array}{rcl} (X_t^{x+h} - X_t^x)^2 & = & h^2 - \int_0^t (X_s^{x+h} - X_s^x)(\phi'(X_s^{x+h}) - \phi'(X_s^x))ds \\ & \leq & h^2 \ , \end{array}$$

proving a.s. continuity of $x \mapsto X_t^x$. Moreover,

$$\begin{array}{rcl} \frac{X_t^{x+h} - X_t^x}{h} & = & 1 - \frac{1}{h}\int_0^t (\phi'(X_s^{x+h}) - \phi'(X_s^x))ds \\ & = & 1 - \frac{1}{h}\int_0^t \int_0^1 \frac{d}{d\lambda}[\phi'(X_s^x + \lambda(X_s^{x+h} - X_s^x))]d\lambda\, ds \\ & = & 1 - \int_0^t \frac{X_s^{x+h} - X_s^x}{h}[\int_0^1 \phi''(X_s^x + \lambda(X_s^{x+h} - X_s^x))d\lambda]ds \ . \end{array}$$

It follows that

$$\frac{X_t^{x+h} - X_t^x}{h} = \exp\{-\int_0^t [\int_0^1 \phi''(X_s^x + \lambda(X_s^{x+h} - X_s^x))d\lambda]ds\}$$

and a.s.

$$\lim_{h\to 0} \frac{X_t^{x+h} - X_t^x}{h} = \exp\{-\int_0^t \phi''(X_s^x)ds\} \ .$$

If $f \in C_b^1(I)$, analogously

$$\frac{f(X_t^{x+h}) - f(X_t^x)}{h} = \frac{X_t^{x+h} - X_t^x}{h}\int_0^1 f'(X_t^x + \lambda(X_t^{x+h} - X_t^x))d\lambda \ .$$

Taking expectations and using dominated convergence we get

$$\frac{P_t f(x+h) - P_t f(x)}{h} \overset{h\to 0}{\to} \mathbb{E}[f'(X_t^x)\ \exp\{-\int_0^t \phi''(X_s^x)ds\}] \ .$$

It is straight to prove that this limit is bounded and continuous. $\square$

3 Entropy and semigroups

The relative entropy of a probability P with respect to a probability Q is defined by

$$\begin{array}{rcll} I(P|Q) & = & \int f \ln f\, dQ & \text{if} \quad P = f.Q \\ & = & +\infty & \text{if} \quad P \text{ is not a.c.} \end{array}$$

We go back to measure μ and set for any positive function in $L^1(\mu)$

$$Ent(f) = \int_a^b f \ln f\, d\mu - \int_a^b f\, d\mu\ \ln \int_a^b f\, d\mu \ .$$

When f is the density with respect to μ of the initial variable X_0 in the Langevin equation (1), from the reversibility of μ we deduce that $P_t f$ is the density of the law of X_t. In this case we are interested in the asymptotic behavior of $I(\mathcal{L}(X_t)|\mu) = Ent(P_t f)$ as t goes to infinity and will prove exponential decay under additional hypotheses on the convex function ϕ.

Proposition 1 *Assume that $0 < \alpha \leq f = \gamma + k \leq \beta$, where α, β, γ are positive constants and $k \in C_c^\infty(I)$. Let $f_t = P_t f$ for any $t \geq 0$. Then*
a) for any $u \in C^2([\alpha, \beta])$,

$$\frac{d}{dt} u \circ f_t = u' \circ f_t \, A f_t \qquad \text{in } L^2(\mu)$$

and

$$\frac{d}{dt} \int_a^b u \circ f_t \, d\mu = -\frac{1}{2} \int_a^b u'' \circ f_t \, f_t'^2 \, d\mu$$

b)

$$Ent(f_t) = \frac{1}{2} \int_t^\infty \left(\int_a^b \frac{f_s'^2}{f_s} d\mu \right) ds \ .$$

Proof. a) Positivity of (P_t) entails that $\alpha \leq f_t \leq \beta$, hypoellipticity that $f_t \in C^\infty(I)$ and Lemma 3 that $f_t \in C_b^1(I)$. From Lemma 1 we deduce that

$$\begin{cases} u \circ f_t & \in & D(A) \\ A(u \circ f_t) & = & u' \circ f_t \, A f_t + \frac{1}{2} u'' \circ f_t \, f_t'^2 \end{cases}$$

There exists $M < \infty$ such that for $h > 0$ and $t \geq 0$

$$|u \circ f_{t+h} - u \circ f_t - u' \circ f_t (f_{t+h} - f_t)| \leq M (f_{t+h} - f_t)^2 \ .$$

We remark that

$$f_{t+h} - f_t = \int_0^h A P_v f_t \, dv = \int_0^h P_{v+t} A f \, dv$$

and as Af is bounded, $P_{v+t} A f$ is also bounded. So

$$\int_a^b \left(\frac{u \circ f_{t+h} - u \circ f_t - u' \circ f_t (f_{t+h} - f_t)}{h} \right)^2 d\mu \to 0 \quad \text{as } h \to 0$$

proving that

$$\frac{d}{dt} u \circ f_t = u' \circ f_t \, A f_t$$

in $L^2(\mu)$. As a consequence,

$$\begin{aligned} \frac{d}{dt} \int_a^b u \circ f_t \, d\mu &= \int_a^b u' \circ f_t \, A f_t \, d\mu \\ &= -\frac{1}{2} \int_a^b u'' \circ f_t \, f_t'^2 \, d\mu \end{aligned}$$

where the last equality comes from Lemma 2.
b) From Theorem 6.1 in [3], there exists $C > 0$ and $0 < \rho < 1$ such that for any $x \in I$ and $t \geq 0$,

$$|f_t(x) - \int f\, d\mu| \leq Ce^{-\rho t} .$$

It follows that

$$Ent(f_t) \to 0$$

as $t \to \infty$ and

$$Ent(f_t) = -\int_t^{\infty} \frac{d}{ds}[\int_a^b f_s \ln f_s\, d\mu] ds .$$

Applying a) with $u(x) = x \ln x$, we get

$$\frac{d}{ds}\int_a^b f_s \ln f_s\, d\mu = -\frac{1}{2}\int_a^b \frac{f_s'^2}{f_s} d\mu .$$

$\square$

We set for $t \geq 0$

$$H(t) = \frac{1}{2}\int_a^b \frac{f_t'^2}{f_t} d\mu = -\int_a^b \ln f_t\, Af_t\, d\mu .$$

Proposition 2 *Under the assumptions in Proposition 1, for any $t \geq 0$,*

$$\frac{dH(t)}{dt} = \int_a^b [-2\frac{(Af_t)^2}{f_t} + \frac{f_t'^2}{2f_t^2} Af_t] d\mu .$$

Proof. For $h > 0$,

$$\frac{1}{h}[H(t+h) - H(t)] = -\frac{1}{h}\int_a^b Af_t(\ln f_{t+h} - \ln f_t) d\mu - \frac{1}{h}\int_a^b \ln f_{t+h}(Af_{t+h} - Af_t) d\mu .$$

From Proposition 1 a), as $h \to 0$,

$$-\frac{1}{h}\int_a^b Af_t(\ln f_{t+h} - \ln f_t) d\mu \to -\int_a^b \frac{(Af_t)^2}{f_t} d\mu .$$

Since $Af \in C_c^{\infty}(I) \subset D(A)$,

$$\frac{1}{h}(Af_{t+h} - Af_t) = P_t(\frac{1}{h}(P_h Af - Af)) \overset{h \to 0}{\to} P_t AAf = AAP_t f = AAf_t$$

in $L^2(\mu)$ and from Lemma 1 we have

$$\begin{aligned} \ln f_t &\in D(A) \\ A(\ln f_t) &= \tfrac{Af_t}{f_t} - \tfrac{f_t'^2}{2f_t^2} . \end{aligned}$$

As A is symmetric,

$$\int_a^b \ln f_t AAf_t\, d\mu = \int_a^b A \ln f_t\, Af_t\, d\mu,$$

which concludes the proof. $\square$

4 Curvature-dimension inequality

We introduce a new assumption on function ϕ:

$$(H2): \quad \int_a^b \phi'^2 d\mu = \int_a^b (\frac{\rho'}{2\rho})^2 d\mu < \infty .$$

Then, for any $c \in I$ and $\varepsilon > 0$,

$$\int_c^{b-\varepsilon} \rho'' dx = \int_c^{b-\varepsilon} \frac{\rho'^2}{\rho} dx - 2\int_c^{b-\varepsilon} \phi'' \rho dx$$

converges as $\varepsilon \to 0$ to a limit (finite or $-\infty$). But from $(H1)$ and $(H2)$ $\rho'(b-)$ may only vanish, as well as $\rho'(a+)$.

In this section, we assume $(H1)$ and $(H2)$ hold true and f satisfies the conditions in Proposition 1. We fix $s \geq 0$ and write $g = f_s = P_s f$.

We introduce the "carré du champ" operator of [2]

$$\Gamma_2 g = \frac{1}{4} g''^2 + \frac{1}{2} \phi'' g'^2 .$$

From the hypotheses we know that

$$g'' = 2Ag - \frac{\rho'}{\rho} g' \in L^2(\mu) .$$

We are now in a position to follow the proof of 6.20 and 6.21 in [1].

Lemma 4 *Under the above set of assumptions,*

$$\lim_{\varepsilon\to 0} \int_{a+\varepsilon}^{b-\varepsilon} \frac{\Gamma_2 g}{g} d\mu = \int_a^b (\frac{(Ag)^2}{g} + \frac{3g'^3 g''}{4g^2} - \frac{g'^4}{2g^3}) d\mu .$$

Proof. Simple computations give

$$\frac{\Gamma_2 g}{g} = \frac{(Ag)^2}{g} - \frac{\rho'' g'^2}{4\rho g} - \frac{\rho' g' g''}{2\rho g} .$$

Integrating by parts, we get

$$\begin{array}{rcl} \lim_{\varepsilon\to 0} \int_{a+\varepsilon}^{b-\varepsilon} \frac{\rho'' g'^2}{\rho g} \rho\, dx & = & \lim_{\varepsilon\to 0} [\frac{\rho' g'^2}{g}]_{a+\varepsilon}^{b-\varepsilon} - \int_a^b \rho' (\frac{2g' g''}{g} - \frac{g'^3}{g^2}) d\mu \\ & = & -2\int_a^b \frac{\rho' g' g''}{\rho g} d\mu + \int_a^b \frac{\rho' g'^3}{\rho g^2} d\mu . \end{array}$$

Another integration by parts gives

$$-\frac{1}{4} \int_a^b \frac{\rho' g'^3}{\rho g^2} d\mu = \int_a^b (\frac{3g'^3 g''}{4g^2} - \frac{g'^4}{2g^3}) d\mu . \quad \square$$

Lemma 5 *Under the above set of assumptions,*

$$\int_a^b \frac{Ag\, g'^2}{2g^2} d\mu = -\frac{1}{2} \int_a^b \frac{g'^2 g''}{g^2} d\mu + \frac{1}{2} \int_a^b \frac{g'^4}{g^3} d\mu .$$

Proof. This is just an application of Lemma 2. □

We are now going to introduce an additional hypothesis that is the key to obtain sharp constants in logarithmic Sobolev inequalities and in the related rate of convergence to the invariant measure [2].

Definition 1 *We say that* ϕ *satisfies a* curvature-dimension inequality *with constants* $r > 0$ *and* $m > 1$ *if*

$$CD(r,m) \qquad 2\phi'^2 \leq (\phi'' - r)(m-1)$$

We refer to [1] for comments about this terminology and follow the same way of proof.

Proposition 3 *Assume* ϕ *satisfies* $(H1), (H2), CD(r,m)$ *and* f *satisfies the hypotheses of Proposition 1. Then with* $g = f_s$, $s \geq 0$,

$$\int_a^b \left(\frac{(Ag)^2}{g} - \frac{Ag\, g'^2}{4g^2}\right) d\mu \geq \frac{rm}{m-1} \int_a^b \frac{g'^2}{2g} d\mu$$

Proof. For any x and y in $\mathbb{R}$, we have from $CD(r,m)$

$$x^2 + 2y^2\phi'' \geq 2ry^2 + \frac{1}{m}(x - 2\phi' y)^2 .$$

Let β be a real number that will be defined later on. Applying the inequality to

$$x = \frac{(g'' + \beta \frac{g'}{g})^2}{2\sqrt{g}} \qquad y = \frac{g'}{2\sqrt{g}} ,$$

we get

$$\frac{\Gamma_2 g}{g} + \frac{\beta g'^2 g''}{2g^2} + \frac{\beta^2 g'^4}{4g^3} \geq \frac{rg'^2}{2g} + \frac{(Ag)^2}{mg} + \frac{\beta Ag\, g'^2}{mg^3} + \frac{\beta^2 g'^4}{4mg^3} .$$

Integrating with respect to μ and using Lemma 4 and Lemma 5, we obtain

$$\int_a^b [(1 - \frac{1}{m})\frac{(Ag)^2}{g} - (\frac{3}{2} + \beta + \frac{2\beta}{m})\frac{Ag\, g'^2}{2g^2} + (1 + 2\beta + \beta^2 - \frac{\beta^2}{m})\frac{g'^4}{4g^3}] d\mu \geq r \int_a^b \frac{g'^2}{2g} d\mu .$$

Taking $\beta = -\frac{2m+1}{2(m+2)}$, we have

$$1 + 2\beta + \beta^2 - \frac{\beta^2}{m} \leq 0$$

and the result follows. □

We go back to the entropy of Section 3.

Proposition 4 *Under the same hypotheses as in Proposition 3, for any* $t \geq 0$,

$$Ent(f_t) \leq \frac{m-1}{2rm} \int_a^b \frac{f_t'^2}{2f_t} d\mu \tag{2}$$

and

$$Ent(f_t) \leq Ent(f) e^{-\frac{2rm}{m-1}t} . \tag{3}$$

Proof. From Proposition 2 and Proposition 3

$$\begin{array}{rcl} \frac{dH(s)}{ds} & = & \int_a^b [-2\frac{(Af_s)^2}{f_s} + \frac{1}{2}\frac{f_s'^2}{f_s^2}Af_s]d\mu \\ & \leq & -\frac{2rm}{m-1}H(s) \ . \end{array}$$

For $0 \leq t \leq s < \infty$,

$$H(s) \leq H(t)e^{-\frac{2rm}{m-1}(s-t)}$$

and

$$\begin{array}{rcl} Ent(f_t) & = & \int_t^\infty H(s)ds \\ & \leq & \frac{m-1}{2rm}H(t) \ . \end{array}$$

As

$$\frac{d}{dt}Ent(f_t) = -H(t) \ ,$$

inequality (3) easily follows. □

Inequality (2) in Proposition 4 is a logarithmic Sobolev inequality. To extend this inequality to a larger set of functions we must prove a density property.

5 Density of $C_c^\infty(I)$ in $H^1(\mu)$

Let $H^1(\mu)$ be the Hilbert space of functions in $L^2(\mu)$ with derivative (in the sense of Schwartz distributions) in $L^2(\mu)$, endowed with the usual norm. The proof of the density of C_c^∞ in $H^1(\mu)$ follows the arguments of Theorem 7.2 in [4]. We proceed in two steps, refering to [4] for the first step. Let us denote $C^\infty(\bar{I})$ the set of C^∞ functions on I which are uniformly continuous together with all their derivatives.

Proposition 5 *If ϕ satisfies $(H1)$, then $C^\infty(\bar{I})$ is dense in $H^1(\mu)$.*

Proof. Hypothesis $(H1)$ entails that

$$-\phi'(a+) = \phi'(b-) = \infty$$

and as a consequence there exists $r \in (0, \frac{b-a}{2})$ such that $\rho' > 0$ on the interval $(a, a+r)$ and $\rho' < 0$ on $(b-r, b)$ with $\rho(a+) = \rho(b-) = 0$. This is enough to replace the function $|x|^\delta$ in [4] with ρ and we can check that all the steps of the proof in [4] are still valid. □

Proposition 6 *Assume $(H1)$ and $(H2)$. Then $C_c^\infty(I)$ is dense in $H^1(\mu)$.*

Proof. Let $u \in H^1(\mu)$. For $\eta > 0$, there exists $v \in C^\infty(\bar{I})$ such that

$$\|u - v\| < \frac{\eta}{2} \ .$$

Let $f \in C^\infty(0, \infty)$ with

$$\begin{array}{lll} 0 \leq f(t) \leq 1 & & \\ f(t) = 0 & \text{for} & t \leq \frac{1}{2} \\ f(t) = 1 & \text{for} & t \geq 1 \ . \end{array}$$

Let $c = \sup_t |f'(t)|$. For $h > 0$ and $x \in I$, set

$$\begin{array}{rcl} F_h(x) & = & f(\frac{\rho(x)}{h}) \\ v_h(x) & = & v(x)\, F_h(x) \ . \end{array}$$

Then

$$\begin{array}{lcl} v_h & \in & C_c^\infty(I) \\ F_h'(x) & \leq & c\frac{|\rho'(x)|}{h} \\ (v - v_h)' & = & v'(1 - F_h) - vF_h' \ . \end{array}$$

By dominated convergence, as $h \to 0$,

$$\begin{array}{lcl} \int_a^b (v - v_h)^2 d\mu & \to & 0 \\ \int_a^b v'^2 (1 - F_h)^2 d\mu & \to & 0 \\ \int_a^b v^2 F_h'^2 d\mu & = & \int_a^b v^2 f'^2 (\frac{\rho}{h}) \frac{\rho'^2}{h^2} d\mu \\ & \leq & c^2 \int_a^b v^2 \mathbf{1}_{\{\rho < h\}} (\frac{\rho'}{\rho})^2 d\mu \\ & \to & 0 \end{array}$$

from $(H2)$ and the boundedness of v on I. □

6 Logarithmic Sobolev inequalities and ergodicity

We say that a measure μ on I satisfies a *logarithmic Sobolev inequality* with constant $c > 0$ if for any $h \in H^1(\mu)$, $h \neq 0$,

$$Ent(h^2) \leq c \int_a^b h'^2 d\mu \ .$$

In Proposition 4, we obtained this inequality for $h = \sqrt{f_t}$ and $c = \frac{m-1}{rm}$.

Theorem 1 *Under hypotheses $(H1)$, $(H2)$ and $CD(r, m)$, the measure μ satisfies a logarithmic Sobolev inequality with constant $\frac{m-1}{rm}$.*

Proof. For $g \in C_c^\infty(I)$ and $c > 0$, we consider as in [6, p.41] $g_c = \sqrt{g^2 + c}$. The function $g_c^2 \, \|g_c\|_{L^2(\mu)}^{-2}$ satisfies the hypotheses on $f = f_0$ in Proposition 4, so that

$$\int_a^b g_c^2 \ln g_c^2 \, d\mu - \int_a^b g_c^2 \, d\mu \, \ln \int_a^b g_c^2 \, d\mu \leq \frac{m-1}{rm} \int_a^b g_c'^2 d\mu \ .$$

As $c \to 0$,

$$\begin{array}{lclcl} \int_a^b g_c'^2 d\mu & \to & \int_a^b g'^2 \mathbf{1}_{\{g \neq 0\}} d\mu & \leq & \int_a^b g'^2 d\mu \\ \int_a^b g^2 \ln g^2 d\mu & \leq & \liminf_c \int_a^b g_c^2 \ln g_c^2 d\mu & & \\ \int_a^b g_c^2 d\mu & \to & \int_a^b g^2 d\mu & & \end{array}$$

and finally

$$\int_a^b g^2 \ln g^2 \, d\mu - \int_a^b g^2 \, d\mu \, \ln \int_a^b g^2 \, d\mu \leq \frac{m-1}{rm} \int_a^b g'^2 d\mu \ .$$

Any f in $H^1(\mu)$ may be approximated by a sequence (f_n) in $C_c^\infty(I)$ which may be assumed to converge also μ-a.e., and

$$\begin{array}{rcl}\int_a^b f^2 \ln f^2 d\mu & \leq & \liminf_n \int_a^b f_n^2 \ln f_n^2 d\mu \\ & \leq & \frac{m-1}{rm} \liminf_n \int_a^b f_n'^2 d\mu + \lim_n \int_a^b f_n^2 d\mu \,\ln \int_a^b f_n^2 d\mu \\ & \leq & \frac{m-1}{rm} \int_a^b f'^2 d\mu + \int_a^b f^2 d\mu \,\ln \int_a^b f^2 d\mu \ . \qquad \square\end{array}$$

One of the most striking applications of logarithmic Sobolev inequalities is to the computation of the rate of convergence to the invariant measure for diffusion semigroups.

Theorem 2 *Assume $(H1)$, $(H2)$ and $CD(r,m)$. If the distribution of X_0 has density f with respect to μ and $Ent(f) < \infty$, then for any $t \geq 0$*

$$Ent(P_t f) \leq Ent(f) e^{-\frac{2rm}{m-1}t}$$

Proof. The result has been proved in Proposition 4 for $f = \gamma + k$, $0 < \alpha \leq f \leq \beta$, $k \in C_c^\infty(I)$, $\int_a^b f d\mu = 1$. We only sketch the five steps of the proof:
- a density $f = \gamma + k$, $0 < \alpha \leq f \leq \beta$, $k \in C_c(I)$ is uniformly approximated with functions of the previous type;
- a density $f \in C(I)$, $0 < \alpha \leq f \leq \beta$ is L^2-approximated with functions of the previous type;
- a density f, $0 < \alpha \leq f \leq \beta$, is L^2-approximated with functions of the previous type;
- a density f, $0 \leq f \leq \beta$ is approximated by below truncation with functions of the previous type;
- a nonnegative density is approximated by above truncation with functions of the previous type (see [6, p.55]).

At each step, the entropy inequality is proved to hold true for a larger class of density functions. $\square$

7 Examples

1. Let $I = (0, 2\pi)$ and $\phi(x) = -\gamma \ln \sin \frac{x}{2}$. Hypothesis $(H1)$ is satisfied if $\gamma \geq \frac{1}{2}$ and hypothesis $(H2)$ if $\gamma > \frac{1}{2}$. We can check

$$2\phi'^2 = (\phi'' - \frac{\gamma}{4})2\gamma$$

and ϕ satisfies $CD(\frac{\gamma}{4}, 1+2\gamma)$: in fact the inequality turns into an equality. For $\gamma > \frac{1}{2}$, we have from Theorem 1 and Theorem 2

$$\begin{array}{rcl} Ent(f^2) & \leq & \frac{8}{1+2\gamma} \int_a^b f'^2 d\mu \\ Ent(P_t f) & \leq & Ent(f) e^{-\frac{1+2\gamma}{4}t} \ . \end{array}$$

For $\gamma = \frac{1}{2}$ the first inequality applies as well by taking limits in both sides; inequality (3) in Proposition 4 is a consequence of inequality (2) and extends to each density function as in the proof of Theorem 2.

This result is optimal because the smallest positive eigenvalue of $-A$ is known to be $\frac{1+2\gamma}{8}$. The method is not so powerful as that of Mueller and Weissler [5] who proved the same result for any $\gamma > -\frac{1}{2}$ through purely analytic methods.

2. Now $I = (-1, +1)$, $\phi = -\alpha \ln(1-x) - \beta \ln(1+x)$.
Let $\gamma = \min(\alpha, \beta)$, $\delta = \max(\alpha, \beta)$. Hypothesis $(H1)$ corresponds to $\gamma \geq \frac{1}{2}$ and hypothesis $(H2)$ to $\gamma > \frac{1}{2}$. Then ϕ satisfies $CD(2\gamma, 1+2\delta)$ and for $\gamma > \frac{1}{2}$,

$$\begin{array}{rcl} Ent(f^2) & \leq & \frac{\delta}{\gamma(1+2\delta)} \int_a^b f'^2 d\mu \\ Ent(P_t f) & \leq & Ent(f) e^{-\frac{2\gamma(1+2\delta)}{\delta} t} . \end{array}$$

These results extend to $\gamma = \frac{1}{2}$ as in the previous example. If $\alpha = \beta$, the change of variables $y = \cos \frac{x}{2}$ makes it possible to deduce the above LSI from the previous one.

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