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Maxime LIGONNIÈRE

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THÈSE dirigée par

M. Vincent Bansaye	Professeur, CMAP, École Polytechnique
M. Marc Peigné	Professeur, Institut Denis Poisson, Université de Tours

RAPPORTEURS

M. Gerold Alsmeyer	Professeur, Universität Münster
M. Quansheng Liu	Professeur, LMBA, Université de Bretagne Sud

JURY

M. Pierre Gabriel (*Président du jury*)	Professeur, Institut Denis Poisson, Université de Tours
M. Gerold Alsmeyer	Professeur, Universität Münster
M. Vincent Bansaye	Professeur, CMAP, École Polytechnique
Mme Sara Brofferio	Professeure, LAMA, Université Paris-Est Créteil
M. Quansheng Liu	Professeur, LMBA, Université de Bretagne Sud
M. Marc Peigné	Professeur, Institut Denis Poisson, Université de Tours
Mme Charline Smadi	Chargée de recherche, INRAE et Institut Fourier, Grenoble
M. Vitali Wachtel	Professeur, Universität Bielefeld

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Sommaire

CHAPITRE I

Introduction

I.1 Processus de Galton-Watson : vers l'environnement aléatoire et une infinité de types

Dans cette thèse, nous étudions diverses extensions du processus de Galton-Watson. Avant de nous concentrer sur le cas de processus avec une infinité de types et un environnement aléatoire, nous commençons donc par un panorama de variantes moins complexes de processus de Galton-Watson. Nous tentons entre autres de présenter quelques questions émergeant naturellement lors de leur étude, les approches classiques pour les résoudre et les résultats principaux déjà obtenus sur ces modèles parents. On pourra trouver un aperçu plus complet de ces modèles et résultats dans [AN72a ; KV17 ; Mod71 ; Har68].
De nombreux progrès majeurs sur l'étude des processus de Galton-Watson utilisent de manière centrale des résultats importants issus d'autres domaines des probabilités, notamment l'étude des marches aléatoires réelles ou des produits de matrices aléatoires. Un des résultats principaux de cette thèse s'inscrit d'ailleurs dans le domaine des produits d'opérateurs aléatoires. Cette introduction historique est donc également l'occasion de présenter quelques résultats issus de ces théories et leur application à l'étude des processus de Galton-Watson. Nous choisissons dans cette introduction historique de les introduire au moment où ils se révèlent utiles pour l'étude de processus de Galton-Watson, bien qu'ils aient souvent été développés indépendamment et qu'ils soient loin de se réduire à de simples outils pour le branchement.

I.1.1 Processus en environnement constant

Cas monotype. Le processus dit de Galton-Watson (ou de Bienaymé-Galton-Watson) a été introduit séparément par Bienaymé [Bie45] et Watson [Gal74] respectivement en 1845 et 1875 dans l'objectif de mieux comprendre l'extinction des noms de familles chez les nobles. Plus exactement, les auteurs remarquent que de nombreuses familles nobles ou bourgeoises s'éteignent par manque de descendants et qu'une théorie tentante pour expliquer ce phénomène est l'idée d'une baisse de la fertilité et une hausse de l'abstinence des nobles, elles mêmes dues à une vie trop confortable et trop concentrée sur l'activité intellectuelle. Ils souhaitent désavouer cette théorie en illustrant et quantifiant le fait qu'à fertilité moyenne donnée, la simple variabilité du nombre d'enfants entre les couples et en particulier le fait qu'une certaine proportion de couples ne font aucun descendant (plus exactement aucun descendant masculin) peut amener à l'extinction de la lignée.
Pour cela, ils se fixent pour chaque entier k une "proportion" p_k de couples faisant k enfants masculins et calculent, en utilisant la série génératrice $\sum_{k\geqslant 0} p_k x^k$, la probabilité que la famille soit éteinte après n générations. Ils concluent de ce développement que si l'on a initialement un grand nombre de familles, il est tout à fait possible que la population totale se maintienne voire explose en temps long et qu'en même temps la plupart des noms de familles s'éteignent en un temps très court : la plupart des familles disparaissent et parmi celles qui survivent, certaines comptent un très grand nombre de représentants. Au delà de ce cas pratique, on observe dans de nombreuses autres situations de modélisation (notamment en biologie et en physique) des systèmes de particules où, par des mécanismes de reproduction et de division, chaque particule est récursivement remplacée par un nombre variable de nouvelles particules. Le processus de Galton-Watson

est un modèle très élémentaire permettant de représenter ce type de situations. On le définit de la manière suivante.

Définition 1. Soit L une variable aléatoire à valeurs entières et $(L^{k,n})_{k,n\in\mathbb{Z}_+}$ une famille de variables indépendantes de même loi que L et Z_0 un entier. Un processus (Z_n) satisfaisant la relation de récurrence

$$Z_{n+1} = \sum_{k=1}^{Z_n} L^{k,n} \tag{I.1}$$

est appelé processus de Galton-Watson.

Dans cette définition, Z_0 compte le nombre d'individus en vie au temps 0 et la variable $L^{k,n}$ compte le nombre d'enfants du k-ième individu de la génération n. Ainsi Z_n représente le nombre d'individus de la génération n (ou encore le nombre d'individus en vie au temps n). On appelle loi de reproduction du processus (Z_n) la loi de L (et parfois L elle même, par abus de langage).
Un tel processus aléatoire est clairement une chaîne de Markov à valeurs dans $\mathbb{Z}_+$, dont un état absorbant est 0 : si $Z_N = 0$ a un temps N donné alors $Z_n = 0$ pour tout $n \geqslant N$. L'événement $\bigcup_{n\geqslant 0}\{Z_n = 0\}$ correspond donc à l'événement où la suite (Z_n) est nulle à partir d'un certain rang. On appelle cet événement "extinction" et sa probabilité, notée q, est au coeur de l'étude du processus de Galton-Watson. On note $\phi(x) = \sum_{k=0}^{+\infty} \mathbb{P}[L = k]x^k$ la fonction génératrice de L et $m = \mathbb{E}[L] = \phi'(1)$, $\sigma^2 = \mathrm{Var}(L)$. Le comportement du processus (Z_n) est alors très bien connu. En regroupant divers résultats issus de la littérature des années 1960, on aboutit au

Théorème A ([Har68 ; HSV67 ; KNS66 ; KS66])**.** Considérons un processus de Galton-Watson (Z_n), dont la loi de reproduction L a pour moyenne m, pour variance (possiblement infinie) σ^2 et pour fonction génératrice ϕ. Alors

i) La probabilité d'extinction q est le plus petit point fixe dans $[0, 1]$ de ϕ.

ii) Si $m < 1$ alors $q = 1$. De plus il existe $c \in [0, +\infty)$ tel que $\lim_{n\to\infty} m^{-n}\mathbb{P}[Z_n \neq 0] = c$. Enfin, $c > 0$ si et seulement si $L\log(L)$ est intégrable.

iii) Si $m = 1$ et $\mathbb{P}[L = 1] < 1$ alors $q = 1$. De plus, si $\sigma^2 < +\infty$ alors $\lim_{n\to\infty} n\mathbb{P}[Z_n \neq 0] = \frac{2}{\sigma^2}$.

iv) Si $m > 1$ alors $q < 1$. De plus la quantité $W_n = m^{-n}Z_n$ converge presque surement vers une variable aléatoire $W \in [0, \infty)$. Si $L\log(L)$ est intégrable alors $\mathbb{P}[W > 0] = 1 - q > 0$.

Ce théorème justifie la séparation de 3 régimes en fonction de la valeur de m. Lorsque $m < 1$, le processus est sous-critique : l'extinction est presque sûre et le temps d'extinction a une queue de distribution exponentielle. Lorsque $m = 1$, on est dans le régime critique : en dehors du cas trivial où chaque individu a exactement un enfant (dans lequel il n'y a alors pas d'aléatoire), la population s'éteint presque sûrement, mais à vitesse polynomiale. Dans les régimes sous-critique et critique, il existe également des résultats, connus sous le nom de loi limite de Yaglom, caractérisant la limite de la loi de la taille de la population au temps n, conditionnellement à ce qu'elle ne soit pas encore éteinte. Lorsque $m > 1$, la population a une probabilité positive de survivre et même de subir une croissance dite malthusienne, à vitesse m^n. Ces résultats ont historiquement été obtenus à l'aide de techniques analytiques impliquant les fonctions génératrices de (Z_n) et de L. Néanmoins, des approches probabilistes peuvent partiellement permettre d'illustrer autrement ces régimes. En particulier, la moyenne de notre processus est $\mathbb{E}[Z_n] = Z_0 m^n$. Le régime sous-critique correspond au cas où $\lim_{n\to\infty} \mathbb{E}[Z_n] = 0$, dans lequel la méthode du premier moment permet de retrouver l'extinction presque sûre. Dans le régime critique, la quantité $\mathbb{E}[Z_n]$ est bornée, donc par le lemme de Fatou $\liminf Z_n < +\infty$ et presque sûrement, il existe un entier K tel que $Z_n \in \{0, \dots K\}$ infiniment souvent. En supposant en plus que $\mathbb{P}(L = 1) < 1$, on a $\mathbb{P}[L = 0] > 0$ et des arguments de chaînes de Markov permettent de conclure à l'extinction presque sûre du processus. Le régime surcritique est à son tour caractérisé par la propriété $\lim_{n\to+\infty} \mathbb{E}[Z_n] = +\infty$. Le fait que $q < 1$ ne se déduit cependant pas de manière immédiate à partir de cette propriété.

Processus à d types. Le Théorème A procure une description relativement précise du processus de Galton-Watson. Diverses variantes de ce processus ont ensuite été définies dans le but de relâcher le caractère indépendant et identiquement distribué des descendances des individus. Une première façon d'affaiblir cette hypothèse est d'introduire une notion de type. On peut dans un premier temps se restreindre au cas où le nombre de types possible est fini et égal à un entier d. Pour tout vecteur $x \in \mathbb{R}^d$, on note $x = (x(1), \dots, x(d))$ et $\|x\| = |x(1)| + \dots + |x(d)|$. On peut alors représenter une population à d types par un vecteur $z = (z(1), ..., z(d)) \in \mathbb{Z}_+^d$, où $z(i)$ représente le nombre d'individus de type i dans la population et $\|z\|$ représente la taille de la population. Si e_i désigne le i-ème vecteur de la base canonique, la population $z = e_i$ est donc composée d'un seul individu qui est de type i. Enfin, si x, y sont deux vecteurs de $\mathbb{R}^d$, $\langle x, y\rangle = \sum_{i=1}^d x_i y_i$ désigne leur produit scalaire. Cette notation intervient lorsque x représente une population, et y un vecteurs de poids : $\langle x, y\rangle$ représente une notion de masse de la population, obtenue en donnant aux individus de type i un poids $y(i)$. Choisissons maintenant d variables aléatoires $L_1, \dots, L_d$ à valeurs dans $\mathbb{Z}_+^d$. La descendance d'un individu de type i sera distribuée comme la variable L_i. On peut écrire $L_i = (L_i(1), \dots, L_i(d))$ où $L_i(j)$ désigne le nombre d'individus de type j dans la descendance d'un individu de type i.

Définition 2. Considérons un entier d et d variables aléatoires $L_1, \dots L_d$ à valeurs dans $\mathbb{Z}_+^d$. Soit $\left(L_i^{k,n}\right)_{\substack{1\leqslant d\\ k,n\geqslant 0}}$ une famille de variables aléatoires indépendantes, telles que pour tout $k, n \in \mathbb{Z}_+$, pour tout $1 \leqslant i \leqslant d$ on ait $L_i^{k,n} \stackrel{d}{=} L_i$. Soit $Z_0 \in \mathbb{Z}_+^d$. On appelle processus de Galton-Watson à d-types un processus aléatoire à valeurs dans $\mathbb{Z}_+^d$ vérifiant la relation de récurrence

$$Z_{n+1} = \sum_{i=1}^{d} \sum_{k=1}^{Z_n(i)} L_i^{k,n}.$$

Remarquons tout d'abord que dans le cas où $d = 1$, on retrouve la Définition 1 du processus du Galton-Watson dit monotype (*single type* en anglais). L'étude des processus de Galton-Watson à d-types se fait de manière similaire à celle à 1-type. En particulier, la notion de moyenne de la loi de reproduction joue encore un rôle majeur. Cette fois, la moyenne est une matrice positive $M = (\mathbb{E}[L_i(j)])_{1\leqslant i,j\leqslant d} = (\mathbb{E}[Z_1(j)|Z_0 = e_i])_{1\leqslant i,jd}$. La moyenne de la population au temps n est cette fois donnée par $\mathbb{E}[Z_n|Z_0] = Z_0 M^n$, où Z_n est considéré comme un vecteur ligne. La compréhension de (Z_n) repose alors sur celle du comportement de la suite des puissances de M, pour laquelle on a recours au théorème de Perron-Frobenius. Sous l'hypothèse que M est positive régulière, c'est-à-dire que M^{n_0} a tous ses coefficients non nuls à partir d'un certain rang n_0, on sait que le rayon spectral λ de M est une valeur propre associée à des vecteurs propres L, R (respectivement à gauche et à droite) à coordonnées strictement positives. La matrice M vérifie la propriété d'ergodicité géométrique

$$M^n = \lambda RL(1 + \underset{n\to\infty}{o}(\rho^n))$$

pour un nombre $\rho \in [0,1)$. Le comportement asymptotique de la moyenne $\mathbb{E}[Z_n]$ sera donc déterminé par le triplet d'éléments propres (λ, L, R) et en particulier par la position par rapport à 1 du rayon spectral de M. On note $\mathbf{q} = (\mathbf{q}(1), \ldots, \mathbf{q}(d))$ le vecteur des probabilités d'extinction, définies par $\mathbf{q}(i) = \mathbb{P}\left[\bigcup_{n\geqslant 0}\{Z_n = 0\}|Z_0 = e_i\right]$. Si x, y sont deux vecteurs de $\mathbb{R}^d$, on note $x < y$ lorsque $x(i) < y(i)$ pour tout i. On dispose alors du théorème suivant :

Théorème B ([Har68])**.** Soit (Z_n) un processus de Galton-Watson à d-types. On suppose que la matrice moyenne M est positive régulière et on note λ son rayon spectral.

- (Régime sous-critique) Si $\lambda < 1$, $\mathbf{q} = 1$.
- (Régime critique) Si $\lambda = 1$ et $\mathbb{P}[\|L_i\| = 1] < 1$ pour au moins un type $i \in \{1, \ldots, d\}$ alors $\mathbf{q} = 1$.
- (Régime surcritique) Si $\lambda > 1$ alors $\mathbf{q} < 1$.

En plus de ces résultats sur l'extinction, on dispose par ailleurs de divers résultats similaires à ceux du Galton-Watson à 1-type, énoncés de façon détaillée dans [Har68 ; AN72b ; Mod71]. En particulier, dans les régimes sous-critique et critique, on dispose d'asymptotiques sur le temps d'extinction et d'un résultat de type Yaglom sur la convergence en loi de la population conditionnellement à la survie [Jiř57 ; Mul63]. Nous ne détaillons pas ces résultats ici, mais nous concentrons plutôt sur le théorème dit de Kesten-Stigum [KS66]. Ce théorème étudie l'événement de survie dans le régime surcritique. Il établit que presque sûrement, sur cet événement, d'une part la taille de la population croît géométriquement à taux λ (ce qu'on appelle croissance malthusienne de la population) et d'autre part le vecteur donnant les fréquences de chaque type dans la population converge vers un vecteur propre de la matrice moyenne.

Théorème C (Kesten-Stigum, [KS66])**.** Soit (Z_n) un processus de Galton-Watson à d-types, de matrice moyenne M positive régulière. Si λ, L, R sont les éléments propres de Perron-Frobenius de M, normalisés de sorte que $\langle L, R\rangle = 1$, il existe une variable aléatoire $W \in [0, \infty)$ telle que

$$\lambda^{-n} Z_n \underset{n\to\infty}{\longrightarrow} W\langle Z_0, R\rangle L \quad \mathbb{P}\text{-a.s.}$$

La preuve initiale de ce théorème repose sur des arguments de troncature. Une preuve alternative de ce théorème, utilisant des arguments de changement de mesures et une construction spinale, dont les idées ont été réinvesties dans de nombreux contextes, figure

dans [KLPP97 ; Ath00].
Le cas où la matrice moyenne n'est pas positive régulière est plus complexe et est traité dans [Sev51 ; Mod71], ou plus succintement dans [Har68, Section II.10].

Extension à une infinité de types. L'hypothèse de la finitude du nombre de types n'est pas cruciale pour la définition des processus de Galton-Watson multitypes, on peut en effet les définir avec un ensemble de type de cardinalité quelconque, comme cela est fait dans [Mod71]. On choisit alors de représenter l'ensemble des types possibles par un espace mesurable $(\mathbb{X}, \mathcal{X})$ et l'ensemble des populations possibles par l'ensemble $\mathcal{N} = \{\sum_{i=1}^{p} \delta_{x_i}, p \in \mathbb{Z}_+, x_1, \dots x_p \in \mathbb{X}\}$. Pour chaque type $x \in \mathbb{X}$, on choisit une loi de probabilité $\mathcal{L}_x$ à valeurs dans $\mathcal{N}$. On munit $\mathcal{N}$ de la tribu engendrée par les applications de la forme $\mu \in \mathcal{N} \mapsto \mu(A) \in \mathbb{Z}_+$, pour $A \in \mathcal{X}$. On peut alors définir un processus de Galton-Watson multitype (dit "général").

Définition 3. Soit $(\mathcal{L}_x)_{x\in\mathbb{X}}$ une famille de lois de probabilités sur $\mathcal{N}$, $\left(N_x^{k,n}\right)_{k,n\in\mathbb{Z}_+,x\in\mathbb{X}}$ une famille de variables indépendantes telles que $N_x^{k,n} \underset{n\to\infty}{\sim} \mathcal{L}_x$ pour tous x, k, n et $Z_0 \in \mathcal{N}$. On appelle processus de Galton-Watson multitype général le processus (Z_n) à valeurs dans $\mathcal{N}$ satisfaisant la relation de récurrence

$$Z_{n+1} = \sum_{x\in\mathbb{X}} \sum_{i=1}^{Z_n(x)} N_x^{k,n}. \tag{I.2}$$

Avec cette définition, la population au temps n est représentée par la mesure aléatoire Z_n. Pour tout ensemble mesurable $A \subset \mathbb{X}$, la quantité $Z_n(A)$ compte le nombre d'individus dont le type est dans A. Pour toute fonction f mesurable sur $\mathbb{X}$, la quantité $Z_n(f)$ est elle une notion de masse (potentiellement algébrique si f n'est pas positive) de la population, où chaque individu de type x est compté avec une masse $f(x)$. En particulier, en notant $\mathbb{1}$ la fonction constante en 1 sur $\mathbb{X}$, la quantité $Z_n(\mathbb{1}) = Z_n(\mathbb{X})$ compte la taille de la population au temps n. Le nombre d'individus de type x dans la population au temps n correspond à la masse de l'atome $\{x\}$, que l'on note $Z_n(x) = Z_n(\{x\})$ avec un léger abus de notation. Une population contenant un individu de type x est notée δ_x et l'événement $\{Z_n = 0\}$ décrit la situation où il n'y a aucun individu en vie au temps n.
En particulier, puisque les lois $\mathcal{L}_x$ sont à valeurs dans $\mathcal{N}$, l'ensemble des mesures ponctuelles avec un nombre fini d'atomes, ensemble stable par sommes finies, la propriété $Z_n \in \mathcal{N}$ est bien héréditaire : si $Z_n \in \mathcal{N}$, $Z_n(x)$ est nul sauf pour un nombre fini de valeurs de x, la somme (I.2) n'a qu'un nombre fini de termes non nuls et Z_{n+1} est bien à valeurs dans $\mathcal{N}$.
Dans le cas où $\mathbb{X}$ est fini ou dénombrable, une mesure ponctuelle μ sur $\mathbb{X}$ est entièrement déterminée par ses atomes. On peut donc représenter Z_n par le vecteur $(Z_n(x))_{x\in\mathbb{X}}$. Si $\mathbb{X}$ est fini, en notant $\mathbb{X} = \{x_1, \dots, x_d\}$, ce vecteur est précisément $(Z_n(x_1), \dots Z_n(x_d)) \in (\mathbb{Z}_+)^d$, c'est donc ce vecteur qui satisfait la relation de récurrence énoncée à la Définition 2 dans le paragraphe précédent. Lorsque l'ensemble des types est dénombrable, on peut, comme dans le cas ou $\mathbb{X}$ est fini, représenter la moyenne du processus par la matrice infinie $M = (\mathbb{E}[Z_1(y)|Z_0 = \delta_y])_{x,y\in\mathbb{X}}$. La n-ième puissance de cette matrice correspondra toujours à la moyenne du processus au temps n au sens où $\mathbb{E}[Z_n(y)|Z_0 = \delta_x]$ sera le coefficient de coordonnées (x, y) dans la matrice M^n.

Dans le cas où $\mathbb{X}$ n'est plus nécessairement dénombrable, on peut faire agir les mesures de $\mathcal{N}$ sur des ensembles de fonctions tests. Ainsi, pour toute fonction mesurable positive $f : \mathbb{X} \mapsto \mathbb{R}_+$, on peut calculer $Z_1(f)$ et poser $Mf(x) = \mathbb{E}[Z_1(f)|Z_0 = \delta_x]$. Sous des hypothèses d'intégrabilité, cela permet d'étendre la définition de M en un opérateur sur des espaces de fonctions et de mesures. Par exemple, si $\sup_{x\in\mathbb{X}} \mathbb{E}[Z_1(\mathbb{1})|Z_0 = \delta_x] < +\infty$, on étend M en un opérateur linéaire sur l'ensemble $\mathcal{B}(\mathbb{X})$ des fonctions mesurables bornées sur $\mathbb{X}$, puis par dualité en un opérateur linéaire sur l'ensemble $\mathcal{M}(\mathbb{X})$ des mesures signées sur $\mathbb{X}$ en posant $(\mu M)(f) = \mu(Mf)$. La encore, on obtient la relation $\mathbb{E}[Z_n(f)|Z_0] = Z_0 M^n(f)$. On a ainsi étendu la notion de moyenne d'un processus de Galton-Watson au cas où l'ensemble des types n'est pas nécessairement fini ; la moyenne $\mathbb{E}[Z_n]$ est reliée à la matrice ou l'opérateur M^n. En dimension infinie, différentes approches permettent d'obtenir des résultats de type Perron-Frobenius qui permettent de comprendre le comportement asymptotique des puissances de cet opérateur :

- Si un opérateur positif M agit de manière compacte sur un espace de Banach, le théorème de Krein-Rutman permet d'obtenir l'existence d'un vecteur propre associé à son rayon spectral et une propriété d'ergodicité.

- Si l'opérateur M, ou une de ses puissances, est uniformément positive, au sens où il existe un nombre $K > 0$ et, pour chaque x et f, des nombres $\lambda(f)$ et $h(x)$, tels que $K^{-1} \leqslant \frac{M^n(f)(x)}{\lambda(f)h(x)} \leqslant K$, on peut appliquer des techniques de contractions projectives liées à une distance dite de Hilbert. Cette distance est présentée dans notre Chapitre II et son lien avec l'obtention d'éléments propres et de résultats d'ergodicité est développé dans [Bir57]. Dans le cas dénombrable, l'hypothèse d'uniforme positivité implique entre autres que si un coefficient de la matrice M^n est nul alors toute la ligne ou toute la colonne associée est nulle.

- Dans le cas ou l'ensemble $\mathbb{X}$ est dénombrable, M peut être représenté par une matrice infinie. Il existe une théorie de Perron-Frobenius des matrices de dimension dénombrable, présentée par exemple dans [Sen06] et que nous résumons à la fin de la sous-section III.4.5.

- Si la mesure $\delta_x M$ charge une certaine région de l'espace $\mathbb{X}$ avec suffisamment de masse uniformément en x alors des techniques de contraction de type Doeblin-Harris-Lyapunov peuvent être utilisées. Ces techniques ont initialement été introduites pour obtenir des propriétés d'ergodicité géométrique sur des chaînes de Markov dont le noyau de transition $P(x, dy)$ vérifie des minorations de la forme $P(x, dy) \geqslant c\nu(dy)$, pour une certaine mesure ν et une constante c ne dépendant pas de x. Ces techniques permettent donc originellement d'obtenir des résulats d'ergodicitié pour des opérateurs conservatifs, c'est-à-dire qui vérifient $P\mathbb{1} = \mathbb{1}$. Dans [DM02 ; CV16 ; BCG20 ; BCGM22], des arguments de transformation de Doob sont utilisés pour construire, à partir d'opérateurs positifs non conservatifs, un semi-groupe d'opérateurs conservatifs et une chaîne de Markov associée, (Y_n) à valeurs dans $\mathbb{X}$; lorsque M est la matrice moyenne d'une loi de reproduction, cette chaîne (Y_n) s'interprète comme la suite des types des individus dans la lignée ancestrale d'un individu "typique". En s'appuyant sur des minorations de type Doeblin-Harris ou des variantes "moins uniformes" associées à des fonctions de Lyapunov, les auteurs proposent alors des conditions sur M pour que le noyau de transition de la chaîne de Markov (Y_n) soit géométriquement ergodique et en déduisent des propriétés d'ergodicité sur l'opérateur non conservatif M.

Malheureusement, ces approches sont diverses et ne permettent pas d'obtenir un théorème de Perron-Frobenius tout à fait général, ni des résultats généraux sur les processus de Galton-Watson associés, mais plutôt des résultats sous diverses hypothèses plus ou moins restrictives. Par exemple, dans [Har68, Chap.10], l'auteur fait le choix de travailler sous des hypothèses d'uniforme positivité et démontre l'existence d'un rayon spectral λ ainsi qu'une propriété d'ergodicité géométrique pour M. Avec quelques hypothèses complémentaires, il en déduit l'extinction presque sûre du processus quand $\lambda \leqslant 1$ et sa survie ainsi qu'un théorème de Kesten-Stigum quand $\lambda > 1$. Un autre théorème de type Kesten-Stigum est démontré dans [AH76] en supposant une hypothèse d'ergodicité géométrique assez forte de la moyenne.
Dans le cas d'un ensemble de types dénombrable, on peut également mentionner des résultats de [Moy67] qui établissent un théorème de type Kesten-Stigum sous des hypothèses $\mathcal{L}^2$, ainsi que divers articles étudiant des cas particulier, comme [Sag12] qui se concentre sur le cas où les lois de reproductions des individus sont linéaires fractionnaires. Cet exemple de loi de reproductions est couramment utilisé dans la littérature des processus de Galton-Watson (y compris en environnement aléatoire) car il présente l'avantage que de nombreuses quantités se calculent explicitement par des opérations sur les fonctions génératrices. On peut voir d'autres exemples de son utilisation dans [Als21 ; LPD18 ; Har68 ; Pol74]. Enfin, le cas du nombre de type dénombrable a été utilisé pour illustrer un nouvel effet majeur par rapport aux processus à d-types : la différence entre *extinction locale* et *extinction globale*. On appelle *extinction locale* l'événement où chaque type s'éteint à partir d'un certain temps, ce qui correspond à l'événement $\bigcap_{y\in\mathbb{X}}\bigcup_{N\in\mathbb{Z}_+}\bigcap_{n\geqslant N}\{Z_n(y)=0\}$. L'événement d'extinction $\bigcup_{n\geqslant 0}\{Z_n=0\}$, ou plus exactement d'*extinction globale* est donc inclus dans l'extinction locale. Si l'ensemble des types est fini, ces deux événements sont égaux. Dans le cas où le nombre de types est infini, ce n'est plus vrai en général : une population peut survivre en se déplaçant de manière transiente dans l'espace des types ! Divers articles, notamment [Zuc11 ; HLN13 ; BH19 ; BDH19] comparent de manière plus détaillée extinction locale et globale et construisent des exemples de processus pour lesquels ces deux événements d'extinction ne coïncident pas. Ils mettent en particulier en évidence le fait qu'en notant λ la valeur propre donnée par la théorie de Perron-Frobenius des matrices infinies, la condition $\lambda \leqslant 1$ garantit que l'extinction locale est presque sûre mais ne garantit pas qu'il en va de même pour l'extinction globale. Ils proposent par ailleurs une condition nécessaire et suffisante pour que l'extinction globale soit presque sûre sur la classe des processus de Galton-Watson dits "lower Hessenberg". Trouver une telle condition générale pour les processus de Galton-Watson avec un nombre dénombrable de types est une question encore ouverte à notre connaissance.

I.1.2 Processus monotype en environnement aléatoire

Une autre façon de relâcher l'hypothèse d'indépendance et identique distribution des lois de reproduction des individus est d'ajouter une notion d'environnement changeant, qui permet d'introduire une inhomogénéité temporelle dans le processus. Présentons d'abord cette notion dans le cadre des processus monotypes.

On fixe un espace mesurable $(\mathcal{E}, \mathscr{E})$ représentant les environnements possibles et pour chaque élément $e \in \mathcal{E}$, une loi de reproduction $\mathcal{L}_e$ sur $\mathbb{Z}_+$, qui représente la loi du nombre de descendants d'un individu si il se reproduit dans l'environnement e.

On peut alors fixer une suite d'environnements $(\xi_n)_{n\geqslant 0}$ (à ce stade déterministe) de sorte que ξ_n représente l'état de l'environnement au temps n : c'est dans cet environnement et donc avec la loi $\mathcal{L}_{\xi_n}$, que se reproduiront les individus de la génération n.

On définit alors le processus de Galton Watson en environnement variable (GWVE) par

$$Z_{n+1} = \sum_{i=1}^{Z_n} L_{\xi_n}^{k,n},$$

où les variables $\left(L_e^{k,n}\right)_{e\in\mathcal{E},k,n\in\mathbb{Z}_+}$ sont des variables indépendantes telles que $L_e^{k,n} \underset{n\to\infty}{\sim} \mathcal{L}_e$. On note dans la suite $m(e)$ la moyenne de la loi $\mathcal{L}(e)$ et $m_k = m(\xi_k)$

Ces processus ont été étudiés par exemple dans [Jag74]. En particulier, on y voit que le comportement asymptotique de la suite

$$m_{0,n} = \mathbb{E}[Z_n] = Z_0 m_0 \dots m_{n-1}$$

joue un rôle déterminant pour établir des résultats similaires à ceux mentionnés dans le cas de l'environnement constant. Plus précisément, Jagers établit d'une part l'extinction presque sûre du processus sous l'hypothèse $\liminf_{n\to\infty} m_{0,n} = 0$ par un argument de premier moment. D'autre part, lorsque $\lim_{n\to\infty} m_{0,n} > 0$, sous des hypothèses sur le second moment des lois de reproduction, il établit la survie du processus et son explosion à la même vitesse que $m_{0,n}$.
Les méthodes d'étude de la quantité $m_{0,n}$ dépendent largement des hypothèses que l'on souhaite imposer sur la suite (ξ_n). Divers exemples sont étudiés dans la littérature, notamment celui des suites périodiques. Dans cette thèse, nous nous concentrerons uniquement sur le cas où ces environnements sont choisis aléatoirement, de manière indépendante des variables $(L_e^{k,n})$.

Définition 4. Soit $\bar{\xi} = (\xi_n)$ un processus aléatoire à valeurs dans $\mathcal{E}$ et $(L_e^{k,n})$ une famille de variables indépendantes et indépendantes de $\bar{\xi}$, de sorte que $L_e^{k,n} \underset{n\to\infty}{\sim} \mathcal{L}_e$. On fixe $Z_0 \in \mathbb{Z}_+$. Le processus (Z_n) défini par la relation de récurrence

$$Z_{n+1} = \sum_{i=1}^{Z_n} L_e^{k,n}$$

est appelé processus de Galton-Watson en environnement aléatoire et noté GWRE.
On parle d'un processus de Galton-Watson en environnement i.i.d.(resp. stationnaire ergodique) si la suite $\bar{\xi}$ est i.i.d.(resp. stationnaire ergodique).

Lorsqu'on travaille en environnement aléatoire, deux couches d'aléas se superposent : le choix aléatoire d'une suite d'environnements (qu'on appelle aléa environnemental) et la reproduction aléatoire des individus une fois la suite d'environnement fixée (qu'on appelle aléa démographique). On appelle alors loi *annealed* la loi jointe des variables $(L_e^{k,n})$ et de $\bar{\xi}$, cette loi est notée $\mathbb{P}$. On appelle loi "quenched" du processus la loi de (Z_n) conditionnellement à la suite environnementale $\bar{\xi}$, que l'on note $\mathbb{P}_{\bar{\xi}}$. Remarquons que la loi *quenched* de

(Z_n) est donc la loi d'un GWVE, dans la réalisation choisie de la suite environnementale $\bar{\xi}$.
Pour cette raison, la notion habituelle de probabilité d'extinction $q = \mathbb{P}\left[\bigcup_{n\geqslant 0}\{Z_n = 0\}\right]$ peut être qualifiée de probabilité d'extinction *annealed*. Elle est à distinguer de la probabilité d'extinction *quenched*, définie comme la variable aléatoire $\bar{\xi}$-mesurable

$$q_{\bar{\xi}} = \mathbb{P}\left[\bigcup_{n\geqslant 0}\{Z_n = 0\}\middle|\,\bar{\xi}\right].$$

Comme on peut s'y attendre, on a $q = \mathbb{E}\left[q_{\bar{\xi}}\right]$. En particulier, si $q = 1$ alors $q_{\bar{\xi}} = 1$, $\mathbb{P}(d\bar{\xi})$-p.s., c'est-à-dire que le processus s'éteint presque sûrement dans presque toute suite d'environnement. Il n'est cependant pas évident que la condition $q < 1$ entraine $q_{\bar{\xi}} < 1$ dans presque toute suite d'environnement.
On peut trouver dans le livre [KV17] une description complète et récente des résultats connus sur ces processus. Nous nous contentons ici de rappeler quelques résultats fondamentaux.
La première étude des GWRE, dans le cas d'un environnement i.i.d, est faite dans [SW69], elle est suivie par une étude en environnement stationnaire ergodique qui figure dans [AK71b ; AK71a]. Dans ce cas, les auteurs énoncent une caractérisation complète de l'extinction et définissent donc des régimes. De plus, ils proposent des estimations asymptotiques à la fois sur la probabilité *quenched* de survivre jusqu'au temps n dans le régime sous-critique et sur la taille de la population dans le régime surcritique.

Théorème D ([AK71b])**.** Soit (Z_n) un GWRE. On suppose ici que $\bar{\xi}$ est une suite de variables iid. On suppose que $E\left[|\log m_0|\right] < \infty$. Alors :

- (Régime surcritique) Si $\mathbb{E}\left[\log m(\xi_0)\right] > 0$ alors $q_{\bar{\xi}} < 1$ $\mathbb{P}(d\bar{\xi})$-p.s. De plus, il existe une variable aléatoire W à valeurs dans $\mathbb{R}^+$ telle que $\mathbb{P}$-p.s.

$$W_n = \frac{Z_n}{m_{0,n}} \underset{n\to+\infty}{\longrightarrow} W.$$

  Si de plus $\mathbb{E}\left[\frac{\mathbb{E}\left[L^{\xi_0}\log L^{\xi_0}|\xi\right]}{m(\xi_0)}\right] < \infty$ alors $W > 0$ presque sûrement sur l'événement de survie.

- (Régime critique) Si $\mathbb{E}\left[\log m(\xi_0)\right] = 0$ alors $q = 1$.

- (Régime sous-critique) Si $\mathbb{E}\left[\log m(\xi_0)\right] < 0$ alors $q = 1$. Si de plus $\mathbb{E}_\mu\left[\frac{L^\xi \log L^\xi}{m(\xi)}\right] < \infty$ alors

$$\frac{\mathbb{P}\left[Z_n \neq 0|\,\bar{\xi}\right]}{m_{0,n}} \longrightarrow a(\bar{\xi}) \in (0,1] \quad \mathbb{P}(d\bar{\xi})\text{- p.s.}$$

On remarque que ce critère d'extinction repose sur le signe de $\mathbb{E}[\log(m(\xi_0))]$ où $m(\xi_0)$ correspond à la moyenne *quenched* du nombre d'individus en vie au temps 1 lorsque $Z_0 = 1$. Cela s'explique de la manière suivante. Conditionnellement au processus environnemental, (Z_n) est un GWVE dont le critère d'extinction est alors lié à la quantité

$m_{0,n}$. Or $m_{0,n}$ est ici une variable aléatoire $\bar{\xi}$-mesurable, qui s'exprime comme

$$m_{0,n} = \mathbb{E}[Z_n|(\xi_n)] = Z_0 m_0 \dots m_{n-1} = \exp\left(\log(Z_0) + \sum_{i=0}^{n-1} \log(m_i)\right).$$

Dans cas où la suite (ξ_n) est stationnaire ergodique, le théorème ergodique de Birkhoff permet donc d'obtenir une loi des grands nombres de la forme

$$\lim_{n\to\infty} m_{0,n}^{\frac{1}{n}} = \exp\left(\mathbb{E}\left[\log m(\xi_0)\right]\right) \quad \mathbb{P}(d\bar{\xi})\text{-p.s.}$$

Si la suite $\bar{\xi}$ est i.i.d, $\log(m_{0,n})$ est une marche aléatoire sur $\mathbb{R}$ dont la dérive est donnée par $\mathbb{E}\left[\log m(\xi_0)\right]$. Les régimes sous-critique et critique introduits dans le théorème précédent correspondent alors exactement au cas où $\liminf_{n\to\infty} m_{0,n} = 0$; c'est une conséquence du critère d'extinction de Jagers pour les Galton-Watson en environnement varié. On obtient même dans le régime sous-critique une estimation de la probabilité d'extinction quenched du processus.

Quelques propriétés nouvelles liées à l'environnement aléatoire apparaissent lorsqu'on s'intéresse au comportement annealed du processus. La moyenne *annealed* du processus (Z_n) est donnée par $\mathbb{E}[Z_n] = \mathbb{E}[m_{0,n}]$. Dans le cas où $\bar{\xi}$ est i.i.d, on a $\mathbb{E}[Z_n] = \mathbb{E}[m(\xi_0)]^n$. Remarquons que par convexité, on a $\exp(\mathbb{E}[\log(m(\xi_0))]) \leqslant \mathbb{E}[m(\xi)]$, qui n'est en général pas une égalité. On peut ainsi trouver des GWRE pour lesquels on aura à la fois $\mathbb{E}[\log(m(\xi_0))] < 0$ et $\mathbb{E}[m(\xi_0)] > 1$. Dans ce cas, la moyenne de la taille de la population au temps n tend vers l'infini à vitesse géométrique mais la population finit par s'éteindre. Ce type de phénomène est caractéristique de l'environnement aléatoire : en environnement fixe, la moyenne de la taille de la population tend vers l'infini dans le régime surcritique et la population peut survivre ; en régime sous-critique, cette moyenne tend vers 0 et la population s'éteint. Ce découplage entre comportement en moyenne et comportement presque sûr peut être observé dans un système très simple à deux environnements : un environnement dit standard, très probable (disons de probabilité $p < 1$), dans lequel la loi de reproduction a une moyenne $m_+ > 1$, et un environnement exceptionnel, dans lequel tous les individus meurent (ou en tout cas la moyenne de la loi de reproduction est $m_- < 1$). Le taux de croissance moyen de la population est $\mathbb{E}[m(\xi_0)] = pm_+ + (1-p)m_- > pm_+$ et peut donc être choisi arbitrairement grand en prenant m_+ assez grand, indépendamment de m_-. Néanmoins si l'événement exceptionnel est suffisamment intense, au sens où m_- est nul ou suffisamment petit, il suffira à mener la population à l'extinction car $\mathbb{E}[\log m(\xi_0)] = p\log(m_+) + (1-p)\log(m_-)$ sera alors négatif.

Une autre spécificité majeure de l'environnement aléatoire réside dans le comportement asymptotique de la probabilité de survie *annealed* $\mathbb{P}[Z_n > 0]$ dans les régimes critique et sous-critique. En effet, l'ajout de l'aléatoire environnemental à l'aléa démographique pour construire le GWRE fait que la survie anormalement longue d'une population pourrait être due soit à une atypicité de l'aléa environnemental, soit à une atypicité de l'aléa démographique. Dans le premier cas, on observerait une suite environnementale anormalement favorable à la survie de la population. Dans le second cas, la suite environnementale tirée serait plutôt typique, mais conditionnellement à cette suite, les variables de reproductions tirées seraient anormalement grandes. Comprendre la queue de distribution du temps d'extinction requiert donc une comparaison fine des déviations de l'aléa environnemental et de l'aléa démographique. Cette étude a été menée à bien à la fin des années 1990, en utilisant des asymptotiques précises sur les marches aléatoires conditionnées, notamment issues de [KKS75]. Elle a mené à l'exhibition de plusieurs vitesses d'extinction qui n'existaient pas en environnement constant. Ces nouvelles vitesses correspondent à des processus pour lesquels la façon la plus probable pour la population de survivre pendant un temps atypiquement long correspond à la réalisation d'une suite d'environnements anormalement favorables, ou encore à une déviation vers le haut de la marche aléatoire $\log(m_{0,n})$. Nous ne détaillons pas ici les hypothèses d'intégrabilité sous lesquelles ces résultats ont été établis. On obtient par exemple, dans le régime critique, que la queue de distribution du temps d'extinction est proportionnelle à $n^{-1/2}$, ce qui diffère de la vitesse n^{-1} obtenue dans le régime critique d'un Galton-Watson en environnement constant. Cette asymptotique correspond précisément à la probabilité de tirer une suite environnementale suffisamment favorable pour entrainer la survie de la population, et le fait que la population survive anormalement longtemps est donc un événement imputable à l'aléa environnemental.

Théorème E ([Koz77 ; GK01 ; GL01])**.** Soit (Z_n) un GWRE i.i.d.critique, au sens où $\mathbb{E}[\log(m(\xi_0))] = 0$. Sous certaines conditions d'intégrabilité, il existe $\beta > 0$ tel que

$$\mathbb{P}\left[Z_n > 0\right] \underset{n\to\infty}{\sim} \frac{\beta}{\sqrt{n}} \tag{I.3}$$

Dans le régime sous-critique, différents auteurs ont contribué à faire émerger trois sous-régimes dans lesquels on observe des vitesses d'extinctions différentes. Dans le régime fortement sous-critique, on retrouve une asymptotique similaire à celle observée en environnement constant, mais les deux autres cas font apparaître des nouvelles vitesses : dans le régime faiblement sous-critique, la survie de la population est un événement environnemental ; dans le régime sous critique intermédiaire, cette survie nécessite une déviation simultanée de l'aléa environnemental et de l'aléa démographique.

Théorème F ([Liu93 ; GL01 ; DH97 ; Gei03])**.** Soit (Z_n) un processus de Galton-Watson en environnement i.i.d, sous-critique, au sens où $\mathbb{E}[\log(m(\xi_0))] < 0$. Sous des bonnes conditions d'intégrabilité, on a :

- (Régime fortement sous-critique) Si $\mathbb{E}[m(\xi_0)\log m(\xi_0)] < 0$ alors il existe $c > 0$ tel que
$$\mathbb{P}\left[Z_n > 0\right] \underset{n\to\infty}{\sim} c\left(\mathbb{E}[m(\xi_0)]\right)^n. \tag{I.4}$$
- (Régime sous-critique intermédiaire) Si $\mathbb{E}[m(\xi_0)\log m(\xi_0)] = 0$ alors il existe $c > 0$ tel que
$$\mathbb{P}\left[Z_n > 0\right] \underset{n\to\infty}{\sim} \frac{c}{\sqrt{n}}\left(\mathbb{E}\left[m(\xi_0)\right]\right)^n. \tag{I.5}$$
- (Régime faiblement sous-critique) Si $\mathbb{E}\left[m(\xi_0)\log m(\xi_0)\right] > 0$ alors il existe $c > 0$ et $0 < \gamma < \mathbb{E}\left[m(\xi_0)\right]$ tel que
$$\mathbb{P}\left[Z_n > 0\right] \underset{n\to\infty}{\sim} \frac{c}{n^{\frac{3}{2}}}\gamma^n. \tag{I.6}$$

On a jusqu'ici présenté des GWRE où l'environnement était supposé stationnaire ergodique voire i.i.d. Mentionnons qu'il existe néanmoins des articles établissant des résultats similaires dans le cas d'une suite d'environnements markovienne, par exemple [LY10 ; GLP17 ; Lau17].

I.1.3 Processus de Galton-Watson à d types en environnement aléatoire et produits de matrices

Les deux extensions du processus de Galton-Watson que sont la notion de types et l'environnement aléatoire peuvent être combinées pour définir un processus de Galton-Watson multitype en environnement aléatoire (MGWRE). L'essentiel de la littérature existante à ce sujet s'est concentrée sur le cas d'un nombre de type fini. Pour définir ce processus, on garde un espace mesurable $(\mathcal{E}, \mathscr{E})$ et un processus $(\xi_n)_{n\geqslant 0}$ à valeurs dans $\mathcal{E}$ décrivant les environnements successifs rencontré par le système. On choisit une famille de lois de probabilités $(\mathcal{L}_{i,e})_{\substack{1\leqslant i\leqslant d\\ e\in\mathcal{E}}}$ sur $\mathbb{Z}_+^d$ et une famille de variables indépendantes $\left(L_{i,e}^{k,n}\right)_{1\leqslant i\leqslant d, e\in\mathcal{E}, k,n\in\mathbb{Z}_+}$ de sorte que $L_{i,e}^{k,n} \underset{n\to\infty}{\sim} \mathcal{L}_{i,e}$.

Définition 5. Soit $Z_0 \in \mathbb{Z}_+^d$. On appelle processus de Galton-Watson à d-types en environnement aléatoire le processus $(Z_n)_{n\geqslant 0}$ sur $\mathbb{Z}_+^d$ défini par la relation

$$Z_{n+1} = \sum_{i=1}^{d} \sum_{k=1}^{Z_n(i)} L_{i,\xi_n}^{k,n}.$$

Dans ce cadre, on peut définir des probabilités d'extinction *quenched* et *annealed* notées respectivement $\mathbf{q}_{\bar{\xi}}$ et $\mathbf{q}$, qui sont encore des vecteurs. On a toujours $\mathbf{q}_{\bar{\xi}}(i) = \mathbb{P}\left[\bigcup_{n\geqslant 0}\{Z_n = 0\} \middle| \bar{\xi}, Z_0 = e_i\right]$ et $\mathbf{q} = \mathbb{E}[\mathbf{q}_{\bar{\xi}}]$. On introduit les matrices

$$M(e) = (\mathbb{E}[L_{i,e}(j)])_{1\leqslant i,j\leqslant d} = (\mathbb{E}[Z_1(j)|Z_0 = e_i, \xi_0 = e])_{1\leqslant i,j\leqslant d}\,,$$

et on note $M_k = M(\xi_k)$ et $M_{k,n} = M_k \dots M_{n-1}$. On peut alors montrer que la moyenne *quenched* du processus s'écrit :

$$\mathbb{E}\left[Z_n|\bar{\xi}\right] = Z_0 M_{0,n}.$$

Ainsi, on s'attend à ce que la compréhension de $(Z_n)_{n\geqslant 0}$ repose sur celle du processus $(M_{0,n})_{n\geqslant 0}$, qui n'est rien d'autre qu'une marche aléatoire sur l'ensemble des matrices positives. Au delà d'apparaître au coeur de l'étude des MGWRE, les marches aléatoires sur $\mathcal{M}_d(\mathbb{R})$ sont des objets intéressants à part entière et font l'objet d'une littérature fournie qui continue à se développer [GR85 ; AB17 ; ABI17 ; BPP21 ; ABP24 ; Pén24]. Une première étape majeure dans l'étude de ce types d'objets est l'article [FK60]. En considérant une suite de matrices positives aléatoires $(M_n)_{n\geqslant 0}$, qu'ils supposent stationnaire et mélangeante, et sous l'hypothèse $\mathbb{E}[\log^+\|M_0\|] < +\infty$, les auteurs introduisent l'exposant de Lyapunov

$$\lambda = \inf_{n\geqslant 1} n^{-1}\mathbb{E}\left[\log\|M_{0,n}\|\right] = \lim_{n\to\infty} n^{-1}\mathbb{E}\left[\log\|M_{0,n}\|\right]$$

et démontrent une loi des grands nombres sur la norme du produit $M_{0,n}$ sous la forme : $\lim_{n\to\infty} n^{-1}\log\|M_{0,n}\| = \lambda$. L'équivalence des normes en dimension finie garantit que ce résultat ne dépend pas de la norme choisie. Par ailleurs, à l'aide du lemme sous-additif de Kingman, on peut retrouver ce résultat en remplaçant l'hypothèse de mélange de la suite $(M_n)_{n\geqslant 0}$ par son ergodicité. Une loi des grands nombres similaire pour les coefficients $(M_{0,n}(i,j))_{1\leqslant i,j\leqslant d}$ du produit $M_{0,n}$ est également établie dans [FK60] sous la forme $\lim_{n\to\infty} n^{-1}\log M_{0,n}(i,j) = \lambda$, sous l'hypothèse forte qu'avec probabilité 1, tous les coefficients de la matrice M_0 sont dans un intervalle déterministe $[C,D]$, pour des constantes positives C, D. Ces résultats ont permis de définir des régimes pour le MGWRE à d-types : il est sous-critique lorsque $\lambda < 0$, critique si $\lambda = 0$ et surcritique quand $\lambda > 0$. La probabilité d'extinction dans ces différents régimes est alors étudiée dans [AK71b], complété par [Kap74] pour le régime critique :

Théorème G ([AK71b ; Kap74])**.** Soit (Z_n) un MGWRE à d-types en environnement i.i.d. On suppose qu'il existe des constantes $C, D > 0$ telles que pour tous i, j, k, on ait

$$C \leqslant M_0(i,j) = \mathbb{E}[Z_1(j)|Z_0 = e_i, \xi_0] \leqslant D \quad \mathbb{P}(d\bar{\xi})\text{-p.s.}$$

et

$$0 \leqslant \mathbb{E}[Z_1(j)Z_1(k)|Z_0 = e_i, \xi_0] \leqslant D \quad \mathbb{P}(d\bar{\xi})\text{-p.s.}$$

On suppose enfin que $\mathbb{E}\left[\left|\log\left(\sum_{i=1}^d \mathbb{P}[Z_1 \neq 0|Z_0 = e_i, \bar{\xi}]\right|\right)\right] < +\infty$. On a

- Si $\lambda \leqslant 0$ alors $\mathbf{q}_{\bar{\xi}} = 1$, $\mathbb{P}\left[d\bar{\xi}\right]$-p.s.
- Si $\lambda > 0$ alors $\mathbf{q}_{\bar{\xi}} < 1$ $\mathbb{P}\left[d\bar{\xi}\right]$-p.s.

Pour obtenir une étude asymptotique plus fine de taille et de la répartition des types dans la population dans le régime surcritique, il a fallu attendre l'obtention de résultats fins d'ergodicité sur les produits de matrices aléatoires, qui ont notamment été obtenus dans [Hen97]. Pour $M \in \mathcal{M}_d(\mathbb{R})$, notons $M > 0$ si M a tous ses coefficients strictement positifs. Dans [Hen97], Hennion travaille sous l'hypothèse d'irréductibilité

Hypothèse I.1. - Presque sûrement, M_0 n'a ni ligne ni colonne nulle
- Il existe N tel que $\mathbb{P}[M_{0,N} > 0] > 0$,

Cette hypothèse permet d'une part de garantir qu'avec probabilité 1, pour tout k et pour n assez grand la matrice $M_{k,n}$ admet des vecteurs propres strictement positifs à gauche et à droite $L_{k,n}, R_{k,n}$ associés au rayon spectral $\rho_{k,n}$. Dans la suite, on considère à la fois des vecteurs lignes et des vecteurs colonnes de $\mathbb{R}^d$. En particulier, les vecteurs propres à gauche $(L_{k,n})_{k,n}$ sont des vecteurs lignes et les vecteurs propres à droite $(R_{k,n})_{k,n}$ sont représentés comme des vecteurs colonnes. Les vecteurs lignes sont munis de la norme $\|x\|_1 = \sum_{i=1}^d |x(i)|$ et les vecteurs colonnes de la norme $\|\cdot\|_\infty$. Si $x = (x_1, \dots, x_d)$ est un vecteur ligne et $y = (y_1, \dots, y_d)$ est un vecteur colonne, $\langle x, y\rangle = xy = \sum_{i=1}^d x_i y_i$ est donc le crochet de dualité. Les matrices de $\mathcal{M}_d(\mathbb{R})$ sont elles équipées de la norme d'opérateur associée à ces structures d'espaces vectoriels normés sur $\mathbb{R}^d$:

$$|||M||| = \sup_{x, |x|_\infty = 1} \|Mx\|_\infty = \sup_{1\leqslant i\leqslant d} \sum_{j=1}^d \|M(i,j)\| = \sup_{x, \|x\|_1 = 1} \|xM\|_1.$$

On adopte la convention de renormalisation $\|R_{k,n}\|_\infty = 1$ et $\langle L_{k,n}, R_{k,n}\rangle = 1$. D'autre part, l'hypothèse I.1 garantit qu'avec probabilité 1, pour n assez grand, les actions projectives, respectivement à gauche et à droite, du produit $M_{k,n}$ sur les simplexes $\{x \in \mathbb{R}_+^d | \|x\|_1 = 1\}$ et $\{x \in \mathbb{R}_+^d, \|x\|_\infty = 1\}$ sont contractantes pour la distance de Hilbert précédemment mentionnée. Cette contraction permet d'obtenir la convergence de la suite $R_{k,n}$ vers un vecteur aléatoire h_k. De plus, elle permet de montrer que la suite (h_k) forme une "fonction harmonique temps espace", au sens de [BCN99], c'est-à-dire que $M_k h_{k+1} = \lambda_k h_{k+1}$, pour une suite aléatoire de réels positifs (λ_k). Ces fonctions harmoniques temps espace généralisent la notion de fonction harmonique à un cadre inhomogène en temps. Cela permet à Hennion d'obtenir une forme d'ergodicité du produit $(M_{k,n})$ qui se formule de la manière suivante :

Théorème H ([Hen97])**.** Considérons une suite de matrices positives $(M_n)_{n\geqslant 0}$, qu'on suppose stationnaire ergodique et qui satisfait l'hypothèse I.1. On a alors

i) $\left|r_{0,n}^{-1} M_{0,n}(i,j) - L_{0,n}(j) R_{0,n}(i)\right| = O(\rho^n)$ pour un certain $\rho < 1$ déterministe.

ii) Pour tout k, la suite $(R_{k,n})_{n\geqslant k}$ converge presque sûrement vers une limite $h_k > 0$ telle que $\|h_k\|_\infty = 1$.

iii) Avec probabilité 1, les réels $\lambda_n = \|M_n h_{n+1}\|_\infty$ vérifient la relation
$$M_n h_{n+1} = \lambda_n h_n.$$

iv) Pour tout k, la suite $(L_{k,n}/\|L_{k,n}\|_1)_{n\geqslant k}$ converge en loi vers une variable aléatoire π à valeurs dans le simplexe $\{x \in \mathbb{R}_+^d | \|x\|_1 = 1\}$.

v) Si $\mathbb{E}[\log^+ \|\!|M_0|\!\|] < +\infty$, alors on a
$$\lim_{n\to\infty} n^{-1}\log(\|\!|M_{0,n}|\!\|) = \lim_{n\to\infty} n^{-1}\log(r_{0,n}) = \lim_{n\to\infty} n^{-1}\log(M_{0,n}(i,j)) = \lambda \quad \mathbb{P}\text{-p.s.},$$
où $\lambda = \inf_{n\geqslant 1} n^{-1}\mathbb{E}\left[\log(\|\!|M_{0,n}|\!\|)\right]$ est l'exposant de Lyapunov de la suite $(M_{0,n})$.

Lorsque l'on applique ces résultats au semi-groupe de la moyenne *quenched* d'un MGWRE, la propriété d'ergodicité *i)* implique que dans presque toute réalisation de la suite environnementale, au temps n, la moyenne (prise sur l'aléa démographique) d'une population descendant d'un individu de type i au temps 0 est équivalente à $r_{0,n}h_0(i)L_{0,n}$. On remarque que d'une part la taille de cette population est asymptotiquement d'ordre $r_{0,n}L_{0,n}h_0(i)$: la dépendance de cette taille de population en la population initiale est contenue dans le terme $h_0(i)$, elle est donc séparée de sa dépendance en temps. De plus, par *iv)*, à l'échelle géométrique, cette taille de population est d'ordre $\exp(n\lambda)$: c'est donc le signe de λ qui gouverne le fait que cette taille explose ou tende vers 0. D'autre part, la fréquence des différents types dans la population moyenne est asymptotiquement décrite par le vecteur $L_n/\|L_n\|$, elle ne dépend pas de la population de départ et converge en distribution : en temps long, en moyenne *quenched*, la distribution des types suit une trajectoire ergodique.
En combinant *i)* et *ii)*, on montre que
$$\frac{h_k(i)}{h_k(j)} = \lim_{n\to+\infty} \frac{\|e_i M_{k,n}\|_1}{\|e_j M_{k,n}\|_1}.$$

En termes de modèle de populations, $h_k(i)$ quantifie donc la masse de population moyenne émanant d'un individu de type i en temps long, comparativement à celle venant des autres types. C'est ce qui est appelé la valeur reproductive d'une population, et qui apparaît souvent dans la littérature en écologie, voir par ex. [KC05, Chap.9.1] ou [BL15 ; Lio18]. Le prolongement naturel de ces résultats est de se demander s'ils ne constituent qu'une description en moyenne *quenched* de la population, ou si on observe bien, dans le régime surcritique, sur presque toute réalisation où la population survit, que $\|Z_n\|_1$ croît à vitesse $\exp(n\lambda)$ et que le vecteur donnant les fréquences des types $Z_n/\|Z_n\|_1$ s'approche de celui correspondant à la population moyenne *quenched* $L_n/\|L_n\|_1$. Cette question est abordée par [GLP23], sous la forme d'un théorème de type Kesten-Stigum. Pour établir ce résultat, les auteurs définissent les tribus

$$\mathcal{F}_N = \sigma\left(\bar{\xi}, \left(L_{x,\xi_n}^{k,n}\right)_{1 \leqslant x \leqslant d, k \in \mathbb{Z}_+, n \leqslant N}\right)$$

qui forment une filtration et les variables

$$W_n = \frac{\langle Z_n, h_n \rangle}{\lambda_{0,n} Z_0(h_0)},$$

où $\lambda_{0,n} = \|M_{0,n} h_{n+1}\|_\infty = \prod_{i=0}^{n-1} \lambda_i$. Le processus $(W_n)_{n\geqslant 0}$ généralise la martingale obtenue dans le cas monotype et son étude mène au

Théorème I ([GLP23])**.** Soit (Z_n) un MGWRE à d-types. On suppose que la suite des matrices moyennes de ce processus $(M_n)_{n\geqslant 0}$ vérifie l'hypothèse I.1, que $\mathbb{E}[\log^+ |||M_0|||] < \infty$ et que le processus est surcrtitique, i.e. $\lambda > 0$. Alors

i) le processus $(W_n)_{n\geqslant 0}$ est une martingale positive par rapport à la filtration $(\mathcal{F}_n)_{n\geqslant 0}$, qui converge presque sûrement vers une variable aléatoire $W \in [0, \infty)$. On a de plus $\mathbb{E}[W|\bar{\xi}] \leqslant 1$ $\mathbb{P}(d\bar{\xi})$-p.s.

ii) S'il existe une constante $C > 1$ telle que pour tout $1 \leqslant i \leqslant d$, on ait

$$\sum_{n\geqslant 0} \mathbb{E}\left[\frac{\langle Z_n, h_{n+1}\rangle}{\lambda_n h_n(i)} \mathbb{1}_{\langle Z_n, h_{n+1}\rangle \geqslant C^n} | Z_n = e_i, \bar{\xi}\right] < \infty \quad \mathbb{P}(d\bar{\xi})\text{-p.s.} \tag{I.7}$$

alors
$$\mathbb{P}[W = 0 | Z_0 = e_i, \bar{\xi}] = \mathbf{q}_{\bar{\xi}}(i) < 1, \mathbb{P}[d\bar{\xi}]\text{-p.s.}.$$

De plus, dans ce cas, sur l'événement de survie, on a presque sûrement

$$\lim n^{-1} \log \|Z_n\|_1 = \lambda.$$

iii) On suppose de plus que (ξ_n) est une suite i.i.d. et qu'il existe une constante D telle que

$$\frac{\max_{i,j} M_0(i,j)}{\min_{i,j} M_0(i,j)} \leqslant D \quad \mathbb{P}(d\bar{\xi})\text{-p.s.} \tag{I.8}$$

Alors,
$$\left\| \frac{Z_n}{\|Z_n\|} - \frac{L_{0,n}}{\|L_{0,n}\|} \right\| \xrightarrow[n\to\infty]{\mathbb{P}[\cdot|W>0]} 0. \tag{I.9}$$

Dans les cas précédents, la non dégénérescence de la martingale typique est obtenue sous une condition d'intégrabilité de type $L\log^+ L$ sur la loi de reproduction. C'est aussi le cas ici, puisque les auteurs montrent que si

$$\mathbb{E}\left[\frac{\mathbb{E}\left[Z_1(h_1)\log^+[Z_1(h_1)]|Z_0=\delta_x\right]}{M_0h_1(x)}\right] = \mathbb{E}\left[\frac{Z_1(h_1)\log^+[Z_1(h_1)]}{M_0h_1(x)}\middle| Z_0=\delta_x\right] < \infty \tag{I.10}$$

est vérifiée pour tout x alors (I.7) est vraie également. Par ailleurs, lorsque (I.8) est vérifiée et que l'environnement (ξ_n) est i.i.d, (I.7) et(I.10) sont équivalentes et sont nécessaires pour que la martingale (W_n) soit non dégénérée. Sous ces hypothèses, les auteurs établissent donc à la fois que sur l'événement de survie, la population croît exponentiellement vite à taux λ et que, toujours sur l'événement de survie, la fréquences des différents types dans la population rejoint la trajectoire donnée par la suite des vecteurs propres $\left(\frac{L_{0,n}}{\|L_{0,n}\|}\right)_{n\geqslant 0}$ qui ne dépendent que la suite environnementale $\bar{\xi}$. Remarquons que la répartition des types ne se stabilise pas vers une répartition constante par rapport au temps. Ce n'est pas étonnant, puisqu'elle n'a pas de raison de se moyenner et est très sensible aux derniers environnements rencontrés. Néanmoins, asymptotiquement, les fréquences des différents types ne dépendent ni de la population initiale Z_0, ni de l'aléa démographique. Sous les conditions citées ci-dessus, la convergence (I.9) a lieu en probabilité, conditionnellement à la survie de la population. Les auteurs proposent cependant des conditions de moments renforcées sous lesquelles cette convergence est presque sûre. D'autres résultats étendant notamment les convergences précédentes en $\mathcal{L}^p$ apparaissent dans [GLP22 ; GLN24a].
La question des temps d'extinction *annealed* pour des MGWRE a également été abordée récemment, et les premiers travaux sur ce sujet sont même antérieurs au Théorème I. Là encore, la compréhension a nécessité des théorèmes limites conditionnels sur la moyenne *quenched* de la population, qui est cette fois liée à des processus de la forme $S_n^x = \log \|xM_{0,n}\|_1$. Contrairement au cas du GWRE monotype, $(S_n^x)_{n\geqslant 0}$ n'est pas exactement une marche aléatoire, même lorsque l'environnement est i.i.d. En effet, cette quantité se décompose sous la forme

$$S_n^x = \sum_{k=0}^{n-1} \log\left(\|(x\cdot M_{0,k})M_k\|_1\right)$$

où $x \cdot M_{0,k} = \frac{xM_{0,k}}{\|xM_{0,k}\|_1}$ est le résultat de l'action projective de la matrice $M_{0,k}$ sur le vecteur x. Ainsi, le k-ième incrément dans la somme précédente ne dépend plus seulement de la matrice M_k, mais de toute les matrices précédentes à travers le terme $x \cdot M_{0,k}$. C'est donc suite a des progrès majeurs dans la compréhension des comportements limites de produits de matrices aléatoires conditionnés sur des événements de la forme $\{\inf_{k\leqslant n} S_k^x \geqslant a\}$, obtenus notamment dans [Cam18 ; GLL18 ; GLL20 ; PP24], en utilisant notamment des stratégies dessinées dans [DW15]. En utilisant ces résultats, la vitesse d'extinction $\mathbb{P}[Z_n > 0] \underset{n\to\infty}{\sim} cn^{-1/2}$ a été obtenue dans [LPD18 ; VD18] pour le régime critique. Dans le régime sous-critique, on peut définir $\rho = \lim_{n\to\infty} \mathbb{E}[|||M_{0,n}|||]^{\frac{1}{n}}$ et on retrouve de manière similaire au cas du GWRE sous-critique trois sous régimes

- Le régime fortement sous-critique, étudié dans [VW18], dans lequel $\mathbb{P}[Z_n \neq 0] \underset{n\to\infty}{\sim} c\rho^n$, pour une bonne constante $c > 0$.
- Le régime sous-critique intermédiaire, étudié dans [VD20], dans lequel $c_1 n^{-1/2}\rho^n \leqslant \mathbb{P}[Z_n \neq 0] \leqslant c_2 n^{-1/2}\rho^n$, pour deux bonnes constantes c_1, c_2.
- Le régime faiblement sous-critique, étudié dans [PP23], dans lequel $c_1 n^{-1/2}\rho_*^n \leqslant \mathbb{P}[Z_n \neq 0] \leqslant c_2 n^{-1/2}\rho_*^n$, pour deux bonnes constantes c_1, c_2 et un réel $\rho_* < \rho$ qui ne dépend que de la loi des produits de matrices $(M_{0,n})_{n\geqslant 0}$.

Les constantes c, c_1, c_2 dépendent dans chacun des cas de la population initiale Z_0. Les trois articles précédents font l'hypothèse d'une suite i.i.d.d'environnements et supposent en plus une hypothèse de la forme (I.8) sur les matrices moyennes.

Il ressort des résultats précédemment présentés que dans le cadre des processus de Galton-Watson monotypes ou avec un nombre fini de types, que ce soit en environnement constant ou aléatoire (i.i.d.ou au moins ergodique), trois questions primordiales sont bien comprises. On sait d'abord caractériser le fait que l'extinction du processus soit presque sûre ou non. Lorsque l'extinction est presque sûre, on comprend bien à quelle vitesse elle doit arriver. Au contraire, lorsque la population peut survivre, on sait bien la décrire sur l'événement de survie. D'autres questions plus fines ont été étudiées, et le sont encore aujourd'hui, comme par exemple l'établissement de théorèmes centraux limites [LN68 ; Afa01 ; AGKV05 ; LPP21] ou de résultats de grandes déviations sur pour des MGWRE à d-types [HL12 ; GLM17 ; GLN24b].
Dans le cas processus de Galton-Watson avec une infinité de types, les résultats dont on dispose sont moins généraux, déjà en environnement constant, et à fortiori en environnement variables ou aléatoire. On peut néanmoins mentionner [BCN99], qui établit diverses stratégies puissantes pour obtenir et étudier des martingales dans le cas d'un processus en environnement variable avec une infinité de types. Néanmoins, le manque d'une théorie de Perron-Frobenius suffisamment puissante et générale et de résultats d'ergodicité pour des produits d'opérateurs aléatoires en dimension infinie n'a pour l'instant pas permis d'obtenir une classification, des résultats de type Kesten-Stigum et des résultats sur les temps d'extinction de MGWRE avec une infinité de types.

I.1.4 D'autres modèles de populations structurées

Avant de nous consacrer à l'étude des MGWRE avec une infinité de types, nous souhaitons replacer les processus de Galton-Watson dans un cadre plus large. En effet, les processus de Galton-Watson ne sont qu'un exemple de la grande classe des processus de branchement, qui contient notamment de nombreux processus en temps continu, avec un temps indexé par $\mathbb{R}_+$ et des individus se reproduisant à des instants aléatoires (par exemple déterminés par un processus de Poisson). Les processus de branchement ne sont eux mêmes qu'un modèle de populations parmi d'autres. Les langages des équations différentielles (EDO), équations aux dérivées partielles (EDP), équations aux dérivées partielles

stochastiques (EDS) permettent de définir des modèles de populations dans lesquels les individus ne sont plus représentés individuellement mais en tant qu'une masse globale, dont l'évolution est régie par des mécanismes intégro-différentiels. Les processus de branchement en temps continu et les modèles EDP/EDS sont en particulier souvent utilisés pour modéliser des populations structurées selon un ou plusieurs traits prenant leur valeur dans un ensemble infini (souvent un intervalle de $\mathbb{R}$ ou un domaine de $\mathbb{R}^d$.).
Ces objets ont par exemple été régulièrement utilisés pour étudier des systèmes de croissance-fragmentation, c'est-à-dire des systèmes de particules ou d'individus, souvent d'origine biologique ou physique, où chaque individu a une taille qui croît au cours de sa vie, jusqu'à un instant de division où l'individu se scinde en plusieurs individus plus petits. La somme des tailles des enfants est généralement égale à la taille du parent avant la division. Ces processus peuvent naturellement être utilisés pour modéliser des phénomènes de division cellulaire [GBWZ24], des situations de dynamiques épidémiques [Ber23 ; BGY23], où les individus représentent alors des *clusters*, ou encore des questions plus abstraites de cartes aléatoires [LR20].
La dispersion spatiale d'une population peut également être modélisée par des modèles où l'ensemble des sites occupés par les individus peut être infini [BS24 ; BBBCB24 ; Eth08]. Le mouvement brownien branchant, objet probabiliste bien connu et très intensément étudié, également en lien avec les EDP de Fischer-KPP, intervient par exemple assez naturellement pour modéliser des dynamiques d'invasion spatiale [Roq24]. Ce n'est cependant pas son seul cadre d'application aux modèles de population, puisqu'il intervient naturellement comme modèle de l'évolution d'un trait génétique au cours du temps dans [RS21].
Les populations structurées en âge ont également été largement étudiées et modélisées avec des modèles de branchement en temps continu et des modèles EDP [Jag75 ; Ner81]. L'activité de recherche fournie autour de ces modèles illustre bien l'importance de modéliser des systèmes de particules structurés par un trait ou un type qui peut prendre une infinité de valeurs. Des variantes inhomogènes en temps ont déjà été introduites dans certains de ces modèles, ce qui correspond à notre notion de processus en environnement varié. Le cas d'un environnement évoluant périodiquement a été particulièrement étudié [JN85 ; LCDH15 ; FM21 ; CN20 ; CAG24 ; BCG20]. On peut également mentionner des travaux étudiant une population où la fitness d'un individu dépend d'une distance entre leur type x et un type optimal x_{opt}, qui se déplace dans un espace abstrait au cours du temps selon $x_{\text{opt}}(t) = x_0 + ct$ [CHMT22 ; HMT23].
Ainsi, l'intérêt de l'étude de modèles de populations avec une infinité de types est clairement établi et il existe de nombreux tels modèles qui sont bien compris, à la fois en environnement constant, mais aussi avec certaines formes d'inhomogénéité temporelle. Néanmoins, même au delà du cadre des processus de Galton-Watson, il semble exister dans la littérature peu de résultats concernant des modèles de populations avec une infinité de types et un environnement fluctuant aléatoirement. C'est donc l'objectif de cette thèse de proposer des résultats de ce type. Vue la richesse de la littérature sur les processus de Galton-Watson à d types en environnement aléatoire, il nous a semblé pertinent de choisir ces modèles. Néanmoins, nos techniques et nos résultats pourraient probablement être adaptés pour étudier d'autres formes de modèles de populations structurés utilisant du branchement ou des EDP/EDS en environnement aléatoire.

I.2 Présentation du modèle et résultats principaux

I.2.1 Présentation du modèle

Nous choisissons donc d'étudier dans cette thèse des processus de Galton-Watson multitypes avec un nombre infini de types. Pour les définir, munissons nous d'abord de deux espaces mesurables $(\mathcal{E}, \mathscr{E})$ et $(\mathbb{X}, \mathcal{X})$. Comme précédemment, $\mathcal{E}$ est l'ensemble des environnements auxquels les individus peuvent être exposés. L'espace $\mathbb{X}$ est lui l'ensemble des types possibles des individus. Comme dans le cas des processus de Galton-Watson multitypes en environnement constant, une population de k individus dont les types sont $(x_1, \dots x_k) \in \mathbb{X}^k$ est représentée par une mesure ponctuelle sur $\mathbb{X}$, de la forme $\sum_{i=1}^k \delta_{x_i}$, et on note $\mathcal{N} \subset \mathcal{M}(\mathbb{X})$ l'ensemble des telles mesures ponctuelles obtenues comme somme finie de masses de Dirac.
Pour chaque type $x \in \mathbb{X}$ et chaque environnement $e \in \mathcal{E}$, on considère une loi de probabilité $\mathcal{L}_{x,e}$ sur $\mathcal{N}$, qui décrit la descendance aléatoire d'un individu de type x dans l'environnement e. Soit $\bar{\xi} = (\xi_n)_{n \geqslant 0} \in \mathcal{E}^{\mathbb{Z}_+}$ une suite stationnaire et ergodique d'environnements. On introduit enfin une famille $\left(N_{x,e}^{k,n}\right)_{x \in \mathbb{X}, e \in \mathcal{E}, k, n \in \mathbb{Z}_+}$ de variables aléatoires indépendantes, telles que $N_{x,e}^{k,n} \underset{n \to \infty}{\sim} \mathcal{L}_{x,e}$.

Définition 6. On appelle processus de Galton-Watson multitype en environnement aléatoire (MGWRE) un processus (Z_n) à valeurs dans $\mathcal{N}$ vérifiant la relation de récurrence

$$Z_{n+1} = \sum_{x \in \mathbb{X}} \sum_{k=1}^{Z_n(x)} N_{x,\xi_n}^{k,n}.$$

Comme dans le cas du GWRE, on s'intéresse à la moyenne *quenched* du processus. Cette moyenne sera définie grâce à des fonctions-tests. Pour toute fonction $f : \mathbb{X} \to \mathbb{R}_+$ mesurable, $Z_n(f)$ est la variable aléatoire réelle obtenue en appliquant la mesure ponctuelle aléatoire Z_n à la fonction f. Cette quantité est la masse de la population en vie au temps n, calculée en attribuant un poids $f(x)$ à chaque individu de type x. On définit alors pour tout $x \in \mathbb{X}$

$$M_{k,n} f(x) = \mathbb{E}[Z_n(f) | \bar{\xi}, Z_k = \delta_x]$$

l'espérance de cette masse. Notons $\mathbb{1}$ la fonction constante égale à 1 sur $\mathbb{X}$ et supposons que $\sup_{x \in \mathbb{X}} M_{k,n}(\mathbb{1}) < \infty$. La définition de $M_{k,n} f$ peut alors être étendue à toute fonction f mesurable et bornée et $M_{k,n}$ est alors une application linéaire et positive sur l'ensemble $\mathcal{B}(\mathbb{X})$ des fonctions mesurables et bornées sur $\mathbb{X}$. De plus, en posant

$$\mu M_{k,n} f = \int M_{k,n} f(x) d\mu(x),$$

on définit une action à gauche de $M_{k,n}$ sur l'espace $\mathcal{M}(\mathbb{X})$ des mesures signées sur $\mathbb{X}$, c'est à dire des différences de deux mesures positives finies. Dans la suite, on munit $\mathcal{B}(\mathbb{X})$ de la norme $\|f\|_\infty = \sup_{x \in \mathbb{X}} |f(x)|$ et $\mathcal{M}(\mathbb{X})$ de la norme en variation totale $\|\mu\|_{TV} = \mu^+(\mathbb{1}) + \mu^-(\mathbb{1})$, où $\mu = \mu^+ - \mu^-$ est la décomposition de Hahn-Jordan de la mesure signée μ. Ainsi, pour tous $k \leqslant n$, la quantité $M_{k,n}$ est un opérateur linéaire borné à la fois sur $\mathcal{B}(\mathbb{X})$

et sur $\mathcal{M}(\mathbb{X})$, de norme $|||M_{k,n}||| = \|M_{k,n}\mathbb{1}\|_\infty$. De plus, la famille d'opérateurs $(M_{k,n})_{k\geqslant N}$ vérifie la propriété de semi-groupe $M_{k,N} = M_{k,n}M_{n,N}$, pour tous $0 \leqslant k \leqslant n \leqslant N$. En particulier, en notant $M_n = M_{n,n+1}$, on peut écrire $M_{k,n} = M_k \dots M_{n-1}$.

I.2.2 Résultats principaux

Le premier objectif de cette thèse est d'énoncer un théorème d'ergodicité similaire au Théorème H pour des produits d'opérateurs de dimension infinie. Dans cette optique, nous avons d'abord effectué un travail préliminaire [Lig23], qui constitue le Chapitre II de cette thèse. Nous y étudions les méthodes de l'article de [Hen97] qui permettent de démontrer le Théorème H et qui sont basées sur la distance de Hilbert. Plus précisément, nous étendons cette distance sur des espaces de dimension infinie, comme cela est fait dans [Bir57], et dans ce cadre nous obtenons que l'uniforme positivité d'un opérateur, qui est une condition suffisante bien connue pour qu'il soit contractant en distance de Hilbert, est aussi en fait nécessaire.

Matrices et opérateurs contractants en distance de Hilbert Dans son étude, Hennion considère des matrices appartenant à l'ensemble

$$\mathcal{M}^+ = \{M \in \mathcal{M}_d(\mathbb{R}_+) | M \text{ n'a ni ligne ni colonne nulle}\}.$$

Les actions projectives à gauche et à droite des matrices de $\mathcal{M}_+$ sur les vecteurs de $\mathcal{S} = \{x \in \mathbb{R}_+^d | \|x\| = 1\}$ sont bien définies, par $(x, M) \mapsto x \cdot M = xM/\|xM\|$ et $(M, x) \mapsto M \cdot x = Mx/\|Mx\|$. Hennion s'inspire de la distance dite de Hilbert, présentée par exemple dans [Bir57], en la modifiant légèrement afin de munir S d'une variante bornée de cette distance. Il explicite le coefficient de Lipschitz commun $c(M)$ des deux applications $x \mapsto x \cdot M$ et $x \mapsto M \cdot x$ relativement à cette distance en fonction des coefficients de M, montre que $c(M) \leqslant 1$ pour toute matrice de $\mathcal{M}_+$ et que $c(M) < 1$ si tous les coefficients de M sont strictement positifs. De plus, on dispose de la propriété de sous-multiplicativité $c(MN) \leqslant c(M)c(N)$ et on en déduit, par le lemme de Kingman, que lorsque la suite (M_n) est ergodique, la quantité $c(M_{0,n})^{\frac{1}{n}}$ converge presque sûrement vers $c_\infty = \exp\left(\inf_{N\geqslant 1} N^{-1}\mathbb{E}\left[\log\left(c(M_{0,N})\right)\right]\right)$. L'hypothèse I.1 assure d'une part que toutes les matrices (M_n) sont dans $\mathcal{M}_+$, cet ensemble étant stable par produit, il contient également les produits $(M_{0,n})$. D'autre part, elle garantit que pour N assez grand, avec probabilité positive, $M_{0,N}$ a tous ses coefficients positifs, donc $c(M_{0,N}) < 1$. Ceci entraine donc que $c_\infty < 1$ et que $c(M_{0,n})$ tend géométriquement vite vers 0. Cette convergence est une propriété de contraction et est centrale pour établir les propriétés d'ergodicité présentées dans le Théorème H.
Pour étendre les résultats d'Hennion en dimension infinie, il était donc naturel de chercher à munir des espaces de dimension infnie d'une distance similaire et de chercher à caractériser les applications linéaires dont l'action projective est contractante pour cette distance. C'est l'objectif du II de cette thèse. Plus précisément, en s'appuyant sur [Bir57 ; BK53], on rappelle une construction existante de cette distance dans un cadre de dimension infinie. Une condition suffisante sur un endomorphisme linéaire positif M d'un sous espace $E \subset \mathbb{R}^{\mathbb{X}}$ pour que l'action projective $M : f \mapsto M(f)/\|Mf\|$ soit strictement contractante est alors son uniforme positivité, définie de la manière suivante :

Définition 7. Soit $E \subset \mathbb{R}^{\mathbb{X}}$, $M : E \mapsto E$ une application linéaire positive et $A > 1$. On dit que M est A-uniformément positif s'il existe $h \in E \cap (\mathbb{R}_+)^{\mathbb{X}}$ tel que, pour tout $f \in E$, il existe $b(f) \geqslant 0$ tel que
$$A^{-1}b(f)h \leqslant Mf \leqslant Ab(f)h. \tag{I.11}$$
S'il existe A tel que M est A-uniformément positif, on dit que M est uniformément positif et on note $M \gg 0$.

Le caractère suffisant de cette condition apparaissait déjà dans [Bir57], mais nous n'avons pas trouvé de preuve de son caractère nécessaire dans la littérature. Nous l'avons donc établi et prouvé dans la Proposition II.9.
Dans le cas où E est de dimension dénombrable, un opérateur M admet une représentation matricielle . En appliquant la définition précédente aux fonctions indicatrices $f = \mathbb{1}_x$, $x \in \mathbb{X}$, on comprend que pour que M soit uniformément positif tous les coefficients nuls de sa représentation matricielle doivent faire partie d'une ligne nulle ou d'une colonne nulle. Dans le cas d'un MGWRE, le coefficient (i, j) du produit $M_{0,n}$ représente le nombre moyen de descendants de type j au temps n d'un individu de type i au temps 0. Ainsi, la i-ème ligne de cette matrice est nulle lorsque les individus de type i au temps 0 n'ont presque sûrement aucun descendant au temps n, tandis que la colonne j est nulle lorsqu'aucun individu au temps 0 ne peut avoir de descendant de type j au temps n. Ainsi, pour que $M_{0,n}$ soit uniformément positif, il faut que pour chaque paire de types (i, j), si d'une part les individus de type i au temps 0 peuvent avoir des descendants au temps n et d'autre part que les individus de type j peuvent exister au temps n (en partant d'individus d'un certain type possiblement différent de i au temps 0), alors les individus de type i au temps 0 peuvent avoir des descendants de type j au temps n. La propriété
$$\mathbb{P}[\exists n \geqslant 1, c(M_{0,n}) < 1] = \mathbb{P}[\exists n \geqslant 1, M_{0,n} \gg 0] > 0$$
est donc une propriété d'irréductibilité très forte, puisqu'elle est uniforme en temps : elle implique l'existence d'un temps n tel que, en dehors de types stériles ou inatteignables, chaque type d'individu au temps 0 doit pouvoir engendrer des individus de tous les types au temps n. Pourtant, lorsque l'on choisit de considérer un système avec une infinité de types, il est naturel que certaines paires de types (i, j) soient "loin" l'une de l'autre, au sens où il faut un grand nombre de générations (possiblement non borné en i et j) pour obtenir un individu de type j à partir d'un individu de type i. Cela se comprend naturellement lorsque le type représente une position spatiale (il faut un long temps pour parcourir une grande distance), une notion de taille (il faut un temps long pour que les individus grossissent beaucoup) ou d'âge (il faut un temps long pour que les individus vieillissent beaucoup). Lorsqu'on étudie un tel système, les produits de matrices $M_{0,n}$ ne seront pas uniformément positifs et l'obtention de résultats d'ergodicité pour le semi groupe ne pourra pas se faire avec la contraction de Hilbert. Nous souhaitons que nos résultats d'ergodicité puissent s'appliquer à de tels opérateurs et le Chapitre II montre que nous devrons donc avoir recours à d'autres techniques de contraction pour les établir.
Pour préciser et rendre plus concrète cette idée, nous introduisons dans le Chapitre III un prototype de processus de Galton-Watson multitype qui ne vérifie pas de propriété d'uniforme positivité, que nous appelons processus de Leslie-Galton-Watson. Il nous servira dans les Chapitres IV et V d'exemple récurrent et nous nous attacherons alors à vérifier

que les techniques que nous développons permettent bien de l'étudier.

Processus de Leslie-Galton-Watson Présentons succinctement ce modèle. Il modélise une population structurée en âge : le type d'un individu représente son âge, compté comme le nombre de pas de temps écoulés depuis sa naissance. L'espace des types est donc $\mathbb{X} = \mathbb{Z}_+$. La descendance d'un individu de type x est donc constitué d'un certain nombre aléatoire de nouveaux nés, dont le type est donc 0, et de au plus 1 individu de type $x+1$, selon que l'individu initial de type x survit (et donc vieillit) ou meurt. De nombreux modèles de population permettant de modéliser la reproduction d'individus et leur vieillissement existent et sont étudiés dans la littérature, notamment à travers le formalisme des processus de Crump-Mode-Jagers. Les définitions que nous adoptons ici placent notre modèle dans la classe des processus de Galton-Watson multitypes ; nous l'appelerons dans la suite processus de Leslie-Galton-Watson ou plus succinctement LGW. En effet, la moyenne de la loi de reproduction associée à ce processus peut être représentée matriciellement par une matrice infinie dite matrice de Leslie infinie, ayant la forme suivante

$$M = \begin{pmatrix} f_0 & s_0 & 0 & 0 & \dots \\ f_1 & 0 & s_1 & 0 & \dots \\ f_2 & 0 & 0 & s_2 & \ddots \\ f_3 & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{I.12}$$

Dans cette matrice, le coefficient f_x représente le nombre moyen de nouveaux nés dans la descendance d'un individu de type x et s_x représente la probabilité qu'un individu de type x survive (et soit donc remplacé par un individu de type $x+1$ au pas de temps suivant). On peut se convaincre, soit à partir de la définition du modèle, soit par un argument de récurrence, qu'un produit $M_0 \dots M_{n-1}$ de n de matrices de Leslie infinies ne sera jamais uniformément positif : sur la ligne i, les coefficients dont l'indice de colonne est supérieur à $i+n+1$ sont nécessairement nuls, puisque l'âge de l'individu le plus âgé de population augmente au plus de 1 à chaque pas de temps. Les processus de Leslie-Galton-Watson forment donc bien un prototype de processus qu'on ne peut pas étudier par des méthodes fondées sur la contraction en distance de Hilbert. Par contre, ce modèle présente la propriété que des individus de tous les types peuvent avoir des enfants de type 0. Cette structure de renouvellement en 0 rend ce modèle particulièrement adapté à l'utilisation des techniques de contraction de type Doeblin mentionnées précédemment. De plus, ce cas particulier présente l'avantage que les lignées généalogiques qui ne reviennent pas en 0 ont une structure explicite, ce qui permettra de les manipuler facilement.
Avant de nous concentrer sur l'étude de ce modèle en environnement aléatoire dans les Chapitres IV et V, il nous a semblé pertinent de détailler son étude en environnement constant : c'est l'objet du Chapitre III. On travaille dans la totalité de ce chapitre sous l'hypothèse suivante.

Hypothèse I.2. Pour tout $n \geqslant 0$, $0 < s_n \leqslant 1$ et $f_n \geqslant 0$. De plus l'ensemble $\mathfrak{F} = \{n \geqslant 0 | f_n > 0\}$ est infini.

Comme nous étudions un processus de Galton-Watson avec une infinité de types, nous ne disposons pas d'une méthode systématique pour déterminer pour quelles valeurs des coefficients f_x, s_x le processus s'éteint presque sûrement ou non. Pour caractériser cette extinction presque sûre, nous introduisons un processus auxiliaire qui décrit un sous arbre obtenu en ne conservant que les individus de type 0. Cet arbre est en fait un Galton-Watson monotype dont la loi de reproduction a pour moyenne

$$m = \sum_{n\geqslant 0} s_0 \dots s_{n-1} f_n.$$

Lorsque ce sous arbre est infini, le processus LGW ne s'éteint pas. Lorsque ce sous arbre est fini alors à partir d'un certain temps, il n'y aura plus d'individus de type 0 dans la population du LGW. Elle s'éteindra donc, à moins qu'un des individus de type 0 survive infiniment longtemps, ce qui n'est possible que si $\prod_{i=0}^{+\infty} s_k > 0$. Ces observations mènent au

Theorème 1. *Soit (Z_n) un processus de Leslie-Galton-Watson satisfaisant l'hypothèse I.2. Alors le vecteur $\mathbf{q} = (q(x))_{x\in\mathbb{X}}$ des probabilités d'extinction vérifie :*

i) Si $\prod_{n\in\mathbb{Z}_+} s_n > 0$ ou $m > 1$ alors $\mathbf{q} < 1$,

ii) Si $\prod_{n\in\mathbb{Z}_+} s_n = 0$ et $m \leqslant 1$ alors $\mathbf{q} = 1$.

Dans un second temps, nous étudions la matrice moyenne M de ce processus, caractérisons complètement l'existence d'éléments propres positifs à gauche ou à droite pour cette matrice et en proposons une expression explicite lorsqu'ils existent. Nous montrons entre autres qu'il y a au plus une valeur propre λ_0 associé à des vecteurs propres positifs à gauche et à droite. Une telle valeur propre est caractérisée comme l'unique antécédent de 1, s'il y en a un, par la fonction

$$g : \lambda \mapsto \sum_{n\geqslant 0} \frac{s_0 \dots s_{n-1} f_n}{\lambda^{n+1}}.$$

Nous déduisons enfin une caractérisation de l'ergodicité de la matrice M sous la forme suivante.

Theorème 2. *Soit (Z_n) un processus de Leslie-Galton-Watson, dont on note M la matrice moyenne. On suppose que M satisfait l'hypothèse I.2. On suppose de plus que* $\mathrm{pgcd}(\mathfrak{F} + 1) = 1$. *Alors pour tout $\lambda > 0$, les deux assertions suivantes sont équivalentes*

i) $g(\lambda) = 1$ et $g'(\lambda) < \infty$.

ii) Il existe $h \in \mathbb{R}_+^{\mathbb{Z}_+}, h \neq 0$ et $\pi \in \mathcal{M}(h), \pi \neq 0$ tels que pour tout $\mu \in \mathcal{M}(h)$ on ait

$$\lim_{n\to\infty} \left\| \lambda^{-n}\mu M^n - \frac{\mu(h)}{\pi(h)}\pi \right\|_{\mathcal{M}(h)} = 0. \tag{I.13}$$

Lorsque les deux assertions du Théorème 2 sont vraies, alors $\lambda = \lambda_0$ et les quantités h et π sont bien des λ-vecteurs propres de M, pour lesquels on dispose d'une expression explicite. De plus, dans ce cas, on peut montrer que pour chaque type y, on la quantité $\mathbb{E}[Z_n(y)|Z_0 = \delta_x]$ croît ou décroît géométriquement à taux λ_0 au premier ordre. Ceci

justifie de définir des régimes sous-critique, critique, surcritique selon la position de λ_0 par rapport à 1. Enfin, pour chaque valeur de λ associée à un vecteur propre à droite h_λ, on peut procéder à une transformée de Doob, ce qui revient à définir une chaîne de Markov à valeurs dans $\mathbb{Z}_+$, dont les transitions sont données par la matrice de coefficients $P^\lambda(i,j) = M(i,j)\frac{h_\lambda(j)}{\lambda h_\lambda(i)}$. Cette chaîne de Markov décrit la trajectoire dans l'espace des types d'une lignée typique. Lorsque cette chaîne de Markov satisfait une condition de minoration de Doeblin, ce qui revient à supposer que les individus de tous âges ont suffisamment de nouveaux-nés dans leur descendance, nous montrons de plus que la convergence (I.13) se fait à vitesse géométrique.
Cette étude détaillée du semi groupe nous permet d'aboutir à des résultats précisant la vitesse d'extinction du processus en régime sous-critique et décrivant la population sur l'événement de survie dans le régime surcritique.

Theorème 3. *Considérons un processus de Leslie-Galton-Watson de matrice moyenne M, satifaisant I.2. Supposons de plus que $\inf_{n\geqslant 0} f_n > 0$ et qu'il existe un réel $\lambda_0 > 0$ tel que $g(\lambda_0) = 1$. Alors*

i) (Régime sous-critique) Si $\lambda_0 < 1$,

$$\mathbb{P}[Z_n \neq 0] = \underset{n\to\infty}{O}(\lambda_0^n),$$

pour toute population initiale Z_0.

ii) (Régime surcritique) Si $\lambda_0 > 1$ et que de plus

- $\mathrm{pgcd}(\mathfrak{F} + 1) = 1$
- *h est bornée*
- $x \mapsto \mathbb{E}[Z_1(h)^2 | Z_0 = \delta_x] \in \mathcal{B}_h$,

alors il existe une variable aléatoire positive et intégrable W telle que $\mathbb{P}[W > 0] > 0$ et

$$\mathbb{1}_{W>0}\frac{Z_n(f)}{\lambda_0^n} \xrightarrow[n\to\infty]{a.s.} W\frac{\pi(f)}{\pi(h)},$$

pour toute fonction f bornée et toute population initiale $Z_0 \neq 0$.

Produits d'opérateurs positifs aléatoires Après avoir étudié la distance de Hilbert et les processus de Leslie-Galton-Watson en environnement constant, nous abordons le premier objectif principal de ce manuscrit : obtenir un résultat d'ergodicité pour des produits d'opérateurs positifs aléatoires sur des espaces de dimension infinie. Cette étude est présentée dans le Chapitre IV.
Dans ce chapitre, nous considérons un espace mesurable $\mathbb{X}$ et définissons un ensemble $\mathcal{K}^+$ d'opérateurs linéaires positifs agissant à la fois sur l'espace $\mathcal{M}(\mathbb{X})$ des mesures signées sur $\mathbb{X}$ et de manière duale sur l'ensemble $\mathcal{B}(\mathbb{X})$ des fonctions mesurables bornées sur $\mathbb{X}$. Comme précédemment, $\mathcal{M}(\mathbb{X})$ est muni de la norme de variation totale $\|\cdot\|_{TV}$, notée parfois $\|\cdot\|$ pour alléger. L'espace des fonctions mesurables bornées $\mathcal{B}(\mathbb{X})$ est lui muni de

la norme $\|f\|_\infty = \sup_{x\in\mathbb{X}} |f(x)|$. Nous pouvons alors munir $\mathcal{K}^+$ de la norme d'opérateur définie par

$$|||M||| = \|M\mathbb{1}\|_\infty = \sup_{\substack{\mu\in\mathcal{M}(\mathbb{X}) \\ \mu\geqslant 0 \\ \|\mu\|_{TV}=1}} \|\mu M\|.$$

Soit de plus $(\Omega, \mathcal{A}, \mathbb{P}, \theta)$ un système dynamique, au sens où $\theta : \Omega \to \Omega$ est une application mesurable, préservant la mesure de probabilité $\mathbb{P}$. Enfin, nous considérons une application $M : \Omega \to \mathcal{K}^+$ et notons $M_n(\omega) = M \circ \theta^n(\omega)$ pour tout $\omega \in \Omega$. Ainsi, la suite (M_n) est une suite stationnaire d'opérateurs de $\mathcal{K}^+$. Enfin, nous définissons les produits $M_{k,n} = M_k \dots M_{n-1}$ pour tous $k < n$, avec la convention $M_{k,k} = \mathrm{Id}$. Ces produits vérifient la propriété de semi-groupe $M_{k,N} = M_{k,n}M_{n,N}$ pour $0 \leqslant k \leqslant n \leqslant N$. Enfin, on définit $m_{k,n}(x) = \|\delta_x M_{k,n}\|_{TV}$ et on remarque que $|||M_{k,n}||| = \|m_{k,n}\|_\infty$.
Nos premières hypothèses sur ces objets sont

Hypothèse I.3. Le système dynamique $(\Omega, \mathcal{A}, \mathbb{P}, \theta)$ est ergodique.

On rappelle qu'un système dynamique est dit ergodique lorsque tout ensemble $A \in \mathcal{A}$ tel que $\theta^{-1}(A) = A$ vérifie $\mathbb{P}(A) \in \{0, 1\}$.

Hypothèse I.4. Pour tout $\omega \in \Omega$, la fonction $x \mapsto m_{0,1}(x)$ est strictement positive et bornée.

Hypothèse I.5. $\mathbb{E}\left[\log^+ \|m_{0,1}\|_\infty\right] < \infty$.

En particulier, $\|m_{k,n}\|_\infty = |||M_{k,n}|||$. Par sous multiplicativité de la norme $|||\cdot|||$, sous les hypothèses I.3 et I.5, on a $\mathbb{E}\log^+(|||M_{k,n}|||) < \infty$ pour tous $k \leqslant n$. Cela permet de définir l'exposant de Lyapunov de la suite (M_n) comme pour des produits de matrices $d \times d$, par $\lambda = \inf_{n\geqslant 1} n^{-1}\mathbb{E}[\log |||M_{0,n}|||]$ et d'obtenir, sous l'hypothèse I.3, la loi des grands nombres $\lim_{n\to\infty} n^{-1} \log |||M_{0,n}||| = \lambda$, par le lemme sous additif de Kingman.

Comme nous l'avons précédemment expliqué, les résultats d'ergodicité que nous cherchons à obtenir se fondent sur une notion de contraction projective. En dimension finie, cette contraction est obtenue par rapport à la distance de Hilbert, ce que nous ne souhaitons pas faire en dimension infinie pour éviter de devoir travailler sous des hypothèses d'uniforme positivité. Nous préférons obtenir une contraction en montrant une propriété d'ergodicité géométrique sur une chaîne de Markov auxiliaire, grâce à une minoration de Doeblin. Pour quantifier ces contractions, nous introduisons la notion de "constantes de couplage admissibles".

On appelle "constantes de couplage admissibles" une application mesurable (ν, c, d) : $\omega \in \Omega \mapsto (\nu_\omega, c(\omega), d(\omega)) \in \mathcal{M}_1(\mathbb{X}) \times [0,1]^2$ telle que pour $\mathbb{P}$-presque tout $\omega \in \Omega$,

i) pour tout $x \in \mathbb{X}$ et tout $f \in \mathcal{B}_+(\mathbb{X})$, le couple $(\nu_\omega, c(\omega))$ vérifie

$$\delta_x M_0(\omega)(f) \geqslant c(\omega)\|\delta_x M_0(\omega)\|\nu_\omega(f) \tag{I.14}$$

ii) pour tout $n \geqslant 0$, le couple $(\nu_\omega, d(\omega))$ vérifie

$$\nu_\omega(m_{1,n}) \geqslant d(\omega)|||M_{1,n}||| \tag{I.15}$$

La propriété *i)* garantit qu'au temps $n = 1$, une fraction c la masse de la mesure $\mu M_{0,n}$ est distribuée selon la mesure ν, qui est indépendante de μ. Ceci introduit une forme de couplage et d'oubli de la distribution initiale μ. La propriété *ii)* permet de garantir que cette masse qui a été distribuée indépendamment de μ au premier pas de temps reste une proportion non-négligeable de la masse totale après application des matrices $M_1, \dots M_n$. Quand un triplet de constantes de couplage admissible est défini, on introduit la variable aléatoire

$$\gamma(\omega) = c(\omega)d(\omega).$$

En prenant $c = d = 0$ et une application mesurable quelconque $\omega \mapsto \nu_\omega$, on obtient un triplet de constantes de couplages admissibles, mais dans ce cas $\gamma = 0$. Notre hypothèse principale est donc

Hypothèse I.6. Il existe des constantes de couplage admissibles $\omega \mapsto (\nu_\omega, c(\omega), d(\omega))$ tel que $\mathbb{P}[\gamma > 0] > 0$.

On note

$$\bar{\gamma} = \mathbb{E}[\log(1 - \gamma)] \in [0, 1]$$

et on remarque que $\bar{\gamma} < 1$ sous l'hypothèse I.6.

Sous ces hypothèses, nous obtenons le théorème

Theorème 4. *Soit $M : \Omega \longrightarrow \mathcal{K}^+$ une application mesurable vérifiant les hypothèses I.3,I.4 and I.6. Alors,*

i) $\mathbb{P}(d\omega)$-presque sûrement, il existe une fonction aléatoire $h_0 \in \mathcal{B}(\mathbb{X})$ telle que pour tout $\delta \in (\bar{\gamma}^{\mathbb{P}[\gamma>0]}, 1)$, pour n assez grand, pour toutes mesures finies $\mu_1, \mu_2 \in \mathcal{M}_+(\mathbb{X})\backslash\{0\}$,

$$\left\| \mu_1 M_{0,n} - \frac{\mu_1(h_0)}{\mu_2(h_0)} \mu_2 M_{0,n} \right\|_{TV} \leqslant \delta^n \|\mu_1 M_{0,n}\|_{TV}. \tag{I.16}$$

Une telle fonction h est unique à une constante multiplicative près.

ii) Il existe une mesure de probabilité Λ sur l'espace $\mathcal{M}_1(\mathbb{X})$, telle que pour toute mesure de probabilité μ sur $\mathbb{X}$, le processus à valeur mesure $(\mu \cdot M_{0,n})$ converge en loi vers la mesure de probabilité Λ dans l'espace $\mathcal{M}_1(\mathbb{X})$ muni de la norme de variation totale.

iii) En supposant de plus que l'hypothèse I.5 est vérifiée, on a pour presque tout $\omega \in \Omega$ et toute mesure finie, positive et non nulle μ,

$$\frac{1}{n} \log \|\mu M_{0,n}\|_{TV} \underset{n\to\infty}{\longrightarrow} \inf_{N \geqslant 1} \frac{1}{N} \mathbb{E}\left[\log |||M_{0,N}|||\right] = \lambda \in [-\infty, \infty). \tag{I.17}$$

Ce théorème étend les conclusions du Théorème H à un cadre de dimension infinie, à quelques différences près. Tout d'abord, remarquons que lorsque $\mathbb{X}$ est fini, les espaces $(\mathcal{M}(\mathbb{X}), \|\cdot\|_{TV})$, $(\mathcal{B}(\mathbb{X}), \|\cdot\|_\infty)$ et $(\mathcal{K}^+, |||\cdot|||)$ se ramènent bien à $(\mathbb{R}^d, \|\cdot\|_1)$, $(\mathbb{R}^d, \|\cdot\|_\infty)$,$(\mathcal{M}_d(\mathbb{R}_+), |||\cdot|||)$ considérés par Hennion. Notons ensuite que le Théorème H utilise les éléments propres $r_{0,n}, L_{0,n}, R_{0,n}$ de Perron-Frobenius du produit de matrices $M_{0,n}$ et établit des résultats sur les coefficients de la matrice $M_{0,n}$. Dans notre contexte, nous ne savons pas si le produit $M_{0,n}$ admet de tels éléments propres. De plus, nous ne pouvons pas établir de résultats aussi précis qu'Hennion sur les quantités $\mu M_{0,n} f$, pour tous μ et f, car notre méthode ne permet d'obtenir que des résultats sur la mesure $\mu M_{0,n}$, avec des erreurs en distance de variation totale : si la mesure $\mu M_{0,n}$ charge des zones de $\mathbb{X}$ sur lesquelles f est petite ou nulle alors $\mu M_{0,n} f$ sera très petit devant $\|\mu M_{0,n}\|_{TV}$ et nous n'aurons qu'une information très grossière sur cette quantité. Pour mettre en évidence le lien entre les Théorèmes H et 2, il convient de déduire du Théorème H des propriétés sur des vecteurs de la forme $xM_{0,n}$ qui ne font pas apparaître les vecteurs propres $R_{k,n}, L_{k,n}$. Dans le cadre de dimension finie du Théorème H, le vecteur h_0 est caractérisé par

$$\frac{\langle x, h_0 \rangle}{\langle y, h_0 \rangle} = \lim_{n\to\infty} \frac{\|xM_{0,n}\|_1}{\|yM_{0,n}\|_1} \quad \mathbb{P}(d\xi)\text{-p.s.}$$

et on peut obtenir la même relation sur h à partir de (I.16). Notre fonction h généralise donc bien le vecteur h_0 en dimension infinie.
De plus, on peut déduire de Théorème H *i), ii)* une propriété de la forme

$$\lim_{n\to\infty} \left\| xM_{0,n} - \frac{\langle x, h_0 \rangle}{\langle y, h_0 \rangle} yM_{0,n} \right\|_1 = \underset{n\to\infty}{O}(\|xM_{0,n}\|_1 \rho^n) \quad \mathbb{P}(d\xi)\text{p.s},$$

qui est semblable à l'assertion *i)* de notre Théorème 4. La convergence en loi *ii)* de notre théorème correspond à l'assertion *iv)* du Théorème H en remarquant que $L_{0,n}$ et $xM_{0,n}$ ont asymptotiquement des directions proches en conséquence de l'assertion *i)* du Théorème H,
Enfin, en dimension finie, la convergence (I.17) correspond à $\lim n^{-1} \log(\|xM_{0,n}\|_1) = \lambda$, propriété qui se déduit elle-même aisément de la loi des grands nombres sur les coordonnées $\lim_{n\to\infty} n^{-1} \log(M_{0,n}(i,j)) = \lambda$. Nous n'avons pas énoncé dans notre Théorème 4 le fait que les vecteurs propres (h_n) sont des vecteurs propres temps espace, i.e. $M_n h_{n+1} = \lambda_n h_n$, car nous ne le prouvons pas dans le Chapitre IV, elle est néanmoins vraie et prouvée dans le Chapitre V.

On s'intéresse dans la suite du Chapitre IV au cas où la suite d'opérateurs (M_n) n'est pas seulement ergodique mais i.i.d, de loi $\mathcal{P}$. On introduit alors les hypothèses

Hypothèse I.7. $\mathbb{E}\left[|\log \|m_{0,1}\|_\infty|\right] < \infty$.

Hypothèse I.8. Il existe un triplet de constantes de couplages admissible (ν, c, d) tel que $\mathbb{E}\,|\log(\gamma)| < \infty$.

Ces hypothèses renforcent respectivement I.5 et I.6. Dans un premier temps, elles permettent de relier l'exposant de Lyapunov Λ et la mesure limite Λ sur l'espace projectif $\mathcal{M}_1(\mathbb{X})$.

Theorème 5. *Considérons une suite i.i.d. (M_n) d'éléments de $\mathcal{K}^+$ ayant la loi $\mathcal{P}$ et supposons que les hypothèses I.3, I.4, I.7 et I.8 sont vérifiées. Alors la convergence presque sûre (I.17) est aussi une convergence dans l'espace $\mathcal{L}^1(\Lambda \otimes \mathbb{P})$, c'est-à-dire*

$$\int_{\mathcal{M}_1(\mathbb{X})\times(\mathcal{K}^+)^n} \left| \frac{1}{n} \log \|\mu M_{0,n}\| - \lambda \right| d\Lambda(\mu) d\mathcal{P}^{\otimes n}(M_0, \cdots, M_{n-1}) \underset{n\to\infty}{\longrightarrow} 0.$$

En conséquence,

$$\lambda = \int_{\mathcal{M}_1(\mathbb{X})\times\mathcal{K}^+} \log\|\mu M\| d\Lambda(\mu) d\mathcal{P}(M).$$

Grâce au Théorème 4, on comprend que $\|\mu M_{0,n}\|_{TV}$ tend vers $+\infty$ (resp. vers 0) lorsque $\lambda > 0$ (resp. $\lambda < 0$). Dans le cas $\lambda = 0$, le Théorème 4 ne donne aucune information sur le comportement asymptotique de cette masse, si ce n'est qu'il est sous-géométrique au sens où $\lim_{n\to\infty} (\|\mu M_{0,n}\|_{TV})^{\frac{1}{n}} = 1$. Les hypothèses renforcées I.7 et I.8 permettent ainsi de comprendre mieux ce régime. On introduit pour présenter ces résultats la notion de Null-Homology. On dit que la loi $\mathcal{P}$ satisfait la propriété de Null-Homology lorsqu'il existe une application $\psi : \mathcal{M}_1(\mathbb{X}) \mapsto \mathbb{R}$ telle que $d(\Lambda \otimes \mathcal{P})(\mu, M)$-p.s., on ait

$$\log\|\mu M\| = \psi(\mu \cdot M) - \psi(\mu). \tag{NH}$$

Lorsque (NH) est vérifiée, soit $\mu_0 \underset{n\to\infty}{\sim} \Lambda$ et notons $a < b \in [-\infty, +\infty]$ les infinimum et supremum respectifs du support de la variable $\psi(\mu_0)$. Le comportement de $\|\mu M_{0,n}\|$ dépend alors du fait que $\mathcal{P}$ vérifie ou non la propriété de Null Homology. Précisons cette affirmation.

Theorème 6. *Considérons une suite i.i.d.(M_n) d'éléments de $\mathcal{K}^+$ de loi $\mathcal{P}$, qui vérifie les hypothèses I.3, I.4, I.5 et I.8. Supposons de plus que $\lambda = 0$. Alors si (NH) n'est pas vérifiée, on a*

$$\liminf_{n\to\infty} \log\|\mu M_{0,n}\| = -\infty \text{ et } \limsup_{n\to\infty} \log\|\mu M_{0,n}\| = +\infty \tag{OSC}$$

pour tout $\mu \in \mathcal{M}_+(\mathbb{X})\backslash\{0\}$. Si au contraire (NH) est vérifiée alors on a

$$a = -\infty \Leftrightarrow \mathbb{P}\text{-p.s., pour tout } \mu \in \mathcal{M}_+(\mathbb{X})\backslash\{0\},\ \liminf_{n\to\infty} \log\|\mu M_{0,n}\| = -\infty$$

et

$$b = +\infty \Leftrightarrow \mathbb{P}\text{-p.s., pour tout } \mu \in \mathcal{M}_+(\mathbb{X})\backslash\{0\},\ \limsup_{n\to\infty} \log\|\mu M_{0,n}\| = +\infty.$$

Un théorème de Kesten-Stigum pour des MGWRE avec une infinité de types
Le dernier chapitre (Chapitre V) de cette thèse est consacré à l'établissement d'un théorème de Kesten-Stigum pour des MGWRE avec une infinité de types. On se munit donc d'un tel processus et on cherche à appliquer les résultats du Chapitre IV. Pour cela, on suppose que la suite environnementale (ξ_n) est stationnaire ergodique de loi $\mathbb{P}$, on prend donc $\Omega = \mathcal{E}^{\mathbb{Z}_+}$ et on pose $M(\bar{\xi})(f)(x) \mapsto \mathbb{E}[Z_1(f)|Z_0 = \delta_x, \bar{\xi}]$ pour tout $\omega = \bar{\xi} \in \Omega$. Sous l'hypothèse que pour tout $e \in \mathcal{E}$ on a $\sup_{x\in\mathbb{X}} \mathbb{E}[\|Z_1\|_{TV}|Z_0 = \delta_x, \xi_0 = e] < +\infty$, on peut vérifier que $M(\bar{\xi})$ est bien un opérateur borné sur $\mathcal{B}(\mathbb{X})$ et par dualité sur $\mathcal{M}(\mathbb{X})$. On se trouve donc dans le cadre du Chapitre IV dont on reprend les notations dans ce qui suit.

Nous supposons dans la suite que l'application M satisfait les hypothèses I.3, I.4, I.7 et I.8. L'exposant de Lyapunov $\lambda = \inf_{n\geqslant 1} n^{-1}\mathbb{E}[\log|||M_{0,n}|||]$ du processus est bien défini. Le Théorème 4 est satisfait et nous extrayons certains résultats intermédiaires de l'article [Lig25] (i.e. le Chapitre IV de cette thèse) que nous regroupons dans le théorème suivant :

Theorème 7. *Considérons un MGWRE vérifiant les hypothèses I.3, I.4, I.7, I.8. Alors les trois assertions suivantes sont vraies $\mathbb{P}(d\bar{\xi})$-p.s.*

i) Pour tout $k \geqslant 0$, la suite $\left(\frac{m_{k,n}}{\|m_{k,n}\|}\right)_{n\geqslant k}$ converge uniformément sur $\mathbb{X}$ vers une limite $h_k \in \mathcal{B}(\mathbb{X})$.

ii) Pour tout $\eta > \tilde{\eta}$, pour tout $n \geqslant k \geqslant 0$ et tous $\mu_1, \mu_2 \in \mathcal{M}_+(\mathbb{X})\backslash\{0\}$,

$$\left\| \mu_1 M_{k,n} - \frac{\mu_1(h_k)}{\mu_2(h_k)} \mu_2 M_{k,n} \right\|_{TV} \leqslant \frac{4}{\gamma_{n-1}} \prod_{i=k}^{n-1} (1-\gamma_i) \|\mu_1 M_{k,n}\| = \underset{n\to\infty}{O} (\eta^n \|\mu_1 M_{k,n}\|). \tag{I.20}$$

iii) Pour toute mesure positivie, finie et non nulle μ sur $\mathbb{X}$,

$$\lim_{n\to\infty} \frac{1}{n} \log \|\mu M_{0,n}\| = \lim_{n\to\infty} \frac{1}{n} \log |||M_{0,n}||| = \inf_{n\geqslant 1} \frac{1}{n} \mathbb{E}\left[\log |||M_{0,n}|||\right] = \lambda \in [-\infty, \infty). \tag{I.21}$$

Une conséquence de (I.21) est que $\lim_{n\to\infty} \left(\mathbb{E}[\|Z_n\| | \bar{\xi}, Z_0]\right)^{\frac{1}{n}} = e^{\lambda}$, $\mathbb{P}(d\bar{\xi})$-a.s. En particulier, si $\lambda < 0$, on a $\lim_{n\to\infty} \mathbb{E}[\|Z_n\| | \bar{\xi}, Z_0] = 0$, ce qui garantit l'extinction presque sûre du processus. Sous les hypothèses du Théorème 6, en dehors du cas où (NH) est vérifiée et $a > -\infty$, on a $\liminf_{n\to\infty} \mathbb{E}[\|Z_n\| | \bar{\xi}, Z_0] = 0$, $\mathbb{P}(d\bar{\xi})$-p.s, ce qui, la encore, garantit que $\mathbf{q} = 1$. On appelle ces régimes respectivement sous-critique et critique. On souhaite étudier principalement dans ce chapitre le régime surcritique et on introduit donc l'hypothèse :

Hypothèse I.9. $\lambda > 0$

Notre premier résultat est l'extension à notre contexte de la martingale obtenue dans [GLP23]. Pour cela, nous commençons par remarquer que la famille de fonctions h_k vérifie une propriété de fontion harmonique espace temps : en posant $\lambda_k = \|M_k h_{k+1}\|_\infty$, on peut montrer que

$$M_k h_{k+1} = \lambda_k h_k \quad \mathbb{P}(d\bar{\xi})\text{-p.s.}$$

De manière similaire à [GLP23], on définit alors

$$W_n = \frac{Z_n(h_n)}{\lambda_{0,n} Z_0(h_0)},$$

où $\lambda_{k,n} = \lambda_k \ldots \lambda_{n-1}$. Le premier résultat majeur du Chapitre V est alors le

Theorème 8. *Soit (Z_n) un MGWRE qui satisfait les hypothèses I.3, I.4, I.7 et I.8. Alors, conditionnellement à la suite environnementale $\bar{\xi}$, le processus (W_n) est une martingale.*

Notons que ce résultat ne requiert pas que le processus (Z_n) soit surcritique. Cette martingale W_n étant positive, elle converge presque sûrement vers une limite $W = \lim_{n\to\infty} W_n \in [0, \infty)$. On s'intéresse dans la suite à la dégénérescence de cette martingale et on introduit pour cela l'hypothèse

Hypothèse I.10. Il existe une fonction croissante et positive f telle que $\frac{1}{x\log(x)f(x)}$ est intégrable en $+\infty$ et

$$\mathbb{E}\left[\sup_{x\in\mathbb{X}} \frac{\mathbb{E}_{\bar{\xi}}\left[Z_1(h_1)\log[Z_1(h_1)] f(Z_1(h_1)) | Z_0 = \delta_x\right]}{M_0 h_1(x)}\right] < \infty.$$

On dispose alors du théorème

Theorème 9. *Soit* (Z_n) *un MGWRE satisfaisant les hypothèses I.3, I.4, I.7, I.8, I.9 et I.10. Alors pour toute population initiale* $Z_0 \in \mathcal{N}$, *on a*

$$\mathbb{P}_{\bar{\xi}}[W > 0|Z_0] > 0 \text{ et } \mathbb{E}_{\bar{\xi}}[W|Z_0] = 1 \quad \mathbb{P}(d\bar{\xi})\text{-p.s.}$$

Ainsi, sous l'hypothèse I.10, la martingale W_n tend vers une limite strictement positive avec probabilité non nulle. On prouve dans le Chapitre V que $\lim_{n\to+\infty} (\lambda_{0,n})^{\frac{1}{n}} = e^{\lambda}$. Ceci garantit que sur l'événement $\{W > 0\}$, on a $\lim_{n\to+\infty} (Z_n(h_n))^{\frac{1}{n}} = e^{\lambda}$. Comme h_n est bornée, cela implique que $\liminf_n (\|Z_n\|_{TV})^{\frac{1}{n}} \geqslant e^{\lambda} > 1$, on en déduit donc que la taille de la population peut bien exploser géométriquement. Nous n'avons pas pu démontrer en général que l'événement $\{W > 0\}$ sur lequel a lieu cette explosion coïncide avec l'événement de survie $\{\forall n \geqslant 0, Z_n \neq 0\}$. Nous conjecturons même que ce résultat est faux en général lorsque le nombre de types est infini. Nous discutons précisément des raisons de ces difficultés à la fin de l'introduction du Chapitre V

Notre dernier résultat permet de comprendre la répartition des types dans la population sur l'événement $\{W > 0\}$ en précisant le comportement asymptotique de $Z_n(f)$ pour une large de classe de fonctions f. Ce résultat est obtenu grâce à des calculs de seconds moments et nécessite donc l'hypothèse suivante.

Hypothèse I.11.

$$\mathbb{E}\left[\log^+\left(\sup_{x\in\mathbb{X}} \mathbb{E}_{\bar{\xi}}\left[\,\|Z_1\|^2\middle|\, Z_0 = \delta_x\right]\right)\right] < \infty.$$

Theorème 10. *Soit* (Z_n) *un MGWRE vérifiant les hypothèses I.3, I.4, I.7, I.8, I.9, I.10 et I.11. Alors on a*

i) pour toute fonction $f \in \mathcal{B}(\mathbb{X})$ *et pour tout* $\varepsilon > 0$, *on a*

$$Z_n(f) = W_n Z_0 M_{0,n}(f) + \underset{n\to+\infty}{o}\left(\left(\max\left(e^{\lambda}\tilde{\eta}, \sqrt{e^{\lambda}}\right) + \varepsilon\right)^n\right) \quad \mathbb{P}\text{-p.s.} \tag{I.22}$$

ii) Pour toute fonction $f \in \mathcal{B}(\mathbb{X})$, $f \geqslant 0$, *si* $\mathbb{E}\left[\log^-(\nu M_{1,N} f)\right] < +\infty$ *pour un certain* $N \geqslant 1$ *alors*

$$\lim_{n\to\infty} \frac{Z_n(f)}{Z_0 M_{0,n}(f)} = W \quad \mathbb{P}\text{-p.s.} \tag{I.23}$$

et de plus, sur l'événement $\{W > 0\}$, *on a*

$$\lim_{n\to\infty} (Z_n(f))^{\frac{1}{n}} = e^{\lambda}. \quad \mathbb{P}\text{-p.s.}$$

En prenant $f = \mathbb{1}$ dans (I.22), pour laquelle

$$\lim_{n\to\infty} (Z_0 M_{0,n}\mathbb{1})^{\frac{1}{n}} = \lim_{n\to\infty} (\|Z_0 M_{0,n}\|_{TV})^{\frac{1}{n}} = e^{\lambda} > \max\left(e^{\lambda}\tilde{\eta}, \sqrt{e^{\lambda}}\right),$$

on obtient $\lim_{n\to\infty} \|Z_n\|_{TV}/\mathbb{E}[\|Z_n\|_{TV}|\bar{\xi}, Z_0] = W$. Ceci montre que la taille asymptotique de la population sur une réalisation du processus (Z_n) est donnée par sa moyenne quenched et la limite de la martingale (W_n), on a en particulier $\lim_{n\to\infty} (\|Z_n\|_{TV})^{\frac{1}{n}} = e^\lambda$ presque sûrement sur l'événement $\{W > 0\}$.

Pour décrire plus précisément la répartition des types sur une réalisation du processus (Z_n), on souhaite pouvoir procéder de même avec une classe plus large de fonctions f. On aura alors $Z_n(f) \underset{n\to\infty}{\sim} W Z_0 M_{0,n}(f)$ sur l'événement $\{W > 0\}$, pour toute fonction f telle que

$$\liminf_{n\to\infty} \left(Z_0 M_{0,n}(f)\right)^{\frac{1}{n}} > \max\left(e^\lambda \tilde{\eta}, \sqrt{e^\lambda}\right) \quad \mathbb{P}(d\bar{\xi})\text{-p.s.}. \tag{I.24}$$

Toute fonction f telle que $\inf_{x\in\mathbb{X}} f(x) > 0$ vérifie clairement cette condition, mais l'assertion *ii)* nous permet de trouver une classe plus large et plus intéressante de fonctions la vérifiant également. En particulier, dans le cas d'un processus de Leslie-Galton-Watson en environnement aléatoire, sous une hypothèse d'intégrabilité raisonnable, toute fonction non nulle et positive la vérifie, ce qui nous permet d'obtenir des asymptotiques pour l'effectif d'individus de chaque type dans une réalisation de ce processus.

I.3 Perspectives

Diverses perspectives dans le prolongement des résultats de cette thèse peuvent être envisagées. Nous en présentons brièvement trois.

Application à des contextes de croissance-fragmentation Comme nous l'avons détaillé précédemment, de nombreux modèles de populations structurées avec une infinité de types ont été largement étudiés en environnement constant et leur généralisation à l'environnement aléatoire pourrait être envisagé. Une classe d'exemple sur lesquelles nos méthodes seraient susceptibles de s'appliquer est celle des modèles de croissance-fragmentation. Dans ces situations, le mécanisme de fragmentation permet de créer, à partir d'un individu de grande taille, deux individus de taille plus petite. Si le mécanisme de fragmentation est tel qu'un individu de taille T est typiquement fragmenté en deux individus de tailles typiques $(K, T-K)$, où K ne dépend pas de T, on obtient une structure d'irréductibilité compatible avec des hypothèses de type Doeblin puisque les individus de toute taille ont des descendants de la même taille. Il serait donc intéressant d'appliquer nos méthodes, éventuellement en les adaptant si nécessaire, pour étudier ce genre de système et obtenir des descriptions presque sûres de la répartition des tailles en temps long dans une population subissant des mécanismes de croissance fragmentations.

Propriété d'ergodicité sous des hypothèses plus faibles Dans certains modèles de croissance-fragmentation, la fragmentation d'un individu de taille T donne plutôt typiquement des individus de tailles $(\alpha T, (1-\alpha)T)$, où α ne dépend pas de T. On ne dispose alors pas d'une structure d'irréductibilité de type Doeblin, car il faudra plusieurs étapes de divisions pour obtenir un individu de petite taille à partir d'un individu de grande taille. Néanmoins, ce nombre de division croît assez lentement avec la taille de l'individu initial. Dans de tel modèles où le temps nécessaire pour revenir dans une région de l'espace donné à partir d'un individu dépend du type de l'individu, mais où cette dépendance est contrôlée, les approches de type Doeblin peuvent être modifiées de sorte à obtenir des résulats d'ergodicités moins uniformes. Ces méthodes ont été initiées dans [BCGM22] dans un cadre homogène en temps. Il serait intéressant de les adapter à un cadre inhomogène en temps. Cela permettrait d'obtenir des résultats d'ergodicité pour des produits d'opérateurs aléatoires sous des conditions plus faibles, et peut être d'en déduire des résulats de type Kesten-Stigum sur des modèles qui sont hors du champ d'application de cette thèse.

Etude de temps d'extinction annealed On a développé dans cette introduction la littérature fournie caractérisant la queue de distribution annealed du temps d'extinction de divers processus de Galton-Watson en régime critique et sous critique. En particulier, en environnement aléatoire, des effets nouveaux et intéressants apparaissent du fait de l'interaction de l'aléa environnemental et de l'aléa démographique. Une direction de recherche intéressante serait donc de tenter de répondre aux mêmes questions dans le cas de processus avec une infinité de types. Cela nécessite d'élargir les méthodes de [DW15 ; Cam18 ; PP24] à un cadre de dimension infinie, ce qui pose de nombreux défis à relever.

CHAPITRE II

Sur la contraction des opérations linéaires pour la distance de Hilbert

Ce chapitre est tiré de la note [Lig23], consacrée à l'étude d'une variante de la métrique projective de Hilbert en dimension infinie. Plus exactement, nous redéfinissons entièrement cette métrique, rappelons et reprouvons ses propriétés principales, qui figurent dans la littérature. Cette métrique a la propriété remarquable de rendre contractantes les actions projectives associées à des applications linéaires positives. Cette propriété est au coeur des résultats d'ergodicité pour des produits de matrices positives aléatoires en dimension finie établis par Hennion [Hen97]. On cherche donc une condition nécessaire et suffisante pour que l'action projective associé à un opérateur linéaire positif sur un espace de dimension infinie soit strictement contractante. Une condition suffisante déjà présente dans la littérature est la notion d'uniforme positivité de l'opérateur. Nous n'avons pas trouvé dans la littérature de preuve de son caractère nécessaire. Nous le prouvons ici.

Let $\mathbb{X}$ be a set of arbitrary cardinality and consider a vector space $E \subset \mathbb{R}^{\mathbb{X}}$. The projective space $\mathfrak{P}$ associated with E is defined as the set of equivalence classes $\mathfrak{P} = E/\mathcal{R}$, where $\mathcal{R}$ is the equivalence relation such that for any $f, g \in E$

$$f\mathcal{R}g \Leftrightarrow \exists b \in \mathbb{R}_+^*, f = bg$$

Let $\Pi : E \longrightarrow \mathfrak{P}$ be the canonical projection. The Hilbert metric d_H, defined for example in [BK53], is a distance on the projective image $\Pi(C)$ of the positive cone $C = E \cap \mathbb{R}_+^{\mathbb{X}} \backslash \{0\}$. Any linear map M on E which is positive, in the sense that $M(C) \subset C$, generates a projective action on $\Pi(C)$. We say that the projective action $M : \Pi(C) \longrightarrow \Pi(C)$ associated with a linear map M is k-contracting, with $k < 1$, when it is k-Lipschitz with respect to d_H. When M is k-contracting for some $k < 1$, we say that it is contracting, or strictly contracting. As proved in [Bir57], the projective action on $\Pi(C)$ of any positive bounded linear map is 1-Lipschitz with respect to d_H. We say that a positive, linear map M on E is A-uniformly positive for some $A > 1$ when there exists $h \in E \cap (\mathbb{R}_+)^{\mathbb{X}}$ such that, for any $f \in E$, there exists $b(f) \geqslant 0$ satisfying

$$A^{-1}b(f)h \leqslant Mf \leqslant Ab(f)h. \tag{II.1}$$

Birkhoff [Bir57] shows that uniformly positive maps are contracting with respect to the Hilbert metric. This is useful to prove the existence a fixed point for the projective action of such a map, that is an eigenvector for its linear action. It also allows to study the ergodicity properties of semi-groups of uniformly positive maps.

In this note, we focus on a bounded variant d of this Hilbert distance, introduced in a finite dimensional setup in [Hen97]. As shown in [Hen97], any matrix with non negative coefficients is 1-Lipschitz with respect to d, and an explicit formula allows to compute the contraction rate of a matrix in terms of its coefficients is provided. This yields a sufficient condition for a matrix to be strictly contracting with respect to d. In this note, we provide an elementary construction of this pseudo Hilbert distance d in any dimension, as well as a study of some of its properties. In particular, we prove in Proposition II.7 that all linear, positive maps are 1-Lipschitz and in Proposition II.9 that the contracting maps with respect to d are exactly the uniformly positive ones. To the best of our knowledge, it was not yet proven in the literature that uniform positivity is not only sufficient but also necessary for a map to be strictly contracting with respect either to the pseudo-Hilbert or the Hilbert metric.

II.1 Defining a pseudo-Hilbert distance

Consider the partial order $\leqslant$ defined by $f \leqslant g \Leftrightarrow \forall x \in \mathbb{X}, f(x) \leqslant g(x)$.

For any $f, g \in C$, we define

$$\aleph(f, g) = \sup\{b \geqslant 0 | bf \leqslant g\} = \inf\left\{\frac{g(x)}{f(x)}, x \in \mathbb{X}, f(x) \neq 0\right\} \in [0, \infty],$$

$$m(f, g) = \aleph(f, g)\aleph(g, f).$$

Notice that the set $\{b \geqslant 0 | bf \leqslant g\}$ clearly is a sub-interval of $\mathbb{R}_+$, which contains 0.

Moreover, it holds

Lemma II.1. *For any $f, g, h \in C$, any $\alpha, \beta > 0$*

i) $\aleph(f,g) < \infty$ *and* $\{b \geqslant 0 | bf \leqslant g\} = [0, \aleph(f,g)]$.

ii) $m(f,g) = m(g,f)$

iii) $m(\alpha f, \beta g) = m(f,g)$,

iv) $m(f,g)m(g,h) \leqslant m(f,h)$.

v) $0 \leqslant m(f,g) \leqslant 1$.

vi) $m(f,g) = 1 \Leftrightarrow f\mathcal{R}g$. *If* $m(f,g) = 1$, *then* $f = \aleph(g,f)g$.

Proof. *i)* Since $0 \notin C$, $f \neq 0$, thus, there exists $x \in \mathbb{X}$ such that $f(x) > 0$. For b large enough, $bf(x) > g(x)$ which prevents $bf \leqslant g$. Since $\{b \geqslant 0|bf \leqslant g\}$ is an interval, this implies that it is bounded, i.e. $\aleph(f,g) < \infty$. Thus $bf \leqslant g$, for any $b \in [0, \aleph(f,g))$. This yields $\aleph(f,g)f \leqslant g$, thus

$$\aleph(f,g) \in \{b \geqslant 0|bf \leqslant g\} = [0, \aleph(f,g)].$$

ii) This symmetry property is straightforward from the definition of $m(f,g)$.

iii) This derives directly from

$$\aleph(\alpha f, \beta g) = \frac{\beta}{\alpha}\aleph(f,g).$$

iv) Combining $\aleph(f,g)f \leqslant g$, and $\aleph(g,h)g \leqslant h$, we get

$$\aleph(g,h)\aleph(f,g)f \leqslant \aleph(g,h)g \leqslant h,$$

thus $\aleph(g,h)\aleph(f,g) \in \{b \geqslant 0|bf \leqslant h\} = [0, \aleph(f,h)]$. This yields $\aleph(g,h)\aleph(f,g) \leqslant \aleph(f,h)$. For a similar reason, it holds $\aleph(g,f)\aleph(h,g) \leqslant \aleph(h,f)$. Multiplying these two inequalities yields $m(f,g)m(g,h) \leqslant m(f,h)$.

v) Notice that taking $f = h$ in *iv)* yields

$$m(f,g)m(g,f) = m(f,g)^2 \leqslant m(f,f).$$

Since $m(f,f) = 1$, then $m(f,g)^2 \leqslant 1$, which implies that $m(f,g) \leqslant 1$.

vi) From *iii)*, it is clear that if $f\mathcal{R}g$ then $m(f,g) = m(f,f) = 1$. Suppose now that $m(f,g) = 1$. This implies that $\aleph(f,g) \neq 0, \aleph(g,f) \neq 0$ and $\aleph(f,g)^{-1} = \aleph(g,f)$. Moreover, it holds

$$\aleph(f,g)f \leqslant g$$

and

$$\aleph(g,f)g \leqslant f.$$

Thus
$$\aleph(f,g)\aleph(g,f)f = f \leqslant \aleph(g,f)g \leqslant f.$$
Therefore, $f = \aleph(g,f)g$.

□

We define the pseudo-Hilbert metric d by setting
$$d(\bar{f},\bar{g}) = \frac{1-m(f,g)}{1+m(f,g)}$$
for any halflines $\bar{f},\bar{g} \in \Pi(C)$ and any points $f \in \bar{f}$, $g \in \bar{g}$ of those halflines.

Proposition II.2. *d is a well defined distance on $\Pi(C)$. It is bounded by 1.*

Proof. Assertion *iii)* of Lemma II.1 implies that $m(f,g)$ only depends on the equivalence classes $\Pi(f), \Pi(g)$ and not on the choice of f and g inside those classes. Moreover since $m(f,g) \in [0,1]$, and $\phi : [0,1] \longrightarrow [0,1], s \mapsto \frac{1-s}{1+s}$ is well defined, then d is well defined. The map d is clearly symmetric, nonnegative and bounded by 1. Since the map ϕ is strictly decreasing, continuous, $\phi(0) = 1$ and $\phi(1) = 0$, we obtain
$$d(\bar{f},\bar{g}) = 0 \Leftrightarrow m(f,g) = 1 \Leftrightarrow f\mathcal{R}g \Leftrightarrow \bar{f} = \bar{g}.$$
Moreover, it can be checked that $\phi(st) \leqslant \phi(s) + \phi(t)$, for any $s,t \in [0,1]$. Combining this with the fact that ϕ is decreasing and point *iv)* of Lemma II.1 yields the triangular inequality. □

Remark II.3. In many references such as [Bir57], one rather considers the Hilbert metric $d_H(\bar{f},\bar{g}) = |\log m(f,g)|$. We prefer using the pseudo Hilbert metric because its boundedness makes it more convenient.

Proposition II.4. *For any $\bar{f} \neq \bar{g} \in \Pi(C)$, the following claims hold.*

- *There exists $f \in \bar{f}, g \in \bar{g}$ such that the intersection of the line (f,g) with the cone C is reduced to a line segment $[u,v]$, with $u \neq v$. Fix now such points f,g,u,v.*

- *For any $h \in C$, if h is coplanar with u and v (or equivalently with f and g), then the line segment $[u,v]$ intersects the vector line $(0,h)$ in a single point of the cone.*

- *For any point $h \neq 0$ coplanar with u and v, with coordinates $h = h_1u + h_2v$ then $h \in C$ iff $h_1 \geqslant 0$ and $h_2 \geqslant 0$.*

Proof. Let $f \in \bar{f}$ and $g \in \bar{g}$. For any $t \in \mathbb{R}$, we note
$$h_t = tf + (1-t)g \in E.$$
Consider $S = \{t \in \mathbb{R} | h_t \in C\}$. S is clearly an non empty interval in $\mathbb{R}$ since $0,1 \in S$ and C is convex. In order to prove that S is a segment of line, it is enough to show that there

exists $x, x' \in \mathbb{X}$ such that $f(x) < g(x)$ and $g(x') < f(x')$. Indeed, if this is the case, since $h_t = g + t(f - g)$, then

$$h_t(x) \underset{t\to+\infty}{\longrightarrow} -\infty \text{ and } h_t(x') \underset{t\to-\infty}{\longrightarrow} +\infty \ .$$

Let us now prove that we can choose $f \in \bar{f}$ and $g \in \bar{g}$ such that there exists $x, x' \in \mathbb{X}$ satisfying $f(x) < g(x)$ and $f(x') > g(x')$. Consider two points $\tilde{f} \in \bar{f}, \tilde{g} \in \bar{g}$. Since $\bar{f} \neq \bar{g}$, in particular $\tilde{f} \neq \tilde{g}$, thus there exists $x \in \mathbb{X}$ such that $\tilde{f}(x) \neq \tilde{g}(x)$. Then in particular, $\tilde{f}(x)$ or $\tilde{g}(x)$ is nonzero. Without loss of generality, suppose $\tilde{g}(x) \neq 0$. Then, since $\bar{f} \neq \bar{g}$, there exists $x' \in \mathbb{X}$ such that

$$\tilde{f}(x') \neq \frac{\tilde{f}(x)}{\tilde{g}(x)}\tilde{g}(x').$$

If $\tilde{g}(x') = 0$, then for any $b > 0$, $b\tilde{f}(x') > 0 = \tilde{g}(x')$, and for $b > 0$ small enough, $b\tilde{f}(x) < \tilde{g}(x)$, thus setting $f = b\tilde{f}$ and $g = \tilde{g}$ suffices. In the contrary, if $\tilde{g}(x') \neq 0$, then,

$$\frac{\tilde{f}(x')}{\tilde{g}(x')} \neq \frac{\tilde{f}(x)}{\tilde{g}(x)}.$$

Once again, without loss of generality, suppose that

$$\frac{\tilde{f}(x')}{\tilde{g}(x')} > \frac{\tilde{f}(x)}{\tilde{g}(x)}.$$

Then there exists $b > 0$ such that

$$\frac{b\tilde{f}(x')}{\tilde{g}(x')} > 1 > \frac{b\tilde{f}(x)}{\tilde{g}(x)}.$$

Then, setting $f = b\tilde{f}$, $g = \tilde{g}$, we get $f(x) < g(x)$ and $f(x') > g(x')$, thus S is bounded and $(f, g) \cap C$ is a nonempty line segment $[u, v]$. Moreover, $f \neq g$ and $f, g \in [u, v]$, thus $u \neq v$. Finally, let $h \in C$, suppose that h is coplanar with f, g. Note that all points of the vector line $(0, h)$ have either only nonnegative coordinates or only nonpositive coordinates. Thus $f - g$ does not belong to this line. This proves that the lines (f, g) and $(0, h)$ are not parallel. Since they are coplanar, they are secant. Moreover, their intersection point $q = f + t(g - f)$ has either all nonnegative (in which case this point is in $[u, v] \subset C$), or all nonpositive coordinates. If $t = 0$ then $q \in C$, thus in $q \in [u, v]$. If $t > 0$, since $f(x) < g(x)$, then q as a positive coordinate $q(x)$. Similarly, if $t < 0$, the coordinate $q(x')$ is positive. Thus if $t \neq 0$, then the coordinates of q are not all nonpositive, consequently they are all nonnegative and $q \in C \cap (f, g) = [u, v]$. Let now $h \neq 0$ be a point coplanar with u and v. Since $u, v \in C$, if $h_1, h_2 \geqslant 0$, then $h(x) = h_1 u(x) + h_2 v(x) \geqslant 0$ for any $x \in \mathbb{X}$, thus $h \in C$. Conversely, if $h \in C$ the semi-line $\bar{h}$ intersects $[u, v]$. Thus there exists a point $\tilde{h} \in \bar{h} \cap [u, v]$. This point $\tilde{h}$ has non negative coordinates in the basis (u, v) of the plan. Since h is positively colinear with $\tilde{h}$, then $h_1, h_2 \geqslant 0$. □

Proposition II.5. *Let $\bar{f}, \bar{g} \in \Pi(C)$. Consider a line $\Delta \in E$ that intersects both $\bar{f}$ and $\bar{g}$ and suppose that $\Delta \cap C$ is a line segment $[u, v]$, with $u \neq v$. Then any $f \in \bar{f}$ and $g \in \bar{g}$*

are coplanar with u and v. Moreover, noting $f = f_1 u + f_2 v$, $g = g_1 u + g_2 v$, it holds :

$$d(\bar{f}, \bar{g}) = \left| \frac{f_1 g_2 - f_2 g_1}{f_1 g_2 + f_2 g_1} \right|, \tag{II.2}$$

with the convention $0/0 = 0$.

Remark II.6. Applying the triangular inequality in (II.2) yields $d(\bar{f}, \bar{g}) \leqslant 1$ for any $\bar{f}, \bar{g}$. Moreover, $d(\bar{f}, \bar{g}) = 1$ iff exactly one of the quantities $g_1 f_2$, $f_1 g_2$ equals 0. This implies in particular that $\bar{f}$ or $\bar{g}$ is contained in the edge of C.

Proof. Since $\Delta = (u, v)$, then Δ is included in the vector plane spanned by u, v. The line Δ intersects the vector semi-lines $\bar{f}, \bar{g}$, thus they are included in the plan spanned by u, v. Note that the quantity

$$R = \left| \frac{f_1 g_2 - f_2 g_1}{f_1 g_2 + f_2 g_1} \right|$$

is invariant by $(f, g) \mapsto (\alpha f, \beta g)$, for any positive real numbers α, β. Therefore, it suffices to show that this equality is satisfied for a well chosen pair (f, g) of elements of $\bar{f}, \bar{g}$, which we chose to be the intersection points of Δ with $\bar{f}, \bar{g}$. This implies that $f, g \in [u, v]$. We also remark that R is invariant by exchanging the coordinates (f_1, g_1) with (f_2, g_2). Hence, we can moreover assume, without loss of generality, that u, f, g, v are in this order, i.e. that $f_1 \geqslant g_1$ and thus $g_2 = 1 - g_1 \geqslant 1 - f_1 = f_2$. It remains to prove that

$$d(\bar{f}, \bar{g}) = \frac{f_1 g_2 - f_2 g_1}{f_1 g_2 + f_2 g_1}.$$

This holds in the case where $f_1 = g_1 = 0$ or in the case where $f_2 = g_2 = 0$, with the convention $0/0 = 0$. Thus, since $f_1 \geqslant g_1$ and $f_2 \leqslant g_2$, it remains to study the case $f_1, g_2 > 0$. For any $b > 0$, the last point of Proposition II.4 yields

$$g \leqslant bf \Leftrightarrow g - bf = (g_1 - bf_1)u + (g_2 - bf_2)v \in C \Leftrightarrow g_1 \geqslant bf_1 \text{ and } g_2 \geqslant bf_2.$$

Since $f_1 \geqslant g_1$ and $f_2 \leqslant g_2$, it holds $f_1 g_2 \geqslant f_2 g_1$ and thus

$$\frac{f_1}{g_1} \geqslant \frac{f_2}{g_2},$$

with the convention $f_1/g_1 = \infty$ if $g_1 = 0$. Thus, it holds

$$g \leqslant bf \Leftrightarrow b \leqslant \frac{f_2}{g_2}$$

and

$$\aleph(f, g) = \frac{f_2}{g_2} \in (0, +\infty).$$

Similarly,

$$\aleph(g, f) = \frac{g_1}{f_1} \in (0, +\infty).$$

Finally, it holds

$$d(\bar{f}, \bar{g}) = \frac{1 - \frac{g_1 f_2}{f_1 f_2}}{1 + \frac{g_1 f_2}{f_1 f_2}} = \frac{f_1 f_2 - g_1 f_2}{f_1 f_2 + g_1 f_2}.$$

□

II.2 Contraction properties of positive linear mappings

Let M be a linear mapping on E, we assume that M is positive, in the sense that $M(C) \subset C$. Then M induces a projective action on $\Pi(C)$, defined by $M \cdot \Pi(f) = \Pi(Mf)$, for any $f \in C$.

Proposition II.7. *Let*

$$c(M) = \sup\left\{d(M \cdot \bar{f}, M \cdot \bar{g}), \bar{f}, \bar{g} \in \Pi(C)\right\} \leqslant 1.$$

Then for all $\bar{f}, \bar{g} \in \Pi(C)$, it holds

$$d(M \cdot \bar{f}, M \cdot \bar{g}) \leqslant c(M) d(\bar{f}, \bar{g}). \tag{II.3}$$

Proof. Let $\bar{f}, \bar{g} \in C$. Suppose $M \cdot \bar{f} \neq M \cdot \bar{g}$, otherwise, inequality II.3 is obvious. Let $f \in \bar{f}, g \in \bar{g}$, $x \in M \cdot \bar{f}$ and $y \in M \cdot \bar{g}$ such that $(f, g) \cap C$ and $(x, y) \cap C$ are line segments. Let a, b and a_1, b_1 be the respective extreme points of $(f, g) \cap C$, $(x, y) \cap C$. Notice that (a, b) and (a_1, b_1) are respectively the bases of two vector planes between which M acts as an linear isomorphism. Let us decompose, f, g, Ma and Mb as follows :

$$f = f_1 a + f_2 b$$

$$\begin{aligned} g &= g_1 a + g_2 b \\ Ma &= \alpha a_1 + \gamma b_1 \\ Mb &= \beta a_1 + \delta b_1. \end{aligned}$$

Then

$$\begin{aligned} Mf &= (f_1\alpha + f_2\beta) a_1 + (f_1\gamma + f_2\delta) b_1 \\ Mg &= (g_1\alpha + g_2\beta) a_1 + (g_1\gamma + g_2\delta) b_1. \end{aligned}$$

The line (a_1, b_1) intersects both $M \cdot \bar{f}$ and $M \cdot \bar{g}$, and $(a_1, b_1) \cap C = [a_1, b_1]$. Moreover, $Mf \in M \cdot \bar{f}$ and $Mg \in M \cdot \bar{g}$. Thus, by Proposition II.5, it holds

$$d(M \cdot \bar{f}, M \cdot \bar{g}) = \left| \frac{(f_1\alpha + f_2\beta)(g_1\gamma + g_2\delta) - (f_1\gamma + f_2\delta)(g_1\alpha + g_2\beta)}{(f_1\alpha + f_2\beta)(g_1\gamma + g_2\delta) + (f_1\gamma + f_2\delta)(g_1\alpha + g_2\beta)} \right|.$$

Simple calculations yields

$$d(M \cdot \bar{f}, M \cdot \bar{g}) = \left| \frac{(\alpha\delta - \beta\gamma)(f_1 g_2 - f_2 g_1)}{(\alpha\delta + \beta\gamma)(f_1 g_2 + f_2 g_1)} \right| = \left| \frac{\alpha\delta - \beta\gamma}{\alpha\delta + \beta\gamma} \right| d(\bar{f}, \bar{g}).$$

By Proposition II.4, since the points Ma and Mb, are coplanar with $\bar{f}, \bar{g}$, the vector lines $(0, Ma)$ and $(0, Mb)$ intersect the segment $[a_1, b_1]$. Thus, we can apply Proposition II.5 :

$$d(M \cdot \Pi(a), M \cdot \Pi(b)) = \left| \frac{\alpha\delta - \beta\gamma}{\alpha\delta + \beta\gamma} \right|.$$

Since $d(M \cdot \Pi(a), M \cdot \Pi(b)) \leqslant c(M)$, we can finally conclude :

$$d(M \cdot \bar{f}, M \cdot \bar{g}) \leqslant c(M) d(\bar{f}, \bar{g}).$$

□

Corollary II.8. *Consider a linear map $M : E \to E$ such that $M(C) \subset C$. Then the projective action of M on $\Pi(C)$ is $c(M)$-Lipschitz with respect to the distance d.*

We recall that a linear positive mapping $M : E \longrightarrow E$ is A-uniformly positive whenever there exists $h \in C$, such that for any $f \in C$, there exists $b(f) \in \mathbb{R}_+$ such that :

$$\frac{1}{A} b(f) h \leqslant Mf \leqslant A b(f) h.$$

Notice that the inequality remains valid replacing A by $A' \geqslant A$. We note

$$A^*(M) = \inf\{A \geqslant 1 | M \text{ is } A\text{-uniformly positive}\}.$$

Then the following characterisation of strictly contracting applications holds :

Proposition II.9. *Consider a linear map $M : E \to E$, such that $M(C) \subset C$. Then $c(M) < 1$ if and only if M is a uniformly positive linear mapping. Moreover when $c(M) < 1$, $c(M) = \psi(A^*(M))$ where $\psi : s \in [1, \infty) \mapsto \frac{1-s^{-2}}{1+s^{-2}} = \phi(s^{-2}) \in [0, 1)$.*

Proof. First, assume that M is A-uniformly positive for some A. Let $f, g \in C$. Let $b(f), b(g) \in \mathbb{R}$ so that

$$A^{-1} b(f) h \leqslant Mf \leqslant A b(f) h$$
$$A^{-1} b(g) h \leqslant Mg \leqslant A b(g) h.$$

Let $f' = Mf / b(f)$ and $g' = Mg / b(g)$. Then

$$h \leqslant f', g' \leqslant Ah,$$

hence

$$A^{-1} g' \leqslant h \leqslant A f' \text{ and } A^{-1} f' \leqslant h \leqslant A g'.$$

Thus $\aleph(f', g'), \aleph(g', f') \geqslant A^{-1}$ and $m(f', g') \geqslant A^{-2}$. Since $\phi : s \in [0, 1] \mapsto \frac{1-s}{1+s} \in [0, 1]$ is decreasing and one-to one, then ψ is increasing and one-to-one. As a consequence,

$$d(M \cdot \bar{f}, M \cdot \bar{g}) = \phi(m(f, g)) = \phi(m(f', g')) \leqslant \phi(A^{-2}) = \psi(A) < 1.$$

Since $c(M) = \sup\left\{d(M \cdot \bar{f}, M \cdot \bar{g}), \bar{f}, \bar{g} \in \Pi(C)\right\}$, this yields $c(M) \leqslant \psi(A) < 1$. Since ψ is continuous, this yields $c(M) \leqslant \psi(A^*(M))$.
Conversely, suppose that $0 \leqslant c(M) < 1$. We want to show that M is uniformly positive.

Since ψ is one-to-one from $[1,\infty)$ onto $[0,1)$, let us set $A = \psi^{-1}(c(M))$, thus $c(M) = \frac{1-A^{-2}}{1+A^{-2}}$. Let $\bar{f},\bar{g} \in \Pi(C)$ and choose $f' \in M\cdot\bar{f}, g' \in M\cdot\bar{g}$, such that $(f',g')\cap C$ is a segment of line with endpoints u,v. We assume that $f' = su+(1-s)v$ and $g' = tu+(1-t)v$ with $0 \leqslant s \leqslant t \leqslant 1$. Since we have assumed that $c(M) < 1$, then $d(M\cdot\bar{f}, M\cdot\bar{g}) \leqslant c(M) < 1$. Therefore, f' and g' are distinct from u,v (ie $0 < s,t < 1$) and

$$d(M\cdot\bar{f}, M\cdot\bar{g}) = \frac{1-m(f',g')}{1+m(f',g')} \leqslant c(M) = \frac{1-A^{-2}}{1+A^{-2}}.$$

Since $s \mapsto \frac{1-s}{1+s}$ is strictly decreasing on $(0,1)$, this yields

$$m(f',g') \geqslant \frac{1}{A^2}.$$

Let us choose $r > 0$ such that $\aleph(rf',g') = r\aleph(f',g') = A^{-1}$, and set $f'' = rf', g'' = g'$. Then it holds,

$$A^{-2} \leqslant m(f',g') = m(f'',g'') = \aleph(f'',g'')\aleph(g'',f'') = A^{-1}\aleph(g'',f'').$$

Thus $\aleph(f'',g''), \aleph(g'',f'') \geqslant A^{-1}$, which yields $f'' \leqslant Ag''$, and $g'' \leqslant Af''$. In other words : if $c(M) < 1$, then there exists $A > 0$ such that, for any $\bar{f},\bar{g} \in \Pi(C)$, there exists $f'' \in M\cdot\bar{f}, g'' \in M\cdot\bar{g}$, satisfying $A^{-1}g'' \leqslant f'' \leqslant Ag''$. Let us now consider an arbitrary function $g \in C$, and fix $h = Mg$. For each $f \in C$, there exists two positive reals $\alpha(f),\beta(f) > 0$ such that $f'' = \alpha(f)Mf$ and $g'' = \beta(f)Mg = \beta(f)h$ satisfy

$$A^{-1}g'' \leqslant f'' \leqslant Ag''.$$

This yields :

$$A^{-1}\frac{\beta(f)}{\alpha(f)}f \leqslant Mf \leqslant A\frac{\beta(f)}{\alpha(f)}h.$$

In other words, setting $b(f) = \frac{\beta(f)}{\alpha(f)}$,

$$A^{-1}b(f)h \leqslant Mf \leqslant Ab(f)h.$$

Thus, M is A-uniformly positive, and $A^*(M) \leqslant A = \psi^{-1}(c(M))$, thus $\psi(A^*(M)) \leqslant c(M)$. □

II.3 Examples

II.3.1 The finite dimensional case

Let us focus in this subsection on the case of a finite set $\mathbb{X} = \{1,\dots,d\}$. We set $E = \mathbb{R}^d$, $C = (\mathbb{R}_+)^d\backslash\{0\}$, and consider $d\times d$ matrices with non-negative coefficients, acting linearly on column vectors. In this case the definition of uniform positivity is equivalent to considerations on the zero-coefficients of M :

Proposition II.10. *Let $M = (M_{ij})_{1\leqslant i,j\leqslant d}$ be a $d\times d$ matrix with non-negative coefficients.*

Then M is uniformly positive if and only if for each $i, j \in \{1, \dots d\}$ such that $M_{ij} = 0$, either the whole i-th row or the whole j-th column is zero.

Proof. Let us assume that M is uniformly positive, then there exists $A > 0$, and two functions $x \mapsto h(x), y \mapsto b(y)$ satisfying for every column vector y and row vector x

$$A^{-1}h(x)b(y) \leqslant xMy \leqslant Ah(x)b(y). \tag{II.4}$$

Thus if $M_{i,j} = e_i M e_j = 0$, either $h(e_i) = 0$ or $b(e_j) = 0$. This implies respectively that the j-th row or the i-th column of M is 0.
Conversely, if we know that M has no zero coefficients except in whole zero columns or rows, then letting i_0, j_0 be the respective indexes of a non-zero row and a non-zero column, then setting $h(e_i) = M_{ij_0}$ and $b(e_j) = M_{i_0 j}$,and

$$0 < A = \max\left(\max_{i,j} \frac{M_{ij}}{M_{ij_0}M_{i_0j}}, \max_{i,j} \frac{M_{ij_0}M_{i_0j}}{M_{ij}}\right) < \infty,$$

we prove that (II.4) is satisfied for any vectors x, y of the canonical basis. It can be extended to the whole positive cone by linearity. $\square$

This provides a natural necessary and sufficient condition for a matrix to be a strictly contracting projective map on the projective cone $\Pi(C)$.

Corollary II.11. *Let M be a $d \times d$ matrix with non negative coefficients. Then*

i) $M(C) \subset C$ if and only if there is a nonzero coefficient on each column of M

ii) Suppose that $M(C) \subset C$. Then $c(M) < 1$ if and only if for each i, j such that $M_{ij} = 0$, the whole j-th column of M is 0.

Additionally, the following Proposition, stated and proved in [Hen97, Lemma 10.7] claims that the constant $c(M)$ can be explicitly computed in terms of the coefficients of M.

Proposition II.12. *Let M is be a $d \times d$ matrix, with non positive coefficients, and no zero-column. Then*

$$c(M) = \sup\{d(M \cdot x, M \cdot y), x, y \in \Pi(C)\} = \max\{d(M \cdot e_i, M \cdot e_j), i, j \in \{1, \dots n\}\},$$

where $(e_i)_{1 \leqslant d}$ are the unit vectors of the canonical basis of $\mathbb{R}^d$. As a consequence, the constant $c(M)$ can be explicitly expressed in terms of the coefficients of $M = (M_{i,j})_{1 \leqslant i,j \leqslant d}$ as follows :

$$c(M) = \max_{1 \leqslant i,j,k,l \leqslant d} \frac{|M_{ki}M_{lj} - M_{kj}M_{li}|}{M_{ki}M_{lj} + M_{kj}M_{li}}. \tag{II.5}$$

with the convention $0/0 = 1$.

One can easily check that the necessary and sufficient condition for $c(M) < 1$ provided in Corollary II.11 is consistent with the expression of $c(M)$ in (II.5).

II.3.2 The density case

Let $\mathbb{X} = [0,1]$ and let E be the set of continuous real functions on $\mathbb{X}$. Let us consider linear maps of the form

$$M_K f(x) = \int K(x,y) f(y) dy,$$

where $(x,y) \mapsto K(x,y)$ is a continuous map from $[0,1]$ to $\mathbb{R}_+$. Under these assumptions M_K is a bounded linear operator on E. Assume additionally that

- For each $x, y \in \mathbb{R}$, $K(x,y) \geqslant 0$
- For each $y \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that $K(x,y) > 0$.

These conditions ensure that M_K is a positive operator on E, satisfying $M_K(C) \subset C$.

Proposition II.13. *Under these assumptions $c(M_K) < 1$ if and only if there exists $g_1, g_2 \in \mathbb{R}_+^{\mathbb{X}}, A \in \mathbb{R}_+^*$ such that*

$$A^{-1} g_1(x) g_2(y) \leqslant K(x,y) \leqslant A g_1(x) g_2(y). \tag{II.6}$$

When (II.6) holds for some A, noting A^* the infimum of the values of A satisfying (II.6), then it holds

$$c(M_K) = \psi^{-1}(A^*).$$

By continuity and compacity, it is sufficient to have $K(x,y) > 0$ for all x, y in order for equation (II.6) to hold. However, it is clearly not necessary, since any product of the form $K(x,y) = g_1(x) g_2(y) g_3(x,y)$ with g_1, g_2, g_3 continuous and non negative, and g_3 positive on $[0,1]^2$ satisfies (II.6) and is therefore uniformly positive.

Proof. If there exists A, g_1, g_2 satisfying equation (II.6), then for any $x \in \mathbb{X}$, any $f \in E$

$$A^{-1} g_1(x) \int g_2(y) f(y) dy \leqslant M_K f(x) \leqslant A g_1(x) \int g_2(y) f(y) dy,$$

thus M_K is A-uniformly positive with $h = g_1$ and $b(f) = \int g_2(y) f(y) dy$.
Conversely, if M_K is A-uniformly positive, let $h \in \mathbb{R}^{\mathbb{X}}$ and for each $f \in E$, $b(f) > 0$ such that

$$A^{-1} b(f) h \leqslant Mf \leqslant K b(f) h.$$

Let us note, for $\varepsilon > 0$ and $y \in \mathbb{X}$,

$$g_{\varepsilon,y} = \frac{\mathbb{1}_{[y-\varepsilon, y+\varepsilon] \cap \mathbb{X}}}{\int \mathbb{1}_{[y-\varepsilon, y+\varepsilon] \cap \mathbb{X}}(z) dz}.$$

Then it holds, for any $x, y \in \mathbb{X}$, by continuity of $y \mapsto K(x,y)$

$$K(x,y) = \lim_{\varepsilon \to 0} M_K(g_{\varepsilon,y})(x).$$

The uniform positivity assumption yields however that

$$A^{-1} M_K(g_{\varepsilon,y})(x) \leqslant h(x) b(g_{\varepsilon,y}) \leqslant A M_K(g_{\varepsilon,y})(x).$$

Therefore

$$A^{-1}K(x,y) \leqslant h(x) \limsup_{\varepsilon\to 0} b(g_{\varepsilon,y}) \leqslant AK(x,y).$$

Setting $g_1 = h$ and $g_2(y) = \limsup_{\varepsilon\to 0} b(g_{\varepsilon,y})$, we get

$$A^{-1}g_1(x)g_2(y) \leqslant K(x,y) \leqslant Ag_1(x)g_2(y).$$

This proves that M_K is A-uniformly positive if and only if (II.6) holds for some functions g_1, g_2. Proposition II.9 allows to conclude the proof. $\square$

CHAPITRE III

Un modèle de population structurée en âge

Dans ce chapitre, on définit un cas particulier de processus de Galton-Watson multitype en environnement aléatoire. Cet exemple permet de décrire une population structurée en âge, où l'âge d'un individu est défini comme le nombre de générations qui se sont écoulées depuis sa naissance. Dans cette thèse, nous ne mettons pas de borne sur l'âge maximal des individus, et l'ensemble des âges possibles est donc $\mathbb{Z}_+$. La matrice moyenne associée à la loi de reproduction est donc une matrice infinie, dont les lignes et les colonnes sont indexées par $\mathbb{Z}_+$, qui est une version infinie dimensionnelle des matrices dites de Leslie, elles mêmes introduites et utilisées depuis plusieurs décennies pour la modélisation de populations structurées en âge.
Dans notre étude, le principal intérêt de ce modèle jouet est de fournir un exemple typique de processus de Galton-Watson avec une infinité de types qui manque de propriétés d'uniforme positivité. En environnement aléatoire, son étude ne semble donc pas possible par une extension à la dimension infinie des techniques de [Hen97], fondées sur la distance de Hilbert. Dans les chapitres IV et V, ces processus seront le prototype d'exemples auxquels nous voulons pouvoir appliquer les techniques que nous développons. Nous manipulerons donc beaucoup ce processus en environnement aléatoire dans les chapitres suivants. Au préalable, nous consacrons ce chapitre à son étude en environnement constant.

III.1 Introduction

III.1.1 Definition of the model

The Leslie-Galton-Watson process is a specific class of multitype Galton Watson processes where we keep track of the age of the individuals. The type of an individual will therefore be its age, that we represent as the integer number of time steps since its birth. We take thus $\mathbb{X} = \mathbb{Z}_+$, endowed with the σ-algebra $\mathcal{X} = \mathcal{P}(\mathbb{Z}_+)$.

At each time step, each individual may die or survive. When an individual of age x survives, it is replaced at the next time step by an individual of age $x+1$. It can additionally create a random number of new individuals of age 0, which we call *newborns*. The offspring of an individual of type x can therefore be described by a couple of random variables (S_x, F_x), where

- S_x is a Bernoulli variable of parameter s_x, encoding its survival or death : if $S_x = 0$, the individual dies, if $S_x = 1$ it survives.
- F_x is an integer valued variable of mean f_x, encoding the number of newborns in its offspring.

This offspring can equivalently be represented by the random measure $N_x = F_x\delta_0 + S_x\delta_{x+1}$ or by a random vector L_x such that $L_x = (0, \dots, 0) \in \mathbb{Z}_+{}^k$ if $(S_x, F_x) = (0, k)$ and $L_x = (x+1, 0, \dots, 0) \in \mathbb{Z}_+^{k+1}$ if $(S_x, F_x) = (1, k)$. We note $\mathcal{L}_x$ the probability distribution of N_x, it is the offspring distribution of an individual of type x. The mean matrix associated with the offspring distribution $\mathcal{L}_x$ has thus the following shape

$$M = \begin{pmatrix} f_0 & s_0 & 0 & 0 & \dots \\ f_1 & 0 & s_1 & 0 & \dots \\ f_2 & 0 & 0 & s_2 & \ddots \\ f_3 & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{III.1}$$

Remark III.1. The mean matrix (III.1) is clearly an infinite dimensional generalization of the finite dimensional Leslie matrices, introduced in [Les45], which can alternatively be obtained by truncating a matrix of the form of (III.1) and keeping only its first d rows and columns. These finite dimensional Leslie matrices are classically used in matrix population models to describe a population divided into a finite number of age classes. A more recent overview of their known properties can be found for example in [Cas10].

Matrices in the form of (III.1) are known as "infinite Leslie matrices", they have been introduced and studied in [Dem72; GL09; ABO14].

In this chapter, we shall study this population model in a constant environment e. Therefore, at each time n, the offspring of each individual of type x follows the probability distribution $\mathcal{L}_x$, independently of n.

Definition III.2. *For any choice of variables (F_x, S_x) such that $s_x > 0$ for all $x \in \mathbb{Z}_+$, we call **Leslie-Galton-Watson Process** (LGW for short) the time homogeneous and multitype Galton-Watson process with offspring distributions $(\mathcal{L}_x)_{x\geqslant 0}$.*

In this chapter, we consider a LGW $(Z_n)_{n\geqslant 0}$. We recall that the random variables $(Z_n)_{n\geqslant 0}$ are point measures on the discrete type space $\mathbb{X} = \mathbb{Z}_+$. Therefore for each $n \geqslant 0$, the measure Z_n is characterized by the vector $(Z_n(k))_{k\geqslant 0} \in \mathbb{R}_+^{\mathbb{Z}_+}$, where we note for short $Z_n(k) = Z_n(\{k\})$. We note 0 the null measure on $\mathbb{Z}_+$. The event $\{Z_n = 0\}$ describes thus the situation where there is no individual alive at time n : the population is said to be extinct at time n. We emphasize that the event $\{Z_n = 0\}$ must not be confused with the event $\{Z_n = \delta_0\}$ which states that the population at times n contains exactly one individual, who has age 0. As usual when studying Galton-Watson processes, we are particularly interested in the (global) extinction event and its probability, defined by

$$\mathbf{Ext} = \bigcup_{n\geqslant 0}\{Z_n = 0\} \text{ and } \mathbf{q} = (\mathbf{q}(x))_{x\in\mathbb{X}} = (\mathbb{P}[\mathbf{Ext}\,|Z_0 = \delta_x])_{x\in\mathbb{X}}\,.$$

III.1.2 Results and related literature

To the best of our knowledge, the LGW process has never been explicitly studied, though there exists a lot of related literature we should mention.
First of all, from a modelling perspective, the main application of this process is the description of the evolution of a population taking into account the age of its individuals. There are several other examples of models serving the same purpose, the most famous ones being the Bellman-Harris and Crump-Mode-Jagers processes, respectively introduced in [Har68] and [CM68] which have been extensively studied afterwards, see e.g. [Ner81; JS08; Sev64; Sev68]. Processes with similar dynamics arise in different biological contexts, in particular in growth fragmentation models, which also have been the subject of many works [Ber23; BGY23; FM21].
Secondly, even if the LGW was not specifically studied previously, there exists more general results which apply to this process. In particular, there is a very proficient literature dealing with Galton-Watson processes with countably many types. Assuming the mean matrix M is irreducible and aperiodic, Sagitov [Sag12] uses the Perron Frobenius theory of countable matrices [Sen06] to extend the well-known classification of Galton-Watson processes with finitely many types into three regimes (subcritical, critical, supercritical) to the case of an countable number of types. The role of the spectral radius of the mean matrix M is played by the convergence parameter $N(M)$ of M, defined as $N(M) = \limsup_{n\geqslant 0} M^n(i,j)^{1/n}$ which does not depend on (i,j) since M is irreducible. The quantity $N(M)^{-1}$ is the convergence radius of the series $\sum_{n\geqslant 0} s^n M^n(i,j)$, $i,j \geqslant 0$. More precisely, if $N(M) < 1$ (respectively $= 1$, > 1) the process is said to be subcritical, (respectively critical, supercritical). As explained in [Zuc11], this classification is however insufficient to decide whether there is almost sure extinction or survival with positive probability of the population. Indeed, the position with respect to 1 of $N(M)$ only characterizes whether $\tilde{\mathbf{q}} = 1$ or $\tilde{\mathbf{q}} < 1$, where

$$\tilde{\mathbf{q}} = (\tilde{\mathbf{q}}(x))_{x\in\mathbb{X}} = \left(\mathbb{P}\left[\bigcap_{y\in\mathbb{X}}\bigcup_{N\geqslant 0}\bigcap_{n\geqslant N}\{Z_n(y) = 0\}|Z_0 = \delta_x\right]\right)_{x\in\mathbb{X}}.$$

The quantity $\tilde{\mathbf{q}}(x)$ is the probability of local extinction starting from one individual of type x, that is the probability that each type y goes extinct asymptotically. When the

number of types is finite, local extinction is equivalent with extinction hence $\mathbf{q} = \tilde{\mathbf{q}}$. If the number of types is countable, we only know that extinction implies local extinction, thus $\mathbf{q} \leqslant \tilde{\mathbf{q}}$. Finding sufficient conditions for $\mathbf{q} = \tilde{\mathbf{q}}$ is much more difficult. Therefore, Sagitov ensures the survival in the supercritical regime, but does not proves the a.s. extinction of subcritical or critical regimes.
The classification of countable MGWs into three regimes is sharpened using the notion of R-recurrence and R-transience : when M is irreducible, either all the series $\sum_{n\geqslant 0} N(M)^{-n} M^n(i,j), i,j \in \mathbb{Z}_+$ converge, and the LGW process is called $N(M)^{-1}$-transient, or they all diverge, in which case the process is $N(M)^{-1}$-recurrent. In some regimes, some strong results have been established, such that for example, a Kesten-Stigum type result in the case of a supercritical and $N(M)^{-1}$-recurrent process appearing in [Moy67]. However, this sharpened classification does not seem to yield general necessary and sufficient conditions for almost sure extinction on Galton-Watson with countably many types.
In [BH19], the authors focus on lower Hessenberg Galton-Watson processes, that is, Galton-Watson processes with countably many types and a mean matrix M satisfying $M(i,j) = 0$ for any $j > i+1$. In this particuliar setup, they provide a necessary and sufficient criterion for $\mathbf{q} = \tilde{\mathbf{q}}$, and thus for a.s. extinction. The LGW clearly is an example of lower Hessenberg GW process, however the criterion of [BH19] is not really explicit in terms of the coefficients of the mean matrix and thus is not easy to check on LGW, moreover it involes second moments of the offspring distributions.
Finally, there is literature focusing on the properties of infinite Leslie matrices, without seeing them as the mean matrix of a Galton-Watson process. In [Dem72] such matrices are introduced for the first time. Under pretty restrictive conditions (including $\lim_{n\to\infty} s_n = 0$), the author proves that M acts on the left as a compact operator on $\mathcal{L}^2(\mathbb{Z}_+)$ and proves the existence of an eigenvalue λ and an eigenvector π such that $\pi M = \lambda M$. In [GL09], the conditions of [Dem72] are weakened, the authors additionally obtain the quasi-compactness of the operator associated with M on some well chosen weighted $\mathcal{L}^1$ space, and thus an ergodicity property of the form

$$\|M^n - \lambda^n h\pi\| = \underset{n\to\infty}{O}(\rho^n)$$

where $0 < \rho < \lambda$. Such ergodicity results, together with methods inspired from [Har68], might imply interesting results on the LGW. However, these ergodicity properties are obtained under some pretty restrictive assumptions, which are additionally unexplicit in terms of the coefficients $(f_n, s_n)_{n\geqslant 0}$. Indeed, the main assumption of [GL09], is $\limsup_{n\to\infty} s_n < \liminf_{k\to\infty} r([M]_k)$, where $r([M]_k)$ refers to the spectral radius of the $k \times k$ matrix obtained keeping the first k rows and columns of M. Finally, in [ABO14], a different point of view is used to compute many quantities related to the powers of M. In particular the convergence parameter $N(M)$ is characterized as the only root of a certain power series in [ABO14, Corollary 42].
In this chapter, rather than presenting a patchwork of applications of the aforementioned papers to the LGW, we choose to present a unified approach, which describes the LGW relying heavily on its specific structure. In particular, we try to exhibit conditions and criteria which are as weak as possible, and as explicit as possible in terms of the reproduction laws $(\mathcal{L}_x, x \in \mathbb{Z}_+)$ and the coefficients $(f_x, s_x)_{x\in\mathbb{Z}_+}$.

In particular, we always assume

Ass. III.1. For all $x \in \mathbb{Z}_+$, $0 < s_x \leqslant 1$ and $f_x \geqslant 0$. Moreover, the set $\mathfrak{F} = \{x \in \mathbb{Z}_+ | f_x > 0\}$ is infinite.

This condition is strictly weaker than what is assumed in [Dem72; GL09]. We introduce the quantity

$$m = \sum_{n \geqslant 0} s_0 \cdots s_{n-1} f_n \in [0, +\infty].$$

Our first result is the following characterization of a.s. extinction :

Theorem 1. *Let (Z_n) be a LGW satisfying Assumption III.1. Then :*

i) If $\prod_{n \in \mathbb{Z}_+} s_n > 0$ or $m > 1$ then $\mathbf{q} < 1$,

ii) Conversely if $\prod_{n \in \mathbb{Z}_+} s_n = 0$ and $m \leqslant 1$ then $\mathbf{q} = 1$.

This very elementary characterization of a.s. extinction does not seem to be a direct corollary of the previously mentioned existing results in the literature. To obtain a more precise description of the process, we investigate the mean matrix M, looking for its eigenvalues, left and right eigenvectors, and for ergodicity properties. Our approach involves the series

$$g : \lambda \in [0, +\infty) \mapsto g(\lambda) = \sum_{n \geqslant 0} \frac{s_0 \cdots s_{n-1} f_n}{\lambda^{n+1}} \in (0, +\infty].$$

Moreover, for any positive, measurable and bounded function ψ, we note $\mathcal{M}_+(\psi)$ the set of positive, measures μ with possibly infinte mass but such that $\mu(\psi) < +\infty$. We consider the set $\mathcal{M}_\psi = \{\mu^+ - \mu^-, \mu^+, \mu^- \in \mathcal{M}_+(\psi)\}$, which we endow with the norm $\|\mu\|_{\mathcal{M}(\psi)} = \mu^+(\psi) + \mu^-(\psi)$. Besides a complete description of the left and right eigenelements of M, we establish the following necessary and sufficient ergodicity result.

Theorem 2. *Consider a LGW with mean matrix M, satisfying Assumption III.1. We assume additionally that $\gcd(\mathfrak{F} + 1) = 1$. Then for any $\lambda > 0$, the two following claims are equivalent.*

i) $g(\lambda) = 1$ and $g'(\lambda) < \infty$.

ii) There exists $h \in \mathbb{R}_+^{\mathbb{Z}_+}, h \neq 0$ and $\pi \in \mathcal{M}(h), \pi \neq 0$ such that for any measure $\mu \in \mathcal{M}_+(h)$, it holds

$$\lim_{n \to \infty} \left\| \lambda^{-n} \mu M^n - \frac{\mu(h)}{\pi(h)} \pi \right\|_{\mathcal{M}(h)} = 0. \tag{III.2}$$

Notice that since g is strictly decreasing, there is at most one solution to the equation $g(\lambda) = 1$, which we denote λ_0. As a consequence, (III.2) can only hold for $\lambda = \lambda_0$. Additionally, π and h are, in this case, the left and right λ_0-eigenvectors, respectively, of the mean matrix M and are expressed, up to renormalization, as

$$\pi(n) = \frac{s_0 \ldots s_{n-1}}{\lambda_0^n} \text{ and } h(n) = \frac{1}{\lambda_0} \frac{\sum_{k \geqslant n} \pi(k) f_k}{\pi(n)}.$$

Equation (III.2) also provides an interpretation of λ_0 as average growth rate of the population. It appears in the proof of Theorem 2 that the quantity λ_0 is actually the convergence parameter of M. Thus, following Sagitov [Sag12], we say that the LGW is subcritical (resp. critical, resp. supercritical) whenever $\lambda_0 < 1$ (resp. $= 1, > 1$).
Using a Doeblin minorization, we provide an additional sufficient condition to strengthen (III.2) into an explicit geometric convergence of the form

$$\left\| \lambda^{-n} \mu M^n - \frac{\mu(h)}{\pi(h)} h \right\|_{\mathcal{M}(h)} = \underset{n\to\infty}{O}(\rho^n),$$

for some $\rho < 1$ independent of μ.

Finally, we use these ergodicity results to prove the following estimates on the LGW process $(Z_n)_{n\geqslant 0}$.

Theorem 3. *Consider a LGW with mean matrix M satisfying Assumption III.1. We assume additionally that $\inf_{n\geqslant 0} f_n > 0$ and that there exists $\lambda_0 > 0$ such that $g(\lambda_0) = 1$.*

i) (Subcritical case) If $\lambda_0 < 1$ then

$$\mathbb{P}[Z_n \neq 0] = \underset{n\to\infty}{O}(\lambda_0^n),$$

for any initial population.

ii) (Supercritical case) If $\lambda_0 > 1$, assuming additionally that

- $\gcd(\mathfrak{F} + 1) = 1$
- *h is bounded*
- $\sup_{x\in\mathbb{X}} \frac{\mathbb{E}[Z_1(h)^2 | Z_0 = \delta_x]}{h(x)} < \infty$

then there exists a nonnegative, integrable random variable W such that $\mathbb{P}[W > 0] > 0$ and

$$\mathbb{1}_{W>0} \frac{Z_n(f)}{\lambda_0^n} \xrightarrow[n\to\infty]{a.s.} W \frac{\pi(f)}{\pi(h)},$$

for any bounded function f.

Note that the assumption $\inf_{n\geqslant 0} f_n > 0$ is not minimal to obtain a geometric decay of the survival probability, in particular to obtain such a decay at a rate $\lambda > \lambda_0$. Proposition III.10 provides conditions on real numbers λ under which M admits a right λ-eigenfunction which is bounded from below. If such an eigenfunction exists, for some $\lambda \in (\lambda_0, 1)$, we actually establish in Proposition III.23 that $\mathbb{P}[Z_n \neq 0] = O(\lambda^n)$.

III.1.3 Structure of the chapter

In Section III.2 we prove Theorem 1. In Section III.3, we present a necessary and sufficient condition for the existence of non negative eigenelements, and we describe them. This allows to perform a Doob transformation and derive an auxiliary Markov chain $(Y_n)_{n\geqslant 0}$ which describes a notion of typical lineage of the LGW. As we shall see, this Markov chains encapsulates numerous properties of the first moment of the LGW. We study in Section III.4 the properties of $(Y_n)_{n\geqslant 0}$, and prove in particular Theorem 2. Finally, in Section III.5, we prove Theorem 3. It appears as a direct consequence of Proposition III.23 and III.25, which respectively guarantee the exponential decay of the tail of the extinction time in the subcritical regime and the exponential growth of the size of the surviving population in the supercritical regime.

III.1.4 Recurring examples

Throughout this chapter, to try and build an intuition of our results, we apply them often to two specific class of Leslie matrices. Our first elementary example is

Example A

We assume that $s_n = s$ and $f_n = f$ for any $n \geqslant 0$. Therefore, the mean matrix of the process is

$$M = \begin{pmatrix} f & s & 0 & 0 & \dots \\ f & 0 & s & 0 & \dots \\ f & 0 & 0 & s & \ddots \\ f & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{III.3}$$

We also introduce

Example B

We assume that

i) There exist $a_1, a_2 \geqslant 0$ such that $a_1 \leqslant f_n/(1+n)^\alpha \leqslant a_2$ for all $n \geqslant 0$

ii) $\lim_{n\to\infty} s_n = s$ for some $s \in [0,1]$ and $g(s) > 1$.

Each time the Example A. or B. is mentioned in the sequel, we restrict ourselves to one of these two situations. Sometimes, we temporarily add some specific assumptions on the parameters s, f or α, s.

III.2 Condition for a.s. extinction

A first intuitive approach relying on the particular structure of LGW processes can be used to have a full insight of the almost sure extinction. We recall that $\mathbf{q}(x)$ refers to the extinction probability starting from state x. More precisely

$$\begin{aligned} \mathbf{q}(x) &= \mathbb{P}[\mathbf{Ext}\,|Z_0 = \delta_x] \\ &= \mathbb{P}[\exists n \in \mathbb{Z}_+, Z_n = 0|Z_0 = \delta_x] \\ &= 1 - \lim_{n\to\infty} \mathbb{P}[Z_n \neq 0|Z_0 = \delta_x]. \end{aligned}$$

Moreover, we recall that $m = \sum_{n\in\mathbb{Z}_+} s_0 \dots s_{n-1} f_n \in (0, +\infty]$. The main result proved in this subsection is Theorem 1, which provides a full classification of the LGW between the regimes of extinction with probability 1 and the regimes of survival with positive probability.

On the one hand, our criterion relies on the position with respect to 1 of the parameter m, which happens to be the mean of the offspring distribution of some underlying monotype Galton-Watson process. However, it also involves a more unusual positivity criterion on $\prod_{n\geqslant 0} s_n$. This quantity is actually the probability that an individual of age 0 survives forever. Therefore, if $\prod_{n\geqslant 0} s_n > 0$, and $m \leqslant 1$, the population size stays bounded, but the process may survive, since a few individuals survive for an infinite time duration. In this situation, it is possible to encounter local extinction. This means that each type goes extinct but the process survives (see [Bra18] for a development on this topic). This phenonmenon where the population survives thanks to some transient behavior in the type set allows to break the classical dichotomy stated and proven in [Har68] in the finite type setup : $\mathbb{P}[\lim \|Z_n\| = 0] + \mathbb{P}[\lim \|Z_n\| = +\infty] = 1$, where $\|\cdot\|$ refers to the $\mathcal{L}^1$ norm. Such situations will appear in the proof of our theorem when $\prod s_n > 0$ and $m \leqslant 1$: there is almost sure local extinction, the population size stays bounded almost surely but, with positive probability, the population survives.

Let us apply Theorem 1 on Example A.

Example A: $f_n = f > 0, s_n = s \in [0,1]$

- *If* $s = 1$ *then* $\prod_{n\geqslant 0} s_n = 1$, (Z_n) *survives with probability* 1 *starting from any intial population and* $\mathbf{q} = 0 < 1$.
- *If* $s < 1$ *then* $\prod_{n\geqslant 0} s_n = 0$ *and* $m = \sum_{n\geqslant 0} s_0 \cdots s_{n-1} f_n = \frac{f}{1-s}$. *Hence,* $\mathbf{q} = 1$ *if* $f + s \leqslant 1$, *otherwise* $\mathbf{q} < 1$. *As expected, the survival criterion* $f + s > 1$ *is increasing both in* f *and* s.

Theorem 1 relies on an auxiliary monotype Galton-Watson process $(\widehat{Z}_n)$ counting the individuals of type 0 of (Z_n). To define it, we use the classic Ulam-Harris-Neveu tree-based construction of the process (Z_n), which allows to recover the process (Z_n) as some functional of a random tree, when we assume that (Z_n) is started from one individual. We recall here the most important notations, and refer the reader to [Ber18; De 17] for a complete construction of these objects. Let $\mathcal{U} = \bigcup_{n\geqslant 0} (\mathbb{Z}_+)^n$ denote the Ulam-Harris-Neveu tree, with the convention $(\mathbb{Z}_+)^0 = \varnothing$. We also note $l(u)$ the generation of u, that is $l(u) = n \Leftrightarrow u \in \mathbb{Z}^n$. Two elements $u = (u_1 \dots u_k), v = (v_1, \dots, v_n) \in \mathcal{U}$ can be concatenated to form $u \star v = (u_1 \dots u_k, v_1, \dots, v_n) \in \mathcal{U}$. For any individuals $u, v \in \mathcal{U}$, we note $u \prec v$ if u is an ancestor of v (and $u \neq v$), that is, if there exists $w \in \mathcal{U}$ such that $v = u \star w$. Noting $u = (u_1 \dots u_k) \in \mathbb{Z}_+^k$, $v = (v_1, \dots, v_n) \in \mathbb{Z}_+^n$, we see that

$$u \prec v \Leftrightarrow k < n \text{ and } (u_1, \dots, u_k) = (v_1, \dots, v_k).$$

Any individual u has a finite number of ancestors, which can be arranged increasingly which respect to $\prec$. We call ancestral lineage of u the ordered array of ancestors of u. Our Galton-Watson tree is a random subtree $\mathcal{T}$ of $\mathcal{U}$. We note $\mathbb{G}_n = \{u \in \mathcal{T} | l(u) = n\}$ the random sets of individuals alive at time n in the population, and notice that $\mathcal{T} = \bigcup_{n\geqslant 0} \mathbb{G}_n$. Moreover, each individual u of $\mathcal{T}$ has a random label $X(u) \in \mathbb{X} = \mathbb{Z}_+$ which refers to the age of the individual u. Then, the random measure Z_n can be obtained as

$Z_n = \sum_{u \in \mathbb{G}_n} \delta_{X(u)}$.. When $u \prec v$ are two individuals of our tree, we define the relation $\mathcal{H}_{u,v}$ which holds if there is no individual of age 0 in the ancestral lineage of v after u :

$$\mathcal{H}_{u,v} \Leftrightarrow \forall w \in \mathcal{T}, u \prec w \prec v \Rightarrow X(w) \neq 0. \tag{III.4}$$

Notice that the offspring of an individual of age x contains only individuals of age $x+1$ and 0. Therefore, if $u_k \prec \cdots \prec u_n$ is a lineage of individuals, such that $u_i \in \mathbb{G}_i$ for all i, then the following characterization holds

$$\mathcal{H}_{u_k,u_n} \Leftrightarrow X(u_i) = X(u_k) + i - k \text{ for all } k \leqslant i \leqslant n-1 \tag{III.5}$$
$$\Leftrightarrow X(u_{n-1}) = X(u) + n - 1 - k \tag{III.6}$$

To define our auxiliary process $(\widehat{Z}_n)$, we must consider a LGW (Z_n), started with a population consisting of exactly one individual of age 0 : $\mathbb{G}_0 = \{\varnothing\}$, $X(\varnothing) = 0$, and $Z_0 = \delta_0$. We set

$$\widehat{\mathbb{G}}_0 = \{\varnothing\} = \mathbb{G}_0 \tag{III.7}$$

and for all $n \geqslant 0$

$$\widehat{\mathbb{G}}_{n+1} = \bigcup_{v \in \widehat{\mathbb{G}}_n} \{u \in \mathcal{T} | v \prec u, X(u) = 0 \text{ and } \mathcal{H}_{v,u}\}. \tag{III.8}$$

Then, we define the integer valued process $(\widehat{Z}_n)$ by setting

$$\widehat{Z}_n = |\widehat{\mathbb{G}}_n|. \tag{III.9}$$

Furthermore, for any $k \geqslant 0$, we introduce the integer-valued random variable

$$X_k = \sum_{w \in \mathbb{G}_{k-1}} \mathbb{1}_{X(w)=k-1} \sum_{\substack{u \in \mathbb{G}_k \\ w \preceq u}} \mathbb{1}_{X(u)=0} = |\{u \in \mathbb{G}_k | X(u) = 0 \text{ and } \mathcal{H}_{\varnothing,u}\}|\}.$$

X_k counts the indivudals of age 0 in generation k who have no individual of age 0 in their ancestral lineage besides the root of the tree. Our auxiliary process satisfies the following property.

Proposition III.3. *Let us consider a LGW (Z_n) started from $Z_0 = \delta_0$. Then the process $(\widehat{Z}_n)_{n \geqslant 0}$ is a single-type Galton-Watson process with $\widehat{Z}_0 = 1$ and an offspring distribution given by the law of the generic offpsring variable*

$$\widehat{L} = \sum_{k \geqslant 1} X_k.$$

In particular

$$\mathbb{E}[\widehat{L}] = \sum_{n \geqslant 0} s_0 \dots s_{k-1} f_k = m.$$

Proof of Proposition III.3. If $v \in \mathcal{T}$ and $X(v) = 0$, let

$$\widehat{L}_v = \{u \in \mathcal{T} | v \prec u, X(u) = 0 \text{ and } \mathcal{H}_{v,u}\}$$

denote the set of children of v in the process $(\widehat{Z}_n)_{n\geqslant 0}$. By (III.8), it holds

$$\widehat{Z}_{n+1} = \sum_{v\in\widehat{\mathbb{G}}_n} |\widehat{L}_v|.$$

To prove that the process $(\widehat{Z}_n)$ is a monotype Galton-Watson process with generic offspring variable $\widehat{L}$, we need to show that conditionally on $\widehat{\mathbb{G}}_n$, the $|\widehat{L}_v|, v \in \widehat{\mathbb{G}}_n$ are independent variables distributed as $\widehat{L}$. By (III.6), we derive, for $v \in \mathbb{G}_p$,:

$$\widehat{L}_v = \bigcup_{k\geqslant 1} \{u \in \mathbb{G}_{p+k} | v \prec u, X(u) = 0 \text{ and } X(u^*) = k-1\},$$

where u^* refers to the parent of u in $\mathcal{T}$. This set can be decomposed according to the parent of u, as

$$\widehat{L}_v = \bigcup_{k\geqslant 1} \bigcup_{\substack{w\in\mathbb{G}_{p+k-1}\\ v\prec w\\ X(w)=k-1}} \{u \in \mathbb{G}_{p+k} | w \prec u, X(u) = 0\}.$$

Therefore,

$$\begin{aligned}
|\widehat{L}_v| &= \sum_{k\geqslant 0} \sum_{\substack{v\prec w\\ l(w)=p+k-1}} \mathbb{1}_{X(w)=k-1} \sum_{\substack{w\prec u\\ l(u)=p+k}} \mathbb{1}_{X(u)=0} \\
&= \sum_{k\in\mathbb{Z}_+} \sum_{w\in{\mathbb{Z}_+}^{k-1}} \mathbb{1}_{X(v\star w)=k-1} \sum_{u\in\mathbb{Z}_+} \mathbb{1}_{X(v\star(w\star u))=0}.
\end{aligned}$$

Notice that for $v = \varnothing$, we have $\left|\widehat{L}_\varnothing\right| = \widehat{L}$. Moreover, conditionally on $X(v) = 0$, $|\widehat{L}_v|$ is a function of $(X(v \star w), w \in \mathcal{U})$. Since by the branching property,

$$\mathbf{dist}[((X(v \star w))_{w\in\mathcal{U}} | X(v) = 0] = \mathbf{dist}[((X(w))_{w\in\mathcal{U}} | X(\varnothing) = 0],$$

then $|\widehat{L}_v| \overset{d}{=} |L_\varnothing| = \widehat{L}$ conditionally on $X(v) = 0$. Furthermore, the branching property implies that if $v \nprec v'$ and $v' \nprec v$ then $(X(v \star w), w \in \mathcal{U})$ and $(X(v' \star w), w \in \mathcal{U})$ are mutually independent. If $v, v' \in \widehat{\mathbb{G}}_n$, both of them are of type 0 and have the same number of type 0 ancestors. Hence none of them can be an ancestor of the other one. Therefore, conditionally on $\widehat{G}_n$, the $(|L_v|, v \in \widehat{G}_n)$ are independent variables distributed as $\widehat{L}$. This implies that $(\widehat{Z}_n)_n$ is a monotype Galton-Watson process with generic offspring variable $\widehat{L}$. □

We additionally state the following Lemma, which guarantees that the single type Galton-Watson process $(\widehat{Z}_n)_{n\geqslant 0}$ is not trivial.

Lemma III.4. *Under Assumption III.1, the variable $\widehat{L}$ satisfies*

$$\mathbb{P}[\widehat{L} = 1] < 1.$$

Proof of Lemma III.4. We recall that by Proposition III.3, $\widehat{L} \overset{d}{=} \sum_{i=1}^{+\infty} X_k$, where X_k refers to the number of individuals u in a population started from $Z_0 = \delta_0$ such that

- $u \in \mathbb{G}_k$
- u is of age 0 : $X(u) = 0$
- $X(u_i) = i$ for any $i \leqslant k-1$, where $(u_0 = \varnothing, \dots, u_k = u)$ the ancestral lineage of u,

By Assumption III.1, there exist two ages $k < k'$ such that $f_k, f_{k'} > 0$. With probability

$$s_0 \dots s_{k'-1}\mathbb{P}[\{S_k = 1\} \cap \{F_k \geqslant 1\}] \times s_{k+1} \dots s_{k'-1}\mathbb{P}[\{S_{k'} = 1\} \cap \{F_k \geqslant 1\}] > 0,$$

the original individual survives at least until time k', creating offspring of age 0 both at time k and k'. Therefore, with positive probability, we have $X_{k+1}, X_{k'+1} \geqslant 1$ and $\widehat{L} \geqslant 2$. □

Let $\widehat{\mathbf{Ext}} = \bigcup_{n\geqslant 0}\{\widehat{Z}_n = 0\}$ denote the extinction event for $(\widehat{Z}_n)$. The extinction events $\mathbf{Ext}$ and $\widehat{\mathbf{Ext}}$ are closely linked together by the following lemma. This yields the necessary and sufficient condition for extinction stated in Theorem 1.

Lemma III.5. *i) If $Z_0 = \delta_0$ then $\mathbf{Ext} \subset \widehat{\mathbf{Ext}} = \left\{\sum_{n\geqslant 0} Z_n(0) < +\infty\right\}$.*

ii) If $\prod_{n\geqslant 0} s_n = 0$ then $\left\{\sum_{n\geqslant 0} Z_n(0) < +\infty\right\} \subset \mathbf{Ext}$ up to a $\mathbb{P}$-negligeable event.

Proof of Lemma III.5. Let us assume assume that $Z_0 = \delta_0$. Then the auxiliary process $(\widehat{Z}_n)$ is well defined. The first step of the proof relies on noticing that the extinction of $(\widehat{Z}_n)$ can be characterized by the finiteness of the total number of individuals :

$$\left\{\sum_{n\geqslant 0} \widehat{Z}_n < \infty\right\} = \widehat{\mathbf{Ext}} = \bigcup_{n\geqslant 0}\left\{\widehat{Z}_n = 0\right\}. \tag{III.10}$$

Indeed, 0 is an absorbing state of the Markov chain $(\widehat{Z}_n)$. As a consequence, if $\widehat{Z}_{n_0} = 0$ for some $n_0 \geqslant 0$ then $\widehat{Z}_n = 0$ for all $n \geqslant n_0$ and $\sum_{n\geqslant 0} \widehat{Z}_n = \sum_{n<n_0} \widehat{Z}_n < \infty$. Conversely, if the sum is finite, then it must hold $\lim_{n\to\infty} \widehat{Z}_n = 0$ which implies that $\widehat{Z}_n = 0$ for n large enough. This proves (III.10). Since additionally $\bigcup_{n\in\mathbb{Z}_+} \widehat{\mathbb{G}}_n = \bigcup_{n\geqslant 0}\{u \in \mathbb{G}_n | X(u) = 0\}$, it holds

$$\sum_{n\geqslant 0} \widehat{Z}_n = \left|\bigcup_{n\geqslant 0} \widehat{\mathbb{G}}_n\right| = \left|\bigcup_{n\geqslant 0}\{u \in \mathbb{G}_n | X(u) = 0\}\right| = \sum_{n\geqslant 0} Z_n(0) \tag{III.11}$$

Equations (III.10) and (III.11) yield

$$\widehat{\mathbf{Ext}} = \left\{\sum_{n\geqslant 0} Z_n(0) < +\infty\right\}.$$

Similarly as in (III.10), it holds

$$\mathbf{Ext} = \left\{\sum_{n\geqslant 0} Z_n(\mathbb{1}) < \infty\right\}. \tag{III.12}$$

Moreover, it is clear that $Z_n(0) \leqslant Z_n(\mathbb{1})$, thus by (III.12), $\sum_{n\geqslant 0} \widehat{Z}_n \leqslant \sum_{n\geqslant 0} Z_n(\mathbb{1})$. Putting this together with (III.10), (III.11), (III.12) yields

$$\mathbf{Ext} = \left\{\sum_{n\geqslant 0} Z_n(\mathbb{1}) < \infty\right\} \subset \left\{\sum_{n\geqslant 0} \widehat{Z}_n < \infty\right\} = \widehat{\mathbf{Ext}}.$$

This proves *i)*.
For any $1 \leqslant n \leqslant N$, set

$$A_{n,N} = \{Z_N \neq 0\} \cap \{Z_k(0) = 0 \text{ for } k \in [\![n+1, N]\!]\}.$$

$A_{n,N}$ is the event where the population survives until time N, but there is no individual of age 0 between times $n+1$ and N. Therefore, for any x, conditionally on $Z_n = \delta_x$, on $A_{n,N}$, the only individual alive at time n must create a nonempty offspring, but this offspring cannot contain any individual of age 0 : the individual must survive (which happens with probability s_x), and $Z_{n+1} = \delta_{x+1}$. Iterating this reasoning we obtain

$$\mathbb{P}\left[A_{n,N} | Z_n = \delta_x\right] \leqslant s_x \dots s_{x+N-n-1}.$$

Moreover, there is a clear inclusion $A_{n,N+1} \subset A_{n,N}$. Therefore

$$\mathbb{P}\left[\bigcap_{N\geqslant n} A_{n,N} | Z_n = \delta_x\right] = \lim_{N\to\infty} \mathbb{P}\left[A_{n,N} | Z_n = \delta_x\right] \leqslant \prod_{k=n}^{\infty} s_k.$$

As a consequence, if $\prod_{k\geqslant 0} s_k = 0$, then for all $n, x \geqslant 0$

$$\mathbb{P}\left[\bigcap_{N\geqslant n} A_{n,N} | Z_n = \delta_x\right] = 0.$$

Moreover, the event $\bigcap_{N\geqslant n} A_{n,N}$ satisfies

$$\mathbb{P}\left[\bigcap_{N\geqslant 0} A_{n,N} | Z_n\right] \leqslant \sum_{x\geqslant 0} Z_n(x) \mathbb{P}\left[\bigcap_{n\geqslant N} A_{n,N} | Z_n = \delta_x\right] = 0.$$

This clearly proves that $\mathbb{P}[\bigcup_{n\geqslant 0} \bigcap_{N\geqslant n} A_{n,N}] = 0$. Finally, the inclusion

$$\left\{\sum_{n\geqslant 0} Z_n(0) < +\infty\right\} \cap \mathbf{Ext}^c \subset \bigcup_{n\geqslant 0} \bigcap_{N\geqslant n} A_{n,N},$$

yields *ii)*.

□

We have now in hand the main tools to prove Theorem 1.

Proof of Theorem 1. First, suppose that $\prod_{n\in\mathbb{Z}_+} s_n > 0$ and let us prove that $\mathbf{q} < 1$. We set $Z_0 = \delta_x$, and study the number $Z_n(n+x)$ of individuals of age $n+x$ at time n. Notice that

$$\mathbb{P}(Z_n(n+x) > 0 | Z_0 = \delta_x) \geqslant \prod_{i=x}^{n+x-1} s_i \underset{n\to\infty}{\longrightarrow} \frac{\prod_{i\in\mathbb{Z}_+} s_i}{\prod_{i\leqslant x-1} s_i},$$

hence

$$\prod_{n\in\mathbb{Z}_+} s_n > 0 \Rightarrow \forall x \in \mathbb{Z}_+, \lim_{n\to\infty} \mathbb{P}_{Z_0=\delta_x}(Z_n \neq 0) = 1 - \mathbf{q}(x) > 0$$

and thus $\mathbf{q} < 1$.

If $\sum_{n\geqslant 0} s_0 \cdots s_{n-1} f_n = \mathbb{E}[\widehat{L}] > 1$ then the GW-process $(\widehat{Z_n})$ is supercritical, therefore it survives with positive probability. As a consequence, from Lemma III.5, *ii)*, starting from $Z_0 = \delta_0$, the process (Z_n) survives with positive probability, that is $\mathbf{q}(0) < 1$. Suppose now that $Z_0 = \delta_x$. We have assumed that there exists $k > x$ such that $f_k > 0$, thus

$$0 < s_x \dots s_{k-1} f_k \leqslant \mathbb{E}[Z_{k-x+1}(0) | Z_0 = \delta_x].$$

As a consequence $\mathbb{P}[Z_{k-x+1}(0) \geqslant 1 | Z_0 = \delta_x] > 0$, in other words, with positive probability, there exists at least one individual of type 0 at time $k - x + 1$. By the branching property, the subtree descending from such an individual has the same distribution as $\mathcal{T}$ started with $Z_0 = \delta_0$, therefore it survives with positive probability $1 - q(0)$. Thus $1 - q(x) \geqslant \mathbb{P}[Z_{k-x+1}(0) \geqslant 1 | Z_0 = \delta_x](1 - q(0)) > 0$ ie (Z_n) to survive with positive probability. As a consequence $q(x) < 1$. This concludes the proof of *i)*.

Let us assume now that $m \leqslant 1$, $\prod_{n\geqslant 0} s_n = 0$ and $Z_0 = \delta_0$. Then the single type Galton-Watson process $(\widehat{Z}_n)$ is either subcritical or critical. Moreover, by Lemma III.4, it is not trivial. This implies that $(\widehat{Z}_n)$ goes extinct almost surely, that is $\mathbb{P}[\widehat{\mathbf{Ext}}] = 1$. Both the assumptions of Lemma III.5 are satisfied, therefore $\mathbb{P}[\mathbf{Ext}] = \mathbb{P}[\widehat{\mathbf{Ext}}]$, so that $\mathbf{q}(0) = 1$. Let us assume now that we start from a population $Z_0 = \delta_x$. Since $\prod_{n\geqslant x} s_n = 0$, the initial individual survives only during a finite number of generations. All the subtrees descending from their offsprings of type 0 form a finite number of independent copies of $\mathcal{T}$, each started from one individual of type 0, and which are therefore all finite since $\mathbf{q}(0) = 1$. This proves that $\mathbf{q}(x) = 1$ for all x, thus *ii)* is proved. □

III.3 Eigenelements of the mean matrix

We seek now a more precise understanding of the LGW process. In particular, we want to understand the speed of extinction of the process when $\mathbf{q} = 1$, and to obtain asymptotic estimates on Z_n conditionally on survival when $\mathbf{q} < 1$. To do so, we investigate the mean of our process. Since $\mathbb{E}[Z_n | Z_0 = \delta_x] = \delta_x M^n$ for any $x \geqslant 0$, it is natural to look for an ergodicity property on M^n similar to the one given by Perron-Frobenius' theorem when M is a $d \times d$ matrix. The first important question is the existence of positive eigenelements of M. In particular, if there exists a left eigenvector with finite $\mathcal{L}^1$ norm, it represents a distribution of types in the population which is preserved in average. The question of the existence of such eigenvectors is partially discussed in the existing literature. In particular [Dem72] shows the existence of a left eigenvector under the following assumptions

- the set $\mathfrak{F}$ is infinite, $\delta := \gcd(\mathfrak{F} + 1) = 1$
- the sequence $(f_n)_{n\geqslant 0}$ is bounded

- $s_n > 0$ for all n and $\lim_{n\to\infty} s_n = 0$

which can be relaxed as will be shown here. In [GL09], the assumption $\lim s_n = 0$ is replaced by

$$\limsup s_k < \sup_n r(\{M\}_n)$$

where $r(\{M\}_n)$ refers to the spectral radius of the $n \times n$ matrix obtained by truncating the matrix M and keeping only its first n columns and rows. Under this hypotheses, the authors introduce the sets

$$\mathcal{L}^{1,k} = \left\{ u = (u_n)_{n\geqslant 0} \in \mathbb{R}^{\mathbb{Z}_+} \middle| \|u\|_{\mathcal{L}^{1,k}} = \sum_{n\geqslant 0} u_n n^k < \infty \right\}$$

and

$$\mathcal{L}^{+\infty,k} = \left\{ u = (u_n)_{n\geqslant 0} \in \mathbb{R}^{\mathbb{Z}_+} \middle| \|x\|_{\mathcal{L}^{+\infty,k}} = \sup_{n\geqslant 0} \frac{u_n}{n^k} < \infty \right\}$$

with $k \geqslant 0$. They prove that there exists $\lambda > \rho > 0$ and vectors $L \in \bigcap_{k\geqslant 1} \mathcal{L}^{1,k}$ and $R \in \bigcap_{k\geqslant 1} \mathcal{L}^{\infty,k}$ such that for all $k \geqslant 1$ and $u \in \mathcal{L}^{1,k}$

$$\|uM^n - \lambda^n \langle u, R\rangle L\|_{\mathcal{L}^{1,k}} = \underset{n\to\infty}{o}(\rho^n).$$

The restrictive assumptions in both [Dem72] and [GL09] ensure that M is a compact or quasi-compact operator on some well chosen Banach spaces. However, due to the simple structure of the matrix M, a more elementary approach allows to understand the eigenelements of M. In this section, we prove the existence of eigenvectors L and R under weaker conditions and we characterize these elements in Propositions III.6 and III.7.

III.3.1 Eigenfunctions

Notice first that, since the matrix M only has a finite number of non zero coefficients on each row, the linear map $f \mapsto Mf$ is well defined on $\mathbb{R}^{\mathbb{Z}_+}$. We introduce the series

$$g(\lambda) = \sum_{k=0}^{\infty} \frac{s_0 \dots s_{k-1} f_k}{\lambda^{k+1}}. \tag{III.13}$$

and its radius of convergence

$$R := \limsup (s_0 \cdots s_{n-1} f_n)^{\frac{1}{n}} = \inf\{\lambda \in \mathbb{R}_+ | g(\lambda) < +\infty\} \in [0, +\infty]. \tag{III.14}$$

In particular, when $R < \infty$, it holds $g(\lambda) < \infty$ for any $\lambda > R$. For $\lambda < R$, we naturally consider $g(\lambda)$ to be infinite. The same holds for $\lambda = 0$ even if $R = 0$. Moreover, for each $\lambda \in \mathbb{R}$, we introduce the functions $h_\lambda : \mathbb{Z}_+ \mapsto \mathbb{R}$, defined by

$$h_\lambda(n) = \frac{\lambda^n}{s_0 \dots s_{n-1}} \left(1 - \sum_{k=0}^{n-1} \frac{s_0 \dots s_{k-1} f_k}{\lambda^{k+1}} \right). \tag{III.15}$$

Proposition III.6. *Let M be an infinite Leslie matrix, satisfying Assumption III.1. Then*

i) $\mathrm{Ker}\,(M - \lambda\,\mathrm{Id}) = \mathbb{R}h_\lambda$ *for every* $\lambda \in \mathbb{R}$.

ii) The linear map $f \mapsto Mf$ *has nonnegative eigenelements if and only if* $R < \infty$. *In this case, the set* $\{\lambda \geqslant 0 |\,\mathrm{Ker}(M - \lambda\,\mathrm{Id}) \cap \mathbb{R}_+^{\mathbb{Z}_+} \neq \varnothing\}$ *of nonnegative eigenvalues associated with nonnegative eigenfunctions is exactly the interval* $[\lambda_0, +\infty)$ *where*

$$\lambda_0 = \inf\left\{\lambda \in \mathbb{R}^+, g(\lambda) \leqslant 1\right\} \in (0, +\infty).$$

Moreover, for any $\lambda \geqslant \lambda_0$

$$\mathrm{Ker}(M - \lambda\,\mathrm{Id}) \cap \mathbb{R}_+^{\mathbb{Z}_+} = \mathbb{R}_+ h_\lambda.$$

iii) $h_\lambda > 0$ *for all* $\lambda \geqslant \lambda_0$

Let us apply this proposition to our Example A, where R, λ_0, g, h_λ can be explicitely expressed in terms of f and s.

Example A: $f_n = f > 0, s_n = s \in [0, 1]$

In this case, $R = s$ *and for* $\lambda > s$,

$$g(\lambda) = \frac{f}{\lambda - s}.$$

Hence $\lambda_0 = f + s$. *Moreover, for* $\lambda \in \mathbb{R}\backslash\{s\}$, *it holds*

$$h_\lambda(n) = \frac{f + (\lambda - f - s)(\frac{\lambda}{s})^n}{\lambda - s} \tag{III.16}$$

and $h_s(n) = 1 - n\frac{f}{s}$. *In particular* $h_\lambda(0) = 1$ *for any* $\lambda \in \mathbb{R}$ *and*

- *If* $\lambda < s$ *then* $h_\lambda(n) \to \frac{f}{\lambda - s} < 0$ *as* $n \to \infty$.
- *If* $\lambda = s$ *then* $h_\lambda(n) \to -\infty$ *as* $n \to \infty$.
- *If* $s < \lambda < f + s$ *then* $h_\lambda(n) \sim \left(\frac{\lambda}{s}\right)^n \frac{\lambda - \lambda_0}{\lambda - s} \to -\infty$.
- *If* $\lambda = f + s$ *then* $h_\lambda(n) = 1$ *for all* n.
- *If* $\lambda > f + s$ *then* (III.16) *shows that* h_λ *is clearly positive.*

In this example, as stated in Proposition III.6,iii) the function h_λ *is positive iff* $\lambda \geqslant \lambda_0 = f + s$.

Proof of Proposition III.6 . For any function $h : \mathbb{Z}_+ \longrightarrow \mathbb{R}$ and any $n \geqslant 0$, $Mh(n) = s_n h(n+1) + f_n h(0)$. Therefore $Mh = \lambda h$ if and only if

$$h(n+1) = \frac{\lambda}{s_n} h(n) - \frac{f_n}{s_n} h(0) \text{ for } n \in \mathbb{Z}_+ \tag{III.17}$$

This clearly proves that the eigenspace associated with λ has dimension 1. Moreover it is easy to check that the function h_λ indeed satisfies the recursion (III.17). This proves *i)*.
By *i)*, the λ-eigenfunctions of M are colinear with h_λ. Thus, there exists non negative λ-eigenfunctions iff h_λ has constant sign. Since $h_\lambda(0) > 0$, the nonnegative λ-eigenfunctions of M only exist if $h_\lambda \geqslant 0$, in which case they are exactly the functions of the form αh_λ, for $\alpha \geqslant 0$. By (III.15), the function h_λ is nonnegative if and only if

$$g(\lambda) = \sum_{k=0}^{\infty} \frac{s_0 \dots s_{k-1} f_k}{\lambda^{k+1}} \leqslant 1 \tag{III.18}$$

Suppose $R < \infty$, then the function g is defined on (R, ∞), it is a strictly decreasing and tends to 0 at $+\infty$. Thus $\lambda_0 < \infty$, moreover $g(\lambda) \leqslant 1$ if and only if $\lambda \geqslant \lambda_0$. It remains to check that $\lambda_0 > 0$. If $R > 0$, then it simply holds $\lambda_0 \geqslant R > 0$. If $R = 0$, notice that $\lim_{\lambda \to 0} g(\lambda) = +\infty$ thus $\lambda_0 > 0$. This completes the proof of *ii)*
We want to check now that h_λ is strictly positive for $\lambda \geqslant \lambda_0$. The function g is strictly decreasing on (R, ∞), therefore if $\lambda > \lambda_0$, $g(\lambda) < g(\lambda_0) \leqslant 1$, and $h_\lambda(n) > 0$ for all n. Let us take now $\lambda = \lambda_0$. Since the set $\{k \geqslant 0 | f_k > 0\}$ is infinite, then it holds

$$\sum_{k=0}^{n-1} \frac{s_0 \dots s_{k-1} f_k}{{\lambda_0}^{k+1}} < g(\lambda_0) \leqslant 1,$$

hence $h_{\lambda_0}(n) > 0$ for all n. This proves *iii)*. □

In (III.15), the quantity $s_0 \cdots s_{n-1} f_n$ counts the average number of individuals of age 0 at time $n+1$ whose ancestral lineage did not go back to the type 0 before time $n+1$. Hence, the condition $R < \infty$ simply means that this quantity of such individuals is not growing overexponentially.

III.3.2 Eigenmeasures

The map $\mu \mapsto \mu M$ is well defined on the set of nonnegative (possibly infinite) measures $\mu = (\mu(n))_{n \in \mathbb{Z}_+}$ on $\mathbb{Z}_+$. In this subsection, we characterize the eigenelements of this map. Let us introduce, when $R < +\infty$, the measure π defined by

$$\pi(n) = \frac{s_0 \dots s_{n-1}}{\lambda_0^n}. \tag{III.19}$$

Proposition III.7. *Consider an infinite Leslie matrix M satisfying Assumption III.1. Then the map $\mu \mapsto \mu M$ admits a positive eigenvalue if and only if :*

$$R = \limsup (f_n s_0 \dots s_{n-1})^{\frac{1}{n}} < \infty \text{ and } g(R) = \sum_{n \in \mathbb{Z}_+} \frac{s_0 \dots s_{n-1}}{R^{n+1}} f_n \geqslant 1$$

If these conditions are satisfied then for any positive measure $\mu \neq 0$ and any positive real number λ,

$$\mu M = \lambda \mu \Leftrightarrow \lambda = \lambda_0 \text{ and } \exists a > 0, \mu = a\pi.$$

Example A: $f_n = f > 0, s_n = s \in [0, 1]$

> *We recall that* $R = s$, *and* $g(\lambda) = \frac{f}{\lambda - s}$, *thus* $g(s) = +\infty \geqslant 1$. *Moreover, for all* $n \geqslant 0$,
>
> $$\pi(n) = \left(\frac{s}{\lambda_0}\right)^n = \left(\frac{s}{s+f}\right)^n$$
>
> *and one can check that as claimed by Proposition III.7,* $\pi M = (f + s)M$.

Proof of Proposition III.7. The equation $\mu M = \lambda\mu$ is equivalent to

$$\begin{cases} \forall n \geqslant 1, \lambda\mu(n) = s_{n-1}\mu(n-1) & \text{(III.20a)} \\ \lambda\mu(0) = \sum_{n \geqslant 0} f_n \mu(n) & \text{(III.20b)} \end{cases}$$

and in turn to

$$\begin{cases} \mu = \mu(0)\pi & \text{(III.21a)} \\ \mu(0) = \mu(0) \sum_{n \geqslant 0} f_n \frac{s_0 \dots s_{n-1}}{\lambda^{n+1}} = \mu(0) g(\lambda) & \text{(III.21b)} \end{cases}$$

where π is given by (III.19). For any $\lambda > 0$ and $\mu(0) > 0$, condition (III.21b) is equivalent to

$$g(\lambda) = 1. \tag{III.22}$$

Notice that $g(\lambda) = +\infty$ for any $\lambda \geqslant 0$ when $R = +\infty$, in which case the equation $g(\lambda) = 1$ has no solution and $\pi M = \lambda\pi$ implies $\pi = 0$. On the other hand, if $R < \infty$, the function g is strictly decreasing and continuous on $(R, +\infty)$, it tends to 0 at infinity and $g(R)$ can take any value in $(0, +\infty]$. Therefore, the equation $g(\lambda) = 1$ has a solution λ if and only if

$$R < \infty \text{ and } g(R) = \sum_{n \geqslant 0} \frac{s_0 \cdots s_{n-1} f_n}{R^{n+1}} \geqslant 1.$$

In this case the solution is unique and equal to λ_0, it is the only positive eigenvalue of $\pi \mapsto \pi M$, and the associated eigenspace is exactly $\mathbb{R}_+\pi$. This concludes the proof. $\square$

III.3.3 From eigenelements to the Doob transform

When $R < \infty$, for large enough values of λ, h_λ is the unique (up to multiplicative constant) positive λ-eigenfunction. This allows to perform a Doob transform of M as follows.

Definition III.8. *(Doob transform) When* $R < +\infty$, *we introduce the Markov operator* P_λ, *defined for any* $\lambda \geqslant \lambda_0$, *by*

$$P_\lambda(f)(x) = \frac{M(h_\lambda f)(x)}{\lambda h_\lambda(x)}$$

for any bounded, positive function $f : \mathbb{Z}_+ \to \mathbb{R}_+$ *and any* $x \in \mathbb{X}$. *We denote by* $(Y_n^\lambda)_{n \geqslant 0}$ *the Markov chain with transition kernel* P_λ *on* $\mathbb{Z}_+$.

The transition kernel P_λ is represented by the matrix

$$P_\lambda = \begin{pmatrix} \frac{f_0 h_\lambda(0)}{\lambda h_\lambda(0)} & \frac{s_0 h_\lambda(1)}{\lambda h_\lambda(0)} & 0 & 0 & \cdots \\ \frac{f_1 h_\lambda(0)}{\lambda h_\lambda(1)} & 0 & \frac{s_1 h_\lambda(2)}{\lambda h_\lambda(1)} & 0 & \cdots \\ \frac{f_2 h_\lambda(0)}{\lambda h_\lambda(2)} & 0 & 0 & \frac{s_2 h_\lambda(3)}{\lambda h_\lambda(2)} & \ddots \\ \frac{f_3 h_\lambda(0)}{\lambda h_\lambda(3)} & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Until now, M has been defined as a linear map on the sets of functions and measures. In order to describe the asymptotic behavior of the sequence $(M^n)_{n\geqslant 0}$ it is necessary to consider M and P_λ as operators on some topological spaces. Since $\mathbb{X} = \mathbb{Z}_+$, the set of signed measures on $\mathbb{X}$ can be written $\mathcal{M}(\mathbb{X}) = \mathcal{L}^1(\mathbb{Z}_+)$. For any $\mu \in \mathcal{L}^1(\mathbb{Z}_+)$, we write $\mu = \mu^+ - \mu^-$ and $|\mu| = \mu^+ + \mu^-$. Finally, let $\psi : \mathbb{Z}^+ \to \mathbb{R}^+$. We introduce the vector spaces :

$$\mathcal{B}_\psi = \left\{ g \in \mathbb{R}^{\mathbb{Z}_+} \middle| \frac{g}{\psi} \text{ bounded} \right\},$$

$$\mathcal{M}_\psi = \left\{ \mu \in \mathbb{R}^{\mathbb{Z}_+} | \sum_{x\geqslant 0} |\mu(x)\psi(x)| < \infty \right\}$$

$\mathcal{M}_\psi$ can be seen as the set of measures of the form $\mu = \mu^+ - \mu^-$, where μ^+ and μ^- are two positive (possibly infinite) measures on $\mathbb{Z}_+$ such that $\mu^+(\psi) < \infty$ and $\mu^-(\psi) < \infty$. In particular $\mathcal{B}_{\mathbb{1}} = \mathcal{B}(\mathbb{X})$ and $\mathcal{M}_{\mathbb{1}} = \mathcal{M}(\mathbb{X})$. The vector spaces $\mathcal{B}_\psi$, $\mathcal{M}_\psi$ are respectively equipped with the norms

$$\|g\|_{\mathcal{B}_\psi} = \left\| \frac{g}{\psi} \right\|_\infty, \|\mu\|_{\mathcal{M}_\psi} = \sup_{g\in\mathcal{B}_\psi, \|g\|_{\mathcal{B}_\psi}=1} \mu(g).$$

We recall the following classical topological result.

Proposition III.9. *Let M be a Leslie Matrix such that Assumption III.1 holds and $R = \limsup_{n\to\infty} (s_0 \ldots s_{n-1} f_n)^{\frac{1}{n}} < \infty$. Let $\lambda \geqslant \lambda_0$. Then M is a bounded linear operator on the sets $\mathcal{B}_{h_\lambda}$ and $\mathcal{M}_{h_\lambda}$, with norm $|||M||| = \lambda$. The Markov operator P_λ acts linearly on the sets of bounded functions $(\mathcal{L}^\infty(\mathbb{Z}_+), \|\cdot\|_\infty)$ and signed measures $(\mathcal{M}(\mathbb{Z}_+), \|\cdot\|_{TV})$. In particular, the set of probability measures on $\mathbb{X} = \mathbb{Z}_+$ is invariant under P_λ.*

III.3.4 Asymptotic estimates of the eigenfunctions

The results of the next section require some asymptotic estimates on $h_\lambda(n)$ as $n \to \infty$, as well a uniform minorization of $h_\lambda(n)$. The following proposition focuses on this topic.

Proposition III.10. *Assume III.1 holds and $R < +\infty$.*

i) If $g(\lambda) < 1$ then

$$h_\lambda(n) \sim (1 - g(\lambda)) \frac{\lambda^n}{s_0 \cdots s_{n-1}}. \tag{III.23}$$

Consequently, for any

$$\lambda > \min\left(\frac{R}{\liminf_{n\to\infty} f_n^{\frac{1}{n}}}, \limsup_{n\to\infty}\left[(s_0\dots s_{n-1})^{\frac{1}{n}}\right]\right)$$

such that $g(\lambda) < 1$, *it holds*

$$\lim_{n\to\infty} h_\lambda(n) = +\infty \text{ and } \inf_{n\geqslant 0} h_\lambda(n) > 0.$$

ii) If $g(\lambda) = 1$ *then*

$$h_\lambda(n) = \underset{n\to\infty}{o}\left(\frac{\lambda^n}{s_0\cdots s_{n-1}}\right) \tag{III.24}$$

and

$$h_\lambda(n) = \frac{1}{\lambda}\frac{\sum_{k\geqslant n}\pi_k f_k}{\pi_n} \geqslant \frac{f_n}{\lambda}. \tag{III.25}$$

Thus, when $g(\lambda) = 1$, $\inf_{n\geqslant 0} h_\lambda(n) > 0$ *as soon as* $\inf_{n\geqslant 0} f_n > 0$.

iii) If $\lambda_0 \leqslant \lambda < \lambda'$ *then* $h_\lambda = \underset{n\to\infty}{o}(h_{\lambda'})$. *Thus* $\mathcal{B}_{h_\lambda} \subset \mathcal{B}_{h_{\lambda'}}$.

We recall here that, as explained in the proof of Proposition III.6, the condition $g(\lambda) < 1$ in *i)* is satisfied if and only if $\lambda > \lambda_0$ or ($g(R) < 1$ and $\lambda = \lambda_0 = R$). Similarly $g(\lambda) = 1$ is equivalent to $\lambda = \lambda_0$ and $g(R) \geqslant 1$.

Proof of Proposition III.10. Let us recall first (III.15) :

$$\forall n \in \mathbb{Z}_+, h_\lambda(n) = \frac{\lambda^n}{s_0\dots s_{n-1}}\left(1 - \sum_{k=0}^{n-1}\frac{s_0\dots s_{k-1}f_k}{\lambda^{k+1}}\right)$$

and notice that, as n tends to infinity,

$$\sum_{k=0}^{n-1}\frac{s_0\dots s_{k-1}f_k}{\lambda^{k+1}} \underset{n\longrightarrow\infty}{\longrightarrow} g(\lambda).$$

Thus, if $g(\lambda) < 1$, we recover (III.23).
If $\lambda > \limsup_{n\to\infty}\left[(s_0\dots s_{n-1})^{\frac{1}{n}}\right]$ and $g(\lambda) < 1$ then clearly

$$\lim_{n\to+\infty} h_\lambda(n) = (1 - g(\lambda))\lim_{n\to\infty}\frac{\lambda^n}{s_0\dots s_{n-1}} = +\infty.$$

Let us suppose now that $A := \liminf_{n\to\infty} f_n^{\frac{1}{n}} > 0$ and $g(\lambda) < 1$. In this case $f_n > 0$ for n large enough, and

$$\log h_\lambda(n) = n\left(\log(\lambda) - \log\left[(s_0\cdots s_{n-1}f_n)^{\frac{1}{n}}\right] + \log\left(f_n^{\frac{1}{n}}\right)\right) + \underset{n\to\infty}{o}(n). \tag{III.26}$$

Hence, if $\lambda > R/A$, and ε small enough, it holds for n large enough

$$\begin{aligned}\log(\lambda) - \log\left((s_0 \cdots s_{n-1} f_n)^{\frac{1}{n}}\right) + \log\left(f_n^{\frac{1}{n}}\right) &\geqslant \log(\lambda) - (\log(R) + \varepsilon) + \log(A) - \varepsilon \\ &\geqslant \log(\lambda) - \log(R/A) - 2\varepsilon > 0\end{aligned}$$

Hence, by (III.26), $h_\lambda(n) \longrightarrow \infty$ as $n \to \infty$. This completes the proof of *i)*.
If $g(\lambda) = 1$ then

$$\lim_{n\to\infty} \left(1 - \sum_{k=0}^{n-1} \frac{s_0 \dots s_{k-1} f_k}{\lambda^{k+1}}\right) = 0,$$

which proves (III.24) when plugged in (III.15). Moreover, if $g(\lambda) = 1$ then $\lambda = \lambda_0$; noticing that $g(\lambda_0) = 1 = \sum_{k=0}^{+\infty} \frac{s_0 \dots s_{k-1} f_k}{\lambda_0^{k+1}}$, we can rewrite (III.15) as

$$\begin{aligned}h_n(\lambda_0) &= \frac{\lambda_0^n}{s_0 \dots s_{n-1}} \sum_{k=n}^{+\infty} \frac{s_0 \dots s_{k-1} f_k}{\lambda_0^{k+1}} \\ &= \frac{1}{\lambda_0} \frac{\sum_{k=n}^{+\infty} \frac{s_0 \dots s_{k-1} f_k}{\lambda_0^{k+1}}}{\frac{s_0 \dots s_{n-1}}{\lambda_0^{n+1}}} \\ &= \frac{f_n}{\lambda_0} \frac{\sum_{k\geqslant n} \pi_k f_k}{f_n \pi_n} \geqslant \frac{f_n}{\lambda_0},\end{aligned} \tag{III.27}$$

since $\pi_n f_n$ is the first term of the sum in the numerator. This proves (III.25) and concludes the proof of *ii)*.
Let now $\lambda' > \lambda \geqslant \lambda_0$. Suppose first that $g(\lambda) < 1$. Since g is decreasing, it holds $g(\lambda') < 1$ and (III.23) applies to both h_λ and $h_{\lambda'}$. Then

$$\frac{h_\lambda(n)}{h_{\lambda'}(n)} \underset{n\to\infty}{\sim} \frac{1 - g(\lambda)}{1 - g(\lambda')} \left(\frac{\lambda}{\lambda'}\right)^n \underset{n\to\infty}{\longrightarrow} 0,$$

in other words, $h_\lambda(n) = \underset{n\to\infty}{o}(h_{\lambda'}(n))$. If $g(\lambda) = 1$ then $h_\lambda(n) = \underset{n\to\infty}{o}\left(\frac{\lambda^n}{s_0 \dots s_{n-1}}\right)$. Since g is strictly decreasing, we have $g(\lambda') < 1$ and

$$\frac{h_\lambda(n)}{h_{\lambda'}(n)} = \underset{n\to\infty}{o}\left(\frac{1}{1 - g(\lambda')} \left(\frac{\lambda}{\lambda'}\right)^n\right) \underset{n\to\infty}{\longrightarrow} 0.$$

Once again $h_\lambda(n) = \underset{n\to\infty}{o}(h_{\lambda'(n)})$. In particular $h_\lambda/h_{\lambda'}$ is bounded and $\mathcal{B}_{h_\lambda} \subset \mathcal{B}_{h_{\lambda'}}$. □

III.4 Properties of the Doob-transform

In this section, we study the Markov chain $(Y_n^\lambda)_{n\geqslant 0}$. It will provide information on the process $(Z_n)_{n\geqslant 0}$. Let us start by looking at its irreducibility and periodicity properties.

III.4.1 Irreducibility and periodicity

Under mild positivity conditions on the coefficients $(f_n)_{n\geqslant 0}$, the Markov chains $(Y_n^\lambda)_{n\geqslant 0}$ are irreducible and aperiodic. We clarify this claim in the following statement.

Proposition III.11. *Consider a LGW-process satisfying Assumption III.1, assume that $R < +\infty$ and fix $\lambda \geqslant \lambda_0$. Then*

i) The Markov chain $(Y_n^\lambda)_{n\geqslant 0}$ is irreducible.

ii) It is aperiodic if and only if $\delta = \gcd\{k \in \mathbb{Z}_+^* | f_{k-1} > 0\} = \gcd(\mathfrak{F} + 1) = 1$.

Proof of Proposition III.11. Fix $i \in \mathbb{Z}_+$ and $j > i$. Since $s_k > 0$ and $h_\lambda(k) > 0$ for all $k \geqslant 0$ then P_λ^{j-i} satisfies

$$P_\lambda^{j-i}(i,j) \geqslant P_\lambda(i,i+1)\dots P_\lambda(j-1,j) = \frac{s_i \dots s_{j-1} h_\lambda(j)}{\lambda^{j-i} h_\lambda(i)} > 0.$$

since for all k, $s_k > 0$ and $h_\lambda(k) > 0$. Therefore for any states $i < j$, i is connected to j. Assume now $i > j$. By Assumption III.1, the set $\mathfrak{F}$ is infinite, let us thus choose $k \geqslant i$ such that $f_k > 0$. By the previous case, the site i is connected to the site k. The equality $P_\lambda(k,0) = \frac{f_k h_\lambda(0)}{\lambda h_\lambda(k)} > 0$ implies that k is connected to 0, and 0 is connected to j since $0 \leqslant j$, once again using the previous case. This proves that $(Y_n^\lambda)_{n\geqslant 0}$ is irreducible, i.e. *i)*. Let us prove now *ii)*. Since $(Y_n^\lambda)_{n\geqslant 0}$ is irreducible, all the states have the same period. We claim that this period is

$$\gcd\{n \geqslant 1, P_\lambda^n(0,0) > 0\} = \gcd\{k \geqslant 1 | f_{k-1} > 0\} =: \delta.$$

Therefore the Markov chain $(Y_n^\lambda)_{n\geqslant 0}$ is aperiodic if and only if $\delta = 1$. Let us prove this claim. On the one hand if $f_{k-1} > 0$ then

$$\begin{aligned} P_\lambda^k(0,0) &\geqslant \mathbb{P}[(Y_0^\lambda, \dots Y_k^\lambda) = (0,1,\dots,k-1,0) | Y_0^\lambda = 0] \\ &= P_\lambda(0,1)\dots P_\lambda(k-2,k-1) P_\lambda(k-1,0) \\ &= \frac{s_0 \dots s_{k-2} f_{k-1}}{\lambda^k} > 0. \end{aligned}$$

Therefore $\{k \geqslant 1 | f_{k-1} > 0\} \subset \{n \geqslant 1, P_\lambda^n(0,0) > 0\}$ and $\gcd\{n \geqslant 1, P_\lambda^n(0,0) > 0\}$ is a divisor of δ.
On the other hand, if $P_\lambda^n(0,0) > 0$ then there exists $y_0^\lambda = 0, y_1^\lambda, \dots y_{n-1}^\lambda, y_n^\lambda = 0 \in \mathbb{Z}_+$ such that $\mathbb{P}[(Y_0^\lambda, \dots Y_n^\lambda) = (y_0, \dots, y_n)] > 0$. The structure of the matrix P_λ implies that for all k, almost surely $Y_{k+1}^\lambda \in \{0, 1 + Y_k^\lambda\}$. Thus noting $0 = k_0 < \dots < k_p = n$ the values of k such that $y_k = 0$, it holds $y_k = k - k_i$ for any $k_i \leqslant k \leqslant k_{i+1}$, . The path $(y_i)_{0\leqslant i\leqslant n}$ can thus be decomposed into excursions away from 0 as:

$$(y_0, \dots, y_n) = (0,1,\dots,k_1 - 1, 0, \dots, 0, \dots, k_p - k_{p-1} - 1, 0).$$

Therefore, since $\mathbb{P}[(Y_0^\lambda, \dots Y_n^\lambda) = (y_0, \dots, y_n)|Y_0 = y_0] > 0$, it holds

$$\mathbb{P}[Y_{k_i}^\lambda = 0|Y_{k_i-1} = k_i - k_{i-1} - 1] = \frac{f_{k_i - k_{i-1} - 1} h_\lambda(0)}{\lambda h_\lambda(k_i - k_{i-1} - 1)} > 0,$$

for all $1 \leqslant i \leqslant p$. For all $1 \leqslant i \leqslant p$, we derive therefore $f_{k_{i+1}-k_i-1} > 0$ and $\delta | k_{i+1} - k_i$. Hence $\delta | \sum_{i=0}^{p-1} k_{i+1} - k_i = k_p - k_0 = n$. Therefore δ is a divisor of all the elements of the set $\{n \geqslant 1, P_\lambda^n(0,0) > 0\}$ thus δ divides $\gcd\{n \geqslant 1, P_\lambda^n(0,0) > 0\}$. This concludes the proof of *ii)*. $\square$

Example A: $f_n = f > 0, s_n = s \in [0,1]$

In this example, $f_n > 0$ for all n, thus for any $\lambda \geqslant \lambda_0$, the Markov chain $(Y_n^\lambda)_{n \geqslant 0}$ is irreducible and aperiodic.

Remark III.12. Let us notice that the assumption $\delta = 1$ does not depend on the choice of λ. Therefore, when $R < \infty$ and $\delta = 1$, the Markov chains $(Y_n^\lambda)_{n \geqslant 0}$ for $\lambda \geqslant \lambda_0$ are irreducible and have the same period.

III.4.2 Recurrence

In this subsection, we work under the assumptions of Propositions III.6 and III.11. The following proposition studies the recurrence of the Markov chain (Y_n^λ).

Proposition III.13. *Consider a LGW-process satisfying Assumption III.1, such that $R < \infty$, and $\delta = 1$. Let $\lambda \geqslant \lambda_0$. Then (Y_n^λ) is recurrent if and only if $g(\lambda) = 1$. In other words :*

- *For $\lambda > \lambda_0$, $(Y_n^\lambda)_{n \geqslant 0}$ is transient.*
- *$(Y_n^{\lambda_0})$ is recurrent if, and only if, $g(\lambda_0) = 1$ or equivalently $g(R) \geqslant 1$.*

Example A: $f_n = f > 0, s_n = s \in [0,1]$

In this case $R = s$, $g(\lambda) = \frac{f}{\lambda - s}$, thus $g(s) = +\infty$ and $(Y_n^{\lambda_0})_{n \geqslant 0}$ is recurrent.

Proof of Proposition III.13. We fix $\lambda \in [\lambda_0, +\infty)$. Let T be the first return time to 0 of (Y_n^λ) :

$$T = \inf\{n \geqslant 1, Y_n^\lambda = 0\} \cup \{+\infty\}.$$

We remark that

$$\mathbb{P}[T = n+1|Y_0^\lambda = 0] = \mathbb{P}[(Y_0^\lambda, \dots Y_{n+1}^\lambda) = (0, 1, \dots, n, 0)] = \frac{s_0 \cdots s_{n-1} f_n}{\lambda^{n+1}} \tag{III.28}$$

Consequently

$$\mathbb{P}[T < \infty|Y_0 = 0] = \sum_{k=0}^{+\infty} \mathbb{P}[T = k+1|Y_0 = 0] = g(\lambda).$$

Since the Markov chain (Y_n^λ) is irreducible, it is recurrent if and only $g(\lambda) = 1$. We recover Equation (III.22), which was already solved in the proof of Proposition III.7. This equation has solutions if and only if $R < \infty$ and $g(R) \geqslant 1$, in which case its solution is unique and equals λ_0. This concludes the proof. $\square$

The recurrence criterion $g(\lambda) = 1$ given by Proposition III.13 already appeared in Proposition III.7 as a necessary and sufficient condition for λ to be an eigenvalue of $\pi \mapsto \pi M$. This is not entirely surprising, as the following proposition exhibits a clear link between λ-eigenmeasures for M and invariant measures for P_λ. As a consequence, (Y_n^λ) is recurrent if and only if P_λ has a non trivial invariant measure. It is a general fact that a recurrent Markov chain has an invariant measure, however the converse is not always true in general : transient Markov chains on a denumerable state space may possess an invariant measure.

For any measure μ and any positive function f, we note $f \cdot \mu$ the measure with density f with respect to μ.

Proposition III.14. *Consider a LGW-process with mean matrix M, such that Assumption III.1 holds and $R < \infty$. Let $\lambda \geqslant \lambda_0$. Let μ be a positive measure. Then*

$$\mu M = \lambda \mu \Leftrightarrow (h_\lambda \cdot \mu) P_\lambda = h_\lambda \cdot \mu.$$

In particular, when $g(R) \geqslant 1$, the map $\mu \mapsto h_{\lambda_0} \cdot \mu$ realizes a natural bijection between the (non-trivial) set of λ_0-eigenmeasures of M and the set of invariant measures of the recurrent Markov chain $(Y_n^{\lambda_0})$.

Proof of Proposition III.14. Let f be a positive function and μ be a positive measure. By definition of P_λ, it holds

$$\begin{aligned}(h_\lambda \cdot \mu) P_\lambda f &= \sum_{n \geqslant 0} \mu(n) h_\lambda(n) P_\lambda(f)(n). \\ &= \frac{1}{\lambda} \sum_{n \geqslant 0} \mu(n) M(h_\lambda f)(n) \\ &= \frac{1}{\lambda} \mu M(h_\lambda f).\end{aligned}$$

Since $h_\lambda > 0$, we derive $\mu M = \lambda \mu \Leftrightarrow (h_\lambda \cdot \mu) P_\lambda = h_\lambda \cdot \mu$, which concludes the proof. $\square$

III.4.3 Positive recurrence

In this section, we consider a LGW-process such that $R < \infty$ and $g(R) \geqslant 1$. We assume that the assumptions of Proposition III.11 hold. Therefore $(Y_n^{\lambda_0})$ is well defined and recurrent. In the rest of this section, the only eigenvalue of M we focus on is λ_0, thus we note $P = P_{\lambda_0}$, $(Y_n) = (Y_n^{\lambda_0})_{n \geqslant 0}$, $h = h_{\lambda_0}$ to lighten the notations. The next natural question is whether this Markov chain is positive recurrent.

Proposition III.15. *Consider a LGW-process such that Assumption III.1 holds and $R < \infty$. Then the following assertions hold* *i) The recurrent Markov chain* $(Y_n)_{n\geqslant 0} = (Y_n^{\lambda_0})_{n\geqslant 0}$ *is positive recurrent if and only if*

$$g(R) > 1$$

or

$$g(R) = 1 \text{ and } |g'(R)| = R^{-1}\sum_{k=0}^{\infty}\frac{(k+1)s_0\dots s_{k-1}f_k}{R^{k+1}} < +\infty.$$

Assume that one of these conditions is satisfied. Then

ii) The only invariant probability measure of (Y_n) *is* $\tilde{\pi} = \frac{(h\cdot\pi)}{\pi(h)}$.

iii) If additionally $\delta = \gcd(\mathfrak{F}+1) = 1$, *then for any positive measure* $\mu \in \mathcal{M}_+(\mathbb{Z}_+)$ *such that* $\mu(h) < \infty$, *it holds*

$$\left\|\lambda_0^{-n}\mu M^n - \frac{\mu(h)}{\pi(h)}\pi\right\|_{\mathcal{M}(h)} \leqslant \sum_{x\geqslant 0}\mu(x)h(x)\|\delta_x P^n - \tilde{\pi}\|_{TV} \xrightarrow[n\to\infty]{} 0.$$

Example A: $f_n = f > 0, s_n = s \in [0,1]$

In this case, $R = s$ *and* $g(\lambda) = \frac{f}{\lambda - s}$, *thus* $g(R) = +\infty$. *Thus* (Y_n) *is positive recurrent and ii) and iii) apply.*

Before proving Proposition III.15, we state and prove the following Lemma.

Lemma III.16. *Suppose that Assumption III.1 holds,* $R < \infty$ *and that* $\lambda \geqslant R$. *Then there the two following assertions are equivalent.*

i) $g(\lambda) = 1$ *and* $|g'(\lambda)| < +\infty$

ii) $\lambda = \lambda_0$ *and*

$$(g(R) > 1) \text{ or } (g(R) = 1 \text{ and } |g'(R)| < \infty)$$

Proof of Lemma III.16. Assume that *i)* holds. Then $g(R) \geqslant g(\lambda) = 1$ and $\lambda = \lambda_0$ since g is decreasing.
If $g(R) > 1$ then $\lambda = \lambda_0 > R$. Since $g(\lambda)$ is a power series of radius R^{-1} evaluated at λ^{-1}, the real λ_0^{-1} is in the interior of the convergence disk and $|g'(\lambda)| < \infty$.
When $g(R) = 1$, it holds $\lambda = \lambda_0 = R$ thus $|g'(R)| = |g'(\lambda)| < +\infty$.
Conversely, assuming $\lambda = \lambda_0$ and $g(R) \geqslant 1$, it holds $g(\lambda) = 1$. If additionally $g(R) > 1$, then by the previous argument $|g'(\lambda)| < \infty$, since $\lambda = \lambda_0 > R$. If $g(R) = 1$ then $R = \lambda = \lambda_0$, thus $|g'(\lambda)| = |g'(R)|$. $\square$

Proof of Proposition III.15. When $R < +\infty$, if $g(R) < 1$, the chain (Y_n) is transient and therefore not positive recurrent. If $g(R) \geqslant 1$, by Proposition III.13, the Markov chain (Y_n) is recurrent, and by Proposition III.14, the unique invariant measure of P, up to a multiplicative constant, is $h\cdot\pi$. Therefore (Y_n) is positive recurrent if and only if the measure $h\cdot\pi$ is finite, that is if

$$(h\cdot\pi)(\mathbb{1}) = \sum_{n\geqslant 0}\pi(n)h(n) = \pi(h) < +\infty.$$

By (III.34), it holds

$$h(n) = \frac{\lambda_0^n}{s_0 \dots s_{n-1}} \left(1 - \sum_{k=0}^{n-1} \frac{s_0 \dots s_{k-1} f_k}{\lambda_0^{k+1}}\right) = \frac{1}{\pi(n)} \sum_{k \geqslant n} \frac{f_k s_n \dots s_{k-1}}{\lambda_0^{k+1}}$$

Therefore $(h \cdot \pi)(\mathbb{1}) = \sum_{n \geqslant 0} h(n)\pi(n) = \sum_{k=0}^{\infty} \frac{(k+1)s_0 \dots s_{k-1} f_k}{\lambda_0^{k+1}}$. Thus, (Y_n) is positive recurrent if and only if

$$\sum_{k=0}^{\infty} \frac{(k+1)s_0 \dots s_{k-1} f_k}{\lambda_0^{k+1}} = \lambda_0 |g'(\lambda_0)| < +\infty. \tag{III.29}$$

When $g(R) = 1$ then plugging $\lambda_0 = R$ into (III.29) proves that (Y_n) is positive recurrent if and only if

$$\sum_{k=0}^{\infty} \frac{(k+1)s_0 \dots s_{k-1} f_k}{R^{k+1}} < +\infty,$$

which can be rewritten $|g'(R)| < +\infty$. This concludes the proof of *i)*.
When (III.29) holds, the measure $\tilde{\pi} = \frac{\pi \cdot h}{\pi(h)}$ is the only invariant probability measure of (Y_n). This proves *ii)*.
Let μ be a positive measure such that $\mu(h) < \infty$. Since by Proposition III.9, M is a bounded linear operator on $\mathcal{M}(h)$ and $\pi(h) < \infty$, the quantity $\left\|\lambda_0^{-n}\mu M^n - \frac{\mu(h)}{\pi(h)}\pi\right\|_{\mathcal{M}(h)}$ is well defined and finite. Let $f \in \mathcal{B}(h), x \in \mathbb{Z}_+, n \geqslant 0$. By definition of P, it holds $M^n(f) = \lambda^n h P^n \frac{f}{h}$, thus

$$\begin{aligned}
\left|\lambda_0^{-n} M^n f(x) - \frac{\pi(f)}{\pi(h)} h(x)\right| &= h(x) \left|\delta_x P^n \frac{f}{h} - \tilde{\pi}\left(\frac{f}{h}\right)\right| \\
&\leqslant h(x) \left\|\frac{f}{h}\right\|_{\infty} \|\delta_x P^n - \tilde{\pi}\|_{TV}.
\end{aligned}$$

Moreover, by the triangular inequality

$$\begin{aligned}
\left|\lambda_0^{-n}\mu M^n f - \frac{\pi(f)}{\pi(h)}\mu(h)\right| &\leqslant \sum_{x \geqslant 0} \mu(x) \left|\lambda_0^{-n} M^n f(x) - \frac{\pi(f)}{\pi(h)} h(x)\right| \\
&\leqslant \left\|\frac{f}{h}\right\|_{\infty} \sum_{x \in \mathbb{X}} \mu(x) h(x) \|\delta_x P^n - \tilde{\pi}\|_{TV}.
\end{aligned} \tag{III.30}$$

This yields $\left\|\lambda_0^{-n}\mu M^n - \frac{\mu(h)}{\pi(h)}\pi\right\|_{\mathcal{M}(h)} \leqslant \sum_{x \in \mathbb{X}} \mu(x) h(x) \|\delta_x P^n - \tilde{\pi}\|_{TV}$, and it remains to prove that

$$\lim_{n \to \infty} \sum_{x \in \mathbb{X}} \mu(x) h(x) \|\delta_x P^n - \tilde{\pi}\|_{TV} = 0. \tag{III.31}$$

Notice first that for any $x \geqslant 0$, the quantity $\|\delta_x P^n - \tilde{\pi}\|_{TV}$ is bounded by 2 and that $\sum_{x \geqslant 0} \mu(x) h(x) = \mu(h) < \infty$. Moreover, under the assumptions of *iii)*, (Y_n) is an aperiodic, positive recurrent Markov chain on a countable state space, with invariant probability $\tilde{\pi}$. Therefore for all $x \geqslant 0$, it holds $\lim_{n \to \infty} \|\delta_x P^n - \tilde{\pi}\|_{TV} = 0$. Equation (III.31) is thus a consequence of the dominated convergence theorem. This concludes the proof of *iii)*. $\square$

Remark III.17. In this proof, we characterized the positive recurrence of (Y_n) by the finiteness of its invariant measures. However since

$$\frac{s_0 \dots s_{k-1} f_k}{\lambda^{k+1}} = \mathbb{P}[T = k+1 | Y_0 = 0],$$

where T is the first return time of (Y_n) to 0, condition (III.29) can clearly be interpreted as the classic alternative characterization of positive recurrence

$$\mathbb{E}[T | Y_0 = 0] < \infty.$$

Let us prove now Theorem 2, using the previous Proposition.

Proof of Theorem 2. Assume first that $g(\lambda) = 1$ and $g'(\lambda) < \infty$. Then by Lemma III.16 and Proposition III.15, $\lambda = \lambda_0$ and $(Y_n^\lambda) = (Y_n)$ is positive recurrent. Applying Proposition III.15, *iii)*, we obtain the desired decomposition.
Conversely, assume now that an ergodicity property of the form (III.2) holds. We don't suppose a priori that the measure and function involved are actually the eigenmeasure π and the eigenfunction h, therefore we note them respectively ν and ϕ. For any positive measure $\mu \in \mathcal{M}_\phi$ and any positive function $f \in \mathcal{B}_\phi$, we derive

$$\lim_{n \to \infty} \lambda^{-n} \mu M^n f = \frac{\nu(f) \mu(\phi)}{\nu(\phi)}.$$

In particular, we obtain $\lim_{n \to +\infty} \lambda^{-n} M^n(f)(x) = \phi(x) \frac{\nu(f)}{\nu(\phi)}$. Since $\delta_x M = s_x \delta_{x+1} + f_x \delta_0 \in \mathcal{M}(h)$, we also have

$$\lim_{n \to +\infty} \lambda^{-n} (\delta_x M) M^n(f) = (\delta_x M) \frac{\nu(f)}{\nu(\phi)} = (M\phi)(x) \frac{\nu(f)}{\nu(\phi)}.$$

Since moreover $\lambda^{-n}(\delta_x M) M^n(f) = \lambda \times \lambda^{-(n+1)} \delta_x M^{n+1}(f)$, we derive

$$\lim_{n \to +\infty} \lambda^{-n} (\delta_x M) M^n(f) = \lambda \lim_{n \to +\infty} \lambda^{-(n+1)} (\delta_x M) M^{n+1}(f) = \lambda \phi(x) \frac{\nu(f)}{\nu(\phi)}.$$

Thus $\lambda \phi(x) \frac{\nu(f)}{\nu(\phi)} = (M\phi)(x) \frac{\nu(f)}{\nu(\phi)}$. Since $\nu \neq 0$ we chose f such that $\nu(f) \neq 0$ and obtain $M\phi = \lambda \phi$. This yields $\phi = h_\lambda > 0$, up to renormalization, by Proposition III.6.
For any $f \in \mathcal{B}_\phi$, such that $f \geqslant 0$, we have $Mf \leqslant M\phi = \lambda \phi$ thus $Mf \in \mathcal{B}_\phi$. Therefore we write for $x \in \mathbb{X}$

$$\lim_{n \to \infty} \lambda^{-n} \delta_x M^n(Mf) = \nu(Mf) \frac{\phi(x)}{\nu(\phi)}$$

and

$$\lim_{n \to \infty} \lambda^{-n} \delta_x M^n(Mf) = \lambda \lim_{n \to \infty} \delta_x \lambda^{-(n+1)} M^{n+1}(f) = \lambda \nu(f) \frac{\phi(x)}{\nu(\phi)}.$$

Since $\phi > 0$, we get $(\nu M)(f) = \lambda \nu(f)$ for any $f \in \mathcal{B}_\psi$. Notice that $f = \mathbb{1}_y \in \mathcal{B}_\psi$ for any $y \in \mathbb{Z}_+$. This guarantees that $\nu M = \lambda \nu$. By Proposition III.7, we know that $\lambda = \lambda_0$, $g(\lambda) = 1$ and $\nu = \pi$ up to renormalization. In particular, the Markov chain (Y_n^λ) is recurrent with invariant measure $h \cdot \pi$. Moreover, we know that $\nu = \pi \in \mathcal{M}_\phi$

thus $\pi(h) < \infty$, which implies that $h \cdot \pi$ has finite mass. Therefore, the chain (Y_n^λ) is positive recurrent, and by Proposition III.15 and Lemma III.16, it holds $g(\lambda) = 1$ and $|g'(\lambda)| < \infty$. $\square$

III.4.4 Doeblin contraction

In this subsection, we try to strengthen the convergence from Proposition III.15, to obtain an explicit bound on the convergence rate of P^n to $\tilde{\pi}$. To do so, we use techniques based on a Doeblin minorization : if a Markov kernel Q satisfies a Dobelin condition of the form $Q(x, dy) \geqslant c\nu(dy)$ then it is geometrically ergodic. We state this formally in the case of a countable state set in the next Proposition, for which a proof can be found for example in [Str05, Chap. 2].

Proposition III.18 ([Str05])**.** *Let Q be a Markov kernel on the countable space $(E, \mathcal{E})$, and suppose that there exists $c > 0$ and a state $z \in E$ such that, for any $x \in E$:*

$$Q(x, dy) \geqslant c\delta_z(dy).$$

Then :

- *For any signed measures of equal mass ρ_1, ρ_2 on E :*

$$\|\rho_1 Q - \rho_2 Q\|_{TV} \leqslant (1-c)\|\rho_1 - \rho_2\|.$$

- *There exists a unique stationary probability $\tilde{\pi}$ such that, for any probability ρ*

$$\|\rho Q^n - \tilde{\pi}\|_{TV} \leqslant 2(1-c)^n.$$

Given the structure of our model, the only possible choice of state z for which a Doeblin minorization can hold is $z = 0$. Applying the previous Proposition to $(Y_n) = (Y_n^{\lambda_0})$ yields then the following proposition.

Proposition III.19. *Consider a LGW such that Assumption III.1 holds and $R < +\infty$. We also assume that*

$$\inf_{n \geqslant 0} \frac{f_n}{\lambda_0 h_{\lambda_0}(n)} =: c > 0. \tag{III.32}$$

Then the Markov chain (Y_n) satisfies Proposition III.18. In particular, for any positive measure $\mu \in \mathcal{M}_{h_{\lambda_0}}$:

$$\left\| \mu M^n - \lambda_0^n \frac{\mu(h_{\lambda_0})}{\pi(h_{\lambda_0})} \pi \right\|_{\mathcal{M}_{h_{\lambda_0}}} \leqslant 2\mu(h_{\lambda_0})\lambda_0^n(1-c)^n \tag{III.33}$$

Note that if the Markov chain $(Y_n^\lambda)_{n \geqslant 0}$ satisfies Proposition III.18, then it is recurrent positive, $\lambda = \lambda_0$ and $g(R) \leqslant 1$. This is why condition (III.32) cannot hold for $\lambda \neq \lambda_0$.

Example A: $f_n = f > 0, s_n = s \in [0,1]$

In this case the function g and all the related quantities can be explicitly computed : $R = s$, $\lambda_0 = s + f$, $h_{\lambda_0} = 1$ is λ_0-harmonic and P trivially satisfies the Doeblin condition, with $\inf_n \frac{f_n h_{\lambda_0}(0)}{\lambda_0 h_{\lambda_0}(n)} = \frac{f}{s+f}$.

Proof of Proposition III.19. Equation (III.32) implies that

$$\inf_{x \geqslant 0} P_{\lambda_0}(x, 0) = \inf_{n \geqslant 0} \frac{f_n h_{\lambda_0}(0)}{\lambda_0 h_{\lambda_0}(n)} = c > 0$$

since $h_{\lambda_0}(0) = 1$. Thus P_{λ_0} satisfies the assumption of Proposition III.18. As a consequence, $(Y_n^{\lambda_0})_{n \geqslant 0}$ has exactly one stationnary probability measure, it is therefore positive recurrent. We note once again for short $h = h_{\lambda_0}, P = P_{\lambda_0}$. From Theorem III.15, we deduce that the unique invariant probability measure of P is $\tilde{\pi}$. Let μ be a positive measure in $\mathcal{M}_h$, we note $\tilde{\mu} = h \cdot \mu$. Then, for any $f \in \mathcal{B}_h$,

$$\mu M^n f = \mu \left({\lambda_0}^n h P^n \frac{f}{h} \right) = {\lambda_0}^n \tilde{\mu} P^n \frac{f}{h},$$

where $\tilde{\mu}$ has mass $\mu(h)$. Therefore :

$$\|\tilde{\mu} P^n - \mu(h)\tilde{\pi}\|_{TV} \leqslant 2(1-c)^n \mu(h).$$

This yields

$$\left| \mu M^n f - {\lambda_0}^n \frac{\mu(h)}{\pi(h)} \pi(f) \right| \leqslant {\lambda_0}^n \|\tilde{\mu} P^n - \mu(h)\tilde{\pi}\|_{TV} \left\| \frac{f}{h} \right\|_\infty \leqslant 2{\lambda_0}^n (1-c)^n \left\| \frac{f}{h} \right\|_\infty,$$

which implies (III.33). □

Remark III.20. We recall that from (III.27), when $g(\lambda_0) = 1$, it holds

$$h_{\lambda_0}(n) = \frac{\lambda_0^n}{s_0 \ldots s_{n-1}} \sum_{k \geqslant n} \frac{s_0 \ldots s_{k-1} f_k}{\lambda_0^{k+1}} = \frac{f_n}{\lambda_0} \frac{\mathbb{P}[T \geqslant n | Y_0 = 0]}{\mathbb{P}[T = n | Y_0 = 0]}, \tag{III.34}$$

where T is the return time of (Y_n^λ) to zero. Thus the Doeblin condition (III.32) can be rewritten as

$$\sup_{n \geqslant 0} \frac{h_{\lambda_0}(n)}{f_n} = \frac{1}{\lambda_0} \sup_{n \geqslant 0} \frac{\mathbb{P}[T \geqslant n | Y_0 = 0]}{\mathbb{P}[T = n | Y_0 = 0]} < +\infty.$$

This implies in particular that the random time T is stochastically dominated by a random variable with geometric distribution.

When studying more complex systems, the function h_{λ_0} is not always easy to express explicitely. When $f_n > 0$ for all $n \geqslant 0$, we introduce a quantity β which helps providing a more explicit criterion for the Doeblin condition (III.32)

$$\beta := \limsup_k \left[\sup_n \left(\frac{f_{n+k}}{f_n} s_n \ldots s_{n+k-1} \right) \right]^{\frac{1}{k}} \in [R, +\infty].$$

Proposition III.21. *Consider a LGW such that $R < +\infty$, assume that $f_n > 0$ for all n, that $\beta < +\infty$ and that $g(\beta) \in (1, +\infty)$. Then the Doeblin condition (III.32) is satisfied and the conclusions of Proposition III.19 hold.*

Proof of Proposition III.21. Since $R < \infty$, the real λ_0 is the smallest nonnegative number such that $g(\lambda_0) \leqslant 1$. Moreover, taking $n = 0$ in the definition of β shows that $\beta \geqslant R$. Moreover $\beta < \lambda_0$ since $g(\beta) > 1$. Therefore $R < \lambda_0$ and $g(\lambda_0) = 1$. On the one hand, equation (III.34) yields

$$h_{\lambda_0}(n) = \frac{1}{\lambda_0} \sum_{k \geqslant 0} \frac{f_{n+k} s_n \dots s_{n+k-1}}{\lambda_0^k}.$$

Therefore, condition (III.32) can be rewritten as

$$\sup_{n \in \mathbb{Z}_+} \frac{\lambda_0 h_{\lambda_0}(n)}{f_n} = \sup_{n \in \mathbb{Z}_+} \left(\sum_{k \geqslant 0} \frac{1}{\lambda_0^k} \frac{f_{n+k}}{f_n} s_n \dots s_{n+k-1} \right) < \infty.$$

On the other hand

$$\sup_{n \geqslant 0} \left(\sum_{k \geqslant 0} \frac{1}{\lambda_0^k} \frac{f_{n+k}}{f_n} s_n \dots s_{n+k-1} \right) \leqslant \sum_{k \geqslant 0} \frac{1}{\lambda_0^k} \sup_{n \geqslant 0} \left(\frac{f_{n+k}}{f_n} s_n \dots s_{n+k-1} \right) < +\infty$$

since $\beta > \lambda_0$, thus (III.32) is satisfied. □

This criterion may be used on the following example.

Example B: $a_1 \leqslant f_n/(n+1)^\alpha \leqslant a_2$, $\lim_{n \to +\infty} s_n = s \geqslant 0$, $g(s) > 1$.

This example satisfies the assumptions of Propositions III.21 and thus III.19. Indeed, we have

$$\lim_{k \to \infty} f_k^{\frac{1}{k}} = 1$$

and

$$\lim_{n \to \infty} (s_0 \dots s_{n-1})^{\frac{1}{n}} = \exp\left(\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \log(s_k) \right) = s$$

Thus $R = s$. Since we've assumed $g(s) > 1$, it holds $\lambda_0 > R$ and $g(\lambda_0) = 1$. Moreover for all $n, k \geqslant 0$

$$\frac{f_{n+k}}{f_n} s_n \dots s_{n+k-1} \leqslant \left(1 + \frac{k}{n+1}\right)^\alpha \frac{a_2}{a_1} s_n \dots s_{n+k-1}$$

where $\left(1+\frac{k}{n+1}\right)^{\alpha} \leqslant \max(1,(1+k)^{\alpha})$. *As a consequence*

$$\begin{aligned}\beta &= \limsup_{k\to\infty} \sup_{n\geqslant 0} \left(\frac{f_{n+k}}{f_n} s_n \dots s_{n+k-1}\right)^{\frac{1}{k}} \\ &\leqslant \limsup_{k\to\infty} \left[\max(1,(1+k)^{\alpha}) \frac{a_2}{a_1} \sup_{n\geqslant 0}(s_n \dots s_{n+k-1})\right]^{\frac{1}{k}} \\ &\leqslant \limsup_{k\to\infty} \sup_{n\geqslant 0} (s_n \dots s_{n+k-1})^{\frac{1}{k}}.\end{aligned}$$

Let us choose $\varepsilon > 0$ *and* $n_0 \geqslant 0$ *such that* $s_n \leqslant s+\varepsilon$ *for* $n \geqslant n_0$. *Then it holds*

$$\begin{aligned}\beta &\leqslant \limsup_{k\to\infty} \max\left[\max_{n\leqslant n_0}(s_n \dots s_{n+k-1})^{\frac{1}{k}}, s+\varepsilon\right] \\ &\leqslant \max\left[s+\varepsilon, \max_{n\leqslant n_0} \limsup_{k\to\infty} (s_n \dots s_{n+k-1})^{\frac{1}{k}}\right] \\ &\leqslant \max(s+\varepsilon, s) = s+\varepsilon.\end{aligned}$$

This holds for all $\varepsilon > 0$, *therefore* $\beta = s = R$ *and* $g(\beta) = g(R) = g(s) > 1$.

III.4.5 Summary of the results

We sum up the results of Sections III.3 and III.4 in the three following tables. We separate three cases depending on whether $g(R) < 1, = 1, > 1$. Notice that the value of $g(R)$ depends on all the parameters $(s_n, f_n)_{n\geqslant 0}$, it is not only an asymptotic property.

Case 1 : $\mathbf{g(R) > 1}$

$\lambda < \lambda_0$	$\lambda = \lambda_0$	$\lambda > \lambda_0$
$g(\lambda) > 1$ h_λ changes sign (Y_n^λ) is not defined M has no λ-eigenmeasure	$g(\lambda_0) = 1, g'(\lambda_0) < +\infty$ $h_{\lambda_0} > 0$ $(Y_n^{\lambda_0})$ is positive recurrent $\pi M = \lambda_0 M$ $\tilde{\pi}$ is an inv. measure of $(Y_n^{\lambda_0})$ $\tilde{\pi}$ is a probability measure	$g(\lambda) < 1$ $h_\lambda > 0$ (Y_n^λ) is transient M has no λ-eigenmeasure $(Y_n^{\lambda_0})$ has no inv. measure

When $g(R) = 1$, this table changes slightly (changes appear in red).

Case 2 : $\mathbf{g(R) = 1}$ (which implies $\lambda_0 = R$)

$\lambda < \lambda$	$\lambda = \lambda_0 = R = 1$	$\lambda > \lambda$
$g(\lambda) = +\infty$ h_λ changes sign (Y_n^λ) is not defined M has no λ-eigenmeasure	$g(\lambda) = 1$ $h_\lambda > 0$ (Y_n^λ) is recurrent (Y_n^λ) is pos. rec. iff $\lvert g'(1)\rvert < \infty$. $\pi M = \lambda M$ $\tilde{\pi}$ is an inv. measure of (Y_n^λ) $\tilde{\pi}(\mathbb{1}) = 1$ iff $\lvert g'(1)\rvert < \infty$	$g(\lambda) < 1$ $h_\lambda > 0$ (Y_n^λ) is transient M has no λ-eigenmeasure (Y_n^λ) has no inv. measure

The last (and most degenerate case) is $g(R) < 1$, in which case the table becomes

Case 3 : $\mathbf{g(R) < 1}$ (which implies $\lambda_0 = R$)

$\lambda < \lambda_0$	$\lambda = \lambda_0 = R$	$\lambda > \lambda_0$
$g(\lambda) = +\infty$ h_λ changes sign (Y_n^λ) is not defined M has no λ-eigenmeasure	$g(\lambda) < 1$ $h_\lambda > 0$ (Y_n^λ) is transient M has no λ-eigenmeasure (Y_n^λ) has no inv. measure	$g(\lambda) < 1$ $h_\lambda > 0$ (Y_n^λ) is transient M has no λ-eigenmeasure (Y_n^λ) has no inv. measure

Perron-Frobenius theory for countable matrices Let us explain now the links between our results with existing results coming from the Perron-Frobenius theory of countable matrices. As developed in [Sen06], for a general, irreducible, countable matrix M, with nonnegative entries, for each $z > 0$, the series $S_{i,j}(z) = \sum_{n \geqslant 0} z^n M^n(i,j), i,j \in \mathbb{Z}_+$ (where $M^n(i,j)$ refers to the coefficient on the i-th row and j-th column of M^n) either all converge or all diverge. Therefore, they all have the same radius of convergence r. We set $N(M) = r^{-1}$ and call $N(M)$ the converence parameter of M. Let us assume now that the convergence radius r is positive, that is $N(M) < +\infty$. The matrix is said to be r-recurrent (respectively r-transient) if the series $S_{i,j}(r)$ diverge (respectively converge). By [Sen06, Theorem 6.2], if M is r-recurrent, it admits non-negative eigenvectors (X, Y) respectively on the left and on the right. The matrix M is additionally either r-positive recurrent (resp. null recurrent) if $\langle X, Y\rangle < \infty$ (resp. $= \infty$). If M is aperiodic and r-positive recurrent then it enjoys an ergodicity property of the form :

$$\lim_{n\to\infty} N(M)^{-n} M^n(i,j) = \frac{X_j Y_i}{\langle X, Y\rangle}.$$

Moreover, the techniques used in [Sen06] essentially rely on a Doob transform, which transforms M into a stochastic matrix, when M is r-recurrent.
Moreover, one can define the quantities

$$l_i^0 = 0, l_i^n = \sum_{\substack{k_0 = k_n = i \\ k_1, \ldots, k_{n-1} \neq i}} M(k_0, k_1) \ldots M(k_{n-1}, k_n),$$

for $n \geqslant 1$, as well as the series $L_i(z) = \sum_{n \geqslant 0} z^n l_i^n$. Rearranging the terms in the sums, one can show that

$$S_{i,i}(z) = \sum_{n \geqslant 0} M^n(i,i)(z) = \sum_{n \geqslant 0} L_i(z)^n.$$

As a consequence, $S_{i,i}(z) < \infty \Leftrightarrow L_i(z) < 1$, so that for a given value of z, either $L_i(z) < 1$ for all i, or $L_i(z) \geqslant 1$ for all i. Thus, for any i, the radius of convergence of $S_{i,i}$ is $r = \inf\{z \geqslant 0 | L_i(z) \leqslant 1\} = \sup\{z \geqslant 0 | L_i(z) \geqslant 1\}$ since L_i is increasing. Thus $L_i(z) < 1$ for any $z < r$, and $L_i(r) \leqslant 1$. If $L_i(r) = 1$, then $S_i(r) = +\infty$ and M is r-recurrent. If $L_i(r) < 1$ then $S_i(r) < \infty$ and M is r-transient.

Many of our results might have been obtained applying this theory to the infinite Leslie matrix. In particular, setting $z = \frac{1}{\lambda}$, it holds $g(\lambda) = L_0(z)$, as well as $r = 1/\lambda_0$, thus $\lambda_0 = N(M)$. In particular the following results are consequences of this PF theory :

- The existence of left and right positive λ-eigenvectors for M when $g(\lambda) = 1$ (that is when M is λ_0-recurrent).

- The ergodicity of M when M is aperiodic and λ_0-positive recurrent, that is when the scalar product of its left and right λ_0-eigenelements is finite.

Our calculations have nevertheless some advantages and provide results which do not seem to be direct consequences of general results of this theory. First of all, we perform a full and explicit characterization of the eigenlements of M, as well as various asymptotics results on them. We prove for example that the condition $g(\lambda) = 1$ is not only sufficient but also necessary for M to have a left positive eigenvector. The same does not hold for right eigenvectors : M has a right positive λ-eigenvector for any $\lambda \geqslant \lambda_0$. This notable asymetry does not seem to appear in Seneta's book. Moreover, we exhibit the formal link between r-positive recurrence, r-null recurrence and r-transience and the (positive or null)-recurrence or transcience of the Markov chain (Y_n^λ). We provide the necessary and sufficient condition $|g'(\lambda)| < +\infty$ for positive recurrence. According to the tables in the previous subsection, in our context, a mere calculation of R, $g(R)$ (and $g'(R)$ if $g(R) = 1$) is enough to determine whether M will be λ_0-transient, positive recurrent or null recurrent. Finally, we obtain a necessary and sufficient condition for ergodicity of M and a sufficient condition for geometric ergodicity.

III.5 Consequences for the Galton-Watson process

III.5.1 Classification in regimes and extinction

The usual classification of Galton-Watson processes in regimes was extended in [Sag12], replacing the spectral radius of the mean matrix by the convergence parameter $N(M)$ of M. In Subsection III.4.5, we recall the definition of this quantity and show that in our case, $N(M) = \lambda_0$. This yields the following definition :

Definition III.22. *Consider a LGW-process $(Z_n)_{n \geqslant 0}$, with $R < \infty$. We say that (Z_n) is*

- *subcritical if* $\lambda_0 < 1$

- *critical if* $\lambda_0 = 1$

- *supercritical if* $\lambda_0 > 1$.

As explained in the introduction of the chapter, the convergence parameter $N(M)$ actually chacracterizes the a.s. local extinction and not the global extinction. Indeed, by [Zuc11, Thme 4.1], the condition $\lambda_0 > 1$ is prevents local extinction and thus guarantees that the population survives with positive probability : if $\lambda_0 > 1$, then $\mathbf{q} \leqslant \tilde{\mathbf{q}} < 1$ thus $\mathbf{q} < 1$. However, if $\lambda_0 \leqslant 1$, we only have $\mathbf{q} \leqslant \tilde{\mathbf{q}}(x) = 1$: each age class must go extinct for n large enough but the population might survive anyway if a few individuals live forever. However per Theorem 1, the global survival happens with positive probability if and only if $m > 1$ or $\prod_{n \geqslant 0} s_n > 0$.
Let us clarify the different possible configurations of λ_0, m and $\prod_{n \geqslant 0} s_n$. The position with respect to 1 of $m = g(1)$ can be easily deduced depending on $g(R)$ and λ_0, by the following reasoning.
If $g(R) \geqslant 1$ then we know that $g(\lambda_0) = 1$. Therefore M is λ_0-recurrent, and since g is decreasing, it holds

$$\begin{aligned} g(1) = 1 &\Rightarrow \lambda_0 = 1 \\ g(1) > 1 &\Rightarrow \lambda_0 > 1 \\ g(1) < 1 &\Rightarrow \lambda_0 < 1. \end{aligned}$$

In other words, $m = g(1)$ and λ_0 are on the same side of 1.
When $g(R) < 1$, M is λ_0-transient, it holds $\lambda_0 = R$ and

- either $R > 1$, in which case $m = g(1) = +\infty$ and $\lambda_0 = R > 1$,

- or $R = \lambda_0 \geqslant 1$, in which case $m = g(1) \leqslant g(R) < 1$.

It is therefore not true anymore that $g(1) < 1 \Rightarrow \lambda_0 < 1$. These facts are summed up in the two following tables :

Case 1 : M is λ_0-recurrent i.e. $\mathbf{g(R) \geqslant 1}$

Subcritical case, $\lambda_0 < 1$	Critical case, $\lambda_0 = 1$	Supercritical case, $\lambda_0 > 1$
$m < 1$ $\mathbf{q} = 1 \Leftrightarrow \prod_{n \geqslant 0} s_n = 0$	$m = 1$ $\mathbf{q} = 1 \Leftrightarrow \prod_{n \geqslant 0} s_n = 0$	$m > 1$ $\mathbf{q} \leqslant \tilde{\mathbf{q}} < 1$

Case 2 : M is λ_0-transient i.e. $\mathbf{g(R) < 1}$, thus $\lambda_0 = R$

Subcritical case, $\lambda_0 < 1$	Critical case, $\lambda_0 = 1$	Supercritical case, $\lambda_0 > 1$
$m < 1$ $\mathbf{q} = 1 \Leftrightarrow \prod_{n \geqslant 0} s_n = 0$	$m = g(1) = g(R) < 1$ $\mathbf{q} = 1 \Leftrightarrow \prod_{n \geqslant 0} s_n = 0$	$m = +\infty$ $\mathbf{q} \leqslant \tilde{\mathbf{q}} < 1$

In the rest of this section, we apply the results from Sections III.3 and III.4 pertaining to the eigenelements and ergodicity of M to sharpen our understanding of the LGW (Z_n). In Subsection III.5.2, we focus on the subcritical case $\lambda_0 < 1$. In this case, we prove that the extinction time of each type is dominated by a geometric random variable. The same holds for the global extinction time under some minorization on the eigenfunction h_λ. In particular, we can partially recover the extinction result from Theorem 1, using our controls under M^n instead of the auxiliary process $\widehat{Z}_n$. In Subsection III.5.3, we focus on the supercritical case. We work here using second moments assumptions, which is stronger than what was used in Theorem 1 to ensure the positivity of the probabilty of survival. Under these moments conditions, we prove the convergence of the usual martingale to some non degenerate limit, and we are also able to establish a Kesten-Stigum theorem, proving both the Malthusian growth of the population and the convergence of the distribution of age in the population on the survival event.

III.5.2 Subcritical case

Proposition III.23. *Consider a LGW such that Assumption III.1 holds and $R < 1$. Then $g(1) < 1$ if and only if $\lambda_0 < 1$, in which case, for any $\lambda \in [\lambda_0, 1)$, it holds*

i) for any $x, y \in \mathbb{Z}_+$,

$$\mathbb{P}[Z_n(y) > 0 | Z_0 = \delta_x] \leqslant \frac{h_\lambda(x)}{h_\lambda(y)} \lambda^n, \tag{III.35}$$

ii) if $\prod_{n \geqslant 0} s_n = 0$, then $\mathbf{q}(x) = \mathbb{P}[\exists n \geqslant 0, Z_n = 0 | Z_0 = \delta_x] = 1$ for any $x \geqslant 0$,

iii) if $\inf_{n \geqslant 0} h_\lambda(n) > 0$, then $\mathbb{P}[\|Z_n\| > 0 | Z_0 = \delta_x] \leqslant \frac{h_\lambda(x)}{\inf h_\lambda} \lambda^n$ for any $x \geqslant 0$.

By Borel Cantelli's lemma, Equation (III.35) implies that each type y goes extinct almost surely. *ii)* deduces from this the extinction result stated in Theorem 1, under the additionnal assumption that $R < 1$ (we thus exclude the case where $R = 1$ and $g(1) < 1$, which was covered by Theorem 1). Moreover, *iii)* yields an upper bound on the extinction speed, under the assumption that some harmonic function h_λ is bounded from below. Note that this can be applied for any $\lambda \in [\lambda_0, 1)$. Since the functions h_λ are increasing order as a function of λ, the condition $\inf_{n \geqslant 0} h_\lambda(n) > 0$ might hold only for some values of λ. Note that Proposition III.10 provides some sufficient conditions for $\inf_{n \geqslant 0} h_\lambda(n) > 0$.

Example A: $f_n = f > 0, s_n = s \in [0, 1]$

Assuming that $f + s < 1$, we have

$$\mathbb{P}[\|Z_n\| > 0] \leqslant (f + s)^n \xrightarrow[n \to \infty]{} 0.$$

Indeed, we recall that $R = s$, $g(s) = +\infty$, $g(1) = \frac{f}{1-s}$, and $\lambda_0 = f + s$. If $f < 1 - s$ then $\lambda_0 < 1$ thus i) and ii) apply for any $\lambda \in [\lambda_0, 1)$. Finally, $h_{\lambda_0} = \mathbb{1}$, thus, iii) applies at $\lambda = \lambda_0$, which yields

$$\mathbb{P}[\|Z_n\| > 0] \leqslant (f + s)^n \xrightarrow[n \to \infty]{} 0.$$

In a more general situation, the estimates from Proposition III.10 are useful since they provide sufficient conditions for $\inf_{n \geqslant 0} h_\lambda(n) > 0$.

Example B: $a_1 \leqslant f_n/(n+1)^\alpha \leqslant a_2$, $\lim_{n\to+\infty} s_n = s \geqslant 0$, $g(s) > 1$.

We assume additionally that $g(1) < 1$ to ensure that $\lambda_0 < 1$. Then

$$\mathbb{P}[\|Z_n\| > 0] = O(\lambda^n)$$

holds for any $\lambda > \lambda_0$. If additionally $\alpha \geqslant 0$ then the same property also holds at $\lambda = \lambda_0$.

Indeed , in this situation, since f_n grows polynomially, we have $\liminf(f_n)^{\frac{1}{n}} = 1$, thus for $\lambda \in (\lambda_0, 1)$, it holds $g(\lambda) < 1$ and $\lambda > \lambda_0 \geqslant R/\liminf_{n\to+\infty}(f_n^{1/n} = R = s)$. Thus by Proposition III.10, the quantity $\inf_{n\geqslant 0} h_\lambda(n)$ is nonzero for any $\lambda > \lambda_0$. Proposition III.23,iii) is satisfied for any $\lambda \in (\lambda_0, 1)$.

Additionally, if $\alpha \geqslant 0$, then the sequence (f_n) is bounded away from 0, thus $\inf_{n\geqslant 0} h_{\lambda_0}(n) > 0$ by Proposition III.10,ii) and thus III.23,iii) also holds at $\lambda = \lambda_0$.

Remark III.24. As a consequence of Proposition III.10, the third claim of Proposition III.23 yields a stochastic domination of the time of extinction

$$T_{ext} := \inf\{n \geqslant 0, Z_n = 0\}$$

by a geometric distribution of any parameter $\lambda \in (\lambda_0, 1)$ as soon as $\lambda_0 < 1$ and one of the following condition holds

- $\limsup (s_0 \dots s_{n-1})^{\frac{1}{n}} < 1$

- $R < \liminf(f_n)^{\frac{1}{n}}$.

Proof of Proposition III.23. i) is obtained applying the first moment method to $\mathbb{1}_y$, indeed :

$$\begin{aligned}\mathbb{P}[Z_n(y) > 0|Z_0 = \delta_x] &= \mathbb{P}[Z_n(\mathbb{1}_y) > 0|Z_0 = \delta_x]\\ &= \mathbb{P}[Z_n(\mathbb{1}_y) \geqslant 1|Z_0 = \delta_x]\\ &\leqslant \mathbb{E}[Z_n(\mathbb{1}_y) \geqslant 1|Z_0 = \delta_x]\\ &\leqslant M^n(\mathbb{1}_y)(x)\end{aligned}$$

since $Z_n(\mathbb{1}_y)$ is integer-valued. Moreover, $\mathbb{1}_y \leqslant \frac{h_\lambda}{h_\lambda(y)}$, thus

$$\mathbb{P}[Z_n(y) > 0|Z_0 = \delta_x] \leqslant \frac{1}{h_\lambda(y)} M^n(h_\lambda)(x) = \frac{h_\lambda(x)}{h_\lambda(y)}\lambda^n.$$

Hence, *i)* is proved. Let us move now to the proof of *ii)*. As a consequence of *i)*,

$$\sum_{n\geqslant 0} \mathbb{P}[Z_n(0)|Z_0 = \delta_x] < +\infty.$$

Thus, by Borel-Cantelli's lemma, there exists almost surely a random integer N such that $Z_n(0) = 0$ for $n \geqslant N$, hence $\sum_{n\geqslant 0} Z_n(0) > 0 < \infty$ almost surely. Combined with Lemma III.5,*ii)*, this proves *ii)*.

If $d = \inf_{n\geqslant 0} h_\lambda(n) > 0$ then $\mathbb{1} \leqslant \frac{h_\lambda}{d}$. Thus, once again

$$\mathbb{P}_x[Z_n(\mathbb{1}) > 0] \leqslant \mathbb{E}_x[Z_n(\mathbb{1})] \leqslant d\mathbb{E}_x[Z_n(h_\lambda)] \leqslant dM^n(h_\lambda)(x) \leqslant \frac{h_\lambda(x)}{\inf_{k\geqslant 0} h_\lambda(k)}\lambda^n$$

This proves *iii)*. □

III.5.3 Supercritical case.

We assume here that $\lambda_0 > 1$. Thus, for any $\lambda \geqslant \lambda_0$, it holds

$$\mathbb{E}[Z_n(h_\lambda)|Z_0 = \delta_x] = \lambda^n h_\lambda(x) \longrightarrow \infty. \tag{III.36}$$

We want to obtain more precise estimates that simply the convergence of the average of the quantity $Z_n(f)$. In particular, we would like to be able to establish that for a large class of functions f, with positive probability, the quantity $Z_n(f)$ grows as λ_0^n. In particular, for $f = \mathbb{1}$, this would imply that the population size grows at rate λ_0 on some non trivial event. To do so, we introduce the martingale

$$W_n = \frac{Z_n(h_{\lambda_0})}{\lambda_0^n}.$$

Note that a similar martingale could be introduced using any eigenvalue $\lambda \geqslant \lambda_0$ and the corresponding eigenfunction h_λ. We shall only use the martingale associated with $\lambda = \lambda_0$ in the sequel. This positive martingale almost surely converges to a limit $W \in [0, +\infty)$. On the event $W > 0$, the quantity $Z_n(h_{\lambda_0})$ tends to infinity at a geometric rate $W\lambda_0^n$. In this context, it is possible to establish the following Kesten-Stigum type result.

Proposition III.25. *Suppose (Z_n) is a LGW such that Assumption III.1 holds, $R < +\infty$, $g(1) > 1$, h_{λ_0} is bounded and*

$$x \mapsto \mathbb{E}\left[Z_1(h_{\lambda_0})^2 | Z_0 = \delta_x\right] \in \mathcal{B}_{h_{\lambda_0}}. \tag{III.37}$$

Then

i) the process (Z_n) survives with positive probability,

ii) $\mathbb{P}[W > 0] > 0$,

iii) if the matrix M is aperiodic and positive recurrent, then

$$\mathbb{1}_{W>0}\left|\frac{Z_n(f)}{Z_n(h_{\lambda_0})} - \frac{\pi(f)}{\pi(h_{\lambda_0})}\right| \xrightarrow{\mathcal{L}^2} 0, \tag{III.38}$$

iv) if the Doeblin condition (III.32) holds, then (III.38) holds almost surely and

$$\mathbb{1}_{W>0}\frac{Z_n(f)}{\lambda_0^n} \xrightarrow{a.s.} W\frac{\pi(f)}{\pi(h_{\lambda_0})} \tag{III.39}$$

Point *i)* was actually already proven under weaker assumptions in Section III.2, we present nevertheless an alternative proof here, based on a second moment estimate. Points *ii)* (which actually implies *i)*) and *iii)* are actually proven in [Moy67]. We include a proof for the sake of completeness ; it is also of interest to present these methods in a time homogeneous framework before adapting them to a case with random environment in Chapter V. Finally, under the Doeblin assumption III.32, the geometric ergodicity obtained in Proposition III.21 allows to strengthen *iii)* into the a.s. convergence *iv)*.

In particular, this Theorem can be applied to our examples.

Example A: $f_n = f > 0, s_n = s \in [0,1]$

This example satisfies the assumptions of Proposition III.25,i)-iv) as soon as $f+s>1$ *and* $\sup_{x\geqslant 0}\mathbb{E}[F_x^2] < +\infty$.
Indeed, we recall here that $R = s < \infty$, $g(1) = \frac{f}{1-s} > 1$ *iff* $\lambda_0 = f + s > 1$, *and* $h_{\lambda_0}(x) = 1$ *for all* $x \geqslant 0$. *As a consequence,* h_{λ_0} *is bounded. By Proposition III.19, the Doeblin minorization* (III.32) *also holds. Therefore, it only remains to check the second moment condition* (III.37), *i.e., plugging in* $h_{\lambda_0} = 1$ *into* (III.37), *that*

$$\sup_{x\in\mathbb{Z}_+} \mathbb{E}[\|Z_1\|^2 | Z_0 = \delta_x] < \infty.$$

Conditionally on $Z_0 = \delta_x$, *it holds* $\|Z_1\| = F_x + S_x \leqslant F_x + 1$, *since* $S_x \leqslant 1$ *by definition. Thus*

$$\mathbb{E}[\|Z_1\|^2 | Z_0 = \delta_x] \leqslant 1 + 2\mathbb{E}[F_x] + \mathbb{E}[F_x^2 | Z_0 = \delta_x] = 1 + 2f + \mathbb{E}[F_x^2 | Z_0 = \delta_x].$$

Hence $\mathbb{E}[\|Z_1\|^2 | Z_0 = \delta_x]$ *is bounded as soon as* $\sup_{x\in\mathbb{Z}_+}\mathbb{E}[F_x^2] < +\infty$.

Example B: $a_1 \leqslant f_n/(n+1)^\alpha \leqslant a_2$, $\lim_{n\to+\infty} s_n = s \geqslant 0$, $g(s) > 1$.

This example satisfies the conclusions of Proposition III.25 as soon as $g(1) > 1$, $\alpha \leqslant 0$ *and*

$$\sup_{x\geqslant 0} \frac{\mathbb{E}[F_x^2]}{f_x} < +\infty. \tag{III.40}$$

Indeed, by Proposition III.21, the Doeblin minorization (III.32) *holds. Moreover, as a consequence from* (III.32) *and* (III.25) *the function* h_{λ_0} *satisfies* $\frac{f_n}{\lambda_0} \leqslant h_{\lambda_0}(n) \leqslant \frac{f_n}{c\lambda_0} =: c_2 f_n$ *for all* $n \geqslant 0$. *This implies first that* $h_{\lambda_0}(n)$ *is bounded if* $\alpha \leqslant 0$. *Moreover, the second moment condition* (III.37) *is equivalent to*

$$\sup_{x\in\mathbb{Z}_+} \frac{\mathbb{E}\left[Z_1(h_{\lambda_0})^2 | Z_0 = \delta_x\right]}{f_x} < +\infty.$$

Since $h_{\lambda_0}(x) \leqslant c_2 f_x$ and $S_x \leqslant 1$ almost surely, we derive now

$$\begin{aligned}\frac{\mathbb{E}\left[Z_1(h_{\lambda_0})^2|Z_0=\delta_x\right]}{f_x} &\leqslant c_2\frac{\mathbb{E}\left[(F_x f_0 + S_x f_{x+1})^2\right]}{f_x}\\ &\leqslant c_2\frac{\mathbb{E}\left[F_x^2 f_0^2 + f_{x+1}^2 + 2F_x f_0 f_{x+1}\right]}{f_x}\\ &\leqslant c_2\left(f_0^2\frac{\mathbb{E}\left[F_x^2\right]}{f_x} + \frac{f_{x+1}^2}{f_x} + 2f_0 f_{x+1}\right)\end{aligned}$$

The condition $a_1(x+1)^\alpha \leqslant f_x \leqslant a_2(x+1)^\alpha$, $x \geqslant 0$, thus implies

$$\limsup_{x\to\infty}\frac{\mathbb{E}\left[Z_1(h_{\lambda_0})^2|Z_0=\delta_x\right]}{f_x} \leqslant c_2 f_0^2 \limsup_{x\to\infty}\frac{\mathbb{E}[F_x^2]}{f_x} < +\infty.$$

The proof of Proposition III.25 is based on second moments estimates. To compute them, we define, for any positive fonction g,

$$V(g) : x \mapsto V(g)(x) = \mathbb{E}\left[\sum_{u\neq v\in\mathbb{G}_1} g(X(u))g(X(v))|Z_0=\delta_x\right]$$

and

$$\Sigma_p(g) = \sum_{w\in\mathbb{G}_p}\sum_{\substack{u_0\neq v_0\in\mathbb{G}_{p+1}\\ w<u_0,v_0}}\sum_{\substack{u\in\mathbb{G}_n\\ u_0<u}} f(X(u))\sum_{\substack{v\in\mathbb{G}_n\\ v_0<v}} f(X(v)).$$

We state now a technical but crucial lemma.

Lemma III.26 (Many-to-two Formula)**.** *Let (Z_n) be a LGW-process, with $R<+\infty$, let $\lambda \geqslant \lambda_0$ and $f \in \mathcal{B}_{h_\lambda}$. Additionally assume that $V(h_\lambda) \in \mathcal{B}_{h_\lambda}$. Then*

$$\mathbb{E}[Z_n(f)^2|Z_0=\delta_x] = M^n(f^2)(x) + \lambda^{2n}\sum_{p=0}^{n-1}\frac{1}{\lambda^{p+2}}h_\lambda P_\lambda^p\left[\frac{V\left(h_\lambda P_\lambda^p\left(\frac{f}{h_\lambda}\right)\right)}{h_\lambda}\right]$$

for all $n \geqslant 0$.

Proof of Lemma III.26. Let us write

$$(Z_n(f))^2 = \sum_{u,v\in\mathbb{G}_n} f(X(u))f(X(v) = \sum_{u\in\mathbb{G}_n} f(X(u))^2 + \sum_{u\neq v\in\mathbb{G}_n} f(X(u))f(X(v)),$$

where $\mathbb{G}_n$ refers to the set of individuals alive at time n and when $X(u) \in \mathbb{Z}_+$ is the age of u for any $u \in \mathbb{G}_n$. Moreover,

$$\begin{aligned}\sum_{u\neq v\in\mathbb{G}_n} f(X(u))f(X(v)) &= \sum_{p=0}^{n-1}\sum_{w\in\mathbb{G}_p}\sum_{\substack{u_0\neq v_0\in\mathbb{G}_{p+1}\\ w<u_0,v_0}}\sum_{\substack{u\in\mathbb{G}_n\\ u_0<u}} f(X(u))\sum_{\substack{v\in\mathbb{G}_n\\ v_0<v}} f(X(v))\\ &= \sum_{p=0}^{n-1}\Sigma_p(f).\end{aligned}$$

As a consequence,

$$\begin{aligned}
\mathbb{E}[Z_n(f)^2|Z_0=\delta_x] &= \mathbb{E}\left[\sum_{u\in\mathbb{G}_n} f(X(u))^2|Z_0=\delta_x\right] + \mathbb{E}\left[\sum_{u\neq v\in\mathbb{G}_n} f(X(u))f(X(v))\right] \\
&= M^n(f^2)(x) + \sum_{p=0}^{n-1}\mathbb{E}\left[\Sigma_p(f)|Z0=\delta_x\right]. \qquad \text{(III.41)}
\end{aligned}$$

Moreover, noting $\mathcal{F}_k$ the σ-algebra generated by the p first generations of the Galton-Watson process, we may write

$$\mathbb{E}\left[\Sigma_p(f)|\mathcal{F}_{p+1}\right] = \sum_{w\in\mathbb{G}_p}\sum_{\substack{u_0\neq v_0\in\mathbb{G}_{p+1}\\ w<u_0,v_0}} \mathbb{E}\left[\sum_{\substack{u\in\mathbb{G}_n\\ u_0<u}} f(X(u))\middle|\mathcal{F}_{p+1}\right]\mathbb{E}\left[\sum_{\substack{v\in\mathbb{G}_n\\ v_0<v}} f(X(v))\middle|\mathcal{F}_{p+1}\right]$$

where $\mathbb{E}\left[\sum_{\substack{u\in\mathbb{G}_n\\ u_0<u}} f(X(u))\middle|\mathcal{F}_{p+1}\right]$ and $\mathbb{E}\left[\sum_{\substack{v\in\mathbb{G}_n\\ v_0<v}} f(X(v))\middle|\mathcal{F}_{p+1}\right]$ are the expectation of the f-mass of the population at time n descending respectively from u_0 and v_0. Thus

$$\mathbb{E}\left[\sum_{\substack{u\in\mathbb{G}_n\\ u_0<u}} f(X(u))\middle|\mathcal{F}_{p+1}\right] = M^{n-p-1}(f)(X(u_0)),$$

and a similar statement holds replacing u_0 by v_0. Hence

$$\mathbb{E}\left[\Sigma_p(f)|\mathcal{F}_{p+1}\right] = \sum_{w\in\mathbb{G}_p}\sum_{\substack{u_0\neq v_0\in\mathbb{G}_{p+1}\\ w<u_0,v_0}} M^{n-p-1}(f)(X(u_0))M^{n-p-1}(f)(X(v_0))$$

Therefore

$$\begin{aligned}
\mathbb{E}[\Sigma_p(f)|\mathcal{F}_p] &= \sum_{w\in\mathbb{G}_p}\mathbb{E}\left[\sum_{\substack{u_0\neq v_0\in\mathbb{G}_{p+1}\\ w<u_0,v_0}} M^{n-p-1}(f)(X(u_0))M^{n-p-1}(f)(X(v_0))\middle|\mathcal{F}_p\right] \\
&= \sum_{w\in\mathbb{G}_p} V(M^{n-p-1}(f))(X(w)) \\
&= Z_p(M^{n-p-1}(f))
\end{aligned}$$

by the branching property. We finally get

$$\mathbb{E}[\Sigma_p(f)|Z_0=\delta_x] = M^p(V(M^{n-p-1}(f)))(x).$$

From the definition of P_λ, we know that, when $f\in\mathcal{B}_{h_\lambda}$, it holds

$$M^k(f) = \lambda^k h_\lambda P_\lambda^k\left(\frac{f}{h_\lambda}\right).$$

Thus

$$\mathbb{E}[\Sigma_p(f)] = M^p(V(M^{n-p-1}(f))) = M^p\left(V\left(\lambda^{n-p-1}h_\lambda P_\lambda^{n-p-1}\left(\frac{f}{h_\lambda}\right)\right)\right)$$
$$= \lambda^{2n-2p-2}M^p\left(V\left(h_\lambda P_\lambda^{n-p-1}\left(\frac{f}{h_\lambda}\right)\right)\right)$$

Moreover, the function f/h_λ is assumed to be bounded and P_λ is Markovian, thus $P_\lambda^{n-p-1}\left(\frac{f}{h_\lambda}\right)$ is bounded and $h_\lambda P_\lambda^{n-p-1}\left(\frac{f}{h_\lambda}\right) \in \mathcal{B}_{h_\lambda}$. Since $V(h_\lambda) \in \mathcal{B}_{h_\lambda}$, then the function $V\left(h_\lambda P_\lambda^{n-p-1}\left(\frac{f}{h_\lambda}\right)\right)$ also belongs to $\mathcal{B}_{h_\lambda}$ and

$$\mathbb{E}[\Sigma_p(f)] = \lambda^{2n-p-2}h_\lambda P_\lambda^p\left(\frac{V\left(h_\lambda P_\lambda^{n-p-1}\left(\frac{f}{h_\lambda}\right)\right)}{h_\lambda}\right).$$

Finally, plugging the previous equation in (III.41), we recover

$$\mathbb{E}[Z_n(f)^2|Z_0 = \delta_x] = M^n(f^2)(x) + \sum_{p)0}^{n-1}\lambda^{2n-p-2}h_\lambda P_\lambda^p\left(\frac{V\left(h_\lambda P_\lambda^{n-p-1}\left(\frac{f}{h_\lambda}\right)\right)}{h_\lambda}\right)$$
$$= M^n(f^2)(x) + \lambda^{2n}\sum_{p=0}^{n-1}\frac{1}{\lambda^{p+2}}h_\lambda P_\lambda^p\left(\frac{V\left(h_\lambda P_\lambda^{n-p-1}\left(\frac{f}{h_\lambda}\right)\right)}{h_\lambda}\right).$$

This concludes the proof. □

Proof of Proposition III.25. We work here at $\lambda = \lambda_0 > 1$ and set $h = h_{\lambda_0}$ to lighten the notation. We recall that under the assumptions of the Proposition, the set of nonnegative λ_0-eigenmeasures of M is $\mathbb{R}^+\pi$ and $V(h)(x) \leqslant \mathbb{E}[Z_1(h)^2|Z_0 = \delta_x]$. By (III.37) the function $V(h)$ belongs to $\mathcal{B}_h$. We can then apply Lemma III.26 with $f = h$. As a consequence, for any $n \geqslant 0$, we have

$$\mathbb{E}[Z_n(h)^2|Z_0 = \delta_x] = M^n(h^2)(x) + {\lambda_0}^{2n}\sum_{p=0}^{n-1}\frac{1}{{\lambda_0}^{p+2}}h(x)P_{\lambda_0}^p\left[\frac{V(h)}{h}\right](x). \tag{III.42}$$

On the one hand, h is assumed to be bounded thus $h^2 \leqslant \|h\|_\infty h$ and $M^n(h^2) \leqslant \|h\|_\infty{\lambda_0}^n h$; on the other hand, $V(h) \in \mathcal{B}_h$, thus $\frac{V(h)}{h}$ is bounded. Plugging this into (III.42) yields

$$\mathbb{E}[Z_n(h)^2|Z_0 = \delta_x] \leqslant {\lambda_0}^n\|h\|_\infty h(x) + {\lambda_0}^{2n}\sum_{p=0}^{n-1}\frac{1}{{\lambda_0}^{p+2}}h(x)\left\|\frac{V(h)}{h}\right\|_\infty$$
$$\leqslant {\lambda_0}^n\|h\|_\infty h(x) + {\lambda_0}^{2n-2}\frac{1}{1-{\lambda_0}^{-1}}h(x)\left\|\frac{V(h)}{h}\right\|_\infty. \tag{III.43}$$

Notice that the events $\{Z_n(h) = 0\}$ and $\{Z_n = 0\}$ are equal since h is positive. Therefore we derive *i)* from (III.43) by using Cauchy-Schwarz's inequality:

$$\begin{aligned}
\mathbb{P}[Z_n \neq 0 | Z_0 = \delta_x] &= \mathbb{P}[Z_n(h) > 0 | Z_0 = \delta_x] \\
&\geqslant \frac{\mathbb{E}[Z_n(h)|Z_0 = \delta_x]^2}{\mathbb{E}[Z_n(h)^2|Z_0 = \delta_x]} \\
&\geqslant \frac{{\lambda_0}^{2n} h^2(x)}{{\lambda_0}^n \|h\|_\infty h(x) + \frac{{\lambda_0}^{2n-2}}{1-{\lambda_0}^{-1}} h(x) \left\| \frac{V(h)}{h} \right\|_\infty}.
\end{aligned}$$

Since $\lambda_0 > 1$, it holds

$$\begin{aligned}
\inf_{n \geqslant 0} \mathbb{P}[Z_n \neq 0 | Z_0 = \delta_x] &= \liminf_{n \to \infty} \mathbb{P}[Z_n \neq 0 | Z_0 = \delta_x] \\
&\geqslant \frac{\lambda_0(\lambda_0 - 1) h(x)}{\left\| \frac{V(h)}{h} \right\|_\infty} > 0.
\end{aligned}$$

This proves *i)*.
To prove *ii)*, which is a strictly stronger statement, we notice that the martingale (W_n) is bounded in $\mathcal{L}^2$. Indeed, (III.47) yields $\limsup_{n\to\infty} \mathbb{E}[W_n^2] \leqslant \frac{1}{\lambda_0(\lambda_0 - 1)} Z_0(h) \left\| \frac{V(h)}{h} \right\|_\infty < +\infty$. This implies $\lim_{n\to+\infty} W_n = W$ in $\mathcal{L}^2$ and hence in $\mathcal{L}^1$, which yields in particular

$$\mathbb{E}[W] = \lim \mathbb{E}[W_n] = \mathbb{E}[W] = Z_0(h) > 0,$$

for any initial population Z_0. Hence,

$$\mathbb{P}[W > 0] > 0.$$

This proves *ii)*
Let us now fix $f \in \mathcal{B}(h)$ and set

$$g = f - \frac{\pi(f)}{\pi(h)} h \in \mathcal{B}(h).$$

We introduce the quantity

$$\Delta_n = \frac{Z_n(g)}{{\lambda_0}^n} = \frac{Z_n(f) - \frac{\pi(f)}{\pi(h)} Z_n(h)}{{\lambda_0}^n}.$$

We have

$$\mathbb{E}(\Delta_n) = \frac{M^n(g)}{{\lambda_0}^n} = \frac{M^n(f) - {\lambda_0}^n h \frac{\pi(f)}{\pi(h)}}{{\lambda_0}^n},$$

and by Proposition III.15 yields $\mathbb{E}(\Delta_n) \to 0$. We want to show a $\mathcal{L}^2$ convergence, let us remark that by Lemma III.26, it holds

$$\mathbb{E}[\Delta_n^2 | Z_0 = \delta_x] = \frac{M^n(g^2)(x)}{{\lambda_0}^{2n}} + \sum_{p=0}^{n-1} \frac{1}{{\lambda_0}^{p+2}} h P_{\lambda_0}^p \left[\frac{V\left(h P_{\lambda_0}^{n-p-1}\left(\frac{g}{h}\right)\right)}{h} \right]. \tag{III.44}$$

On the one hand, since $g \in \mathcal{B}(h)$, and $h \in \mathcal{B}(\mathbb{1})$, then the function g^2 belongs to $\mathcal{B}(h^2) \subset \mathcal{B}(h)$, therefore

$$\frac{M^n(g^2)}{{\lambda_0}^{2n}} \leqslant \left\|\frac{g}{h}\right\|_\infty \frac{M^n(h)}{{\lambda_0}^{2n}} = {\lambda_0}^{-n} \left\|\frac{g}{h}\right\|_\infty h \underset{n\to+\infty}{\longrightarrow} 0 \tag{III.45}$$

since $\lambda_0 > 1$. We decompose the study of the second term of (III.44) into four steps

Step 1: Let us first fix p, and study the behavior as n goes to infinity of the function

$$\mathcal{P}_{p,n} = P^{n-(p+1)}\left(\frac{g}{h}\right) = P^{n-(p+1)}\left(\frac{f}{h}\right) - \tilde{\pi}\left(\frac{f}{h}\right).$$

Since P is assumed to be positive recurrent, it holds

$$|\mathcal{P}_{p,n}(x)| \leqslant \|\delta_x P^{n-(p+1)} - \tilde{\pi}\|_{TV} \|f\|_{\mathcal{B}(h)} \underset{n\to+\infty}{\longrightarrow} 0. \tag{III.46}$$

Moreover, the function $\frac{g}{h}$ is bounded, thus $\|\mathcal{P}_{p,n}\|_\infty \leqslant \|g\|_{\mathcal{B}(h)}$.

Step 2: We introduce now the function $\mathcal{V}_{p,n} = V\left(hP^{n-(p+1)}\left(\frac{g}{h}\right)\right) = V\left(h\mathcal{P}_{p,n}\right)$. Notice first that

$$\mathcal{V}_{p,n}(x) = \mathbb{E}\left[\sum_{u\neq v\in\mathbb{G}_1} \left(h\mathcal{P}_{p,n}\right)(X(u))\left(h\mathcal{P}_{p,n}\right)(X(v))|z_0 = \delta_x\right].$$

Moreover, $\mathbb{G}_1$ is almost surely finite, therefore, by (III.46), it holds almost surely

$$\sum_{u\neq v\in\mathbb{G}_1} \left(h\mathcal{P}_{p,n}\right)(X(u))\left(h\mathcal{P}_{p,n}\right)(X(v)) \underset{n\to+\infty}{\longrightarrow} 0 \tag{III.47}$$

Moreover $|\mathcal{P}_{p,n}(x)| \leqslant \|g\|_{\mathcal{B}(h)}$ for any $x \geqslant 0$. As a consequence, it holds

$$\left|\sum_{u\neq v\in\mathbb{G}_1} \left(h\mathcal{P}_{p,n}\right)(X(u))\left(h\mathcal{P}_{p,n}\right)(X(v))\right| \leqslant \|g\|^2_{\mathcal{B}(h)} \sum_{u\neq v\in\mathbb{G}_1} h(X(u))h(X(v)).$$

Therefore, since $\mathbb{E}[\sum_{u\neq v\in\mathbb{G}_1} h(X(u))h(X(v))|Z_0 = \delta_x] = V(h)(x)$ is finite, by dominated convergence, it holds for all $x \geqslant 0$

$$\mathcal{V}_{p,n}(x) \underset{n\to\infty}{\longrightarrow} 0.$$

Moreover $\frac{\mathcal{V}_{p,n}(x)}{h(x)} \leqslant \frac{V(h)(x)}{h(x)}\|g\|^2_{\mathcal{B}(h)}$, where $V(h)/h$ is bounded.

Step 3: By dominated convergence, when p is fixed and $n \to \infty$,

$$P^p\left[\frac{1}{h}V\left(hP^{n-(p+1)}\left(\frac{g}{h}\right)\right)\right] = P^p\left[\frac{\mathcal{V}_{p,n}}{h}\right] \underset{n\to+\infty}{\longrightarrow} 0$$

and

$$\left|P^p\left[\frac{1}{h}V\left(hP^{n-(p+1)}\left(\frac{g}{h}\right)\right)\right]\right| = \left|P^p\left[\frac{\mathcal{V}_{p,n}}{h}\right]\right| \leqslant \|g\|^2_{\mathcal{B}(h)} \left\|\frac{V(h)}{h}\right\|_\infty.$$

Step 4: By a last application of the dominated convergence theorem, since $\lambda_0 > 1$,

$$\sum_{p=0}^{n-1} \frac{1}{{\lambda_0}^{p+2}} hP_{\lambda_0}^p \left[\frac{V\left(hP_{\lambda_0}^{n-p-1}\left(\frac{g}{h}\right)\right)}{h} \right] = h \sum_{p=0}^{\infty} \frac{1}{{\lambda_0}^{p+2}} P^p \left[\frac{\mathcal{V}_{p,n}}{h} \right] \underset{n\to+\infty}{\longrightarrow} 0. \tag{III.48}$$

Combining the estimates (III.45) and (III.48) and plugging them into (III.44) we obtain that the sequence $(\Delta_n)_n$ converges to 0 in $\mathcal{L}^2$, thus in probability. Moreover, since $\lim W_n = W$ almost surely,

$$\mathbb{1}_{W>0} \left| \frac{Z_n(f)}{Z_n(h)} - \frac{\pi(f)}{\pi(h)} \right| = \mathbb{1}_{W>0} \frac{\Delta_n}{W_n} \xrightarrow{\mathbb{P}} 0. \tag{III.49}$$

The fact that $f \in \mathcal{B}(h)$ also yields

$$\mathbb{1}_{W>0} \left| \frac{Z_n(f)}{Z_n(h)} - \frac{\pi(f)}{\pi(h)} \right| = \mathbb{1}_{W>0} \frac{\Delta_n}{W_n} \leqslant \|f\|_{\mathcal{B}(h)} + \frac{\pi(f)}{\pi(h)}$$

Therefore $\left(\mathbb{1}_{W>0}\frac{\Delta_n}{W_n}\right)$ is bounded, hence its square is uniformly integrable. As a consequence the convergence (III.49) also holds in $\mathcal{L}^2$, this yields *iii)*
We suppose now that the Doeblin condition (III.32) is satisfied. Then

$$\|\delta_x P^n - \tilde{\pi}\|_{TV} \leqslant 2(1-c)^n.$$

Therefore

$$\|\mathcal{P}_{p,n}\|_\infty \leqslant 2\|f\|_{\mathcal{B}(h)}(1-c)^{n-(p+1)}$$

and

$$\mathcal{V}_{p,n} \leqslant 4\|f\|^2_{\mathcal{B}(h)} V(h)(1-c)^{2n-2(p+1)}.$$

Consequently

$$P^p \left[\frac{\mathcal{V}_{p,n}}{h} \right] \leqslant 4\|f\|^2_{\mathcal{B}(h)} \|V(h)\|_{\mathcal{B}(h)} (1-c)^{2n-2(p+1)}$$

and

$$\begin{aligned}
\mathbb{E}[\Delta_n^2 | Z_0 = \delta_x] =& \frac{M^n(g^2)(x)}{{\lambda_0}^{2n}} + h(x) \sum_{p=0}^{n-1} \frac{1}{{\lambda_0}^{p+2}} P^p \left[\frac{\mathcal{V}_{p,n}}{h} \right](x) \\
\leqslant& {\lambda_0}^{-n} \|g^2\|_{B_h} h(x) + 4h(x)\|f\|^2_{\mathcal{B}(h)} \|V(h)\|_{\mathcal{B}(h)} (1-c)^{2n+2} \sum_{p=0}^{n-1} \left(\frac{1}{(1-c)^2 \lambda_0} \right)^{p+2}
\end{aligned}$$

where

$$\sum_{p=0}^{n-1} \left(\frac{1}{(1-c)^2\lambda_0} \right)^{p+2} \leqslant n \left(\max\left(1, \frac{1}{\lambda_0(1-c)^2}\right) \right)^{n+2}$$

Thus

$$\mathbb{E}[\Delta_n^2 | Z_0 = \delta_x] = \underset{n\to\infty}{O}((1-c)^{2n} n \alpha^n) = \underset{n\to\infty}{O}\left(n \max\left[(1-c)^2, \frac{1}{\lambda_0} \right]^n \right)$$

where, $\lambda_0 > 1$ and $0 < c < 1$. Therefore, for any $\varepsilon > 0$

$$\sum_{n\geqslant 0} \mathbb{P}[|\Delta_n| \geqslant \varepsilon] \leqslant \sum_{n\geqslant 0} \mathbb{P}[\Delta_n^2 \geqslant \varepsilon^2] \leqslant \frac{1}{\varepsilon^2} \sum_{n\geqslant 0} \mathbb{E}\left[\Delta_n^2\right] < +\infty.$$

By Borel-Cantelli's Lemma, $\Delta_n \underset{n\to\infty}{\longrightarrow} 0$ almost surely. Since

$$\mathbb{1}_{W>0} \left| \frac{Z_n(f)}{Z_n(h)} - \frac{\pi(f)}{\pi(h)} \right| = \mathbb{1}_{W>0} \frac{\Delta_n}{W_n} \text{ and } W_n \underset{n\to+\infty}{\longrightarrow} W \text{ a.s.},$$

we obtain

$$\mathbb{1}_{W>0} \left| \frac{Z_n(f)}{Z_n(h)} - \frac{\pi(f)}{\pi(h)} \right| \xrightarrow[n\to+\infty]{a.s.} 0.$$

This can be rewritten as

$$\mathbb{1}_{W>0} \frac{Z_n(f)}{Z_n(h)} \xrightarrow[n\to+\infty]{a.s.} \frac{\pi(f)}{\pi(h)},$$

where $Z_n(h) = \lambda_0^n W_n$ and $\lim_{n\to+\infty} W_n = W$ a.s.. Hence

$$\mathbb{1}_{W>0} \frac{Z_n(f)}{\lambda_0^n} \xrightarrow[n\to+\infty]{a.s.} W \frac{\pi(f)}{\pi(h)}.$$

□

III.5.4 Proof of Theorem 3

Proof of Theorem 3. Let us first assume that $\lambda_0 < 1$, and $\inf_{n\geqslant 0} f_n > 0$. Then by Proposition III.10 *ii)*, we know that $\inf_{n\geqslant 0} h_{\lambda_0}(n) > 0$. Therefore Proposition III.23 applies at $\lambda = \lambda_0$ and we get the desired result *i)*
The additional assumptions of *ii)* clearly allow to apply Proposition III.25 to obtain the desired result *ii)*

□

CHAPITRE IV

Ergodicité de Produits d'opérateurs aléatoires

Le contenu de ce chapitre provient de l'article [Lig25]. Nous y considérons une suite aléatoire, stationnaire et ergodique (M_n), constituée d'opérateurs linéaires positifs sur des espaces de mesures signées. Sous des conditions de type Doeblin, nous établissons des résultats d'ergodicité pour les produits $M_{0,n} = M_0 \dots M_{n-1}$ associés à ces opérateurs. Plus précisément, nous montrons l'existence presque sûre d'une suite réelle (r_n), d'une suite de mesures de probabilités (π_n), d'une fonction positive h, et d'un réel $\delta < 1$ déterministe tel que pour toute mesure signée μ, on ait

$$\|\mu M_{0,n} - r_n \mu(h) \pi_n\|_{TV} \leqslant \delta^n \|\mu M_{0,n}\|_{TV},$$

où $\|\cdot\|_{TV}$ est la norme de variation totale sur l'espace sous-jacent. Le triplet $(h, (r_n), (\pi_n))$ est aléatoire mais ne dépend pas de la mesure μ choisie. Par ailleurs, ces éléments ont un comportement asymptotique remarquable : $n^{-1}\log(r_n)$ converge presque sûrement vers une quantité déterministe appelée Exposant de Lyapounov du processus (M_n), et la suite de mesures de probabilités aléatoires (π_n) converge faiblement vers une mesure de probabilité aléatoire. Ces résultats étendent les résultats d'Hennion [Hen97] à un cadre de dimension infinie, en utilisant des techniques différentes. En effet, nous remplaçons une contraction projective fondée sur la distance de Hilbert par une contraction en variation totale obtenue par des techniques impliquant une contraction de Doeblin sur une chaine de Markov sous jacente. Nous incluons une comparaison détaillée de nos hypothèses obtenues avec ces techniques par rapport à celles d'Hennion et appliquons nos techniques au cas des produits de matrices de Leslie infinies.

IV.1 Introduction

IV.1.1 General introduction

The study of products of random linear operators can be traced back to the seminal article of [FK60], studying products of the form

$$M_{0,n} = M_0 \dots M_{n-1},$$

where $(M_n)_{n\geqslant 0}$ is a stationary and metrically transitive sequence of $p\times p$ real or complex random matrices. Under a mild irreducibility assumption, the authors exhibit a strong law of large numbers on the norm of $M_{0,n}$, in the form

$$\lim_{n\to\infty} \frac{1}{n} \log \|\mu M_{0,n}\| = \lambda,$$

where λ is a deterministic number called Lyapunov exponent of the sequence $(M_n)_{n\geqslant 0}$, defined as

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}\left[\log \|M_{0,n}\|\right] = \inf_{n\geqslant 1} \frac{1}{n} \mathbb{E}\left[\log \|M_{0,n}\|\right],$$

where the norm $\|\cdot\|$ can be chosen to be any submultiplicative norm. Under additional positivity and boundedness assumptions on the entries of the matrices (M_n), [FK60] also prove a strong law of large numbers for the entries $M_{0,n}(i,j)$ of the products : for any $i,j\in\{1,\dots,d\}$, almost surely,

$$\lim_{n\to\infty} \frac{1}{n} \log M_{0,n}(i,j) = \lambda.$$

These estimates on the behavior of the entries of $M_{0,n}$ were then extended to the case of products of invertible matrices, see e.g. [GL01] and [BL85]. These works rely on a careful study of the action of invertible matrices on the projective space $\mathcal{P}(\mathbb{R}^d)$.
To strengthen the results from [FK60] on products of matrices with non negative entries and relax their assumptions, [Hen97] studied the action of $\mathcal{M}_d(\mathbb{R}_+)$ on the projective space $\mathcal{P}(\mathbb{R}_+^d)$, endowed with the so called (pseudo)-Hilbert distance d_H previously defined in [BK53] and [Bir57]. This distance is particularly well adapted to this problem, since the contraction coefficient of the projective action of a matrix with respect to d_H is explicit in terms of its entries, in particular, any matrix with nonnegative entries is 1-contracting and any matrix with positive entries is strictly contracting. Under the assumption that almost surely, for n large enough, $M_{0,n}$ has all positive entries, Hennion obtains the asymptotic decomposition

$$M_{0,n}(i,j) = \lambda_n R_n(i) L_n(j) + \underset{n\to\infty}{o}(\lambda_n),$$

where λ_n is the dominant eigenvalue of $M_{0,n}$ and L_n, R_n are the associated left and right eigenvectors, with the normalizations $\|R_n\| = 1$ and $\langle L_n, R_n\rangle = 1$. Moreover $(R_n)_{n\geqslant 0}$ almost surely converges to a random vector R, $(L_n/\|L_n\|)_n$ converges in distribution, and $(n^{-1}\log\lambda_n)_{n\geqslant 1}$ almost surely converges to the Lyapunov exponent λ.
Such results have important implications, in particular in the field of populations dynam-

ics. Indeed, a population composed of d types of individuals, evolving in a fluctuating environment, without interacting which each other, can be modelled by a linear model of the form

$$x_n = x_{n-1}M_{n-1}, \tag{IV.1}$$

where x_n is a row vector of $\mathbb{R}_+^d$ encoding the mass of individuals of each type at time n and $M_{n-1} = (M_{n-1}(i,j))_{1\leqslant i,j\leqslant d}$ is a random matrix encoding the rates at which individuals of each type i create individuals of each type j between times $n-1$ and n. In such a time-inhomogeneous population model, the understanding of the asymptotics of x_n amounts to the understanding of the matrix product $M_{0,n}$.
Moreover, such products also appear in the study of multitype Galton-Watson processes in random environment (MGWRE), which were introduced in [AK71b]. They are a generalization of Galton-Watson processes to the case where the distribution of the (random) offspring of an individual depends on a notion of type and on a random environment that changes through time. When conditioning a MGWRE on the environment sequence, one obtains a so-called quenched population model, which satisifies (IV.1), where x_n is the expectation of the population conditionally on the environmental sequence. The value of the Lyapunov exponent λ of the underlying matrix product separates three regimes of the MGWRE : subcritical ($\lambda < 0$), critical ($\lambda = 0$), supercritical ($\lambda > 0$). These three regimes have different properties. In particular, when $\lambda \leqslant 0$, the MGWRE goes extinct with probability 1, when $\lambda > 0$, the MGWRE survives with positive probability. This separation between regimes was established in [AK71b] and [Kap74], using results from [FK60]. More recent advances in the study of random matrix products - in particular Hennion's article- were key to the last developments of the theory of MGWRE in random environments, see e.g. [Cam18; LPD18; GLP23].
Products of random infinite dimensional operators have also been the subject of some investigation. In the case where the involved operators are the transition matrices of some Markov chain, in other words if they are conservative (in the sense that $M\mathbb{1} = \mathbb{1}$), ergodicity results are obtained by [Cog84], completed with some more precise results in [Ore91]. Various limit theorems (law of large numbers, existence of Lyapunov exponents, central limit theorem, local central limit theorem, Large Deviation principle) have also been obtained on products of ergodic sequences of infinite dimensional operators using spectral techniques by [DFGV18]. The stability of the Lyapunov exponents under perturbation is studied in [AFGV22; FGQ19]. Ergodicity of products of random operators was also obtained in [Kif96] in the case where they act on some compact space $\mathbb{X}$.
In this paper, we would like to obtain similar ergodicity results for products of infinite dimensional positive operators, thus extending the results of [Kif96] without any topological assumption on $\mathbb{X}$, in particular without compacity. We have in particular in mind applications to population models with an infinite number of types. We first consider such a set $\mathbb{X}$, typically infinite, endowed with a σ-algebra $\mathcal{X}$, and build a set $\mathcal{K}^+$ of positive linear operators acting both on the space of signed measures $\mathcal{M}(\mathbb{X})$ on the left and the space of measurable bounded functions $\mathcal{B}(\mathbb{X})$ on the right. Then, we let (M_n) be a stationary, ergodic sequence of elements of $\mathcal{K}^+$ and define the products $M_{0,n} = M_0 \cdots M_{n-1}$. The approach of [Hen97] can be extended to this infinite dimensional setup. Indeed, it is possible to define the Hilbert distance d_H on the projective positive cone of an infinite dimensional vector space and to obtain a nice characterisation of the operators that are

(strictly) contracting with respect to d_H. We refer the reader to [Lig23] for a proof of these facts. However, as we explain in Section IV.4, this characterisation leads to stronger positivity assumptions in an infinite dimensional context than it did in the finite dimensional one. For example, such an extension of Hennion's approach would not be able to deal with products of infinite Leslie matrices that we present in Section IV.5.
For this reason, we use a different contraction method to obtain a projective contraction. This method aims at extending the Doeblin contraction techniques for Markov operators to a product of non conservative operators (that is operators M such that $M\mathbb{1} \neq \mathbb{1}$ in general). To do so, we consider the auxiliary family of Markov operators $P^N_{k,n}$, defined for each $k \leqslant n \leqslant N$ as

$$\delta_x P^N_{k,n} f = \frac{\delta_x M_{k,n}(f m_{n,N})}{m_{k,N}(x)},$$

where, for $x \in \mathbb{X}$,

$$m_{k,n}(x) = \delta_x M_{k,n}\mathbb{1}.$$

These Markov operators are related to the projective action $\mu \cdot M_{k,n} = \frac{\mu M_{k,n}}{\|\mu M_{k,n}\|}$ of $M_{k,n}$ on measures by :

$$\delta_x \cdot M_{k,n} = \delta_x P^n_{k,n} = \delta_x P^n_{k,k+1} \cdots P^n_{n-1,n},$$

for any $x \in \mathbb{X}$. We provide sufficient conditions for the Markov operators $(P^n_{k,k+1})_{k<n}$ to satisfy Doeblin minorations of the form $\delta_x P(f) \geqslant c\nu(f)$, which guarantees that they are contracting in total variation. This allows to obtain a notion of projective contraction of the $M_{k,n}$ on the set of signed measures. Such auxiliary operators were already introduced and already applied to study both homogeneous semi-groups of operators, e.g. in [DM02; CV16] and inhomogeneous ones, e.g. in [BCG20]. We consider here a random sequence of operators (M_n), i.e a discrete time, random, time-inhomogeneous semi group, and assume this sequence is stationary and ergodic. The stationary and ergodic framework allows us to provide more explicit assumptions, and we obtain a finer asymptotic analysis of the sequence $(M_{0,n})_{n\geqslant 0}$. Namely, we prove that the following almost sure approximation in total variation holds for n large enough

$$\|\mu M_{0,n} - \mu(h) r_n \pi_n\|_{TV} \leqslant \delta^n \|\mu M_{0,n}\|_{TV}, \tag{IV.2}$$

where $\delta < 1$, h is a random bounded function, r_n is a positive random number and (π_n) is a sequence of random probability measures on $\mathbb{X}$, which are all independent of the measure μ. We prove additionally that $(n^{-1}\log(r_n))$ converges almost surely to the Lyapunov exponent of the process $M_{0,n}$, and that the sequence (π_n) of random probabilities converges in distribution with respect to the total variation topology towards a random probability measure Λ on $\mathcal{M}_1(\mathbb{X})$. Additionally, we show in Theorem 5 that when the sequence (M_n) is i.i.d, the probability distribution Λ and the Lyapunov exponent λ are related as follows :

$$\lambda = \int \log \|\mu M\| d\Lambda(\mu) d\mathcal{P}(M),$$

where $\mathcal{P}$ refers to the law of the operators (M_n), thus extending a result stated in [BL85] in finite dimension. Finally, still under the assumption that (M_n) is i.i.d, we show in

Theorem 6 that, when $\lambda = 0$, it holds almost surely, for any $\mu \in \mathcal{M}_+(\mathbb{X}) - \{0\}$,

$$\limsup_{n\to\infty} \log \|\mu M_{0,n}\| = -\liminf_{n\to\infty} \|\mu M_{0,n}\| = +\infty, \tag{OSC}$$

except in a situation knowed as Null-Homology.
These results should allow to extend many known results on MGWRE with a finite type set to a class of MGWRE with an infinite type set $\mathbb{X}$. In particular, our results imply that when the Lyapunov exponent λ is nonpositive, outside of Null-Homology, the quenched population size $\mu M_{0,n}\mathbb{1}$ satisfies $\liminf_{n\to\infty} \mu M_{0,n}\mathbb{1} = 0$ almost surely. By a classical first moment argument, this is a sufficient condition for the almost sure extinction of the population. The survival of the population when $\lambda > 0$ is a more delicate problem and will be the object of a forecoming article.

IV.1.2 Framework and notations

Let $(\mathbb{X}, \mathcal{X})$ be a measurable set of arbitrary cardinality, such that for all $x \in \mathbb{X}$, the singleton $\{x\} \in \mathcal{X}$. We denote by $\mathcal{B}(\mathbb{X})$ the Banach space of bounded measurable functions on $\mathbb{X}$, endowed with the supremum norm, and $\mathcal{B}_+(\mathbb{X})$ the cone of nonnegative functions of $\mathcal{B}(\mathbb{X})$. The vector space of signed measures, noted $\mathcal{M}(\mathbb{X})$, and the cone of nonnegative elements of $\mathcal{M}(\mathbb{X})$, noted $\mathcal{M}_+(\mathbb{X})$, are endowed with the total variation norm $\|\cdot\|_{TV}$. Note that $\|\mu\|_{TV} = \mu(\mathbb{X})$ for any $\mu \in \mathcal{M}_+(\mathbb{X})$. Let $\mathcal{M}_1(\mathbb{X})$ be the set of probability measures on $(\mathbb{X}, \mathcal{X})$. For any measurable set $A \in \mathcal{X}$, let $\mathbb{1}_A \in \mathcal{B}(\mathbb{X})$ be the indicator function of A. For short, we note $\mathbb{1} = \mathbb{1}_{\mathbb{X}} \in \mathcal{B}(\mathbb{X})$ the unit function on $\mathbb{X}$. We also note $\delta_x \in \mathcal{M}_1(\mathbb{X})$ the Dirac measure at x.

Let $\mathcal{K}^+$ be the set of maps Q of the form:

$$Q : \mathbb{X} \times \mathcal{X} \longrightarrow \mathbb{R}^+,$$

such that, for any $x \in \mathbb{X}$, for any $A \in \mathcal{X}$, the map $x \mapsto Q(x, A)$ is measurable, the map $A \mapsto Q(x, A)$ is a positive and finite measure on $(\mathbb{X}, \mathcal{X})$ and $|||Q||| := \sup_{x\in\mathbb{X}} Q(x, \mathbb{X}) < \infty$.

Such a map $Q \in \mathcal{K}^+$ naturally operates on $\mathcal{B}(\mathbb{X})$ by setting, for any $f \in \mathcal{B}(\mathbb{X})$, and any $x \in \mathbb{X}$,

$$Qf(x) = \int_{\mathbb{X}} f(y)Q(x, dy).$$

Note that $|Qf(x)| \leqslant \|f\|_\infty |||Q|||$, thus $Qf \in \mathcal{B}_+(\mathbb{X})$ as soon as $f \in \mathcal{B}_+(\mathbb{X})$ and Q acts as a bounded positive operator with norm $|||Q|||$ on $\mathcal{B}(\mathbb{X})$. Moreover, for any positive measure $\mu \in \mathcal{M}_+(\mathbb{X})$, and any $Q \in \mathcal{K}^+$, the positive measure μQ on $\mathbb{X}$ is well defined by setting, for any nonnegative function f,

$$\mu Q(f) = \mu(Qf) = \int_{\mathbb{X}} Qf(x)\mu(dx).$$

Note that μQ has indeed finite mass $\mu Q(\mathbb{1}) = \mu(Q\mathbb{1}) \leqslant |||Q|||\mu(\mathbb{1}) < \infty$ since μ is assumed to be a finite measure. This action can therefore naturally be extended to the set of signed measures $\mathcal{M}(\mathbb{X})$, where Q acts as a bounded linear operator with norm $|||Q|||$.

Thus, the elements of $\mathcal{K}^+$ operate as positive linear operators both on the sets of bounded measurable functions and on the set of signed measures on $\mathbb{X}$, with a duality relation between these two actions. Moreover, it is also possible to define a projective action $\cdot$ of $\mathcal{K}^+$ onto the projective space associated with $\mathcal{M}_+(\mathbb{X})$, ie the set of probability measures $\mathcal{M}_1(\mathbb{X})$, by setting, for any $\mu \in \mathcal{M}_+(\mathbb{X})$ and any $M \in \mathcal{K}^+$ such that $\mu M \neq 0$,

$$\mu \cdot M = \frac{\mu M}{\|\mu M\|_{TV}} \in \mathcal{M}_1(\mathbb{X}).$$

Finally, the set $\mathcal{K}^+$ is naturally endowed with an associative, non commutative product, defined by : for any $Q_1, Q_2 \in \mathcal{K}^+$, any $x \in \mathbb{X}$ and any $A \in \mathcal{X}$,

$$Q_1Q_2(x, A) = \int_y Q_1(x, dy)Q_2(y, A).$$

This product is compatible with the left and right actions defined above, in other words, for any $Q_1, Q_2 \in \mathcal{K}^+$, any $\mu \in \mathcal{M}_+(\mathbb{X})$, and $f \in \mathcal{B}_+(\mathbb{X})$,

$$\mu(Q_1Q_2) = (\mu Q_1)Q_2 \text{ and } Q_1Q_2(f) = Q_1(Q_2 f),$$

and whenever $\mu \in \mathcal{M}_+(\mathbb{X})$ and $\mu Q_1 Q_2 \neq 0$,

$$\mu \cdot (Q_1Q_2) = (\mu \cdot Q_1) \cdot Q_2.$$

The operator norm $|||\cdot|||$ satisfies the submultiplicativity relation

$$|||Q_1Q_2||| \leqslant |||Q_1||| \, |||Q_2|||.$$

Remark IV.1. In the case of a finite or countable set $\mathbb{X}$, any measure on $\mathbb{X}$ is atomic thus an operator $Q \in \mathcal{K}^+$ on $\mathbb{X}$ corresponds to a matrix indexed by $\mathbb{X}$ with nonnegative entries. The product of operators of $\mathcal{K}^+$ corresponds to the matrix product and the respective left and right actions of $\mathcal{K}^+$ on signed measures and bounded functions correspond to the product of matrices respectively with the vectors of $\ell^1(\mathbb{X})$ (seen as row vectors) and of $\ell^\infty(\mathbb{X})$ (seen as column vectors).

We consider a dynamical system $(\Omega, \mathcal{A}, \mathbb{P}, \theta)$, where $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability set and $\theta : \Omega \longrightarrow \Omega$ is a measurable transformation, which preserves the probability measure $\mathbb{P}$, i.e $\mathbb{P} \circ \theta^{-1} = \mathbb{P}$.
Let $M : \Omega \longrightarrow \mathcal{K}^+$ be a measurable map. We denote as $\mathbb{N}_0$ the set of nonnegative integers and note, for each $n \in \mathbb{N}_0$,

$$M_n = M \circ \theta^n.$$

Note that the sequence $(M_n)_{n \in \mathbb{N}_0}$ is stationary. For each $k < n$, $\omega \in \Omega$, let us define the random product

$$M_{k,n}(\omega) = M_k(\omega) \cdots M_{n-1}(\omega) = (M \circ \theta^k(\omega)) \cdots \big(M \circ \theta^{n-1}(\omega)\big) \in \mathcal{K}^+.$$

with the convention $M_{k,k}(\omega) = \mathrm{Id} \in \mathcal{K}^+$. Notice that $M_{k,k+n}(\omega) = M_{0,n} \circ \theta^k(\omega)$. The operators satisfy the following semi group property : for any $k \leqslant n \leqslant N$, any $\mu \in \mathcal{M}(\mathbb{X})$, any $f \in \mathcal{B}(\mathbb{X})$

$$\mu M_{k,N}(\omega)f = \mu M_{k,n}(\omega)M_{n,N}(\omega)f. \tag{IV.3}$$

Moreover, for any $x \in \mathbb{X}$, $k \leqslant n$, $\omega \in \Omega$, we set

$$m_{k,n}(x,\omega) = \delta_x M_{k,n}(\omega)\mathbb{1} = \|\delta_x M_{k,n}(\omega)\|_{TV}.$$

Notice in particular that for any positive measure μ,

$$\|\mu M_{k,N}\| = \mu(m_{k,N}) = \mu M_{k,N}\mathbb{1} = \mu M_{k,n}m_{n,N}.$$

Let us point out additionally that $\|\!|M_{k,n}(\omega)|\!\| = \sup_{x\in\mathbb{X}} m_{k,n}(x,\omega) = \|m_{k,n}(\cdot,\omega)\|_\infty$, and that for any $k \leqslant n \leqslant N$, $\|\!|M_{k,N}|\!\| \leqslant \|\!|M_{k,n}|\!\|\|\!|M_{n,N}|\!\|$. Finally, to shorten the notations, we often omit the dependence in ω, writing for example $m_{k,n}(x) = m_{k,n}(x,\omega)$, and $\|\!|M_{k,n}|\!\| = \|m_{k,n}\|_\infty = \sup_{x\in\mathbb{X}} m_{k,n}(x,\omega)$.

IV.1.3 Assumptions

We list here several hypotheses that will be used in the rest of the article.

Ass. IV.1. The dynamical system $(\Omega, \mathcal{A}, \mathbb{P}, \theta)$ is ergodic.

We recall that a dynamical system is ergodic when any measurable set $A \in \mathcal{A}$ such that $\theta^{-1}(A) = A$ satisfies $\mathbb{P}(A) \in \{0,1\}$.

Ass. IV.2. For almost all $\omega \in \Omega$, the function $x \mapsto m_{0,1}(x,\omega)$ is a positive function.

By stationarity, **Ass. IV.2** implies that $\mathbb{P}(d\omega)$-almost surely, the product $M_{k,n}(\omega)$ is a continuous, non zero, positive linear operator. We introduce the integrability property

Ass. IV.3. $\mathbb{E}\left[\log^+ \|m_{0,1}\|_\infty\right] < \infty$.

In particular, recalling that $\|m_{k,n}\|_\infty = \|\!|M_{k,n}|\!\|$, by submultiplicativity of the norm $\|\!|\cdot|\!\|$, **Ass. IV.1** and **Ass. IV.3** imply that $\mathbb{E}\log^+(\|\!|M_{k,n}|\!\|) < \infty$ for all $k \leqslant n$.

We call admissible coupling constants a measurable map

$$(\nu, c, d) : \omega \in \Omega \mapsto (\nu_\omega, c(\omega), d(\omega)) \in \mathcal{M}_1(\mathbb{X}) \times [0,1]^2$$

such that for $\mathbb{P}$-almost any $\omega \in \Omega$,

i) for all $x \in \mathbb{X}$ and all $f \in \mathcal{B}_+(\mathbb{X})$, the couple $(\nu_\omega, c(\omega))$ satisfies

$$\delta_x M(\omega)(f) \geqslant c(\omega)\|\delta_x M(\omega)\|\nu_\omega(f) \tag{IV.4}$$

ii) for all $n \geqslant 0$, the couple $(\nu_\omega, d(\omega))$ satisfies

$$\nu_\omega(m_{1,n}) \geqslant d(\omega)\|\!|M_{1,n}|\!\| \tag{IV.5}$$

When an admissible triplet is defined, we define the random variable

$$\gamma(\omega) = c(\omega)d(\omega).$$

Note that taking $c = d = 0$ and any measurable map $\omega \mapsto \nu_\omega$ defines an admissible triplet, however in this case $\gamma = 0$. Our main assumption is therefore

Ass. IV.4. There exists an admissible triplet $\omega \mapsto (\nu_\omega, c(\omega), d(\omega))$ such that $\mathbb{P}[\gamma > 0] > 0$.

IV.2 Statement of the results and structure of the paper

IV.2.1 Main results

Set

$$\bar{\gamma} := \exp\left(\mathbb{E}\left[\log(1-\gamma)\right]\right) \in [0,1],$$

and notice that **Ass. IV.4** yields $\bar{\gamma} < 1$. Under the previous assumptions, we prove the following Theorem:

Theorem 4. *Let $M : \Omega \longrightarrow \mathcal{K}^+$ be a measurable map and assume that Assumptions **Ass. IV.1**,**Ass. IV.2** and **Ass. IV.4** hold. Then,*

i) $\mathbb{P}(d\omega)$-almost surely, there exists a random function $h \in \mathcal{B}(\mathbb{X})$ such that, for any $\delta \in (\bar{\gamma}^{\mathbb{P}[\gamma>0]}, 1)$, for n large enough, for any finite measures $\mu_1, \mu_2 \in \mathcal{M}_+(\mathbb{X}) - \{0\}$,

$$\left\| \mu_1 M_{0,n} - \frac{\mu_1(h)}{\mu_2(h)} \mu_2 M_{0,n} \right\|_{TV} \leqslant \delta^n \|\mu_1 M_{0,n}\|. \qquad \text{(IV.6)}$$

Such a function h is unique up to a multiplicative constant.

ii) There exists a probability measure Λ on the space $\mathcal{M}_1(\mathbb{X})$, such that for any probability measure μ, the sequence of random probability measures $(\mu \cdot M_{0,n})$ converges in distribution towards Λ, in the space $\mathcal{M}_1(\mathbb{X})$, endowed with the total variation norm.

*iii) Assuming additionally **Ass. IV.3**, for almost any $\omega \in \Omega$ and any finite, positive, non-zero measure μ,*

$$\frac{1}{n} \log \|\mu M_{0,n}\| \underset{n\to\infty}{\longrightarrow} \inf_{N\in\mathbb{N}} \frac{1}{N} \mathbb{E}\left[\log \||M_{0,N}\||\right] = \lambda \in [-\infty, \infty). \qquad \text{(IV.7)}$$

Note that the estimate (IV.2) can be derived from Theorem 4 by a choice of an arbitrary measure μ_2, and by setting

$$\pi_n = \mu_2 \cdot M_{0,n}, \;\; r_n = \frac{\|\mu_2 M_{0,n}\|}{\mu_2(h)},$$

The rest of this paper focuses on the independent case, that is, the case where the sequence of operators (M_n) is i.i.d, with a law called $\mathcal{P}$. This can be obtained by setting $(\Omega, \mathcal{A}, \mathbb{P})$ to be the product space $\Omega = (\mathcal{K}^+)^{\mathbb{N}}$, $\mathbb{P} = \mathcal{P}^{\otimes\mathbb{N}}$ and $M : (\mathcal{K}^+)^{\mathbb{N}} \to \mathcal{K}^+, (N_k)_{k\geqslant 0} \mapsto N_0$. In this independent case, we are able to characterize the measure Λ on the projective space $\mathcal{M}_1(\mathbb{X})$ as the only invariant measure left invariant by the projective action of the matrices (M_n). In other words, it is the only invariant probability measure of the Markov chain $(\mu_n)_{n\geqslant 0}$ defined by

$$\mu_n = \frac{\mu_0 M_{0,n}}{\|\mu_0 M_{0,n}\|} \in \mathcal{M}_1(\mathbb{X}).$$

A classical cocyle property yields the decomposition

$$\log \|\mu_0 M_{0,n}\|_{TV} = \sum_{k=0}^{n-1} \log(\|\mu_k M_k\|) = \sum_{k=0}^{n-1} \rho(\mu_k, M_k), \tag{IV.8}$$

where (μ_k, M_k) is a Markov chain on $\mathcal{M}_1(\mathbb{X}) \times \mathcal{K}^+$, with the unique invariant probability measure $\Lambda \otimes \mathcal{P}$. These property allows us to obtain additional insight over the asymptotic behavior of the mass of the measure $\mu M_{0,n}$. To do so, we introduce the following strengthened versions of assumptions **Ass. IV.3** and **Ass. IV.4**:

Ass. IV.3+. $\mathbb{E}\left[|\log \|m_{0,1}\|_\infty|\right] < \infty$.

Ass. IV.4+. There exists an admissible triplet (ν, c, d) such that $\mathbb{E}\,|\log(\gamma)| < \infty$.

Note that **Ass. IV.4+** implies in particular that $\mathbb{P}[\gamma > 0] = 1$. Under these assumptions we are able to link the Lyapunov exponent λ, which governs the exponential growth of the mass of the measure $\|\mu M_{k,n}\|_{TV}$, with the asymptotic projective distribution Λ on $\mathcal{M}_1(\mathbb{X})$.

Theorem 5. *Consider an i.i.d sequence (M_n) of elements of $\mathcal{K}^+$ with law $\mathcal{P}$, suppose assumptions **Ass. IV.1**, **Ass. IV.2**, **Ass. IV.3+** and **Ass. IV.4+** hold. Then, the almost sure convergence (IV.7) also holds in $\mathcal{L}^1(\Lambda \otimes \mathbb{P})$, that is*

$$\int \left| \frac{1}{n} \log \|\mu M_{0,n}\|_{TV} - \lambda \right| d\Lambda(\mu) d\mathcal{P}^{\otimes n}(M_0, \cdots, M_{n-1}) \xrightarrow[n\to\infty]{} 0.$$

As a consequence,

$$\lambda = \int \log \|\mu M\|_{TV} d\Lambda(\mu) d\mathcal{P}(M). \tag{IV.9}$$

Moreover, in the critical case where $\lambda = 0$, the law of large numbers (IV.7) is not enough to know whether $\liminf \|\mu M_{0,n}\|_{TV}$ and $\limsup \|\mu M_{0,n}\|_{TV}$ are 0, a positive real number, or $+\infty$. Answering these questions is the objective of our last theorem. When $\lambda = 0$, the sequence (M_n) is i.i.d and $\mu_0 \sim \Lambda$, the increments $\rho(\mu_k, M_k)$ of the sum (IV.8) are centered and form an ergodic sequence. By analogy with a centered random walk with i.i.d increments, we expect their sum $\log \|\mu_0 M_{0,n}\|_{TV}$ to oscillate between $-\infty$ and $+\infty$. More formally, we use the theory of Markov random walks, and in particular [Als01], which establishes this oscillation property exists, except in some case called Null-Homology. In our context, we say that there is Null Homology when there exists some function $\eta : \mathcal{M}_1(\mathbb{X}) \mapsto \mathbb{R}$ such that $d(\Lambda \otimes \mathcal{P})(\mu, M)$-a.s.

$$\log \|\mu M\|_{TV} = \eta(\mu \cdot M) - \eta(\mu). \tag{NH}$$

When (NH) does not hold, using results from [Als01], we indeed obtain the oscillation of $\log \|\mu M_{0,n}\|_{TV}$ for any initial measure μ. On the contrary, when (NH) holds, it is clear that $\log \|\mu_0 M_{0,n}\|_{TV} = \eta(\mu_n) - \eta(\mu_0)$, thus $\log \|\mu_0 M_{0,n}\|_{TV}$ may or may not oscillate, depending on η. More formally, our result is

Theorem 6. *Consider an i.i.d sequence (M_n) of elements of $\mathcal{K}^+$ with law $\mathcal{P}$, suppose assumptions **Ass. IV.1**, **Ass. IV.2**, **Ass. IV.3** and **Ass. IV.4+** hold. Assume additionally that $\lambda = 0$. Then, if (NH) does not hold, we have*

$$\liminf_{n\to\infty} \log \|\mu M_{0,n}\|_{TV} = -\infty \text{ and } \limsup_{n\to\infty} \log \|\mu M_{0,n}\|_{TV} = +\infty \tag{OSC}$$

for any $\mu \in \mathcal{M}_+(\mathbb{X}) - \{0\}$. If (NH) holds, let $\mu_0 \sim \Lambda$, and let us note $a < b \in [-\infty, +\infty]$ the respective infimum and supremum of the support of the random variable $\eta(\mu_0)$. It holds

$$a = -\infty \Leftrightarrow \mathbb{P} - a.s., \text{ for any } \mu \in \mathcal{M}_+(\mathbb{X}) - \{0\}, \liminf_{n\to\infty} \log \|\mu M_{0,n}\|_{TV} = -\infty$$

and

$$b = +\infty \Leftrightarrow \mathbb{P} - a.s., \text{ for any } \mu \in \mathcal{M}_+(\mathbb{X}) - \{0\}, \limsup_{n\to\infty} \log \|\mu M_{0,n}\|_{TV} = +\infty.$$

The notion of Null Homology already appears in the framework of products of $p \times p$ non-negative matrices, and [Hen97] provides some geometric condition which prevents Null-Homology (see in particular Theorem 5 of [Hen97]). We are unfortunately not able to generalize this condition in the infinite dimensional case.

IV.2.2 Structure of the paper

Section IV.3 contains the proofs of Theorems 4, 5 and 6. More precisely, in Subsection IV.3.1, we recall how the coefficient γ allow to control some contraction rates of the operators $M_{0,n}$. These results are adapted from [BCG20]. In Subsection IV.3.2, we use the ergodic structure, in particular Assumptions **Ass. IV.1** and **Ass. IV.4** to obtain a geometric decay of the error terms that appeared in our previous estimations. In Subsection IV.3.3 we derive the three claims of Theorem 4. Finally, in Subsection IV.3.4, we focus on the case where the sequence (M_n) is i.i.d. In this case, a study of the invariant measures and the ergodicity properties of the Markov chains $(\mu_0 \cdot M_{0,n})_{n\geqslant 0}$ and $(\mu_0 \cdot M_{0,n}, M_n)_{n\geqslant 0}$, allows to prove Theorems 5 and 6.

Section IV.4 is dedicated to a comparison of our results with those obtained based on Hilbert contractions. More precisely, we show how natural conditions coming from Hilbert contractions techniques provide more tractable sufficient conditions for our Assumptions (in particular **Ass. IV.4**), both in finite and infinite dimension.

In Section IV.5, we apply our results to study products of infinite Leslie matrices. This constitutes an example of an interesting class of systems that cannot be studied using the Hilbert metric. More precisely, we provide in Subsection IV.5.2 reasonable sufficient conditions under which a product of Leslie matrices modelling the behavior of an age structured population satisfies assumptions **Ass. IV.1** to **Ass. IV.4**. However, when these conditions are not satisfied, it can be quite difficult to exhibit an admissible triplet such that $\gamma > 0$, even on a deterministic and constant sequence of Leslie matrices. To illustrate this fact, we present in Subsection IV.5.3 an example of system where $\gamma = 0$ even if all the other assumptions are satisfied.

IV.3 Proofs

IV.3.1 Contraction results based on an inhomogeneous Doeblin minoration

Given an admissible triplet (ν, c, d), we define for every $k \geqslant 0$ the $[0,1]$-valued random variables $c_k = c_k(\omega) = c(\theta^k(\omega))$, $d_{k+1} = d_{k+1}(\omega) = d(\theta^k(\omega))$ and $\gamma_k = \gamma_k(\omega) = \gamma(\theta^k(\omega))$, as well as the $\mathcal{M}_1(\mathbb{X})$ valued random variable $\nu_k = \nu_{\theta^k(\omega)}$. Notice that $c = c_0$ and $d = d_1$, that the sequences $(c_k)_{k\geqslant 0}$, $(d_k)_{k\geqslant 1}$, $(\gamma_k)_{k\geqslant 0}$ are all stationary sequences and that for all $k \geqslant 0$,

$$\gamma_k = c_k d_{k+1}.$$

Moreover the random variables ν_k, c_k, d_{k+1} satisfy some time-shifted versions of (IV.4) and (IV.5). Indeed, for any $f \in \mathcal{B}_+(\mathbb{X})$ and any $x \in \mathbb{X}$, it holds

$$\delta_x M_{k,k+1}(f) \geqslant c_k m_{k,k+1}(x) \nu_k(f), \ \mathbb{P}\text{-a.s.} \tag{IV.12}$$

and for any $n \geqslant k$,

$$\nu_k(m_{k+1,n}) \geqslant d_{k+1} |||M_{k+1,n}||| = d_{k+1} \sup_{x\in\mathbb{X}} m_{k+1,n}(x), \ \mathbb{P}\text{-a.s.} \tag{IV.13}$$

The first step towards proving Theorem 4 is establishing

Proposition IV.2. *Suppose* ***Ass. IV.2*** *and let $\omega \mapsto (\nu_\omega, c(\omega), d(\omega))$ be an admissible triplet. Then, $\mathbb{P}(d\omega)$-almost surely, for any $k \leqslant n \leqslant N$ and any finite measures $\mu_1, \mu_2 \in \mathcal{M}_+(\mathbb{X}) - \{0\}$, it holds*

$$\|\mu_1 \cdot M_{k,n} - \mu_2 \cdot M_{k,n}\|_{TV} \leqslant 2 \prod_{i=k}^{n-1} (1 - \gamma_i), \tag{IV.14}$$

and, if $n \geqslant 1$, it also holds $\mathbb{P}(d\omega)$-a.s.

$$\gamma_{n-1} \left| \frac{\mu_1(m_{k,N})}{\mu_1(m_{k,n})} - \frac{\mu_2(m_{k,N})}{\mu_2(m_{k,n})} \right| \leqslant 2 \frac{\mu_2(m_{k,N})}{\mu_2(m_{k,n})} \prod_{i=k}^{n-1} (1 - \gamma_i). \tag{IV.15}$$

This result was already introduced in [BCG20] in a somewhat different setup. We have chosen to state and prove it here for the sake of completeness. Its proof is based on performing a Doeblin minoration on a well-chosen sequence of auxiliary Markov operators

$(P^N_{k,n})$. This Doeblin property yields (IV.14), a contraction property for the projective action of $M_{k,n}$ on the space of measures $\mathcal{M}_+(\mathbb{X})$. We derive then Equation (IV.15), which describes how the growth of the mass $\|\mu M_{k,t}\|$ between times $t = n$ and $t = N$ depends on the initial measure μ.

Let us introduce now the operators $P^N_{k,n}$ upon which we perform the desired Doeblin minoration. Under assumption **Ass. IV.2**, $\mathbb{P}(d\omega)$-almost surely, for any $k \leqslant n \leqslant N$, the functions $x \mapsto m_{k,n}(x)$ and $x \mapsto m_{n,N}(x)$ are positive on $\mathbb{X}$. For each $k \leqslant n \leqslant N$, an operator $P^N_{k,n}(\omega)$ can be defined $\mathbb{P}(d\omega)$-almost surely, as follows : for each $x \in \mathbb{X}$, for each positive measurable $f : \mathbb{X} \longrightarrow \mathbb{R}$,

$$\delta_x P^N_{k,n} f = \frac{\delta_x M_{k,n}(f m_{n,N})}{m_{k,N}(x)},$$

$P^N_{k,n}$ is a positive and conservative operator (i.e. $P^N_{k,n}\mathbb{1} = \mathbb{1}$). Indeed, by Equation (IV.3), for any $x \in \mathbb{X}$,

$$\delta_x P^N_{k,n}\mathbb{1} = \frac{\delta_x M_{k,n} m_{n,N}}{\delta_x M_{k,N}\mathbb{1}} = 1.$$

Moreover, $P^N_{k,n}$ satisfies the relation :

$$P^N_{k,n} = P^N_{k,k+1} \cdots P^N_{n-1,n}.$$

Note that $P^N_{k,n}$ is a matrix when $\mathbb{X}$ is countable and then, for any $x, y \in \mathbb{X}$,

$$P^N_{k,n}(x,y) = \frac{m_{n,N}(y)}{m_{k,N}(x)} M_{k,n}(x,y).$$

These operators satisfy a Doeblin contraction property summed up in

Lemma IV.3. *Assume Assumption **Ass. IV.2** holds and let $\omega \mapsto (\nu_\omega, c(\omega), d(\omega))$ be an admissible triplet. Then $\mathbb{P}(d\omega)$-almost surely, all the $P^N_{k,n}(\omega)$ are well defined, and it holds*

i) For any $n \leqslant N-1$, there exists a random probability measure $\nu_{n,N}$ on $\mathbb{X}$ such that, for any $x \in \mathbb{X}$,

$$\delta_x P^N_{n,n+1} \geqslant c_n d_{n+1} \nu_{n,N} = \gamma_n \nu_{n,N}.$$

ii) For any signed measures ρ_1, ρ_2, of same mass and any $n \leqslant N-1$,

$$\left\|\rho_1 P^N_{n,n+1} - \rho_2 P^N_{n,n+1}\right\|_{TV} \leqslant (1-\gamma_n)\|\rho_1 - \rho_2\|_{TV}.$$

iii) For any $k \leqslant n \leqslant N$ and any signed measures ρ_1, ρ_2 of same mass,

$$\left\|\rho_1 P^N_{k,n} - \rho_2 P^N_{k,n}\right\|_{TV} \leqslant \prod_{i=k}^{n-1}(1-\gamma_i)\left\|\rho_1 - \rho_2\right\|_{TV}.$$

Notice that in this lemma, our single assumption is **Ass. IV.2**. It allows the $(P^N_{k,n})$ to be defined $\mathbb{P}(d\omega)$-almost surely. In particular, **Ass. IV.4** is not assumed, we allow

$\gamma_n(\omega) = 0$, in which case we just obtain that $P^N_{n,n+1}$ is 1-contracting.

Proof of Lemma IV.3. Let $\omega \in \Omega$ such that all the $P^N_{k,n}$ are well defined. For any $x \in \mathbb{X}$ and any $f \in \mathcal{B}_+(\mathbb{X})$, it holds,

$$\delta_x M_{n,n+1}(f m_{n+1,N}) \geqslant c_n m_{n,n+1}(x) \nu_n(f m_{n+1,N}),$$

thus

$$\delta_x P^N_{n,n+1} f = \frac{\delta_x M_{n,n+1}(f m_{n+1,N})}{m_{n,N}(x)} \geqslant c_n \frac{\nu_n(f m_{n+1,N}) m_{n,n+1}(x)}{m_{n,N}(x)},$$

with, by definition of d_{n+1} :

$$d_{n+1} m_{n,N}(x) = d_{n+1} \delta_x M_{n,n+1}(m_{n+1,N}) \leqslant \nu_n(m_{n+1,N}) m_{n,n+1}(x).$$

Therefore,

$$\delta_x P^N_{n,n+1} f \geqslant c_n d_{n+1} \frac{\nu(f m_{n+1,N})}{\nu(m_{n+1,N})} = c_n d_{n+1} \nu_{n,N}(f) = \gamma_n \nu_{n,N}(f),$$

setting

$$\nu_{n,N}(\cdot) = \frac{\nu_n(\cdot\, m_{n+1,N})}{\nu_n(m_{n+1,N})},$$

which is a probability measure. This concludes the proof of *i)*.
Let us prove now *ii)*. This result is a classical consequence of the previous point using the theory of Markov operators. A Markov operator P is said to be δ-Doeblin (with $\delta > 0$) when there exists a probability measure μ such that $\delta_x P f \geqslant \delta \mu(f)$ for any x in the state space and any $f \in \mathcal{B}_+(\mathbb{X})$. Furthermore, such an operator is $1-\delta$ contracting in total variation : for any signed measure ρ of mass 0,

$$\|\rho P\|_{TV} \leqslant (1-\delta)\|\rho\|_{TV}.$$

This property trivially holds for $\delta = 0$: any positive operator satisfies $\delta_x P f \geqslant 0$ when f is a non negative function, and any Markov operator is $1 = (1-0)$-contracting in total variation. In our context, the previous point of the lemma yields that $\mathbb{P}(d\omega)$-almost surely, for any $n \leqslant N-1$, and any $\nu \in \mathcal{M}_1(\mathbb{X})$, the Markov operator $P^N_{n,n+1}$ is γ_n-Doeblin. Therefore, for any $\rho_1, \rho_2 \in \mathcal{M}(\mathbb{X})$, such that $\rho_1(\mathbb{1}) = \rho_2(\mathbb{1})$, noting $\rho = \rho_1 - \rho_2$, it holds

$$\left\|\rho P^N_{n,n+1}\right\|_{TV} \leqslant (1-\gamma_n)\|\rho\|_{TV}.$$

This proves *ii)*, let us move now to *iii)*. Since all the $P^N_{n,n+1}$ are conservative operators, the image of a measure of mass 0 by $P^N_{n,n+1}$ is a measure of mass 0. The equality $P^N_{k,n} = P^N_{k,k+1} \cdots P^N_{n-1,n}$, yields $\left\|\rho P^N_{k,n}\right\|_{TV} \leqslant (1-\gamma_{n-1})\|\rho P^N_{k,n-1}\|_{TV}$. By induction, we deduce

$$\left\|\rho P^N_{k,n}\right\|_{TV} \leqslant \prod_{i=k}^{n-1} (1-\gamma_i)\, \|\rho\|_{TV}\,.$$

This concludes the proof. □

Proof of Proposition IV.2. Let us prove first Inequality (IV.14). Applying Lemma IV.3, *iii)* with $n = N$ and $\rho = \delta_x - \delta_y$, we get, $\mathbb{P}(d\omega)$-almost surely

$$\left\|\delta_x P^n_{k,n} - \delta_y P^n_{k,n}\right\|_{TV} \leqslant \prod_{i=k}^{n-1}(1-\gamma_i)\left\|\delta_x - \delta_y\right\|_{TV} \leqslant 2\prod_{i=k}^{n-1}(1-\gamma_i).$$

Hence, for any $f \in \mathcal{B}(\mathbb{X})$ and $x, y \in \mathbb{X}$,

$$\left|\frac{\delta_x M_{k,n}(f)}{m_{k,n}(x)} - \frac{\delta_y M_{k,n}(f)}{m_{k,n}(y)}\right| \leqslant 2\left\|f\right\|_\infty \prod_{i=k}^{n-1}(1-\gamma_i).$$

Let μ_1 and μ_2 be two positive measures. The inequality

$$\left|\delta_x M_{k,n}(f) - m_{k,n}(x)\frac{\delta_y M_{k,n}(f)}{m_{k,n}(y)}\right| \leqslant 2m_{k,n}(x)\left\|f\right\|_\infty \prod_{i=k}^{n-1}(1-\gamma_i),$$

yields, after integrating with respect to $\mu_1(dx)$:

$$\left|\mu_1 M_{k,n}(f) - \mu_1(m_{k,n})\frac{\delta_y M_{k,n}(f)}{m_{k,n}(y)}\right| \leqslant 2\mu_1(m_{k,n})\left\|f\right\|_\infty \prod_{i=k}^{n-1}(1-\gamma_i),$$

so that

$$\left|\mu_1 \cdot M_{k,n}(f) - \frac{\delta_y M_{k,n}(f)}{m_{k,n}(y)}\right| \leqslant 2\left\|f\right\|_\infty \prod_{i=k}^{n-1}(1-\gamma_i).$$

Integrating now with respect to $\mu_2(dy)$, we obtain

$$\left|\mu_1 \cdot M_{k,n}(f) - \mu_2 \cdot M_{k,n}(f)\right| \leqslant 2\left\|f\right\|_\infty \prod_{i=k}^{n-1}(1-\gamma_i),$$

and finally

$$\left\|\mu_1 \cdot M_{k,n} - \mu_2 \cdot M_{k,n}\right\|_{TV} \leqslant 2\prod_{i=k}^{n-1}(1-\gamma_i).$$

Let us move now to the proof of Inequality (IV.15). Applying Inequality (IV.14) to the function $x \mapsto m_{n,N}(x)$, one gets

$$\left|\frac{\mu_1(m_{k,N})}{\mu_1(m_{k,n})} - \frac{\mu_2(m_{k,N})}{\mu_2(m_{k,n})}\right| \leqslant 2\left\|m_{n,N}\right\|_\infty \prod_{i=k}^{n-1}(1-\gamma_i) = 2\left|\!\left|\!\left|M_{n,N}\right|\!\right|\!\right| \prod_{i=k}^{n-1}(1-\gamma_i).$$

By (IV.12) and (IV.13) it holds,

$$d_n \left|\!\left|\!\left|M_{n,N}\right|\!\right|\!\right| \leqslant \nu_{n-1}(m_{n,N})$$

and

$$\mu_2(m_{k,N}) = \mu_2 M_{k,n-1} M_{n-1,n} m_{n,N} \geqslant c_{n-1}\mu_2(m_{k,n})\nu_{n-1}(m_{n,N}).$$

Combining these identities we obtain, $\mathbb{P}(d\omega)$-almost surely,

$$c_{n-1}d_n\left|\frac{\mu_1(m_{k,N})}{\mu_1(m_{k,n})}-\frac{\mu_2(m_{k,N})}{\mu_2(m_{k,n})}\right|\leqslant 2\frac{\mu_2(m_{k,N})}{\mu_2(m_{k,n})}\prod_{i=k}^{n-1}(1-\gamma_i).$$

Thus taking an infimum in ν, this yields, $\mathbb{P}(d\omega)$-almost surely,

$$\gamma_{n-1}\left|\frac{\mu_1(m_{k,N})}{\mu_1(m_{k,n})}-\frac{\mu_2(m_{k,N})}{\mu_2(m_{k,n})}\right|\leqslant 2\frac{\mu_2(m_{k,N})}{\mu_2(m_{k,n})}\prod_{i=k}^{n-1}(1-\gamma_i).$$

This ends the proof. □

IV.3.2 Asymptotic estimates under ergodicity assumptions

For any n such that $\gamma_{n-1}\neq 0$ and any $k\leqslant n$, we define

$$\Gamma_{k,n}=\frac{1}{\gamma_{n-1}}\prod_{i=k}^{n-1}(1-\gamma_i), \tag{IV.16}$$

and we set $\Gamma_{k,n}=+\infty$ when $\gamma_{n-1}=0$. With these notations, Equation (IV.15) can be rewritten

$$\left|\frac{\mu_1(m_{k,N})}{\mu_1(m_{k,n})}-\frac{\mu_2(m_{k,N})}{\mu_2(m_{k,n})}\right|\leqslant 2\Gamma_{k,n}\frac{\mu_2(m_{k,N})}{\mu_2(m_{k,n})}. \tag{IV.17}$$

In this subsection, we use the ergodicity Assumption **Ass. IV.1**, as well as Assumption **Ass. IV.4**, which provide a control on the sequence of Doeblin coefficients (γ_n). In the following lemma, we use these assumptions to establish a geometric decay of both $\prod_{i=k}^{n-1}(1-\gamma_i)$ and $\Gamma_{k,n}$ as $n\to\infty$. More precisely, (IV.19) provides the geometric decay of $\prod_{k=1}^{n-1}(1-\gamma_i)$ using Birkhoff's ergodic theorem. To derive the decay of $\Gamma_{k,n}$ as $n\to\infty$ we need additionally to avoid that γ_{n-1} is either 0 or too close to 0. Under Assumption **Ass. IV.4+**, the coefficients $(\gamma_n)_{n\geqslant 0}$ are almost surely all nonzero, and (IV.21) provides a sufficient control for the geometric decay of $\Gamma_{k,n}$. Under the weaker assumption **Ass. IV.4**, it is possible however that $\gamma_{n-1}=0$ for some values of n. We can however focus on some random times at which γ_{n-1} is greater that some predetermined level $\varepsilon>0$. We define those random times by

$$\{n\}_\varepsilon=\max\{i\leqslant n|\gamma_{i-1}\geqslant\varepsilon\}\cup\{-\infty\} \tag{IV.18}$$

for all $n\geqslant 0$ and all $\varepsilon>0$. (IV.20) establishes that the subsequence of $(\Gamma_{k,n})_{n\geqslant 0}$ associated with the random times $(\{n\}_\varepsilon)_{n\geqslant 0}$ still enjoys some slower geometric decay.

Lemma IV.4. *Assume that assumptions **Ass. IV.1**, **Ass. IV.2** and **Ass. IV.4** hold. Then for any $k\geqslant 0$,*

$$\left(\prod_{i=k}^{k+n-1}(1-\gamma_i)\right)^{\frac{1}{n}}\underset{n\to\infty}{\longrightarrow}\bar{\gamma}<1,\mathbb{P}(d\omega)-\textit{almost surely} \tag{IV.19}$$

Moreover, for all $\varepsilon > 0$ such that $\mathbb{P}[\gamma \geqslant \varepsilon] > 0$, it holds

$$\limsup_{n \to +\infty} \left(\Gamma_{k,\{n\}_\varepsilon}\right)^{\frac{1}{n}} \leqslant \bar{\gamma}^{\mathbb{P}[\gamma \geqslant \varepsilon]} < 1, \mathbb{P}(d\omega) - \textit{almost surely.} \tag{IV.20}$$

Finally, assuming additionally ***Ass. IV.4+****, it holds*

$$\lim_{n \to \infty} \left(\frac{1}{\gamma_n}\right)^{\frac{1}{n}} = 1, \mathbb{P}(d\omega) - \textit{almost surely.} \tag{IV.21}$$

Proof of Lemma IV.4. We recall first that by definition, for any $\omega \in \Omega$

$$(1 - \gamma_i)(\omega) = (1 - \gamma) \circ \theta^i(\omega).$$

Notice then that, by Assumption **Ass. IV.4**, for almost every $\omega \in \Omega$, $\gamma(\omega) \in (0, 1]$. Thus $\log(1 - \gamma(\omega)) \in [-\infty, 0)$ and $\bar{\gamma} = \exp\left(\mathbb{E}\left[\log(1 - \gamma(\omega))\right]\right) \in [0, 1)$.

Thus for any k,

$$\log\left[\left(\prod_{i=k}^{k+n-1} (1 - \gamma_i)\right)^{\frac{1}{n}}\right] = \frac{1}{n} \sum_{i=0}^{n-1} \log(1 - \gamma_k) \circ \theta^i.$$

Since θ is an ergodic map, by Birkhoff's ergodic theorem, for any $k \geqslant 0$,

$$\frac{1}{n} \sum_{i=0}^{n-1} \log(1 - \gamma_{k+i}) \underset{n \to \infty}{\longrightarrow} \mathbb{E}\left[\log(1 - \gamma)\right] = \log \bar{\gamma}, \ \mathbb{P}(d\omega) - \text{ almost surely.}$$

This yields (IV.19). Moreover, under **Ass. IV.4**, for ε small enough, by Birkhoff's ergodic theorem, it holds almost surely

$$\lim_{n \to \infty} \frac{1}{n} \#\{1 \leqslant k \leqslant n | \gamma_{k-1} \geqslant \varepsilon\} = \mathbb{P}[\gamma \geqslant \varepsilon] > 0.$$

However

$$\{n\}_\varepsilon \geqslant \#\{1 \leqslant k \leqslant n | \gamma_{k-1} \geqslant \varepsilon\}$$

by definition of $\{n\}_\varepsilon$. Thus, almost surely

$$\liminf_{n \to \infty} \frac{\{n\}_\varepsilon}{n} \geqslant \mathbb{P}[\gamma \geqslant \varepsilon] > 0,$$

in particular, $\lim_{n \to \infty} \{n\}_\varepsilon = +\infty$ almost surely.

As a consequence, for any $k \geqslant 0$, it holds $\mathbb{P}$-a.s., for n large enough

$$\left(\Gamma_{k,\{n\}_\varepsilon}\right)^{\frac{1}{n}} = \left(\frac{1}{\gamma_{\{n\}_\varepsilon - 1}} \prod_{i=k}^{\{n\}_\varepsilon - 1} (1-\gamma_i)\right)^{\frac{1}{n}} \leqslant$$
$$\left(\frac{1}{\varepsilon}\right)^{\frac{1}{n}} \exp\left(\frac{\{n\}_\varepsilon - k}{n} \frac{1}{\{n\}_\varepsilon - k} \sum_{i=k}^{\{n\}_\varepsilon - 1} \log(1-\gamma_i)\right)$$

where almost surely

$$\lim_{n\to\infty} \frac{1}{\{n\}_\varepsilon - k} \sum_{i=k}^{\{n\}_\varepsilon - 1} \log(1-\gamma_i) = \log(\bar{\gamma}).$$

Hence

$$\limsup_{n\to\infty} \left(\Gamma_{k,\{n\}_\varepsilon}\right)^{\frac{1}{n}} \leqslant \exp\left(\liminf_{n\to\infty} \left(\frac{\{n\}_\varepsilon}{n}\right) \log(\bar{\gamma})\right) \leqslant \bar{\gamma}^{\mathbb{P}[\gamma \geqslant \varepsilon]}.$$

This concludes the proof of (IV.20). Let us move to the proof of Inequality (IV.21). Notice first that since $\gamma_n \leqslant 1$ for all n,

$$\liminf_{n\to\infty} \left(\frac{1}{\gamma_n}\right)^{\frac{1}{n}} \geqslant 1.$$

Let us prove now the converse inequality. Let us define for each $b > 1$,

$$Y_n(b) = \frac{-\log(\gamma_n)}{\log(b)} \geqslant 0,$$

and

$$N_b = \sum_{n\in\mathbb{N}_0} \mathbb{1}_{(1/\gamma_n)^{\frac{1}{n}} > b} = \sum_{n\in\mathbb{N}_0} \mathbb{1}_{Y_n(b) > n}.$$

For a given value of b, the sequence $(Y_n(b))_{n\in\mathbb{N}_0}$ is stationary, thus

$$\mathbb{E}(N_b) = \sum_{n\in\mathbb{N}_0} \mathbb{P}\left[Y_n(b) > n\right] = \sum_{n\in\mathbb{N}_0} \mathbb{P}\left[Y_0(b) > n\right].$$

It is a well known fact that for a nonnegative random variable Y,

$$\mathbb{E}[Y] < \infty \Leftrightarrow \sum_{n\geqslant 0} \mathbb{P}(Y > n) < \infty.$$

By Assumption **Ass. IV.4**, it holds $\mathbb{E}\left[Y_0(b)\right] < \infty$ and $\mathbb{E}\left[N_b\right] < \infty$ for all $b > 1$. Therefore, $\mathbb{P}(d\omega)$-almost surely, $N_b < \infty$, thus $\left(\frac{1}{\gamma_n}\right)^{\frac{1}{n}} \leqslant b$ for n large enough and

$\limsup_{n\to\infty} \left(\frac{1}{\gamma_n}\right)^{\frac{1}{n}} \leqslant 1$. Finally,

$$\lim_{n\to\infty} \left(\frac{1}{\gamma_n}\right)^{\frac{1}{n}} = 1, \ \mathbb{P}(d\omega) - \text{almost surely.}$$

$\square$

Putting the estimates from Lemma IV.4 together with Proposition IV.2, we obtain

Proposition IV.5. *Assume* ***Ass. IV.1***, ***Ass. IV.2***, ***Ass. IV.4*** *hold. Let* $\varepsilon > 0$ *such that* $\mathbb{P}[\gamma \geqslant \varepsilon] > 0$. *Then,* $\mathbb{P}(d\omega)$*-almost surely, for any* $k \in \mathbb{N}_0$*, there exists a bounded, non-negative measurable function* h_k *such that, for* n *large enough, for any* $\mu_1, \mu_2 \in \mathcal{M}_+(\mathbb{X}) - \{0\}$,

$$\left\| \mu_1 M_{k,n} - \frac{\mu_1(h_k)}{\mu_2(h_k)} \mu_2 M_{k,n} \right\|_{TV} \leqslant \Delta^{\varepsilon}_{k,n} \|\mu_1 M_{k,n}\|, \tag{IV.22}$$

where almost surely, for n *large enough*

$$\Delta^{\varepsilon}_{k,n} := \frac{8\Gamma_{k,\{n\}_\varepsilon}}{1 - 2\Gamma_{k,\{n\}_\varepsilon}}.$$

is well-defined and positive, and

$$\limsup_{n\to\infty} \left(\Delta^{\varepsilon}_{k,n}\right)^{\frac{1}{n}} \leqslant \bar{\gamma}^{\mathbb{P}[\gamma \geqslant \varepsilon]} < 1. \tag{IV.23}$$

Furthermore, $\mathbb{P}(d\omega)$*-almost surely, a function* h_k *satisfying* (IV.22) *is unique up to a multiplicative constant.*

Proof of Proposition IV.5. We recall first that by (IV.20), it holds almost surely

$$\lim_{n\to\infty} \Gamma_{k,\{n\}_\varepsilon} = 0. \tag{IV.24}$$

As a consequence, almost surely, for n large enough, $2\Gamma_{k,\{n\}_\varepsilon} < 1$, thus $\Delta^{\varepsilon}_{k,n}$ is indeed well-defined and positive. We also derive (IV.23) from (IV.20) and (IV.24). As a consequence, $\lim_{n\to\infty} \Delta^{\varepsilon}_{k,n} = 0$ almost surely.

Let us assume now that there exists a positive function h_k satisfying Inequality (IV.22). Then, if $x, y \in \mathbb{X}$, setting $\mu_1 = \delta_x, \mu_2 = \delta_y$ and applying this inequality to the constant function $\mathbb{1}$, we almost surely get, for n large enough

$$\left| m_{k,n}(x) - \frac{h_k(x)}{h_k(y)} m_{k,n}(y) \right| \leqslant \Delta^{\varepsilon}_{k,n} m_{k,n}(x) = \underset{n\to\infty}{o}(m_{k,n}(x)).$$

Thus

$$\lim_{n\to\infty} \frac{h_k(x)}{h_k(y)} \frac{m_{k,n}(y)}{m_{k,n}(x)} = 1,$$

which readily implies that

$$\frac{h_k(x)}{h_k(y)} = \lim_{n\to\infty} \frac{m_{k,n}(x)}{m_{k,n}(y)}.$$

This yields the unicity of h_k up to a multiplicative constant, when it exists. Let us now prove the existence of a function h_k satisfying (IV.22).

By Inequality (IV.15), with $\mu_1 = \delta_x, \mu_2 = \delta_y$, one gets, $\mathbb{P}(d\omega)$-almost surely, for any $k \leqslant n \leqslant N$:

$$\left| \frac{m_{k,N}(x)}{m_{k,N}(y)} - \frac{m_{k,n}(x)}{m_{k,n}(y)} \right| \leqslant 2\Gamma_{k,n} \frac{m_{k,n}(x)}{m_{k,n}(y)}, \tag{IV.25}$$

where the right-hand-side term is infinite on the event $\{\gamma_{n-1} = 0\}$. Setting

$$\mathbf{diam}_{k,n}(x, y) = \sup_{N_1,N_2 \geqslant n} \left| \frac{m_{k,N_1}(x)}{m_{k,N_1}(y)} - \frac{m_{k,N_2}(x)}{m_{k,N_2}(y)} \right|,$$

this yields, for any $x, y \in \mathbb{X}$,

$$\mathbf{diam}_{k,n}(x, y) \leqslant 4\Gamma_{k,n} \frac{m_{k,n}(x)}{m_{k,n}(y)}.$$

Exchanging the roles of x, y, one gets :

$$\min\left[\mathbf{diam}_{k,n}(x, y), \mathbf{diam}_{k,n}(y, x)\right] \leqslant 4\Gamma_{k,n}. \tag{IV.26}$$

For all $k \geqslant 0$, $x, y \in \mathbb{X}$, both the sequences $(\mathbf{diam}_{k,n}(x, y))_{n\geqslant k}, (\mathbf{diam}_{k,n}(y, x))_{n\geqslant k}$ are non-increasing, and as a consequence from the definition of $\{n\}_\varepsilon$, $\{n\}_\varepsilon \leqslant n$, thus it holds

$$\begin{aligned} \min\left[\mathbf{diam}_{k,n}(x, y), \mathbf{diam}_{k,n}(y, x)\right] &\leqslant \min\left[\mathbf{diam}_{k,\{n\}_\varepsilon}(x, y), \mathbf{diam}_{k,\{n\}_\varepsilon}(y, x)\right] \\ &\leqslant \left(4\Gamma_{k,\{n\}_\varepsilon}\right) \underset{n\to\infty}{\longrightarrow} 0, \end{aligned} \tag{IV.27}$$

$\mathbb{P}(d\omega)$-almost surely, for n large enough, by (IV.24). Thus one of the sequences

$$(\mathbf{diam}_{k,n}(x, y))_{n\geqslant k}, (\mathbf{diam}_{k,n}(y, x))_{n\geqslant k}$$

has 0 as an adherence value. Since these sequences are non decreasing and positive, they converge, thus, $\mathbb{P}$-a.s., one of them tends to 0. Without loss of generality, suppose that

$$\mathbf{diam}_{k,n}(x, y) \underset{n\to\infty}{\longrightarrow} 0.$$

Then, the sequence of positive real numbers $\left(\frac{m_{k,n}(x)}{m_{k,n}(y)}\right)_{n\geqslant k}$ is a Cauchy sequence, it converges to a nonnegative limit $l_k(x, y)$. Moreover, as a consequence from the definition of $\{n\}_\varepsilon$, for all $k \leqslant n \leqslant N$, it holds $\{n\}_\varepsilon \leqslant n \leqslant N$ and thus $k \leqslant k \wedge \{n\}_\varepsilon \leqslant N$. Applying (IV.25) yields therefore

$$\left| \frac{m_{k,N}(x)}{m_{k,N}(y)} - \frac{m_{k,k\wedge\{n\}_\varepsilon}(x)}{m_{k,k\wedge\{n\}_\varepsilon}(y)} \right| \leqslant 2\Gamma_{k,k\wedge\{n\}_\varepsilon} \frac{m_{k,k\wedge\{n\}_\varepsilon}(x)}{m_{k,k\wedge\{n\}_\varepsilon}(y)}. \tag{IV.28}$$

In particular, letting $N \to \infty$ in (IV.28) proves that the limit $l_k(x,y) = \lim_{N\to\infty} \frac{m_{k,N}(x)}{m_{k,N}(y)}$ satisfies

$$\left| l_k(x,y) - \frac{m_{k,k\wedge\{n\}_\varepsilon}(x)}{m_{k,k\wedge\{n\}_\varepsilon}(y)} \right| \leqslant 2\Gamma_{k,k\wedge\{n\}_\varepsilon} \frac{m_{k,k\wedge\{n\}_\varepsilon}(x)}{m_{k,k\wedge\{n\}_\varepsilon}(y)} \tag{IV.29}$$

Almost surely, it holds moreover for n large enough

$$k \wedge \{n\}_\varepsilon = \{n\}_\varepsilon \text{ and } 2\Gamma_{k,k\wedge\{n\}_\varepsilon} \leqslant \frac{1}{4}.$$

Plugging this into Equation (IV.29) yields, for n large enough

$$\left| \frac{m_{k,\{n\}_\varepsilon}(x)}{m_{k,\{n\}_\varepsilon}(y)} - l_k(x,y) \right| \leqslant \frac{1}{4} \frac{m_{k,\{n\}_\varepsilon}(x)}{m_{k,\{n\}_\varepsilon}(y)}.$$

Since $\frac{m_{k,\{n\}_\varepsilon}(x)}{m_{k,\{n\}_\varepsilon}(y)} > 0$, this implies that $l_k(x,y) = \lim_{n\to\infty} \frac{m_{k,n}(x)}{m_{k,n}(y)} > 0$ and consequently,

$$\frac{m_{k,n}(y)}{m_{k,n}(x)} \underset{n\to\infty}{\longrightarrow} \frac{1}{l_k(x,y)} < \infty.$$

Note that Proposition IV.2 allows to prove that $\mathbb{P}(d\omega)$-almost surely, (IV.25) holds jointly for any $k \leqslant n \leqslant N$ and any $x, y \in \mathbb{X}$, thus so does (IV.27). Thus $\mathbb{P}(d\omega)$-almost surely, all the sequences of the form $\left(\frac{m_{k,n}(x)}{m_{k,n}(y)}\right)_{n\geqslant k}$ for all $k \geqslant 0, x, y \in \mathbb{X}$ converge to a positive limit as $n \longrightarrow \infty$.
Now, let us fix an arbitrary element $x_0 \in \mathbb{X}$, and set $h_k(x) = \lim\limits_{n\to\infty} \frac{m_{k,n}(x)}{m_{k,n}(x_0)}$ for all x. The function h_k is positive and satisfies, for any $x, y \in \mathbb{X}$,

$$\frac{m_{k,N}(x)}{m_{k,N}(y)} \underset{N\to\infty}{\longrightarrow} \frac{h_k(x)}{h_k(y)}.$$

Plugging this into Equation (IV.28), we obtain almost surely, for n large enough

$$\left| \frac{h_k(x)}{h_k(y)} - \frac{m_{k,\{n\}_\varepsilon}(x)}{m_{k,\{n\}_\varepsilon}(y)} \right| \leqslant 2\Gamma_{k,\{n\}_\varepsilon} \frac{m_{k,\{n\}_\varepsilon}(x)}{m_{k,\{n\}_\varepsilon}(y)}. \tag{IV.30}$$

Consequently,

$$\begin{aligned} h_k(x) &\leqslant h_k(y) \left(1 + 2\Gamma_{k,\{n\}_\varepsilon}\right) \frac{m_{k,\{n\}_\varepsilon}(x)}{m_{k,\{n\}_\varepsilon}(y)} \\ &\leqslant h_k(y) \left(1 + 2\Gamma_{k,\{n\}_\varepsilon}\right) \frac{\|m_{k,\{n\}_\varepsilon}\|_\infty}{m_{k,\{n\}_\varepsilon}(y)} < \infty, \end{aligned}$$

by **Ass. IV.2**, which implies that h_k is bounded.
Notice now that for n large enough, taking $N = n$ in (IV.28) yields

$$\left| \frac{m_{k,n}(x)}{m_{k,n}(y)} - \frac{m_{k,\{n\}_\varepsilon}(x)}{m_{k,\{n\}_\varepsilon}(y)} \right| \leqslant 2\Gamma_{k,\{n\}_\varepsilon} \frac{m_{k,\{n\}_\varepsilon}(x)}{m_{k,\{n\}_\varepsilon}(y)} \tag{IV.31}$$

Therefore, combining (IV.31) with (IV.30), we obtain

$$\left|\frac{h_k(x)}{h_k(y)} - \frac{m_{k,n}(x)}{m_{k,n}(y)}\right| \leqslant \left|\frac{m_{k,n}(x)}{m_{k,n}(y)} - \frac{m_{k,\{n\}_\varepsilon}(x)}{m_{k,\{n\}_\varepsilon}(y)}\right| + \left|\frac{m_{k,N}(x)}{m_{k,N}(y)} - \frac{m_{k,\{n\}_\varepsilon}(x)}{m_{k,\{n\}_\varepsilon}(y)}\right|$$
$$\leqslant 4\Gamma_{k,\{n\}_\varepsilon}\frac{m_{k,\{n\}_\varepsilon}(x)}{m_{k,\{n\}_\varepsilon}(y)} \tag{IV.32}$$

Once again, we recall that almost surely, for n large enough

$$2\Gamma_{k,\{n\}_\varepsilon} < 1.$$

Therefore (IV.28) yields $\mathbb{P}$-a.s., for n large enough

$$\frac{m_{k,\{n\}_\varepsilon}(x)}{m_{k,\{n\}_\varepsilon}(y)} \leqslant \frac{1}{1-2\Gamma_{k,\{n\}_\varepsilon}}\frac{m_{k,n}(x)}{m_{k,n}(y)}. \tag{IV.33}$$

Plugging this into (IV.32) finally yields

$$\left|\frac{h_k(x)}{h_k(y)} - \frac{m_{k,n}(x)}{m_{k,n}(y)}\right| \leqslant \frac{4\Gamma_{k,\{n\}_\varepsilon}}{1-2\Gamma_{k,\{n\}_\varepsilon}}\frac{m_{k,n}(x)}{m_{k,n}(y)} = \frac{\Delta^\varepsilon_{k,n}}{2}\frac{m_{k,n}(x)}{m_{k,n}(y)} \tag{IV.34}$$

Moreover, for any positive and finite measure $\mu_1 \in \mathcal{M}_+(\mathbb{X})$, any $y \in \mathbb{X}$, integrating (IV.34) with respect to $\mu_1(dx)$, one gets

$$\left|\frac{\mu_1(h_k)}{h_k(y)} - \frac{\mu_1(m_{k,n})}{m_{k,n}(y)}\right| \leqslant \frac{\Delta^\varepsilon_{k,n}}{2}\frac{\mu_1(m_{k,n})}{m_{k,n}(y)},$$

Thus

$$|m_{k,n}(y)\mu_1(h_k) - \mu_1(m_{k,n})h_k(y)| \leqslant \frac{\Delta^\varepsilon_{k,n}}{2}\mu_1(m_{k,n})h_k(y).$$

Integrating with respect to any positive and finite measure $\mu_2(dy)$, this yields

$$|\mu_2(m_{k,n})\mu_1(h_k) - \mu_1(m_{k,n})\mu_2(h_k)| \leqslant \frac{\Delta^\varepsilon_{k,n}}{2}\mu_1(m_{k,n})\mu_2(h_k),$$

and finally,

$$\left|\frac{\mu_1(h_k)}{\mu_2(h_k)} - \frac{\mu_1(m_{k,n})}{\mu_2(m_{k,n})}\right| \leqslant \frac{\Delta^\varepsilon_{k,n}}{2}\frac{\mu_1(m_{k,n})}{\mu_2(m_{k,n})}. \tag{IV.35}$$

Let us prove now that h_k satisfies Inequality (IV.22). It holds

$$\left\|\mu_1 M_{k,n} - \frac{\mu_1(h_k)}{\mu_2(h_k)}\mu_2 M_{k,n}\right\|_{TV} \leqslant \|\mu_1 M_{k,n} - \mu_1(m_{k,n})\mu_2 \cdot M_{k,n}\|_{TV}$$
$$+ \left\|\mu_1(m_{k,n})\mu_2 \cdot M_{k,n} - \frac{\mu_1(h_k)}{\mu_2(h_k)}\mu_2 M_{k,n}\right\|_{TV}.$$

On the one hand, applying Inequality (IV.14), one has, almost surely, for n large enough

$$\begin{aligned}\|\mu_1 M_{k,n} - \mu_1(m_{k,n})\mu_2 \cdot M_{k,n}\|_{TV} &\leqslant \mu_1(m_{k,n})\, \|\mu_1 \cdot M_{k,n} - \mu_2 \cdot M_{k,n}\|_{TV} \\ &\leqslant 2\prod_{i=k}^{n-1}(1-\gamma_i)\mu_1(m_{k,n}) \\ &\leqslant 2\prod_{i=k}^{\{n\}_\varepsilon - 1}(1-\gamma_i)\mu_1(m_{k,n}) \\ &\leqslant \frac{\Delta^\varepsilon_{k,n}}{2}\mu_1(m_{k,n}).\end{aligned}$$

On the other hand, by Equation (IV.35), it holds $\mathbb{P}$-a.s., for n large enough

$$\begin{aligned}\left\|\mu_1(m_{k,n})\mu_2 \cdot M_{k,n} - \frac{\mu_1(h_k)}{\mu_2(h_k)}\mu_2 M_{k,n}\right\|_{TV} &\leqslant \mu_2(m_{k,n}) \left|\frac{\mu_1(m_{k,n})}{\mu_2(m_{k,n})} - \frac{\mu_1(h_k)}{\mu_2(h_k)}\right| \\ &\leqslant \frac{\Delta^\varepsilon_{k,n}}{2}\mu_1(m_{k,n}).\end{aligned}$$

Finally, for n large enough, it indeed holds almost surely for n large enough

$$\left\|\mu_1 M_{k,n} - \frac{\mu_1(h_k)}{\mu_2(h_k)}\mu_2 M_{k,n}\right\|_{TV} \leqslant \Delta^\varepsilon_{k,n}\|\mu_1 M_{k,n}\|_{TV}.$$

This ends the proof. □

The control provided in (IV.22) only holds for large enough values of n which are random and depend on k. Under the stronger assumption **Ass. IV.4+**, it is possible to obtain a control which is uniform in $k \leqslant n$.

Proposition IV.6. *Assume* ***Ass. IV.1***, ***Ass. IV.2***, ***Ass. IV.4+*** *hold. Then almost surely, for any $k \leqslant n$ and any measure $\mu_1, \mu_2 \in \mathcal{M}_+(\mathbb{X}) - \{0\}$, it holds*

$$\left\|\mu_1 M_{k,n} - \frac{\mu_1(h_k)}{\mu_2(h_k)}\mu_2 M_{k,n}\right\|_{TV} \leqslant 4\Gamma_{k,n}\|\mu_1 M_{k,n}\| = \underset{n\to\infty}{o}(\delta^n\|\mu_1 M_{k,n}\|), \tag{IV.36}$$

for any $\delta \in (\bar{\gamma}, 1)$.

Proof of Proposition IV.6. We start by letting N go to infinity in (IV.25) and obtain, almost surely, for all $k, n \in \mathbb{N}_0$ and all $x, y \in \mathbb{X}$

$$\left|\frac{h_k(x)(x)}{h_k(y)} - \frac{m_{k,n}(x)}{m_{k,n}(y)}\right| 2\Gamma_{k,n}\frac{m_{k,n}(x)}{m_{k,n}(y)}, \tag{IV.37}$$

where by **Ass. IV.4+** almost surely, $\gamma_{n-1} > 0$ for all $n \geqslant 1$. Replacing (IV.34) by (IV.37) in the proof of Proposition IV.5, we obtain first

$$\left|\frac{\mu_1(m_{k,n})}{\mu_2(m_{k,n})} - \frac{\mu_1(h_k)}{\mu_2(h_k)}\right| \leqslant 2\Gamma_{k,n}\frac{\mu_1(m_{k,n})}{\mu_2(m_{k,n})} \tag{IV.38}$$

for any non-zero positives measures μ_1, μ_2. From this we derive both

$$\begin{aligned}\|\mu_1 M_{k,n} - \mu_1(m_{k,n})\mu_2 \cdot M_{k,n}\|_{TV} &\leqslant \mu_1(m_{k,n})\, \|\mu_2 \cdot M_{k,n} - \mu_2 \cdot M_{k,n}\|_{TV} \\ &\leqslant 2 \prod_{i=k}^{n-1}(1-\gamma_i)\mu_1(m_{k,n}) \\ &\leqslant 2\Gamma_{k,n}\mu_1(m_{k,n})\end{aligned}$$

and

$$\begin{aligned}\left\|\mu_1(m_{k,n})\mu_2 \cdot M_{k,n} - \frac{\mu_1(h_k)}{\mu_2(h_k)}\mu_2 M_{k,n}\right\|_{TV} &\leqslant \mu_2(m_{k,n}) \left|\frac{\mu_1(m_{k,n})}{\mu_2(m_{k,n})} - \frac{\mu_1(h_k)}{\mu_2(h_k)}\right| \\ &\leqslant 2\Gamma_{k,n}\mu_1(m_{k,n}),\end{aligned}$$

which we combine to obtain

$$\begin{aligned}\left\|\mu_1 M_{k,n} - \frac{\mu_1(h_k)}{\mu_2(h_k)}\mu_2 M_{k,n}\right\|_{TV} &\leqslant \|\mu_1 M_{k,n} - \mu_1(m_{k,n})\mu_2 \cdot M_{k,n}\|_{TV} \\ &\quad + \left\|\mu_1(m_{k,n})\mu_2 \cdot M_{k,n} - \frac{\mu_1(h_k)}{\mu_2(h_k)}\mu_2 M_{k,n}\right\|_{TV} \\ &\leqslant 4\Gamma_{k,n}\|\mu_1 M_{k,n}\|.\end{aligned}$$

□

IV.3.3 Proof of Theorem 4

Proof of assertion i), Uniform geometric ergodicity. Let us take $k = 0$ in Proposition IV.5. Then, for any $\varepsilon > 0$, $\mathbb{P}(d\omega)$-almost surely, noting $h = h_0$, it holds for any finite and positive measures μ_1, μ_2, on $\mathbb{X}$,

$$\left\|\mu_1 M_{0,n} - \frac{\mu_1(h)}{\mu_2(h)}\mu_2 M_{0,n}\right\|_{TV} \leqslant \Delta^{\varepsilon}_{0,n}\|\mu_1 M_{0,n}\|,$$

where, $\mathbb{P}(d\omega)$-almost surely,

$$\limsup_{n\to\infty} \left(\Delta^{\varepsilon}_{0,n}\right)^{\frac{1}{n}} \leqslant \bar{\gamma}^{\mathbb{P}[\gamma \geqslant \varepsilon]} \in [0, 1).$$

Let now $\delta \in (\bar{\gamma}^{\mathbb{P}[\gamma>0]}, 1)$. For $\varepsilon > 0$ small enough, $\delta > \bar{\gamma}^{\mathbb{P}[\gamma \geqslant \varepsilon]}$. As a consequence, $\mathbb{P}(d\omega)$-almost surely, for n large enough, (depending on ω),

$$\Delta^{\varepsilon}_{0,n} \leqslant \delta^n.$$

Thus, $\mathbb{P}(d\omega)$-almost surely, for any $\delta \in (\bar{\gamma}, 1)$, for n large enough and any positive and finite measures μ_1, μ_2,

$$\left\|\mu_1 M_{0,n} - \frac{\mu_1(h)}{\mu_2(h)}\mu_2 M_{0,n}\right\|_{TV} \leqslant \delta^n \|\mu_1 M_{0,n}\|_{TV}.$$

This proves Equation (IV.6). ☐

Proof of assertion ii). The proof relies on a classical time-reversal technique, see e.g. [Cog84; Ore91], or [Hen97] for a version that is closer to our context. As stated in [CFS82, II.10.4, pp.239-241], the ergodic system $(\Omega, \mathcal{A}, \mathbb{P}, \theta)$ can be extended as an invertible ergodic system $(\overline{\Omega}, \overline{\mathcal{A}}, \overline{\mathbb{P}}, \overline{\theta})$, such that $\Omega \subset \overline{\Omega}$, $\overline{\theta}|_\Omega = \theta$, and $\overline{\theta}$ is a bijective, bimeasurable, measure preserving and ergodic mapping. The definitions of $M_{k,n}$, $c_n, d_n, \nu_n, \gamma_n$ can be naturally extended to all $k \leqslant n$ in $\mathbb{Z}$, and one still has $c_n = c \circ \overline{\theta}^n$, $d_n = d \circ \overline{\theta}^n$, $\gamma_n = \gamma \circ \overline{\theta}^n$ for $n \in \mathbb{Z}$. Assumption **Ass. IV.4** implies that all the $(\gamma_n)_{n\in\mathbb{Z}}$ are almost surely positive and have log-moments.

Therefore, Lemma IV.3 and Proposition IV.2 extend to indexes $k \leqslant n \leqslant N \in \mathbb{Z}$.

For nonnegative $n \leqslant N$, for any positive measures μ_1, μ_2 on $\mathbb{X}$, one has in particular

$$\|\mu_1 \cdot M_{-n,0} - \mu_2 \cdot M_{-n,0}\|_{TV} \leqslant 2 \prod_{i=0}^{n-1} (1 - \gamma_{-i-1}). \tag{IV.39}$$

With $\mu_2 = \mu_1 M_{-N,-n}$, this yields :

$$\|\mu_1 \cdot M_{-n,0} - \mu_1 \cdot M_{-N,0}\|_{TV} \leqslant 2 \prod_{i=0}^{n-1} (1 - \gamma_{-i-1}).$$

Noticing that $\overline{\theta}$ is now an ergodic automorphism of the measured space $\overline{\Omega}$, and applying Birkhoff-Khinchin Ergodic Theorem as stated in [CFS82, Theorem 1, p.11], one gets, for almost any $\omega \in \overline{\Omega}$

$$\frac{1}{n} \sum_{i=0}^{n-1} \log(1 - \gamma_{-i-1}) \circ \overline{\theta}^{-i} \underset{n\to\infty}{\longrightarrow} \mathbb{E}\left[\log(1 - \gamma_{-1})\right] = \mathbb{E}\left[\log(1 - \gamma)\right].$$

Thus

$$\left(\prod_{i=0}^{n-1} (1 - \gamma_{-i-1}) \right)^{\frac{1}{n}} \underset{n\to\infty}{\longrightarrow} \exp\left(\mathbb{E}\left[\log(1 - \gamma)\right]\right) = \bar{\gamma} < 1.$$

Therefore, almost surely, the sequence $(\mu_1 \cdot M_{-n,0})_{n\in\mathbb{N}_0}$ is a Cauchy sequence in the space $\mathcal{M}_1(\mathbb{X})$ of probabilities on $\mathbb{X}$, endowed with the total variation norm. It thus converges almost surely to a random probability π_{μ_1} on $\mathbb{X}$. For any finite, positive non-zero measures μ_1, μ_2, plugging π_{μ_1}, π_{μ_2} into (IV.39), one proves that for almost any ω,

$$\pi_{\mu_1} = \pi_{\mu_2}.$$

Thus, there exists a random probability π, such that, almost surely, for any positive measure μ

$$\lim_{n\to\infty} \|\mu \cdot M_{-n,0} - \pi\|_{TV} = 0.$$

By stationarity of $\overline{\theta}$,

$$\mu \cdot M_{-n,0} \overset{d}{=} \mu \cdot M_{0,n},$$

which proves that, noting Λ the distribution of π,

$$\mu \cdot M_{0,n} \xrightarrow[n\to\infty]{d} \Lambda.$$

□

Before proving the last assertion of Theorem 4, we need to present the following lemma which establishes a sharp lower bound on some triangular inequality.

Lemma IV.7. *Assume* ***Ass. IV.2*** *holds. Then, for any $0 \leqslant k < n$ and any $\mu \in \mathcal{M}_+(\mathbb{X})$*

$$\gamma_k \|\mu M_{0,k+1}\|_{TV} \|\!| M_{k+1,n} |\!\| \leqslant \|\mu M_{0,n}\|_{TV} \leqslant \|\mu M_{0,k+1}\|_{TV} \|\!| M_{k+1,n} |\!\|. \tag{IV.40}$$

Proof of Lemma IV.7. Let $\mu \in \mathcal{M}_+(\mathbb{X}) - \{0\}$ and $0 \leqslant k < n$.

$$\mu M_{0,n} = (\mu M_{0,k}) M_k M_{k+1,n} \geqslant c_k \|\mu M_{0,k+1}\|_{TV} \nu_k M_{k+1,n},$$

thus

$$\|\mu M_{0,n}\|_{TV} = \mu M_{0,n}\mathbb{1} \geqslant c_k \|\mu M_{0,k+1}\| (\nu_k M_{k+1,n})(\mathbb{1}) = c_k \|\mu M_{0,k+1}\|_{TV} \|\nu_k M_{k+1,n}\|_{TV}.$$

By definition of d_{k+1}, it holds additionally

$$\|\nu_k M_{k+1,n}\| \geqslant d_{k+1} \|\!| M_{k+1,n} |\!\|.$$

Combining the two previous inequalities, we get

$$\|\mu M_{0,n}\|_{TV} \geqslant c_k d_{k+1} \|\mu M_{0,k+1}\|_{TV} \|\!| M_{k+1,n} |\!\| = \gamma_k \|\mu M_{0,k+1}\|_{TV} \|\!| M_{k+1,n} |\!\|$$

To obtain the second inequality of the lemma we simply write

$$\begin{aligned}
\|\mu M_{0,n}\| &= \mu M_{0,n}\mathbb{1} \\
&= \mu M_{0,k+1} M_{k+1,n}\mathbb{1} \\
&\leqslant \sup_{x\in\mathbb{X}} \delta_x M_{k+1,n}\mathbb{1} \times \mu M_{0,k+1}\mathbb{1} \\
&\leqslant \|\!| M_{k+1,n} |\!\| \|\mu M_{0,k+1}\|_{TV}.
\end{aligned}$$

□

Proof of assertion iii). We notice first that for any fixed integer, the $\mathbb{P}-$almost sure convergence

$$n^{-1} \log \|\!| M_{0,n} |\!\| \xrightarrow[n\to\infty]{} \lambda := \inf_{N\geqslant 1} \frac{1}{N} \mathbb{E}\left[\log \|\!| M_{0,N} |\!\|\right]$$

is a classical consequence of Kingsman's subbaditive ergodic theorem and the subadditivity property

$$\log \|\!| M_{0,n+p} |\!\| \leqslant \log \|\!| M_{0,n} |\!\| + \log \|\!| M_{n,n+p} |\!\|.$$

Note that applying this theorem requires Assumption **Ass. IV.3** to ensure the integrability of $\log^+ \|\!| M_{0,n} |\!\|$. It is also classical that the same convergence holds for shifted sequences : for any $k \geqslant 0$,

$$n^{-1} \log \|\!| M_{k,n} |\!\| \underset{n\to\infty}{\longrightarrow} \lambda \tag{IV.41}$$

almost surely.

Let now μ be a positive, finite measure on $\mathbb{X}$. Let $k \geqslant 0$. From Lemma IV.7, it holds for $n \geqslant k$

$$\gamma_k \|\mu M_{0,k}\|_{TV} \|\!| M_{k,n} |\!\| \leqslant \|\mu M_{0,n}\|_{TV} \leqslant \|\mu M_{0,k}\|_{TV} \|\!| M_{k,n} |\!\|.$$

Thus

$$\left| \frac{1}{n} \log \|\mu M_{0,n}\|_{TV} - \frac{1}{n} \log \|\!| M_{k,n} |\!\| \right| \leqslant \frac{1}{n} \left(|\log \|\mu M_{0,k}\|| + |\log \gamma_k| \right) \underset{n\to+\infty}{\longrightarrow} 0$$

almost surely on the event $\{\gamma_k > 0\}$. Combining this last estimate with (IV.41) proves that the convergence

$$\frac{1}{n} \log \|\mu M_{0,n}\| \underset{n\to\infty}{\longrightarrow} \lambda$$

holds almost surely on the event $\{\gamma_k > 0\}$, thus also on the event $\bigcup_{k\geqslant 0}\{\gamma_k > 0\}$. Under Assumptions **Ass. IV.1** and **Ass. IV.4**, $\mathbb{P}[\bigcup_{k\geqslant 0}\{\gamma_k > 0\}] = 1$ by Birkhoff's ergodic theorem, thus the convergence (IV.7) holds $\mathbb{P}(d\omega)$-almost surely. □

IV.3.4 The independent case : proof of Theorems 5 and 6

Let us introduce the Markov chain $(\mu_n)_{n\geqslant 0}$ with state space $\mathcal{M}_1(\mathbb{X})$, defined by $\mu_{n+1} = \mu_n \cdot M_n = \mu_0 \cdot M_{0,n+1}$. The process $(\mu_n, M_n)_{n\geqslant 0}$, is then clearly also a Markov chain with state space $\mathcal{M}_1(\mathbb{X}) \times \mathcal{K}^+$ and transition kernel :

$$Qf(\mu, M) = \int f(\mu \cdot M, N) d\mathcal{P}(N).$$

We denote $\mathbb{P}_{\bar{\chi}}$ the law of the Markov chain $((\mu_n, M_n))_{n\geqslant 0}$ when (μ_0, M_0) is distributed according to a measure $\bar{\chi}$ on $\mathcal{M}_1(\mathbb{X}) \times \mathcal{K}^+$. Theorems 5 and 6 rely on the study of the invariant measures and the ergodicity properties of the Markov chains $(\mu_n)_{n\geqslant 0}$ and $(\mu_n, M_n)_{n\geqslant 0}$. In particular, we show that the limit distribution Λ of the Markov chain (μ_n) is its only invariant distribution. This is stated in the following proposition.

Proposition IV.8. *Suppose $(M_n)_{n\geqslant 0}$ is an i.i.d sequence of elements of $\mathcal{K}^+$ distributed according to $\mathcal{P}$, and satisfying the assumptions of Theorem 4. Then*

i) For any initial distribution χ on $\mathcal{M}_1(\mathbb{X})$, the Markov chain $(\mu_n)_{n\geqslant 0}$ converges weakly to Λ.

ii) Λ is the only invariant measure of the Markov chain $(\mu_n)_{n\geqslant 0}$.

iii) $\Lambda \otimes \mathcal{P}$ is the only invariant measure of the Markov chain $(\mu_n, M_n)_{n\geqslant 0}$.

As a consequence, the dynamical systems associated with the Markov chains $(\mu_n)_{n\geqslant 0}$ and $(\mu_n, M_n)_{n\geqslant 0}$ are ergodic.

To prove Proposition IV.8, we need to define the convolution operation $\star$ between probability measures on $\mathcal{M}_1(\mathcal{K}^+)$ as follows : For any $\mathcal{Q}_1, \mathcal{Q}_2 \in \mathcal{M}_1(\mathcal{K}^+)$, $\mathcal{Q}_1 \star \mathcal{Q}_2$ is the law of $N_1 N_2$, where $(N_1, N_2) \sim \mathcal{Q}_1 \otimes \mathcal{Q}_2$. We note, for any $\mathcal{Q} \in \mathcal{M}_1(\mathcal{K}^+)$, $\mathcal{Q}^{\star n}$ the n-th convolution power of $\mathcal{Q}$. As an example, if $N_0, \ldots, N_{n-1}$ are i.i.d with law $\mathcal{Q}$, $\mathcal{Q}^{\star n}$ is simply the distribution of $N_{0,n} = N_0 \cdots N_{n-1}$. Given a probability distribution χ on $\mathcal{M}_1(\mathbb{X})$ and $\mathcal{Q}$ on $\mathcal{K}^+$, we also note $\chi \dot{\star} \mathcal{Q}$ the law of $\mu \cdot N$, where $(\mu, N) \sim \chi \otimes \mathcal{Q}$. These operations, previously defined in [BL85] in a finite dimensional context, satisfy some elementary properties, summed up in the following lemma.

Lemma IV.9. *Let $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3$ be probability measures on $\mathcal{K}^+$ and χ be a probability measure on $\mathcal{M}_1(\mathbb{X})$, it holds*

i) $(Q_1 \star Q_2) \star Q_3 = Q_1 \star (Q_2 \star \mathcal{Q}_3)$,

ii) $(\chi \dot{\star} \mathcal{Q}_1) \dot{\star} \mathcal{Q}_2 = \chi \dot{\star} (\mathcal{Q}_1 \star \mathcal{Q}_2)$,

iii) For each $\mathcal{Q} \in \mathcal{M}_1(\mathcal{K}^+)$, $\chi \mapsto \chi \dot{\star} \mathcal{Q}$ is continuous with respect to the topology of convergence in law on $\mathcal{M}_1(\mathcal{M}_1(\mathbb{X}))$.

Proof of Lemma IV.9. Consider $(N_1, N_2, N_3) \sim \mathcal{Q}_1 \otimes \mathcal{Q}_2 \otimes \mathcal{Q}_3$. It holds

$$N_1 N_2 N_3 = (N_1 N_2) N_3 = N_1 (N_2 N_3),$$

with $(N_1 N_2) N_3 \sim (Q_1 \star Q_2) \star Q_3$ and $N_1 (N_2 N_3) \sim Q_1 \star (Q_2 \star \mathcal{Q}_3)$. This yields *i)*.
Let us prove now point *ii)*. Consider $(\mu, N_1, N_2) \sim \chi \otimes \mathcal{Q}_1 \otimes \mathcal{Q}_2$. It holds

$$\mu \cdot (N_1 N_2) = (\mu \cdot N_1) \cdot N_2,$$

with $\mu \cdot (N_1 N_2) \sim \chi \dot{\star} (Q_1 \star Q_2)$ and $(\mu \cdot N_1) \cdot N_2 \sim (\chi \dot{\star} Q_1) \dot{\star} Q_2$. This yields *ii)*.
Let us move to the proof of *iii)*. Consider a sequence of probability measures (χ_n) on $\mathcal{M}_1(\mathbb{X})$, converging in distribution to χ. Let us show that $(\chi_n \dot{\star} \mathcal{Q})_{n \geqslant 0}$ converges in distribution towards $\chi \dot{\star} \mathcal{Q}$. Let f be a continuous, bounded function on $\mathcal{M}_1(\mathbb{X})$, it holds :

$$\int f(\mu) d(\chi_n \dot{\star} \mathcal{Q})(\mu) = \int \int f(\mu \cdot N) d\chi_n(\mu) d\mathcal{Q}(N) = \int \chi_n(g_N) d\mathcal{Q}(N),$$

where, for each $N \in \mathcal{K}^+$, the function $g_N : \mu \mapsto f(\mu \cdot N)$ is continuous and bounded. Thus

$$\chi_n(g_N) = \int f(\mu \cdot N) d\chi_n(\mu) \to \int f(\mu \cdot N) d\chi(\mu) = \chi(g_N).$$

This yields, by dominated convergence, as $n \to \infty$,

$$\int \chi_n(g_N) d\mathcal{Q}(N) = \int f(\mu) d(\chi_n \dot{\star} \mathcal{Q})(\mu) \longrightarrow \int \chi(g_N) d\mathcal{Q}(N) = \int \int f(\mu \cdot N) d\chi(\mu) d\mathcal{Q}(N),$$

which implies *iii)*.

$\square$

Proof of Proposition IV.8. Let f be a continuous and bounded function on $\mathcal{M}_1(\mathbb{X})$, it holds

$$\chi \dot{\star} \mathcal{P}^{\star n}(f) = \int f(\mu) d(\chi \dot{\star} \mathcal{P}^{\star n})(\mu) = \int \int f(\mu \cdot M_{0,n}) d\chi(\mu) d\mathcal{P}^{\star n}(M_{0,n}).$$

However, for any $\mu \in \mathcal{M}_1(\mathbb{X})$, Theorem 4, *ii*) states that $(\delta_\mu \dot{\star} \mathcal{P}^{\star n})_{n\geqslant 0}$ converges weakly towards Λ. Thus, for any $\mu \in \mathcal{M}_1(\mathbb{X})$, as $n \to \infty$

$$\int f(\mu \cdot M_{0,n}) d\mathcal{P}^{\star n}(M_{0,n}) = (\delta_\mu \dot{\star} \mathcal{P}^{\star n})(f) \underset{n\to\infty}{\longrightarrow} \Lambda(f).$$

By dominated convergence, this yields

$$\chi \dot{\star} \mathcal{P}^{\star n}(f) = \int \int f(\mu \cdot M_{0,n}) d\chi(\mu) d\mathcal{P}^{\star n}(M_{0,n}) \underset{n\to\infty}{\longrightarrow} \Lambda(f),$$

which proves the weak convergence

$$\chi \dot{\star} \mathcal{P}^{\star n} \underset{n\to\infty}{\longrightarrow} \Lambda$$

in the metric space $\mathcal{M}_1(\mathcal{M}_1(\mathbb{X}))$, for any probability distribution χ. This proves *i)*. Since, by Lemma IV.9, *iii)*, the map $\mu \mapsto \mu \dot{\star} \mathcal{P}$ is continuous, this proves that Λ is one of its fixed points, namely :

$$\Lambda = \Lambda \dot{\star} \mathcal{P}.$$

On the other hand, if $\chi \dot{\star} \mathcal{P} = \chi$, the sequence $(\chi \dot{\star} \mathcal{P}^{\star n})_{n\geqslant 0}$ is constant and converges to χ. By unicity of the limit, it holds

$$\chi = \Lambda.$$

This proves that Λ is the only invariant measure of the Markov chain (μ_n), i.e. *ii)*.
Let $(\mu_0, M_0) \sim \Lambda \otimes \mathcal{P}$. Then $\mu_1 = \mu_0 \cdot M_0 \sim \Lambda \dot{\star} \mathcal{P}$, $M_1 \sim \mathcal{P}$ and M_1 is independent of μ_0, M_0 and thus μ_1. Therefore $(\mu_1, M_1) \sim \Lambda \otimes \mathcal{P}$, and $\Lambda \otimes \mathcal{P}$ is thus an invariant measure of the Markov chain $(\mu_n, M_n)_{n\geqslant 0}$.

Conversely, consider now a probability measure $\bar{\chi}$ on $\mathcal{M}_1(\mathbb{X}) \times \mathcal{K}^+$, suppose it is an invariant measure of the Markov chain $((\mu_n, M_n))_{n\geqslant 0}$. The definition of the transition kernel Q implies that $\mu_1 = \mu_0 \cdot M_0$, $M_1 \sim \mathcal{P}$ and M_1 is independent of (μ_0, M_0), and therefore M_1 is independent of μ_1. However the second term (μ_1, M_1) of the Markov chain is distributed according to $\bar{\chi}Q = \bar{\chi}$ by invariance. Thus $\bar{\chi}$ is of the form $\bar{\chi} = \chi \otimes \mathcal{P}$.

Additionally, if $(\mu_0, M_0) \sim \bar{\chi} = \chi \otimes \mathcal{P}$, then $\mu_1 = \mu_0 \cdot M_0 \sim \chi \dot{\star} \mathcal{P}$. But by invariance of $\bar{\chi}$, $\mu_1 \sim \chi$, thus

$$\chi \dot{\star} \mathcal{P} = \chi.$$

By Proposition IV.8, this implies that $\chi = \Lambda$. Finally, this proves that $\Lambda \otimes \mathcal{P}$ is the only invariant measure of the Markov chain $(\mu_n, M_n)_{n\geqslant 0}$. By Corollary 5.12 of [Hai18], since both the processes $(\mu_n)_{n\geqslant 0}$ and $(\mu_n, M_n)_{n\geqslant 0}$ are Markov chains with a unique invariant measure, they are both ergodic. $\square$

One additional lemma is required before proving Theorem 5.

Lemma IV.10. *Let $(M_n)_{n\geqslant 0}$ be an i.i.d sequence of elements of $\mathcal{K}^+$ with law $\mathcal{P}$, satisfying the assumptions* ***Ass. IV.1***, ***Ass. IV.2*** *and* ***Ass. IV.4+***. *The sets*

$$\{\forall \mu \in \mathcal{M}_1(\mathbb{X}), \limsup_{n\to\infty} \log \|\mu M_{0,n}\|_{TV} = +\infty\},$$

$$\{\exists \mu \in \mathcal{M}_1(\mathbb{X}), \limsup_{n\to\infty} \log \|\mu M_{0,n}\|_{TV} = +\infty\}$$

and the event

$$\{\limsup_{n\to\infty} \log |||M_{1,n}||| = +\infty\}$$

coincide up to $\mathcal{P}^{\otimes\mathbb{N}}$-negligible events. A similar statement holds replacing $\limsup$ *by* $-\liminf$ *in the three events.*

Proof of Lemma IV.10. From Lemma IV.7, with $k = 0$, we get

$$\gamma_0 \|\mu M_0\|_{TV} |||M_{1,n}||| \leqslant \|\mu M_{0,n}\|_{TV} \leqslant \|\mu M_0\|_{TV} |||M_{1,n}|||,$$

where $\gamma_0 \|\mu M_0\|_{TV} > 0$ almost surely by **Ass. IV.2** and **Ass. IV.4+**. The lemma is a straightforward consequence of this inequality. □

Let us prove now Theorem 5.

Proof of Theorem 5, i). Let us notice first that when μ is a probability measure, $\rho : (\mu, M) \mapsto \log \|\mu M\|$ satisfies the cocycle property

$$\begin{aligned}
\rho(\mu_0, M_{0,n}) &= \log \|\mu_0 M_{0,n}\|_{TV} \\
&= \sum_{k=0}^{n-1} \log \left(\frac{\|\mu_0 M_{0,k+1}\|_{TV}}{\|\mu_0 M_{0,k}\|_{TV}} \right) \\
&= \sum_{k=0}^{n-1} \log \|(\mu_0 \cdot M_{0,k}) M_k\|_{TV} \\
&= \sum_{k=0}^{n-1} \rho(\mu_k, M_k). \qquad \text{(IV.42)}
\end{aligned}$$

From Equation (IV.42), we derive

$$\frac{1}{n} \log \|\mu_0 M_{0,n}\|_{TV} = \frac{1}{n} \sum_{k=0}^{n-1} \rho(\mu_k, M_k).$$

By Birkhoff's Ergodic Theorem, since (μ_k, M_k) is an ergodic Markov chain, with stationary distribution $\Lambda \otimes \mathcal{P}$, this quantity converges $\mathbb{P}_{\Lambda\otimes\mathcal{P}}$-almost surely and in $\mathcal{L}^1(\mathbb{P}_{\Lambda\otimes\mathcal{P}})$ towards $\int \rho d(\Lambda \otimes \mathcal{P})$ provided ρ is an $\mathcal{L}^1$ function with respect to $\Lambda \otimes \mathcal{P}$. Let us check now this integrability property. Let $(\mu_0, (M_n)_{n\geqslant 0}) \sim \Lambda \otimes \mathcal{P}^{\mathbb{N}}$. Then, applying (IV.40) with $n = 2$ yields

$$\gamma_0 \|\mu_0 M_0\|_{TV} |||M_1||| \leqslant \|\mu_0 M_{0,2}\|_{TV} \leqslant \|\mu_0 M_0\|_{TV} |||M_1|||.$$

Noting $\mu_1 = \mu_0 \cdot M_0$, we get, since $\|\mu_0 M_0\|_{TV} \neq 0$ almost surely,

$$\gamma_0 |||M_1||| \leqslant \|\mu_1 M_1\|_{TV} \leqslant |||M_1|||,$$

thus

$$|\rho(\mu_1, M_1)| \leqslant |\log |||M_1||| | + |\log(\gamma_0)|.$$

Note that $(\mu_1, M_1) \sim \Lambda \otimes \mathcal{P}$, since by definition $\mu_1 = \mu_0 \cdot M_0$ and $(\mu_0, (M_n)_{n\in\mathbb{N}}) \sim \Lambda \otimes \mathcal{P}^{\otimes \mathbb{N}}$. Thus, under **Ass. IV.3+** and **Ass. IV.4+**, it holds

$$\mathbb{E}\left[|\rho(\mu_1, M_1)|\right] = \int |\rho| d(\Lambda \otimes \mathcal{P}) \leqslant \mathbb{E}|\log |||M_1||| | + \mathbb{E}|\log(\gamma_0)| < \infty.$$

This proves that ρ is integrable with respect to $\Lambda \otimes \mathcal{P}$, thus the convergence

$$n^{-1} \log(\|\mu_0 M_{0,n}\|_{TV}) \xrightarrow[n\to\infty]{} \int \rho \, d\Lambda \otimes \mathcal{P}$$

holds in $\mathcal{L}^1(\Lambda \otimes \mathbb{P})$ and $\mathbb{P}_{\Lambda\otimes\mathbb{P}}$-almost surely. Since by Theorem 4, almost surely, for all μ, it holds $n^{-1} \log \|\mu M_{0,n}\|_{TV} \xrightarrow[n\to\infty]{} \lambda$, by unicity of the almost sure limit,

$$\lambda = \int \rho \, d(\Lambda \otimes \mathcal{P}).$$

□

Proof of Theorem 6, ii). Note now $X_n = \rho(\mu_{n-1}, M_{n-1})$, for $n \geqslant 1$. Then it holds, for $n \geqslant 0$, for any probability measure μ_0

$$(\mu_{n+1}, X_{n+1}) = (\mu_n \cdot M_n, \rho(\mu_n, M_n)).$$

Thus, $(\mu_n, X_n)_{n\geqslant 0}$ is a Markov chain on $\mathcal{M}_1(\mathbb{X}) \times \mathbb{R}$ such that

$$\mathbb{P}\left[(\mu_{n+1}, X_{n+1}) \in A \times B | (\mu_n, X_n)\right] = \int \mathbb{1}_A(\mu_n \cdot M) \mathbb{1}_B\left(\rho(\mu_n, M)\right) d\mathcal{P}(M).$$

Thus $S_n = \log \|\mu M_{0,n}\| = X_1 + \cdots + X_n$ is a Markov random walk associated with (μ_n, X_n), in the sense of [Als01]. Suppose that $\lambda = 0$. By Theorem 4, it holds

$$n^{-1} S_n = n^{-1} \log \|\mu M_{0,n}\|_{TV} \xrightarrow[n\to\infty]{} 0,$$

$d\mathbb{P}_{\Lambda\otimes\mathcal{P}^{\otimes\mathbb{N}}}(\mu, (M_n)_{n\geqslant 0})$-almost surely, thus in probability with respect to $\mathbb{P}_{\Lambda\otimes\mathcal{P}^{\otimes\mathbb{N}}}$. Since moreover, (μ_n) is an ergodic Markov chain, the assumptions of [Als01] are satisfied. If there exists a function η such that $\mathbb{P}_{\Lambda\otimes\mathcal{P}}$-almost surely, for $n \geqslant 1$,

$$X_n = \eta(\mu_n) - \eta(\mu_{n-1}), \qquad \text{(IV.43)}$$

then taking $n = 1$ shows that we are in the case of Null Homology (NH). In this case, it holds moreover

$$\log \|\mu_0 M_{0,n}\|_{TV} = X_1 + \cdots + X_n = \eta(\mu_n) - \eta(\mu_0).$$

Thus, almost surely, noting a, b the respective infimum and supremum of the support of $\eta(\mu)$, when $\mu \sim \Lambda$, since the sequence (μ_n) is a stationary and ergodic sequence with law Λ, it holds

$$\liminf_{n\to\infty} \log \|\mu_0 M_{0,n}\|_{TV} = a - \eta(\mu_0), \text{ and } \limsup_{n\to\infty} \log \|\mu_0 M_{0,n}\|_{TV} = b - \eta(\mu_0).$$

Thus the almost sure finiteness of these quantities are respectively equivalent to the finiteness of a and b. If Equation (IV.43) does not hold, then we are in the setup of Theorem 2 or 3 of [Als01]. These two Theorems imply that the Markov Random Walk (S_n) oscillates : $\limsup S_n = +\infty$ and $\liminf S_n = -\infty$ $\mathbb{P}_{\Lambda\otimes\mathcal{P}}$−almost surely. However, by Lemma IV.10, *ii)*, this implies that $\mathcal{P}^{\otimes\mathbb{N}}$-almost surely, for every $\mu \in \mathcal{M}_1(\mathbb{X})$,

$$\limsup_{n\to\infty} \log \|\mu M_{0,n}\|_{TV} = -\liminf_{n\to\infty} \log \|\mu M_{0,n}\|_{TV} = +\infty.$$

This concludes the proof. □

IV.4 Sufficient conditions under uniform positivity assumptions

In the finite dimensional case $\mathbb{X} = \{1, \dots, p\}$, that is when studying products of $p \times p$ matrices, similar (and actually, more complete) results are obtained in [Hen97]. They rely on the very mild assumption

Ass. IV.5. $\mathbb{P}\left[\bigcup_{k\in\mathbb{N}} \left\{M_{0,k} \in \mathring{\mathcal{S}}\right\}\right] = 1,$

where $\mathring{\mathcal{S}}$ refers to the set of $p \times p$ matrices with positive entries. We expect that this approach, based on Hilbert contractions, might be extended in infinite dimensional contexts. This will require to introduce the notion of uniformly positive operators to strengthen the notion of positive matrices, and state an infinite dimensional generalization of **Ass. IV.5**, as we explain in Subsection IV.4.2.

This section aims at comparing our assumptions both with **Ass. IV.5**, and its natural generalization in infinite dimension.

We did not succeed in proving that **Ass. IV.5** alone is enough for our assumptions to hold. However we provide mild additional assumptions that, together with **Ass. IV.5**, constitute sufficient conditions for our assumptions (**Ass. IV.2**, **Ass. IV.3**, **Ass. IV.4**, **Ass. IV.4+**) to hold, and thus for Theorems 4, 5, 6 to apply.

IV.4.1 The finite dimensional case

Let us focus in this subsection on the case where $\mathbb{X}$ is finite, let us note $p = |\mathbb{X}|$. Consider a stationary and ergodic sequence $(M_n)_{n\in\mathbb{N}_0}$ of $p \times p$ matrices with nonnegative entries.

Checking whether Assumptions **Ass. IV.1**, **Ass. IV.2**, **Ass. IV.3** are satisfied is quite straightforward, since these three assumptions only involve the law of the first matrix of the sequence. Let us see now how the additional Assumption **Ass. IV.5** can help exhibit an admissible triplet in order to check that Assumptions **Ass. IV.4** holds.

Lemma IV.11. *Consider a random and stationary sequence of $p \times p$ matrices $M_n = (M_n(x,y))_{x,y\in\mathbb{X}}$ with nonnegative entries, satisfying* ***Ass. IV.2*** *and* ***Ass. IV.5****. Then for any measurable map $\omega \in \Omega \mapsto \nu(\omega) \in \mathcal{M}_1(\mathbb{X})$, there exists a random variable $d : \omega \mapsto d(\omega)$ such that $\mathbb{P}[d > 0] = 1$ and (IV.5) holds.*

Proof. The following decomposition holds : for any $1 \leqslant k \leqslant n$, $x \in \mathbb{X}$, $\omega \in \Omega$

$$m_{1,n}(x) = \delta_x M_{1,n}\mathbb{1} = \delta_x M_{1,k} m_{k,n} = \sum_{z\in\mathbb{X}} M_{1,k}(x,z) m_{k,n}(z).$$

Thus

$$\nu_0(m_{1,n}) = \nu_0 M_{1,n}\mathbb{1} = \sum_{y,z\in\mathbb{X}} \nu_0(y) M_{1,k}(y,z) m_{k,n}(z),$$

where we note $\nu_0 = \nu_\omega$. The fact that ν_0 is a probability measure yields,

$$\begin{aligned} m_{1,n}(x) &\leqslant \sup_{y,z\in\mathbb{X}} \left(\frac{M_{1,k}(x,z)}{M_{1,k}(y,z)}\right) \sum_{y\in\mathbb{X}} \nu_0(y) M_{1,k}(y,z) m_{k,n}(z) \\ &\leqslant \sup_{y,z\in\mathbb{X}} \frac{M_{1,k}(x,z)}{M_{1,k}(y,z)} \nu_0(m_{1,n}), \end{aligned}$$

for any $1 \leqslant k \leqslant n$, with the conventions $\frac{M_{1,k}(x,z)}{M_{1,k}(y,z)} = 0$ as soon as $M_{1,k}(x,z) = 0$ and $\frac{M_{1,k}(x,z)}{M_{1,k}(y,z)} = \infty$ if $M_{1,k}(x,z) \neq 0$ and $M_{1,k}(y,z) = 0$. Thus, for any $n \geqslant k$,

$$\frac{|||M_{1,n}|||}{\nu_0(m_{1,n})} = \frac{\|m_{1,n}\|_\infty}{\nu_0(m_{1,n})} \leqslant \sup_{x,y,z\in\mathbb{X}} \frac{M_{1,k}(x,z)}{M_{1,k}(y,z)}. \tag{IV.44}$$

This yields

$$\inf_{x,y,z\in\mathbb{X}} \frac{M_{1,k}(y,z)}{M_{1,k}(x,z)} \leqslant \inf_{n\geqslant k} \frac{\nu_0(m_{1,n})}{\|m_{1,n}\|_\infty}, \tag{IV.45}$$

and therefore

$$\sup_{k\geqslant 1} \inf_{x,y,z\in\mathbb{X}} \frac{M_{1,k}(y,z)}{M_{1,k}(x,z)} \leqslant \liminf_n \frac{\nu_0(m_{1,n})}{|||M_{1,n}|||}.$$

By Assumption **Ass. IV.5**, $\mathbb{P}(d\omega)$- almost surely, there exists a random integer k_ω such that $M_{1,k_\omega}(\omega) \in \mathring{\mathcal{S}}$. Since $\mathbb{X}$ is finite, we get, for $\mathbb{P}$-almost any ω,

$$1 < \inf_{x,y,z\in\mathbb{X}} \frac{M_{1,k_\omega}(y,z)}{M_{1,k_\omega}(x,z)} < \sup_{k\geqslant 1} \inf_{x,y,z\in\mathbb{X}} \frac{M_{1,k}(y,z)}{M_{1,k}(x,z)} \leqslant \liminf_n \frac{\nu_0(m_{1,n})}{|||M_{1,n}|||}.$$

Assumption **Ass. IV.2** implies moreover that that for all n, $\mathbb{P}$-almost any $\omega \in \Omega$, $\frac{\nu_0(m_{1,n})}{|||M_{1,n}|||} > 0$. Thus, setting

$$d(\omega) = \inf_{n\in\mathbb{N}} \frac{\nu_0(m_{1,n})}{|||M_{1,n}|||},$$

(ν, d) satisfies (IV.3.1) and $d(\omega) > 0$, $\mathbb{P}(d\omega)$-a.s. $\square$

This provides nice sufficient conditions for **Ass. IV.4** or **Ass. IV.4+** to hold.

Proposition IV.12. *Consider a random, stationary sequence of $p \times p$ matrices $M_n = (M_n(x,y))_{x,y \in \mathbb{X}}$, with nonnegative entries, satisfying **Ass. IV.1**, **Ass. IV.3**, **Ass. IV.5**. We assume that there exists a measurable map*

$$\omega \in \Omega \mapsto (\nu_\omega, c(\omega)) \in \mathcal{M}_1(\mathbb{X}) \times [0,1]$$

*such that $\mathbb{P}[c > 0] > 0$ and (IV.4) hold. Then there exists a random variable d such that (ν, c, d) is an admissible triplet with $\mathbb{P}[\gamma > 0] > 0$. Thus assumption **Ass. IV.4** holds and Theorem 4 applies.*
If moreover

i) $\mathbb{E}\left[-\log c\right] < \infty$,

ii) there exists an deterministic integer $N \in \mathbb{N}_0$ such that

$$\mathbb{E}\left[\log \sup_{x,y,z \in \mathbb{X}} \frac{M_{0,N}(x,z)}{M_{0,N}(y,z)}\right] < \infty, \tag{IV.46}$$

iii) $\int \left[\log \nu_\omega(m_{1,2})\right]^- d\mathbb{P}(\omega) < \infty$,

*Then **Ass. IV.4+** also holds.*

Note that since $\mathbb{X}$ is finite, (δ_j, c) satisfies (IV.4) as soon as $c \leqslant \min_{1 \leqslant i \leqslant p} \frac{M_0(i,j)}{\sum_{l=1}^p M_0(i,l)}$, thus **Ass. IV.4** holds if with positive probability, M_0 has a column with only nonzero coefficients.

Proof. By Lemma IV.11, setting

$$d(\omega) = \inf_{n \in \mathbb{N}} \frac{\nu_\omega(m_{1,n})}{\|\!|M_{1,n}|\!\|},$$

$\omega \mapsto (\nu_\omega, d(\omega))$ satisfies (IV.5) and $\mathbb{P}[d > 0] = 1$. Noting $\gamma(\omega) = c(\omega)d(\omega) = c_0(\omega)d_1(\omega)$, we notice that $\mathbb{P}[d_1 > 0] = \mathbb{P}[d > 0] = 1$ and $\mathbb{P}[c > 0] > 0$. It holds thus $\mathbb{P}[\gamma > 0] > 0$: **Ass. IV.4** is satisfied. This proves the first part of the proposition. Let us suppose now that *i)-iii)* hold. Since $\mathbb{E}\left[-\log c\right] < \infty$, then $\mathbb{P}[c > 0] = 1$ and $\mathbb{P}[\gamma > 0] = 1$. Let us now prove now that $\mathbb{E}|\log \gamma| < \infty$. By the inequality

$$|\log \gamma| = -\log(\gamma) \leqslant -\log(c) - \log(d)$$

and hypothesis $\mathbb{E}[|\log(c)|] < \infty$, it remains to check that $\mathbb{E}[|\log(d)|] < \infty$. Inequality (IV.44) implies that $\mathbb{P}(d\omega)$-almost surely, for any $k \geqslant 1$,

$$\begin{aligned}
-\log d = \log \sup_{n \in \mathbb{N}} \frac{\|\!|M_{1,n}|\!\|}{\nu_\omega(m_{1,n})} &\leqslant \max\left(\log \sup_{x,y,z \in \mathbb{X}} \frac{M_{1,k+1}(x,z)}{M_{1,k+1}(y,z)}, \max_{1 \leqslant n \leqslant k} \log \frac{\|\!|M_{1,n}|\!\|}{\nu(m_{1,n})}\right) \\
&\leqslant \log \sup_{x,y,z \in \mathbb{X}} \frac{M_{1,k+1}(x,z)}{M_{1,k+1}(y,z)} + \sum_{1 \leqslant n \leqslant k} \log \frac{\|\!|M_{1,n}|\!\|}{\nu_\omega(m_{1,n})}.
\end{aligned}$$

In particular, setting $k = N$ and applying condition (IV.46), it holds by stationarity

$$\mathbb{E}\left[\log \sup_{x,y,z\in\mathbb{X}} \frac{M_{1,N+1}(x,z)}{M_{1,N+1}(y,z)}\right] < \infty.$$

Consequently, it suffices to prove that $\mathbb{E}\left[\log \frac{|||M_{1,n}|||}{\nu_0(m_{1,n})}\right] < \infty$ for any $n \geqslant 1$. Let us decompose this quantity as

$$\mathbb{E}\left[\log \frac{|||M_{1,n}|||}{\nu_0(m_{1,n})}\right] \leqslant \underbrace{\mathbb{E}\left(\log |||M_{1,n}|||\right)^+}_{A(n)} + \underbrace{\mathbb{E}\left[\left(\log \nu_0(m_{1,n})\right)^-\right]}_{B(n)}. \tag{IV.47}$$

- On the one hand, the inequality $|||M_{1,n}||| \leqslant \prod_{i=1}^{n-1} |||M_{i,i+1}|||$ readily yields

$$A(n) \leqslant \sum_{i=1}^{n-1} \mathbb{E}\left[\log(|||M_{i,i+1}|||)^+\right] = (n-1)\mathbb{E}\left[\log(|||M_{0,1}|||)^+\right] < \infty.$$

- On the other hand, for any $x \in \mathbb{X}$ and $\mathbb{P}(d\omega)$-a.s.

$$m_{1,n}(x) = \delta_x M_{1,2} m_{2,n} \geqslant c_1 m_{1,2}(x) \nu_1(m_{2,n}),$$

where $c_k = c(\theta^k(\omega))$ and $\nu_k = \nu_{\theta^k(\omega)}$. Consequently, integrating with respect to $\nu_0(dx)$, we obtain

$$\nu_0(m_{1,n}) = \nu_0 M_{1,2} m_{2,n} \geqslant c_1(\omega)\nu_0(m_{1,2})\nu_{\theta(\omega)}(m_{2,n}) \quad \mathbb{P}(d\omega)\text{—a.s},$$

which yields, by induction

$$\nu_0(m_{1,n}) \geqslant \prod_{k=1}^{n-1} c_k \nu_{k-1}(m_{k,k+1}).$$

Consequently, $\mathbb{P}(d\omega)$-a.s.,

$$\left(\log \nu_0(m_{1,n})\right)^- \leqslant \sum_{k=1}^{n-1} -\log c_k + \left[\log \nu_{k-1}(m_{k,k+1})\right]^- .$$

By stationarity, we deduce

$$\begin{aligned} B(n) &\leqslant \sum_{k=1}^{n-1} \mathbb{E}\left[-\log c_k\right] + \mathbb{E}\left[\left(\log \nu_{k-1}(m_{k,k+1})\right)^-\right] \\ &= (n-1)\mathbb{E}\left[-\log c_0\right] + \mathbb{E}\left[\left(\log \nu_0(m_{1,2})\right)^-\right] < \infty. \end{aligned}$$

Finally, combining these estimates, we get, for any $n \in \mathbb{N}_0$,

$$\mathbb{E}\left[\log \frac{\|M_{1,n}\|}{\nu_0(m_{1,n})}\right] \leqslant (n-1)\mathbb{E}\left[-\log c_0 + \log\left[\nu_0(m_{1,2})\right]^- + (\log\|m_{0,1}\|_\infty)^+\right] < \infty.$$

□

IV.4.2 Extension in infinite dimension

When $\mathbb{X}$ is infinite, we need to strengthen the notion of positive matrices as follows.

Definition IV.13. *A positive linear map M on $\mathcal{B}(\mathbb{X})$ is uniformly positive if there exists $K \in \mathbb{R}_+^*, h \in \mathcal{B}_+(\mathbb{X})$, such that, for any $f \in \mathcal{B}_+(\mathbb{X})$ there exists $b(f) \in \mathbb{R}_+$, satisfying*

$$\frac{1}{K}b(f)h \leqslant M(f) \leqslant Kb(f)h.$$

Notice that when $\mathbb{X}$ is finite, a matrix of $\mathring{\mathcal{S}}$ is uniformly positive. Moreover, in Hennion's work, assumption **Ass. IV.5** is used as a sufficient condition to obtain projective contraction properties on the product $M_{k,n}$, with respect to a projective distance called the Hilbert distance (once again, see [BK53; Bir57; Lig23] for a complement on this distance). In an infinite dimensional setting, this distance can still be defined, and the projective action associated with a positive operator is contracting if and only if the operator is uniformly positive (a proof of this claim is proposed in [Lig23]). Uniform positivity is therefore the appropriate infinite dimensional generalization of positivity in our context, and condition **Ass. IV.5** can thus naturally be replaced with the restrictive condition

Ass. IV.5'. $\mathbb{P}\left(\bigcup_{n\in\mathbb{N}}\{M_{0,n} \gg 0\}\right) = 1$,

where we note $M \gg 0$ iff M is uniformly positive.

The present subsection aims at comparing our result with the natural extensions of Hennion's work to infinite dimensional settings. For this purpose, the following Lemma extends the idea of Lemma IV.11 to an infinite dimensional setup, assuming **Ass. IV.5'** instead of **Ass. IV.5**.

Lemma IV.14. *Consider a random stationary sequence of elements of $\mathcal{K}^+$, satisfying **Ass. IV.2** and **Ass. IV.5'**. Then for any measurable map $\omega \in \Omega \mapsto \nu(\omega) \in \mathcal{M}_1(\mathbb{X})$, there exists a random variable d such that $\mathbb{P}[d > 0] = 1$ and (IV.5) holds.*

Proof. For almost any ω and any $1 \leqslant k \leqslant n$, $m_{k,n} \in \mathcal{B}(\mathbb{X})$, it holds,

$$\frac{m_{1,n}(x)}{\nu(m_{1,n})} = \frac{\delta_x M_{1,k} m_{k,n}}{\nu_0 M_{1,k} m_{k,n}} \leqslant \sup_{y\in\mathbb{X}, f\in\mathcal{B}(\mathbb{X})} \frac{\delta_y M_{1,k} f}{\nu_0 M_{1,k} f}.$$

Taking a supremum in $x \in \mathbb{X}$, we get, for any $k \leqslant n$,

$$\frac{\|M_{1,n}\|}{\nu_0(m_{1,n})} \leqslant \sup_{y\in\mathbb{X}, f\in\mathcal{B}(\mathbb{X})} \frac{\delta_y M_{1,k} f}{\nu_0 M_{1,k} f}. \tag{IV.48}$$

By **Ass. IV.5'**, let k_ω be a random integer such that $\mathbb{P}(d\omega)$-almost surely, $M_{1,k_\omega}(\omega) \gg 0$. Then, almost surely, there is $K \in \mathbb{R}_+^*$, $h \in \mathcal{B}_+(\mathbb{X})$ such that for any $f \in \mathcal{B}_+(\mathbb{X})$, there exists $b(f) \geqslant 0$, satisfying

$$K^{-1} b(f) h \leqslant M_{1,k_\omega} f \leqslant K b(f) h. \tag{IV.49}$$

From (IV.49), we deduce $K^{-1} m_{1,k_\omega}(x) \leqslant h(x) b(\mathbb{1}) \leqslant K m_{1,k_\omega}(x)$. By **Ass. IV.2**, m_{1,k_ω} is a bounded and positive function, thus so is h. Moreover, $b(\mathbb{1}) > 0$, $\nu_0(m_{1,k}) \leqslant K \nu_0(h) b(\mathbb{1})$, thus $\nu(h) > 0$. Therefore, for any $x \in \mathbb{X}$, any $f \in \mathcal{B}_+(\mathbb{X})$, it holds $\mathbb{P}(d\omega)$ almost surely:

$$\frac{\delta_x M_{1,k_\omega} f}{\nu_0 M_{1,k_\omega} f} \leqslant K^2 \frac{h(x)}{\nu_0(h)} \leqslant K^3 \frac{\|\!|M_{1,k_\omega}|\!\|}{b(\mathbb{1}) \nu_0(h)} \leqslant K^4 \frac{\|\!|M_{1,k_\omega}|\!\|}{\nu_0(m_{1,k_\omega})}. \tag{IV.50}$$

Finally, combining (IV.50) with (IV.48), we get for almost any ω,

$$\limsup_n \frac{\|\!|M_{1,n}|\!\|}{\nu_0(m_{1,n})} \leqslant \sup_{n \geqslant k_\omega} \frac{\|\!|M_{1,n}|\!\|}{\nu_0(m_{1,n})} \leqslant K^4 \frac{\|\!|M_{1,k_\omega}|\!\|}{\nu(m_{1,k_\omega})} < \infty.$$

Since moreover almost surely, for all $n \geqslant 1$, $\frac{\|\!|M_{1,n}|\!\|}{\nu_0(m_{1,n})}$ is finite, then almost surely,

$$\sup_{n \in \mathbb{N}} \frac{\|\!|M_{1,n}|\!\|}{\nu_0(m_{1,n})} < \infty.$$

Let us set

$$d(\omega) = \inf_{n \geqslant 1} \frac{\nu_0(m_{1,n})}{\|\!|M_{1,n}|\!\|}.$$

Then d clearly satisfies (IV.5) and $\mathbb{P}[d > 0] = 1$. □

The uniform positivity property is interesting to deal with many systems where the mass is sufficiently well mixed. We illustrate this on the following example.

Example IV.15. Take $\mathbb{X} = [0,1]^d$, choose an ergodic dymanical system $(\Omega, \mathcal{A}, \mathbb{P}, \theta)$ and associate with each $\omega \in \Omega$ a bounded measurable function $m_\omega : \mathbb{X} \to \mathbb{R}_+$, and a continuous function $Q_\omega : \mathbb{X}^2 \to \mathbb{R}_+^*$, such that $\int_{\mathbb{X}} Q_\omega(x,y) dy = 1$ for all x and all ω. We define M by setting, for each $\omega \in \Omega$ and $f \in \mathcal{B}(\mathbb{X})$,

$$M(\omega)(f) : x \mapsto m_\omega(x) \int_{\mathbb{X}} f(y) Q_\omega(x,y) dy.$$

Notice that $m_\omega(x) = \|\delta_x M(\omega)\|$. Our system clearly satisfies **Ass. IV.1** and **Ass. IV.2**. Assumptions **Ass. IV.3** and **Ass. IV.3+** just translate into a log-integrability assumption on $\omega \mapsto \|m_\omega\|_\infty = \|\!|M(\omega)|\!\|$.

In terms of populations, this model can represent the spatial evolution of a population in the compact domain $\mathbb{X}$. The quantity $m_\omega(x)$ represents the size of the offspring of an individual located at x, and the kernel $Q_\omega(x,y)$ represents the dispersion of its offspring in the domain $\mathbb{X}$. The dependence in ω of these quantities models an time inhomogeneity of these quantities. $M(\omega)$ is clearly uniformly positive, for each ω, since

$$K^{-1} m_\omega \mu(f) \leqslant M(\omega)(f) \leqslant K m_\omega \mu(f),$$

where μ refers to the Lebesgue measure on $\mathbb{X}$, and

$$K = K_\omega = \max\left(\left(\inf_{u,v} Q_\omega(u,v)\right)^{-1}, \sup_{u,v} Q_\omega(u,v)\right) < \infty$$

since $\mathbb{X}$ is compact and Q_ω is continuous and positive. Setting $\nu_\omega = \mu$ and $c_\omega = K_\omega^{-1}$ for all ω, the left hand side inequality implies (IV.4). Moreover, since $M(\omega)$ is uniformly positive, **Ass. IV.5'** holds, thus by Lemma IV.14 there exists a random variable d such that $\mathbb{P}[d > 0] > 0$ and $\omega \mapsto (\nu_\omega, d(\omega))$ satisfy (IV.5). Therefore **Ass. IV.4** holds and Theorem 4 applies.
One can notice additionally that in this context, each matrix of the product is almost surely uniformly positive, therefore the proof of Lemma IV.14 yields the explicit control

$$d(\omega) > \frac{\int M_{1,2}(\mathbb{1})(x)dx}{K_\omega^4 \sup_{x\in\mathbb{X}} M_{1,2}(\mathbb{1})(x)} = \frac{\int m_{\theta(\omega)}(x)dx}{K_\omega^4 \|m_{\theta(\omega)}\|_\infty}.$$

Thus **Ass. IV.4+** reduces to a log-integrability condition both on the coefficient K_ω and on the quotient $\frac{\int m_\omega(x)dx}{\|m_\omega\|_\infty}$.

This example illustrates how Proposition IV.12 from the previous subsection can be adapted, replacing **Ass. IV.5** by **Ass. IV.5'**. To tackle the integrability of $\log\gamma$, one can replace (IV.46) by

$$\mathbb{E}\left[\log \sup_{x\in\mathbb{X}, f\in\mathcal{B}(\mathbb{X})} \frac{\delta_x M_{1,N} f}{\nu_0 M_{1,N} f}\right] < \infty. \tag{IV.51}$$

This yields a counterpart of Proposition IV.12 in a infinite dimensional setup.

IV.5 Application to products of infinite Leslie matrices

The previous section focused on products of matrices with positive entries, and more generally, products of uniformly positive operators. This kind of products can be efficiently studied with methods based on projective contractions relatively to the Hilbert metric. The main interest of our techniques, based on Doeblin contractions, is their application to products of operators which are not uniformly positive. The goal of this section is to illustrate how such products can be studied with our theorems. We have chosen to focus here on a quite simple but natural example with no uniform positivity properties : the infinite Leslie matrices.

IV.5.1 Introduction to Leslie matrices

In this section, we set $\mathbb{X} = \mathbb{N}_0$, thus the operators of $\mathcal{K}^+$ can be represented as infinite matrices. We choose to consider infinite Leslie matrices, which have the following form :

for any $\omega \in \Omega$,

$$M(\omega) = \begin{pmatrix} f_0(\omega) & s_0(\omega) & 0 & 0 & \dots \\ f_1(\omega) & 0 & s_1(\omega) & 0 & \dots \\ f_2(\omega) & 0 & 0 & s_2(\omega) & \ddots \\ f_3(\omega) & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{IV.52}$$

where the entries $(f_k(\omega))_{k\in\mathbb{N}_0}$, $(s_k(\omega))_{k\in\mathbb{N}_0}$ are nonnegative and $\sup_{x\in\mathbb{X}} s_x(\omega) + f_x(\omega) < \infty$. Notice that such a matrix is not uniformly positive, since there are zeros on every row and every column but the first one. Moreover, if Q is a product of k matrices of this shape, the (x, y)-entry $[Q]_{x,y} = 0$ whenever $y \geqslant x + k + 1$. This prevents any product of such matrices from being uniformly positive. This example is therefore a typical situation where **Ass. IV.5'** does not hold.

Such matrices appear naturally when studying the dynamics of a population counting individuals according to their age. The coefficients f_x (respectively s_x) represent the mean number of individuals of age 0 (respectively of age $x + 1$) created by an individual of age x, that is the mean size of the offspring of an individual of age x (respectively the survival rate of individuals of age x). Usually, only a finite number of age classes are defined, thus $\mathbb{X} = [\![0, p]\!]$, and one considers finite versions of such matrices, called Leslie matrices, see for example [Cas10]. However, it is natural to extend their definition to an infinite number of age classes ($\mathbb{X} = \mathbb{N}_0$) obtaining infinite matrices with this shape. Indeed, several articles already study age-structured populations with an unbounded set of possible ages, see e.g. [BCG20; JK22; Oel90]. Therefore, products of random matrices shaped as in (IV.52) model the dynamics of an age structured population evolving in a randomly changing environment which affect their reproductive behavior. This is the kind of matrices we are studying in this section. Let us note from now on

$$s_x^k(\omega) = s_x \circ \theta^k(\omega) \text{ and } f_x^k(\omega) = f_x \circ \theta^k(\omega),$$

so that $(s_x^k, f_x^k)_{x\in\mathbb{X}}$ are the nonzero entries of the random matrix $M_k(\omega) = M \circ \theta^k$. We introduce the quantities

$$d'(\omega) = \sup_{k\in\mathbb{N}_0, x\leqslant y\in\mathbb{X}} \frac{f_y^k(\omega)}{f_x^k(\omega)} \geqslant 1,$$

and

$$d''(\omega) = \sup_{x\in\mathbb{X}, k\in\mathbb{N}_0} \frac{s_x^0(\omega)\cdots s_{x+k}^k(\omega)}{s_0^0(\omega)\dots s_k^k(\omega)} \geqslant 1,$$

which are useful to construct an admissible triplet.

IV.5.2 Ergodic behavior of products of random Leslie matrices

The following proposition provides sufficient conditions for assumptions **Ass. IV.3** and **Ass. IV.4** to hold in the case of products of infinite Leslie matrices.

Proposition IV.16. *Consider a random matrix product with $\mathbb{X} = \mathbb{N}_0$ and suppose that for any $\omega \in \Omega$, $M(\omega)$ is of the form of equation (IV.52), with $\sup_{x\in\mathbb{X}} s_x(\omega) + f_x(\omega) < \infty$.*

Suppose that ***Ass. IV.1*** *is satisfied, and $\mathbb{P}(d\omega)$-almost surely, it holds*

i) $f_x(\omega) + s_x(\omega) > 0$ *for all* $x \in \mathbb{X}$ *;*

ii) $\mathbb{E}\left[\log^+\left(\sup_{x\in\mathbb{X}} s_x + f_x\right)\right] < \infty$ *;*

then ***Ass. IV.2*** *and* ***Ass. IV.3*** *hold. If moreover*

iii) There exists a deterministic real $A > 0$ *such that* $\mathbb{P}(d\omega)$*-almost surely,* $\sup_{x\leqslant y} \frac{f_y^0}{f_x^0} \leqslant A$*,*

iv) $\mathbb{P}[\sup_{x\in\mathbb{X}} \frac{s_x^0}{f_x^0} < \infty, d'' \circ \theta < \infty] > 0$

then Assumption ***Ass. IV.4*** *holds, and so do the conclusions of Theorem 4.*
Finally, if additionally

v) $\mathbb{P}(d\omega)$*-almost surely,* $\sup_{x\in\mathbb{X}} \frac{s_x}{f_x} < \infty$ *and* $\mathbb{E}\left[\log^+(\sup_{x\in\mathbb{X}} \frac{s_x}{f_x})\right] < \infty$*,*

vi) $\mathbb{P}(d\omega)$*-almost surely,* $d''(\omega) < \infty$ *and* $\mathbb{E}\left|\log d''\right| < \infty$*,*

then M satisfies also Assumption ***Ass. IV.4+****.*

In Proposition IV.16, we've reduced Assumptions **Ass. IV.2** to **Ass. IV.4** to a series of conditions on the law of the coefficients of the random matrix M_0, together with finiteness and integrability conditions on d''. The hardest conditions to check are the ones involving d'', since checking them requires to consider the joint law of all the $M_{0,n}$ and not only M_0. We were not able to find a general sufficient condition for the positivity and log-integrability of d''. However, we provide the following quite restrictive sufficient condition.

Remark IV.17*.* Consider a random, stationary sequence of matrices of the form of equation (IV.52), and assume that there exists an integer $x_0 \in \mathbb{X}$, such that almost surely, the sequence $(s_x(\omega))_{x\geqslant x_0}$ is non increasing. Suppose also that almost surely, for all $x \leqslant x_0$, $s_x > 0$. Then, $\mathbb{P}(d\omega)$-almost surely

$$d'' \leqslant \left(\sup_{i\leqslant x_0} \sup_{x\leqslant y\leqslant x_0} \frac{s_y^i}{s_x^i}\right)^{x_0} < \infty.$$

Moreover, if $\mathbb{E}\left|\log \frac{s_y^i}{s_x^i}\right| < \infty$ for any $x \leqslant y \leqslant x_0$, then $\mathbb{E}|\log d''| < \infty$.

In the context of an age structured population, s_x represents the frequency of individuals of age x surviving to the next time step, and thus being replaced by individuals of age $x+1$. Assuming that $(s_x(\omega))_{x\geqslant x_0}$ is decreasing implies that the older individuals get, the more they tend to die, which is a reasonable assumption. However this condition is somewhat unsatisfying in a more general setting.

We split the proof of Proposition IV.16 into several lemmas that involve different groups of assumptions. Notice first that most quantities involved in Assumptions **Ass. IV.2** to **Ass. IV.4** are explicit in terms of the (f_x, s_x). Indeed :

Lemma IV.18. *Consider a product of stationary random Leslie matrices, in the form of equation* (IV.52)*. Then* ***Ass. IV.2*** *and* ***Ass. IV.3*** *are satisfied if and only if all the following conditions hold simultaneously :*

- $\mathbb{P}(d\omega)$*-almost surely, for each* $x \in \mathbb{X}$, $f_x(\omega) > 0$ *or* $s_x(\omega) > 0$
- $\mathbb{E}\left[\log^+\left(\sup_{x\in\mathbb{X}} f_x + s_x\right)\right] < \infty$.

Proof. This lemma is straightforward after noticing that for any $x \in \mathbb{X}, \omega \in \Omega$,

$$m_{0,1}(x, \omega) = f_x(\omega) + s_x(\omega).$$

□

Moreover, in this model, (IV.4) is well behaved and it is quite clear how to construct a non trivial couple (ν, c).

Lemma IV.19. *Consider a product of stationary, random Leslie matrices and assume that* ***Ass. IV.2*** *holds. Consider a map* $\omega \mapsto (\nu_\omega, c(\omega))$*. Then for almost any* ω *such that* $\nu_\omega \neq \delta_0$, (IV.4) *implies* $c(\omega) = 0$*. If* $\nu_\omega = \delta_0$*, then* (IV.4) *is equivalent to*

$$c(\omega) \leqslant \inf_{x\in\mathbb{X}} \frac{f_x(\omega)}{f_x(\omega) + s_x(\omega)} = \left(1 + \sup_{x\in\mathbb{X}} \frac{s_x(\omega)}{f_x(\omega)}\right)^{-1}.$$

Proof. Notice that for any $x \in \mathbb{X}$, $\omega \in \Omega$,

$$\delta_x M_{0,1} = f_x(\omega)\delta_0 + s_x(\omega)\delta_{x+1}.$$

Let $\omega \in \Omega$ such that $\nu_\omega \neq \delta_0$. Then, there exists $k > 0$ such that $\nu_\omega(k) > 0$. In particular (IV.4) implies

$$0 = f_k(\omega)\delta_0(\mathbb{1}_k) + s_k(\omega)\delta_{k+1}(\mathbb{1}_k) \geqslant c(\omega)m_{0,1}(k)\nu(k),$$

By **Ass. IV.2**, almost surely, $m_{0,1}(k) > 0$, which implies that $c(\omega) = 0$. Conversely, if $\nu_\omega = \delta_0$, (IV.4) implies

$$f_x(\omega)\delta_0 + s_x(\omega)\delta_{x+1} \geqslant c(\omega)m_{0,1}(x)\delta_0,$$

which is equivalent to

$$f_x(\omega) \geqslant c(\omega)m_{0,1}(x) = c(\omega)(f_x(\omega) + s_x(\omega))$$

for all $x \in \mathbb{X}$. This yields the desired result. □

As a consequence, we set from now on $\nu_\omega = \delta_0$ and $c_\omega = \left(1 + \sup_{x\in\mathbb{X}} \frac{s_x(\omega)}{f_x(\omega)}^{-1}\right)^{-1}$. Assumption *iv)* of Proposition IV.16 guarantees that $\mathbb{P}[c > 0] > 0$. Let us try now to exhibit a random variable d such that $(\nu = \delta_0, d)$ satisfy (IV.5).

Lemma IV.20. *Consider a product of stationary random Leslie matrices, of the form of equation* (IV.52)*. Set*

$$d(\omega) = \frac{1}{d' \circ \theta(\omega) d'' \circ \theta(\omega)}.$$

Then (δ_0, d) *satisfy* (IV.5)*.*

Proof. Let $n \geqslant 1, x \in \mathbb{X}$, $\omega \in \Omega$, it holds

$$m_{1,n}(x) = \delta_x M_0 \cdots M_{n-1} \mathbb{1} = \sum_{i_0, i_1, \cdots i_n \in \mathbb{N}_0} \delta_x(i_0) M_1(i_0, i_1) \cdots M_{n-1}(i_{n-1}, i_n).$$

Thus

$$m_{1,n}(x) = \sum_{i_1, \cdots i_n \in \mathbb{N}_0} M_1(x, i_1) \cdots M_{n-1}(i_{n-1}, i_n).$$

Let us rearrange this sum according to the first index $k \leqslant n$ such that $i_k = 0$:

$$\begin{aligned} m_{1,n}(x) = &\sum_{k=1}^{n} \sum_{i_1, \cdots i_{k-1} > 0} M_0(x, i_1) \cdots M_{k-1}(i_{k-1}, 0) \sum_{i_{k+1}, \cdots i_n \in \mathbb{N}_0} M_k(0, i_{k+1}) \cdots M_{n-1}(i_{n-1}, i_n) \\ &+ \sum_{i_1, \cdots i_n > 0} M_0(x, i_1) \cdots M_{n-1}(i_{n-1}, i_n). \end{aligned}$$

Notice that

$$\sum_{i_{k+1}, \cdots i_n \in \mathbb{N}_0} M_k(0, i_{k+1}) \cdots M_{n-1}(i_{n-1}, i_n) = m_{k,n}(0).$$

Moreover, the matrices M_k are shaped according to (IV.52). Thus for any $i \geqslant 0, j > 0$, in order for $M_k(i,j)$ to be non zero, one must have $j = i+1$. Thus :

$$\begin{aligned} m_{0,n}(x) = &\sum_{k=1}^{n} M_0(x, x+1) \cdots M_{k-1}(x+k-1, 0) m_{k,n}(0) \\ &+ M_0(x, x+1) \cdots M_{n-1}(x+n-1, x+n). \end{aligned}$$

Therefore

$$m_{0,n}(x) = \sum_{i=0}^{n-1} s_x^0 s_{x+1}^1 \ldots s_{x+i-1}^{i-1} f_{x+i}^i m_{i+1,n}(0) + s_x^0 \ldots s_{x+n-1}^{n-1}.$$

This is true in particular for $x = 0$:

$$m_{0,n}(0) = \sum_{i=0}^{n-1} s_0^0 s_1^1 \ldots s_{i-1}^{i-1} f_i^i m_{i+1,n}(0) + s_0^0 \ldots s_{n-1}^{n-1}.$$

By definition of d', d'', it holds, for any $k \in \mathbb{N}_0$ and any $x \in \mathbb{N}_0$,

$$f_{x+i}^i \leqslant d' f_i^i,$$

and

$$s_x^0 s_{x+1}^1 \ldots s_{x+i-1}^{i-1} \leqslant d'' s_0^0 s_1^1 \ldots s_{i-1}^{i-1}.$$

Therefore, controlling independently each term of the sum yields

$$m_{0,n}(x) \leqslant d'd'' \sum_{i=0}^{n-1} s_0^0 s_1^1 \dots s_{i-1}^{i-1} f_i^i m_{i+1,n}(0) + d'' s_0^0 \dots s_{n-1}^{n-1} \leqslant d'd'' m_{0,n}(0),$$

thus for all $n \geqslant 0$

$$\frac{1}{d'(\omega)d''(\omega)} \leqslant \inf_{x \in \mathbb{X}} \frac{m_{0,n}(0)}{m_{0,n}(x)}.$$

By stationarity,

$$\frac{1}{d'(\theta(\omega))d''(\theta(\omega))} \leqslant \inf_{n \geqslant 1} \frac{m_{1,n}(0)}{\|m_{1,n}\|_\infty}.$$

As a consequence, setting $d(\omega) = (d'(\theta(\omega))d''(\theta(\omega)))^{-1}$ is enough for $\omega \mapsto (\delta_0, d(\omega))$ to satisfy (IV.5). □

Let us focus on $d'(\omega)$.

Lemma IV.21. *Consider a random product of matrices of the form of equation (IV.52), satisfying* ***Ass. IV.1****. Then the random variable d' is $\mathbb{P}(d\omega)$-almost surely finite if and only if there exists $A > 0$ such that*

$$\mathbb{P}\left[\sup_{x \leqslant y} \frac{f_y^0}{f_x^0} \leqslant A\right] = 1. \tag{IV.53}$$

In this case $d'(\omega) \leqslant A$, $\mathbb{P}(d\omega)$-almost surely. If (IV.53) fails, then $d'(\omega) = +\infty$, $\mathbb{P}(d\omega)$-almost surely.

Proof. Notice that

$$d'(\omega) = \sup_{k \in \mathbb{N}_0} \sup_{x \leqslant y} \frac{f_y^k}{f_x^k} = \sup_{k \in \mathbb{N}_0} X \circ \theta^k(\omega).$$

where

$$X(\omega) = \sup_{x \leqslant y} \frac{f_y(\omega)}{f_x(\omega)} = \sup_{x \leqslant y} \frac{f_y^0}{f_x^0}.$$

Since θ is an ergodic mapping, $\sup_{k \in \mathbb{N}_0} X \circ \theta^k$ is $\mathbb{P}(d\omega)$-almost surely equal to the supremum of the support of X. In particular $\sup_{k \in \mathbb{N}_0} X \circ \theta^k$ is finite almost surely if the support of X is bounded. Conversely, if the support of X is unbounded, then $\sup_{k \in \mathbb{N}_0} X \circ \theta^k$ is infinite almost surely. □

Putting these lemmas together allows to prove Proposition IV.16.

Proof of Proposition IV.16. The assumptions *i), ii)* of Proposition IV.16 are the conditions mentioned in Lemma IV.18. Hence, this lemma allows to check **Ass. IV.2** and **Ass. IV.3**. We set now for any ω, $\nu_\omega = \delta_0$

$$c(\omega) = \left(1 + \sup_{x \in \mathbb{X}} \frac{s_x(\omega)}{f_x(\omega)}\right)^{-1}$$

and

$$d(\omega) = \frac{1}{d'(\theta(\omega))d''(\theta(\omega))}.$$

Lemma IV.19 and Lemma IV.20 respectively guarantee that (IV.4) and (IV.5) are satisfied. By Lemma IV.21, assumption *iii)* guarantees that $d'(\omega) < \infty, \mathbb{P}(d\omega)-$almost surely, and by stationarity, the same holds for $d' \circ \theta(\omega)$. As a consequence, since by *iv)* $\mathbb{P}[d'' \circ \theta > 0, c_0 > 0] > 0$, it holds with positive probability $\gamma(\omega) = c(\omega)d(\theta(\omega)) = c(\omega)\left(d'(\theta(\omega))d''(\theta(\omega))\right)^{-1} > 0$ thus $\mathbb{P}[\gamma > 0] > 0$. This proves that *i)-iv)* imply **Ass. IV.4**. If moreover assumptions *v), vi)* hold, then $d'' \circ \theta$ and $\sup_{x \in \mathbb{X}} \frac{s_x(\omega)}{f_x(\omega)}$ are finite $\mathbb{P}(d\omega)-$a.s., thus $\gamma > 0$ almost surely. In this case it also holds

$$\mathbb{E}|\log \gamma| \leqslant \mathbb{E}\left[-\log c\right] + \mathbb{E}\left[-\log d\right].$$

On the one hand,

$$\mathbb{E}\left[-\log c\right] = \int \log\left(1 + \sup_{x \in \mathbb{X}} \frac{s_x(\omega)}{f_x(\omega)}\right) d\mathbb{P}(\omega).$$

Notice that for any positive real variable X, $\log(1+X)$ is integrable as soon as $\log(X)^+$ is integrable. Since we've assumed that

$$\int \log\left(\sup_{x \in \mathbb{X}} \frac{s_x(\omega)}{f_x(\omega)}\right)^+ d\mathbb{P}(\omega) < \infty,$$

then

$$\mathbb{E}[-\log c] < \infty.$$

On the other hand,

$$\mathbb{E}\left[-\log d\right] \leqslant \mathbb{E} \log d' + \mathbb{E} \log d''.$$

Since $\sup_{x \leqslant y} \frac{f_y^0}{f_x^0} \leqslant A$ almost surely, then by stationarity, almost surely,

$$1 \leqslant d' = \sup_{k \in \mathbb{N}_0} \sup_{y \geqslant x} \frac{f_y^k}{f_x^k} \leqslant A.$$

Thus $\log d'$ is bounded and integrable. We have assumed additionally that $\log d''$ was integrable. This is enough to conclude to the integrability of $|\log \gamma|$, which proves assumption **Ass. IV.4+**. □

IV.5.3 A situation where $\gamma = 0$

It was not clear to us how strong an assumption **Ass. IV.4** is, or whether it was hard to find a system breaking it while satisfying all the other assumptions. We shall present here an example of an infinite Leslie matrix, such that $\gamma = 0$ even if all other assumptions are satisfied. This example is in a deterministic environment, that is $|\mathcal{E}| = 1$, $\Omega = \mathcal{E}^{\mathbb{N}_0}$,

$|\Omega| = 1$. The random matrix $M(\omega)$ is therefore constant, and $M_{0,n} = M^n$. Let us set :

$$M = \begin{pmatrix} \delta m(0) & (1-c)m(0) & 0 & 0 & \cdots \\ \delta m(1) & 0 & (1-c)m(1) & 0 & \cdots \\ \delta m(2) & 0 & 0 & (1-c)m(2) & \ddots \\ \delta m(3) & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{IV.54}$$

where $\delta \in (0,1)$, and $m(x) = \delta m(x) + (1-c)m(x)$ is the mean offspring size of an individual of age x. Such a model satisfies **Ass. IV.2** and **Ass. IV.3**, as soon as $x \mapsto m(x)$ is bounded and positive, since $\delta > 0$. The ergodicity and integrability properties are trivially satisfied since this model is in a constant environment. Moreover, Lemma IV.19 applies, therefore setting $\nu = \delta_0$, and $c = \left(1 + \sup_{x\in\mathbb{X}} \frac{s_x}{f_x}\right)^{-1} = \delta$, the couple (ν, c) satisfies (IV.4). Let us prove that we can tune the parameters $x \mapsto m(x)$ and $c = \delta$ in such a way that the only d satisfying (IV.5) is $d = 0$.

Consider a sequence of integers $(\varepsilon_x)_{x\in\mathbb{N}_0} \in \{0,1\}^{\mathbb{N}_0}$, such that :

- There are arbitrarily long subsequences of consecutive 1 in the sequence (ε_x).
- Noting $S_x = \sum_{k=0}^{x-1} \varepsilon_k$, $\frac{S_x}{x} \longrightarrow 0$ as $x \to \infty$.
- There exists $\alpha < 1$ such that for all $x \in \mathbb{N}_0$, $\frac{S_x}{x} \leqslant \alpha$.

Let a be a real number such that $a > 1$. Then, we set, for any $x \in \mathbb{X}$,

$$m(x) = 1 + (a-1)\varepsilon_x.$$

Defined as such, m is a positive and bounded function, thus Assumptions **Ass. IV.2** and **Ass. IV.3** are satisfied. This yields that for any sequence (x_i),

$$\prod_{i=0}^{n-1} m(x_i) = a^{\sum_{i=0}^{n-1} \varepsilon_{x_i}}.$$

Moreover, since this model is in constant environment, $M_{0,n} = M_{1,n+1} = M^n$, thus

$$\begin{aligned} m_{1,n+1}(x) = m_{0,n}(x) &= \sum_{\substack{x_0 = x, \ldots x_n \in \mathbb{N} \\ x_{i+1} \in \{x_i+1, 0\}}} \delta^{N(x_0,\ldots x_n)} (1-c)^{n-N(x_0,\ldots x_n)} \prod_{i=0}^{n-1} m(x_i) \\ &\geqslant (1-c)^n \prod_{i=0}^{n-1} m(x+i), \end{aligned}$$

where $N(x_0, \ldots x_n) = |\{1 \leqslant i \leqslant n | x_i = 0\}|$. Then

$$m_{0,n}(x) \geqslant (1-c)^n a^{\sum_{i=x}^{x+n-1} \varepsilon_i},$$

In particular, x can be chosen such that $\varepsilon_x = \cdots \varepsilon_{x+n-1} = 1$, which implies that

$$\|m_{0,n}\|_\infty \geqslant m_{0,n}(x) \geqslant (a(1-c))^n.$$

On the other hand

$$m_{0,n}(0) \leqslant 2^n \sup_{\substack{x_0=0,\dots x_n\in\mathbb{N}_0 \\ x_{i+1}\in\{x_i+1,0\}}} \prod_{i=0}^{n-1} m(x_i) \leqslant 2^n a^{\sup_{\substack{x_0=0,\dots x_n\in\mathbb{N}_0 \\ x_{i+1}\in\{x_i+1,0\}}} \sum_{i=0}^{n-1} \varepsilon_{x_i}}.$$

A sequence $(x_i)_{0\leqslant i\leqslant n}$ of integers such that $x_0 = 0$ and for each i, $x_{i+1} \in \{x_i + 1, 0\}$ is entirely determined by the sequence $(T_k)_k$ of the lengths of its excursions away from zero. By convention, if there are only p excursions away from zero, we set T_p such that $T_0 + \cdots + T_p = n$ and $T_{p+1} = \cdots = T_n = 0$. The $(x_{T_0+\cdots+T_{i-1}})_{i\leqslant p}$ are the only zero terms in the sequence $(x_1, \dots x_n)$, and $T_0 + \cdots + T_n \leqslant n - 1$. Thus

$$\sup_{\substack{x_0=0,\dots x_n\in\mathbb{N} \\ x_{i+1}\in\{x_i+1,0\}}} \sum_{i=0}^{n-1} \varepsilon_{x_i} \leqslant \sup_{T_0+\dots T_n=n} \sum_{i=0}^{n} S_{T_i} \leqslant \alpha \sum_{i=0}^{n} T_i \leqslant \alpha n,$$

and

$$m_{0,n}(0) \leqslant (2a^\alpha)^n.$$

Hence

$$\frac{\|m_{1,n+1}\|_\infty}{m_{1,n+1}(0)} = \frac{\|m_{0,n}\|_\infty}{m_{0,n}(0)} \geqslant \left(\frac{a(1-c)}{2a^\alpha}\right)^n = \left(\frac{a^{1-\alpha}(1-c)}{2}\right)^n \underset{n\to\infty}{\longrightarrow} \infty,$$

whenever $\frac{a^{1-\alpha}(1-c)}{2} > 1$. For (IV.5) to hold, we must have $d \leqslant \inf_{n\geqslant 1}\left(\frac{m_{1,n}(0)}{\|m_{1,n}\|_\infty}\right)$. Thus for any values of $\alpha, \delta \in (0,1)$, if a is large enough, then (IV.5) implies $d = 0$.

CHAPITRE V

Un théorème de Kesten-Stigum pour des processus de Galton-Watson multitypes en environnement aléatoire avec un nombre de types infini

Ce chapitre est issu de la prépublication [Lig24]. Il s'agit ici de considérer un processus de Galton-Watson multitype, en environnement aléatoire, avec un nombre de types infini. Le chapitre précédent nous permet de comprendre assez finement le comportement en moyenne quenched, c'est à dire en moyennant les effets démographiques, conditionnellement à une suite d'environnements. Sous des hypothèses de type Doeblin, il permet entre autres de définir une notion de régimes dans lesquels notre processus de Galton-Watson aura en moyenne des comportements différents : la taille moyenne de la population explosera ou tendra vers 0 dans presque toute suite d'environnement, selon la position par rapport à 0 de l'exposant de Lyapunov du produit d'opérateurs associé. On se place ici dans le régime surcritique, où cet exposant de Lyapunov est $\lambda > 0$. Alors la taille moyenne de notre population explose comme $\exp(n\lambda)$ en temps long. On veut ici déduire de cette connaissance de la moyenne du processus des informations sur son comportement réel. Pour cela, on présente un théorème de type Kesten-Stigum, qui contient sous diverses hypothèses de moments

- La présentation d'une martingale positive $(W_n)_{n \geqslant 0}$, obtenue en comptant les individus en vie au temps n avec des pondérations bien choisies, dépendant de leur type et de la suite d'environnements
- Le fait qu'avec probabilité positive, la limite de cette martingale n'est pas nulle. Cela garantit la survie de la population et même l'explosion de sa taille sur un événement de probabilité positive.
- En un certain sens, une convergence presque sûre de la distribution des types dans la population vers une trajectoire de répartitions de types déterminée par l'environnement, et qui ne dépend pas des individus présents au temps 0.

V.1 Introduction

Let $(\mathcal{E}, \mathscr{E})$ be a measurable set and $\bar{\xi} = (\xi_n)_{n\in\mathbb{Z}_+}$ be an ergodic sequence with values in $\mathcal{E}$. We refer to $\mathcal{E}$ as the set of environments and to $(\xi_n)_n$ as the environmental process, and note $\theta : \mathcal{E}^{\mathbb{Z}_+} \mapsto \mathcal{E}^{\mathbb{Z}_+}$ the shift application. Let $(\mathbb{X}, \mathcal{X})$ be another measurable space, such that $\{x\} \in \mathcal{X}$ for any $x \in \mathbb{X}$. We define $\mathfrak{X} = \bigcup_{d\geqslant 0} \mathbb{X}^d$, which we endow with the smallest σ-algebra generated by all the $\mathcal{X}^{\otimes d}$, $d \geqslant 0$. $\mathbb{X}$ represents the possible types of the individuals. We also define the set of point measures

$$\mathcal{N} = \left\{ \sum_{i=1}^{d} \delta_{x_i} \middle| d \geqslant 0, (x_1, \dots, x_n) \in \mathbb{X}^n \right\},$$

and endow it with the σ-algebra generated by the maps $\mu \in \mathcal{N} \mapsto \mu(A) \in \mathbb{Z}_+$, for any $A \in \mathcal{X}$. A population with individuals of various types can either be represented as a tuple of elements of $\mathbb{X}$ (i.e. by an element of $\mathfrak{X}$), or by a point measure, that is an element of $\mathcal{N}$. The map

$$\psi : (x_1, \dots, x_d) \in \mathfrak{X} \mapsto \sum_{i=1}^{d} \delta_{x_i} \in \mathcal{N} \tag{V.1}$$

creates a correspondence between those two representations.

We note $\mathcal{M}_1(\mathfrak{X})$ the set of probability measures on $\mathfrak{X}$, and consider, for each $(x, e) \in \mathbb{X} \times \mathcal{E}$, a probability distribution $\mathcal{L}_{x,e} \in \mathcal{M}(\mathfrak{X})$, which represents the law of the random offpsring of an individual of type x living in the environment e. We consider a probability space Ω, on which are defined both the random process $(\bar{\xi})$ as well as an array $\left(L_{x,e}^{k,n} : x \in \mathbb{X}, e \in \mathcal{E}, k, n \geqslant 0\right)$ of independent random variables with values in $\mathfrak{X}$, independent $\bar{\xi}$, such that $L_{x,e}^{k,n}$ is distributed according to $\mathcal{L}_{x,e}$, for any $(x, e) \in \mathbb{X} \times \mathcal{E}$. We moreover assume that $(x, e, \omega) \in \mathbb{X} \times \mathcal{E} \times \Omega \mapsto L_{x,e}^{k,n}(\omega)$ is measurable for all k, n. Finally, we note $N_{x,e}^{k,n} = \psi\left(L_{x,e}^{k,n}\right)$, and define for each $N \geqslant 0$ the σ-algebra $\mathcal{F}_N$ generated by the variables $(L_{x,e}^{k,n})_{x\in\mathbb{X}, e\in\mathcal{E}, k\geqslant 0, 0\leqslant n\leqslant N-1}$.

Definition V.1. *We call multitype Galton-Watson process (Z_n) in ergodic environment $\bar{\xi} = (\xi_n)_{n\geqslant 0}$ with initial population $Z_0 \in \mathcal{N}$ an $\mathcal{N}$-valued process defined through the induction relation*

$$Z_{n+1} = \sum_{x\in\mathbb{X}} \sum_{k=1}^{Z_n(\{x\})} N_{x,\xi_n}^{k,n}. \tag{V.2}$$

For short, we note such a process MGWRE in the rest of the article. As often when studying random processes in random environment, we note $\mathbb{P}$ the annealed law, that is the joint law of $\bar{\xi}$ and (Z_n), and $\mathbb{P}_{\bar{\xi}}$ the quenched law of (Z_n), obtained by conditioning $\mathbb{P}$ on the environmental process $\bar{\xi}$.

We define the quenched first moment operators $M_{k,n}$ by setting

$$M_{k,n}(f)(x) = \mathbb{E}[Z_n(f)|\bar{\xi}, Z_k = \delta_x] = \mathbb{E}_{\bar{\xi}}\left[Z_n(f)|Z_k = \delta_x\right],$$

for any $k \leqslant n$, $x \in \mathbb{X}$ and any measurable bounded function f on $\mathbb{X}$, as well as

$$\mu M_{k,n}(f) = \int M_{k,n}(f)(x)\mu(dx)$$

for any signed measure μ on $\mathbb{X}$ and any measurable bounded function f on $\mathbb{X}$. The operators $(M_{k,n})_{k\leqslant n}$ act therefore both on the space of bounded measurable functions $\mathcal{B}(\mathbb{X})$, endowed with the supremum norm $\|\cdot\|_\infty$, and on the space of signed measures $\mathcal{M}(\mathbb{X})$, endowed with the total variation norm $\|\cdot\|_{TV}$. They satisfy the semigroup relation

$$M_{k,N} = M_{k,n}M_{n,N}$$

for any $k \leqslant n \leqslant N$, and noting $M_n = M_{n,n+1}$, it holds

$$M_{k,n} = M_k \cdots M_{n-1}.$$

Note that $(M_n)_{n\geqslant 0}$ are random operators, since they depend on the random environmental process $\bar{\xi}$. More precisely, the random operator $M_{k,n}$ is $\sigma(\xi_k, \dots \xi_{n-1})$ measurable, and $(M_n)_n$ is an ergodic sequence of operators. We note $m_{k,n}(x) = M_{k,n}(\mathbb{1})(x)$ for any $x \in \mathbb{X}$ and $k \leqslant n$.

V.1.1 Basic assumptions

In the sequel, we work under the following assumptions from Chapter IV which corresponds to the article [Lig25].

Ass. V.1. For almost all $e \in \mathcal{E}$, the function $x \mapsto m_{0,1}(x,e)$ is bounded and positive.

By stationarity, **Ass. V.1** implies that $\mathbb{P}$-almost surely, the product $M_{k,n}$ is a continuous, nonzero, positive linear operator.

Ass. V.2. $\mathbb{E}\left[\log \|m_{0,1}\|_\infty\right] = \mathbb{E}\log\left(|||M_{0,1}|||\right) < \infty$,

where $|||\cdot|||$ is the operator norm defined as $|||M||| = \sup_{x\in\mathbb{X}}\sup_{f\in\mathcal{B}(\mathbb{X}),\|f\|_\infty=1}|M(f)(x)| = \|M\mathbb{1}\|_\infty$. In particular, the submultiplicativity of the norm $|||\cdot|||$, the ergodicity of $(\xi_n)_n$ and **Ass. V.2** imply that $\mathbb{E}\log^+\left(|||M_{k,n}|||\right) < \infty$ for all $k \leqslant n$. As a consequence of **Ass. V.2**, one defines the Lyapunov exponent λ of the random sequence of operators $(M_n)_{n\geqslant 0}$ as

$$\lambda = \inf_{n\geqslant 1} n^{-1}\mathbb{E}\left[\log |||M_{0,n}|||\right] \in [-\infty, +\infty)$$

and prove, using submultiplcativity arguments, that almost surely $\lim_{n\to\infty}\left(|||M_{0,n}|||\right)^{\frac{1}{n}} = e^\lambda$.

We call admissible coupling constants (or admissible triplet for short) a measurable map $(\nu, c, d) : \bar{e} = (e_n)_{n\geqslant 0} \in \mathcal{E}^{\mathbb{N}} \mapsto (\nu_{\bar{e}}, c(\bar{e}), d(\bar{e})) \in \mathcal{M}_1(\mathbb{X}) \times [0,1]^2$ such that, $\mathbb{P}(d\bar{\xi})$-a.s.

i) for all $x \in \mathbb{X}$ and all $f \in \mathcal{B}_+(\mathbb{X})$, the couple $(\nu_{\bar{\xi}}, c(\bar{\xi}))$ satisfies

$$\delta_x M_{0,1}(f) \geqslant c(\bar{\xi})\|\delta_x M_{0,1}\|\nu_{\bar{\xi}}(f) \tag{V.3}$$

ii) for all $n \geqslant 0$, the couple $(\nu_{\bar{\xi}}, d(\bar{\xi}))$ satisfies

$$\nu_{\bar{\xi}}(m_{1,n}) \geqslant d(\bar{\xi})\|\!|M_{1,n}\|\!| \tag{V.4}$$

When an admissible triplet is defined, we define the random variable

$$\gamma = c(\bar{\xi})d(\bar{\xi}).$$

Note that taking $c = d = 0$ and any measurable map $\bar{e} \mapsto \nu_{\bar{e}}$ defines an admissible triplet, however in this case $\gamma = 0$. Our main assumption, which prevents this trivial situation, is therefore

Ass. V.3. There exists a triplet of admissible coupling constants $\bar{e} \mapsto (\nu_{\bar{e}}, c(\bar{e}), d(\bar{e}))$ such that $\mathbb{E}[-\log(\gamma)] < \infty$.

We introduce

$$\tilde{\eta} = \exp(\mathbb{E}[\log(1-\gamma)]) \in [0, 1)$$

and notice that under **Ass. V.3**, we have $0 \leqslant \tilde{\eta} < 1$. As a consequence of results from Chapter IV, i.e. [Lig25], assumptions **Ass. V.1** to **Ass. V.3** guarantee that the semigroup $(M_{k,n})$ satisfies some geometric ergodicity properties at rate $\tilde{\eta}$ and a law of large numbers in the form $\lim_{n\to\infty} (\|\mu M_{0,n}\|_{TV})^{\frac{1}{n}} = e^{\lambda}$. Moreover, these assumptions guarantee the almost sure existence of

$$h_k(x) = \lim_{n\to\infty} \frac{m_{k,n}(x)}{\|m_{k,n}\|} \in (0, 1]$$

for all $x \in \mathbb{X}$. We provide in Section V.2, Theorem 10 a more complete and precise statement of the results from Chapter IV that we use here.
The Lyapunov exponent e^{λ} governs the asymptotic quenched size of the population. It can be used to extend the classification of Galton-Watson processes into three regimes to the case of MGWREs. When $e^{\lambda} < 1$ (in which case we say that the process is subcritical), it holds $\lim_{n\to+\infty} \|\mu M_{0,n}\| = 0$ a.s. for any measure μ. Under some additional assumptions, when $e^{\lambda} = 1$ (that is, in the critical regime), it is proven in Chapter IV that $\liminf_{n\to+\infty} \|\mu M_{0,n}\| = 0, \mathbb{P}(d\bar{\xi})$-a.s. A classical first moment argument guarantees that the property $\liminf_{n\to+\infty} \|\mu M_{0,n}\| = 0, \mathbb{P}(d\bar{\xi})$-a.s is sufficient for the extinction event $\mathbf{Ext} = \{\exists n \geqslant 0, Z_n = 0\}$ to have probability 1. Therefore extinction is a.s. in the subcritical and the critical regime for an MGWRE. In our paper, we rather focus on the supercritical case and assume

Ass. V.4 (Supercriticality)**.** $\lambda > 0$.

We now present our results.

V.1.2 Results

In the supercritical case $\lambda > 0$, one expects that the population survives on an event of positive probability. However, when studying MGWREs with infinitely many types this is not as easy to prove as when $\mathbb{X}$ is finite, and is, to the best of our knowledge, not proven in full generality. In this article, we obtain in particular, under some uniform moment assumption, the survival of the process in the supercritical case, as well as a description of the surviving population, in the form of on a Kesten-Stigum type theorem. The original Kesten-Stigum result was proven in [KNS66] and deals with Multitype Galton-Watson processes with finitely many types, in a constant environment, an alternative proof based on a classical change of measure which inspires some of our techniques is presented in [LPP95]. Many similar results have been developed since on various random models. For example, in the case of general time homogeneous branching processes with infinitely many types, similar results appear in [AH76; Ath00]. [EHK10; LS20; Med24] also present related results on some branching models, respectively branching diffusions, branching brownian motion with absorbtion, and some continuous space branching process with interactions. Closer to our context, Kesten-Stigum theorems on multitype Galton-Watson processes in random environment with finitely many types have recently been established in [GLP23; GLN24a]. Roughly speaking, these results deal with four problems :

- Exhibiting a martingale of the form $W_n = Z_n(f_n)/r_{0,n}$ where $f_n \in \mathcal{B}_+(\mathbb{X})$ and $r_{0,n} \in \mathbb{R}_+$ are $\bar{\xi}$-measurable (deterministic in constant environment)
- Under some $L\log(L)$-integrability condition on the offspring distribution, proving that the almost sure limit $W := \lim_{n\to\infty} W_n$ is not degenerate, in the sense that $\mathbb{P}[W > 0] > 0$ (in random environment, this statement typically becomes $\mathbb{P}_{\bar{\xi}}(W > 0) > 0, \mathbb{P}(d\bar{\xi})$-a.s.).
- Proving that the extinction event $\bigcup_{n\geqslant 0}\{Z_n = 0\}$ coincides with the event $\{W = 0\}$
- Obtaining some almost sure asymptotic estimate on Z_n describing the distribution of the types in the population conditionally on the event $\{W > 0\}$.

We adress those four topics in this article. Our respective contributions regarding each of these points are described in following four subsubsections.

V.1.2.1 The fundamental martingale

Our first result generalizes the martingale obtained in [GLP23] to the case of an infinite type set $\mathbb{X}$. For all $k \geqslant 0$, we put $\lambda_k = \|M_k h_{k+1}\|_\infty$ and $\lambda_{k,n} = \lambda_k \dots \lambda_{n-1}$ for all $k \leqslant n$. Moreover, we set

$$W_n = \frac{Z_n(h_n)}{\lambda_{0,n} Z_0(h_0)}.$$

Our first main result is

Theorem 7 (The fundamental martingale)**.** *Consider a MGWRE satisfying assumptions* ***Ass. V.1*** *to* ***Ass. V.3****. Then, conditionally on the environmental process $(\xi_n)_{n\geqslant 0}$, the process (W_n) is a $(\mathcal{F}_n)_{n\geqslant 0}$-martingale.*

The main argument behind this theorem is the space-time harmonicity of the family (h_k), in the sense that $M_k h_{k+1} = \lambda_k h_k$ for all k. This is the core of Lemma V.6, stated

in Subsection V.3.1.
When $\mathbb{X}$ is finite, the counterpart of h_k is the limit v_k of the sequence of right dominant eigenvectors $(v_{k,n})_{n \geqslant k}$ of the matrices $(M_{k,n})_{n \geqslant k}$. As a consequence, our martingale W_n is similar to the one obtained in finite dimension by [GLP23].
Since the sequence (λ_n) is ergodic, it is straightforward from Birkhoff's ergodic theorem that $(\lambda_{0,n})^{\frac{1}{n}}$ converges almost surely to a constant. We additionally prove in Subsection V.3.1, Prop V.7 that this limit is actually e^{λ}.

V.1.2.2 Non degeneracy of the martingale

By Theorem 7, the process (W_n) is a positive martingale, therefore there exists a random variable $W \in [0, \infty)$ such that it holds

$$W_n \underset{n \to \infty}{\longrightarrow} W \quad \mathbb{P} - \text{a.s.}$$

We wonder now whether $W = 0$ almost surely or $\{W > 0\}$ with positive probability. To perform this study of W, we introduce the assumption

Ass. V.5. There exists a nondecreasing and positive function f such that $\frac{1}{x \log(x) f(x)}$ is integrable at $+\infty$ and

$$\mathbb{E}\left[\sup_{x \in \mathbb{X}} \frac{\mathbb{E}_{\bar{\xi}}\left[Z_1(h_1) \log[Z_1(h_1)] f(Z_1(h_1)) | Z_0 = \delta_x\right]}{M_0 h_1(x)}\right] < \infty.$$

Note that the functions $f(t) = \log^+(t)^{\varepsilon}$, for any $\varepsilon > 0$ satisfy the integrability assumption $\int^{+\infty} \frac{dx}{x \log(x) f(x)} < +\infty$. Let us now state our main result dealing with the limit of the martingale.

Theorem 8 (Non-degeneracy of the fundamental martingale)**.** *Consider a MGWRE satisfying assumptions* **Ass. V.1** *to* **Ass. V.5***. For any initial population $Z_0 \in \mathcal{N}$, it holds*

$$\mathbb{P}_{\bar{\xi}}[W > 0 | Z_0] > 0 \text{ and } \mathbb{E}_{\bar{\xi}}[W | Z_0] = 1 \quad \mathbb{P}(d\bar{\xi})\text{-a.s.}$$

Since $W = 0$ on the extinction event, our result implies in particular that the population survives with positive probability. The non degeneracy of (W_n) usually requires a $L \log L$ moment on the offspring distribution, see [KNS66; LPP95] for example. Our assumption **Ass. V.5** adapts this condition to our context. Indeed, setting $L = Z_1(h_1)$, **Ass. V.5** can be rewritten

$$\mathbb{E}\left[\sup_{x \in \mathbb{X}} \frac{\mathbb{E}_{\bar{\xi}}[L \log^+(L) f(L) | Z_0 = \delta_x]}{\mathbb{E}_{\bar{\xi}}\left[L | Z_0 = \delta_x\right]}\right] < \infty.$$

Remark V.2. When $\mathbb{X}$ is finite, the supremum over the type space $\mathbb{X}$ can be dominated by a sum over all the elements of $\mathbb{X}$, thus condition **Ass. V.5** reduces further to

$$\mathbb{E}\left[\frac{L \log^+(L) f(L)}{\lambda_0 h_0(x)} \middle| Z_0 = \delta_x\right] < \infty \tag{V.5}$$

for any $x \in \mathbb{X}$. This condition is very close to condition H6 of [GLP23]. Indeed, this condition H6 is also an integrability condition on the quantity $L = Z_1(h_1)$, which consists in giving a weigh $h_1(x)$ to individuals of type x alive at time 1. The only difference between (V.5) and condition H6 in [GLP23] is the multiplicative term $f(L)$, which is only a very slight strengthening of the condition since one can choose for example $f(t) = \log^+(t)^\varepsilon$.

In the case of an infinite type set $\mathbb{X}$, the presence of the supremum over $x \in \mathbb{X}$ is the main difficulty in Assumption **Ass. V.5**. We introduce some examples of offspring distributions with various decay rates which satisfy **Ass. V.5** in Subsection V.3.2.

V.1.2.3 Asymptotic type distribution

Our next result deals with the asymptotic type distribution in the population, in the form of almost sure asymptotic estimates of $Z_n(f)$, for a large classe of functions f. It relies on the following uniform second moment assumption :

Ass. V.6.

$$\mathbb{E}\left[\log^+\left(\sup_{x\in\mathbb{X}} \mathbb{E}_{\bar{\xi}}\left[\,\|Z_1\|^2 \middle|\, Z_0 = \delta_x\right]\right)\right] < \infty.$$

Before stating the theorem, we introduce the notation $\log^-(x) = -\min(\log(x), 0) > 0$, so that $\log(x) = \log^+(x) - \log^-(x)$.

Theorem 9. *Assume **Ass. V.1** to **Ass. V.6**. Then*

i) For any $f \in \mathcal{B}(\mathbb{X})$, for any $\varepsilon > 0$, it holds

$$Z_n(f) = W_n Z_0 M_{0,n}(f) + \underset{n\to+\infty}{o}\left(\left(\max\left(\tilde{\eta}e^{\lambda}, e^{\frac{\lambda}{2}}\right) + \varepsilon\right)^n\right) \quad \mathbb{P}\text{-}a.s. \tag{V.6}$$

ii) For any $f \in \mathcal{B}(\mathbb{X})$, $f \geqslant 0$, if $\mathbb{E}\left[\log^-(\nu M_{1,N} f)\right] < +\infty$ for some $N \geqslant 1$ then

$$\lim_{n\to\infty} \frac{Z_n(f)}{Z_0 M_{0,n}(f)} = W \quad \mathbb{P}\text{-}a.s \tag{V.7}$$

and moreover, conditionally on $\{W > 0\}$,

$$\lim_{n\to\infty} (Z_n(f))^{\frac{1}{n}} = e^{\lambda}. \quad \mathbb{P}\text{-}a.s.$$

Note that (V.7) only derives from (V.6) for functions f such that

$$\liminf_{n\to\infty} (Z_0 M_{0,n}(f))^{\frac{1}{n}} > \max\left(\tilde{\eta}e^{\lambda}, e^{\frac{\lambda}{2}}\right). \tag{V.8}$$

(V.8) is a straightforward consequence of the properties of the Lyapunov exponnent λ if $f > c > 0$ for some constant c, however we do not see a reason for it to hold for a general bounded function f. In Subsection V.4.4, we focus on the example of a GWRE with countably many types representing a population structured in age and prove that under some mild integrability assumption on the mean matrix, any nonnegative and nonzero function f actually satisfies the assumption $\mathbb{E}\left[\log^-(\nu M_{1,N} f)\right] < +\infty$ and thus (V.7). Theorem 9 allows us to describe both the size and the distribution of types within the surviving population on the event $\{W > 0\}$, as stated in the following corollary.

Corollary V.3. *Assume **Ass. V.1** to **Ass. V.6**. Then $\mathbb{P}$-a.s., conditionally on $\{W > 0\}$, it holds*

$$Z_n(\mathbb{1}) \underset{n\to\infty}{\sim} W_n \|Z_0 M_{0,n}\|_{TV} \text{ and } \lim_{n\to\infty} (Z_n(\mathbb{1}))^{\frac{1}{n}} = e^{\lambda}. \tag{V.9}$$

Additionally, for any $f \in \mathcal{B}(\mathbb{X})$, conditionally on $\{W > 0\}$, it holds

$$\lim_{n\to\infty} \frac{Z_n(f)}{Z_n(\mathbb{1})} - \frac{Z_0 M_{0,n}(f)}{Z_0 M_{0,n}(\mathbb{1})} = 0 \quad \mathbb{P}\text{-}a.s. \tag{V.10}$$

Claim (V.9) is obtained by plugging $f = \mathbb{1}$ in the previous Theorem. It proves that the population sizes grows almost surely at rate e^{λ} on the event $\{W > 0\}$. Moreover, applying the Theorem to various test functions, we obtain (V.10), which provides insight on the asymptotic distribution of types in the population. As n is large, on the event $\{W > 0\}$, the respective frequencies of any given type in the population Z_n and in the quenched mean $\mathbb{E}_{\bar{\xi}}[Z_n|Z_0]$ become similar.

V.1.2.4 Extinction and explosion

It is common when studying the non degeneracy of the martingale (W_n) of a Galton-Watson process to prove additionally that the non-explosion event $\mathbf{NonExp} = \{W = 0\}$ coincides with the extinction event $\mathbf{Ext} = \{\lim \|Z_n\| = 0\}$ up to a negligible event. In the single-type case, or when studying a multitype GW process with a finite number of types, in constant environment, this relies on noticing that the probability of extinction $\mathbb{P}[\mathbf{Ext}]$ and the probability of non-explosion $\mathbb{P}[\mathbf{NonExp}]$ both are a fixed point of the generating function of the reproduction law. A convexity argument allows to show then that $\mathbb{P}[\mathbf{Ext}] = \mathbb{P}[\mathbf{NonExp}]$. The obvious inclusion $\mathbf{Ext} \subset \mathbf{NonExp}$ concludes the proof. This method can be adapted in ergodic environment, with a variant on the fixed point argument, see [GLP23, p.27] and [Kap74, Prop 3.1]. However, when considering an infinite type set, and even in fixed environment, the generating function might have several fixed points, see e.g. [Bra18; BH19; BDH19]. Therefore we do not expect that **Ext** and **NonExp** coincide in general when $\mathbb{X}$ is infinite. We are however able to prove that they do coincide when the population of a specific type x_0 is unbounded a.s. on survival.

Proposition V.4. *Let (Z_n) be a MGWRE. We assume that **Ass. V.1**-**Ass. V.5** hold, that the environmental sequence $\bar{\xi}$ is i.i.d and that there exists a type $x_0 \in \mathbb{X}$ such that $\mathbb{P}$-a.s, conditionally on the survival event, for any initial population Z_0,*

$$\limsup_{n\to\infty} Z_n(\{x_0\}) = +\infty. \tag{V.11}$$

Then, for any initial population Z_0, the events $\mathbf{Ext} = \{\exists n \geqslant 0, Z_n = 0\}$ *and $\{W = 0\}$ coincide up to a $\mathbb{P}$-negligible event.*

We provide in subsection V.4.5 a specific example where assumption (V.11) holds.

V.1.3 Structure of the paper

In section V.2, we introduce some preliminary results on the quenched mean of the process from a previous article, which are useful in the rest of the paper. Theorem 7 is proved in subsection V.3.1, Theorem 8 is proved in subsection V.3.2, Theorem 9 is proved in subsection V.3.3 and Proposition V.4 is proved in subsection V.3.4. Finally, in Section V.4, we introduce an example of a GWRE with countably many types, representing a population structured in age. This example is an interesting toy model on which we apply our result and discuss our assumptions, in particular Assumption **Ass. V.5** and criterion (V.11).

V.2 Preliminaries : behavior of the quenched mean of the process

As often when studying Galton-Watson processes in random environment, a prerequisite is a good understanding of the quenched mean of the process. In our case, this quenched mean is the random semigroup of nonnegative operators $(M_{k,n})$, which act on $\mathcal{M}(\mathbb{X})$ on the left and $\mathcal{B}(\mathbb{X})$ on the right. We use the assumptions and notations of Chapter IV. In particular, we recall that (ν, c, d) is an admissible triplet, that $\gamma = c(\bar{\xi})d(\bar{\xi})$ and that θ refers to the shift mapping on $\mathcal{E}^{\mathbb{N}}$. We introduce the random variables

$$c_n = c(\theta^n(\bar{\xi})), d_{n+1} = d(\theta^n(\bar{\xi})), \gamma_n = c_n d_{n+1}, \nu_n = \nu_{\theta^n(\bar{\xi})}.$$

in such a way that

$$\nu_0 = \nu, c_0 = c, d_1 = d, \gamma_0 = \gamma.$$

The main result we rely on in this Chapter, which was proven in Chapter IV, i.e. [Lig25], is the following.

Theorem 10 ([Lig25])**.** Under assumptions **Ass. V.1** to **Ass. V.3**, the semigroup $(M_{k,n})_{k\leqslant n}$ satisfies, $\mathbb{P}(d\bar{\xi})$-a.s.,

i) For any $k \geqslant 0$, the sequence $\left(\frac{m_{k,n}}{\|m_{k,n}\|_\infty}\right)_{n\geqslant k}$ converges uniformly on $\mathbb{X}$ to some random limit $h_k \in \mathcal{B}(\mathbb{X})$.

ii) For any $\eta > \tilde{\eta}$, for any $n \geqslant k \geqslant 0$ and any $\mu_1, \mu_2 \in \mathcal{M}_+(\mathbb{X})$,

$$\left\|\mu_1 M_{k,n} - \frac{\mu_1(h_k)}{\mu_2(h_k)}\mu_2 M_{k,n}\right\|_{TV} \leqslant \frac{4}{\gamma_{n-1}}\prod_{i=k}^{n-1}(1-\gamma_i)\|\mu_1 M_{k,n}\|_{TV} \underset{n\to\infty}{=} O(\eta^n \|\mu_1 M_{k,n}\|_{TV}). \tag{V.12}$$

iii) For any finite, positive and non-zero measure μ on $\mathbb{X}$,

$$\lim_{n\to\infty}\frac{1}{n}\log\|\mu M_{0,n}\|_{TV} = \lim_{n\to\infty}\frac{1}{n}\log|||M_{0,n}||| = \inf_{n\geqslant 1}\frac{1}{n}\mathbb{E}\left[\log|||M_{0,n}|||\right] = \lambda \in [-\infty,\infty). \tag{V.13}$$

This provides a notion of ergodicity for the first moment semigroup $(M_{k,n})_{k\leqslant n}$, in the sense that the measure $\mu_1 M_{k,n}$ asymptotically only depends on μ_1 through a multiplicative factor $\mu_1(h_k)$. It generalizes both Perron-Frobenius' theorem and previous results of [Hen97] dealing with products of random positive $d\times d$ matrices to the case of products of random infinite dimensional operators. In particular, by *ii)*, choosing an arbitrary measure $\pi_0 \in \mathcal{M}_+(\mathbb{X})$ and setting $\pi_n = \frac{\pi_0 M_{0,n}}{\|\pi_0 M_{0,n}\|_{TV}}$ defines a sequence of random probability measures such that $\mathbb{P}(d\bar{\xi})$-a.s., for any $\mu \in \mathcal{M}_+(\mathbb{X})$ and any $\eta \in (\tilde{\eta}, 1)$, it holds

$$\begin{aligned}\left\| \mu M_{k,n} - \frac{\mu(h_k)}{\pi_0 M_{0,k} h_k} \|\pi_0 M_{0,n}\|_{TV} \pi_n \right\|_{TV} &\leqslant \frac{4}{\gamma_{n-1}} \prod_{i=k}^{n-1} (1-\gamma_i) \|\mu M_{k,n}\|_{TV} \qquad \text{(V.14)}\\ &= \underset{n\to\infty}{O} (\eta^n \|\pi_0 M_{0,n}\|_{TV}).\end{aligned}$$

In the context of multitype Galton-Watson processes, Theorem 10 thus implies that, as $n \to \infty$,

- the size of the quenched mean of the population $\|\mu M_{k,n}\|_{TV} = \left\|\mathbb{E}[Z_n | Z_k = \mu, \bar{\xi}]\right\|_{TV}$ is close to $\|\pi_0 M_{0,n}\|_{TV}$, up to a multiplicative factor which is constant in time and depends linearly on μ,
- the quantity $\|\pi_0 M_{0,n}\|_{TV}$ is independent of μ and is of order $e^{n\lambda}$ for large n, up to a subexponential factor,
- the direction $\frac{\mu M_{k,n}}{\|\mu M_{k,n}\|_{TV}}$ of the measure $\mu M_{k,n}$, which represents the distribution of the types in the mean population at time n, is close to the probability π_n. Asymptotically, it does therefore not depend on μ.

As a consequence, for any initial measure Z_0, any fonction $f \in \mathcal{B}(\mathbb{X})$ and any $\varepsilon > 0$, it holds $\mathbb{P}(d\bar{\xi})$-a.s.

$$Z_0 M_{0,n}(f) = \frac{Z_0(h_0)}{\pi_0(h_0)} \pi_0 M_{0,n}(f) + o\left((\tilde{\eta} e^{\lambda} + \varepsilon)^n\right).$$

In particular, this yields the following corollary of Theorem 9.

Corollary V.5. *Under assumptions **Ass. V.1** to **Ass. V.6**, for any $f \in \mathcal{B}(\mathbb{X})$, it holds for any $\varepsilon > 0$,*

$$Z_n(f) = W_n \frac{Z_0(h_0)}{\pi_0(h_0)} \pi_0 M_{0,n}(f) + o\left(\left(\max(\tilde{\eta} e^{\lambda}, e^{\frac{\lambda}{2}}) + \varepsilon\right)^n\right) \quad \mathbb{P}\text{-a.s.}$$

In Chapter IV, it is additionally proven that the random sequence of probability measures $\left(\frac{\pi_0 M_{0,n}}{\|\pi_0 M_{0,n}\|_{TV}}\right)_{n\geqslant 0}$ converges weakly to a random measure Λ on the space $\mathcal{M}_1(\mathbb{X})$.

V.3 Proofs

V.3.1 The fundamental martingale

The main reason for which (W_n) is a martingale is the fact that the sequence (h_k) constitutes a family of space time harmonic functions, in the sense of [BCN99].

Lemma V.6. *It holds $M_k h_{k+1} = \lambda_k h_k$ $\mathbb{P}(d\bar{\xi})$-a.s. for all $k \geqslant 0$.*

Proof. Let us fix $k \geqslant 1$. Applying (V.12) to the constant function $\mathbb{1}$, one proves that almost surely, for any measures μ_1, μ_2, it holds

$$\mu_1(m_{k,n}) \underset{n\to\infty}{\sim} \frac{\mu_1(h_k)}{\mu_2(h_k)} \mu_2(m_{k,n}).$$

Therefore,

$$\frac{\mu_1(h_k)}{\mu_2(h_k)} = \lim_{n\to\infty} \frac{\mu_1(m_{k,n})}{\mu_2(h_{k,n})}.$$

As a consequence, applying the previous results to $\mu_1 M_{k-1}$ and $\mu_2 M_{k-1}$, we obtain

$$\frac{\mu_1 M_{k-1} h_k}{\mu_2 M_{k-1} h_k} = \lim_{n\to\infty} \frac{\mu_1(m_{k-1,n})}{\mu_2(h_{k-1,n})} = \frac{\mu_1(h_{k-1})}{\mu_2(h_{k-1})}.$$

Thus with $\mu_1 = \delta_x$ and an arbitrary $\mu_2 \neq 0$, we derive

$$M_{k-1} h_k(x) = h_{k-1}(x) \frac{\mu_2 M_{k-1} h_k}{\mu_2(h_{k-1})}.$$

The functions $M_{k-1} h_k$ and h_{k-1} are therefore $\mathbb{P}(d\bar{\xi})$-a.s. colinear. Since $\|h_k\|_\infty = 1$, letting $\lambda_{k-1} = \|M_{k-1} h_k\|_\infty \in (0, \infty)$ yields $M_{k-1} h_k = \lambda_{k-1} h_{k-1}$ as expected. $\square$

Proof of Theorem 7. To prove that (W_n) is a martingale, we must compute

$$\mathbb{E}\left[W_{n+1} | (\xi_k)_{k\geqslant 0}, \mathcal{F}_n\right] = \mathbb{E}\left[\frac{Z_{n+1}(h_{n+1})}{\lambda_{0,n+1}} \middle| \bar{\xi}, Z_n \right].$$

Notice that h_{n+1} and the coefficients $\lambda_0, \cdots, \lambda_n$ are measurable with respect to the environmental process (ξ_k). Then

$$\begin{aligned}
\mathbb{E}\left[W_{n+1} | \bar{\xi}, \mathcal{F}_n\right] &= \frac{\mathbb{E}\left[Z_{n+1}(h_{n+1}) | \bar{\xi}, Z_n \right]}{\lambda_{0,n+1} Z_0(h_0)} \\
&= \frac{\sum_{x\in\mathbb{X}} \sum_{k=1}^{Z_n(\mathbb{1}_x)} \mathbb{E}\left[N_{x,\xi_n}^{k,n}(h_{n+1}) \middle| \bar{\xi}, Z_n \right]}{\lambda_{0,n+1} Z_0(h_0)} \\
&= \frac{\sum_{x\in\mathbb{X}} \sum_{k=1}^{Z_n(\mathbb{1}_x)} M_n h_{n+1}(x)}{\lambda_{0,n+1} Z_0(h_0)}. \\
&= \frac{Z_n M_n h_{n+1}}{\lambda_{0,n+1} Z_0(h_0)}
\end{aligned} \tag{V.15}$$

By Lemma V.6, this yields

$$\mathbb{E}\left[W_{n+1} | \bar{\xi}, \mathcal{F}_n\right] = \frac{\lambda_n Z_n(h_n)}{\lambda_{0,n+1} Z_0(h_0)} = \frac{Z_n(h_n)}{\lambda_{0,n} Z_0(h_0)} = W_n.$$

$\square$

We provide additionally the following estimate

Proposition V.7. *Under assumptions* ***Ass. V.1*** *to* ***Ass. V.3****, it holds* $\lim_{n\to\infty}(\lambda_{k,n})^{\frac{1}{n}} = e^{\lambda}\quad \mathbb{P}(d\bar{\xi})$*-a.s. for any* $k \geqslant 0$*.*

Proof. Iterating the relation $M_k h_{k+1} = \lambda_k h_k$, proven in Lemma V.6, we get for any $k \leqslant n$

$$M_{k,n}h_n = \lambda_{k,n}h_k.$$

Thus $\lambda_{k,n} = |||M_{k,n}h_n|||$. Since $\|h_n\|_\infty = 1$, this proves that $\lambda_{k,n} \leqslant |||M_{k,n}|||$. By *i)* of Theorem 10, the function h_n is the uniform limit of the sequence $\left(\frac{m_{n,n+p}}{\|m_{n,n+p}\|_\infty}\right)$, as $p \to \infty$. Therefore

$$\lambda_{k,n} = \|M_{k,n}h_n\|_\infty = \left\|M_{k,n}\lim_{p\to\infty}\frac{m_{n,n+p}}{\|m_{n,n+p}\|}\right\| = \lim_{p\to\infty}\left\|\frac{M_{k,n}m_{n,n+p}}{|||M_{n,n+p}|||}\right\|_\infty = \lim_{p\to\infty}\frac{|||M_{k,n+p}|||}{|||M_{n,n+p}|||}.$$

Additionally, if $0 \leqslant k \leqslant n-1$ then for $p \geqslant 0$,

$$M_{k,n+p} = M_{k,n-1}M_{n-1}M_{n,n+p} \geqslant m_{k,n}c_{n-1}\nu_{n-1}M_{n,n+p}.$$

Applying this to the function $\mathbb{1}$, and using the definition of d_n, we get

$$m_{k,n+p} \geqslant m_{k,n}c_{n-1}\nu_{n-1}(m_{n,n+p}) \geqslant m_{k,n}c_{n-1}d_n\nu_{n-1}|||M_{n,n+p}|||.$$

Taking now the supremum norm yields

$$|||M_{k,n+p}||| = \|m_{k,n+p}\|_\infty \geqslant \gamma_{n-1}|||M_{k,n}|||\,|||M_{n,n+p}|||. \tag{V.16}$$

Thus

$$\lambda_{k,n} = \lim_{p\to\infty}\frac{|||M_{k,n+p}|||}{|||M_{n,n+p}|||} \geqslant \gamma_{n-1}|||M_{k,n}|||.$$

We therefore have, for $k < n$

$$\gamma_{n-1}|||M_{k,n}||| \leqslant \lambda_{k,n} \leqslant |||M_{k,n}||| \tag{V.17}$$

The result then derives from Proposition IV.4 and Theorem 10. $\square$

V.3.2 Condition for nondegeneracy of the fundamental martingale

V.3.2.1 Construction of the Galton-Watson tree

In the introduction, we defined the MGWRE as a random measure-valued process. Our proof of the nondegeneracy of (W_n) requires however to consider an underlying tree structure. Thus, let us recall first a construction of a multitype Galton-Watson tree in a random environment. Instead of the array $(L_{x,e}^{k,n})$, we consider a similar array $(L_{x,e}^u)_{x\in\mathbb{X},e\in\mathcal{E},u\in\mathbb{U}}$ indexed by the Ulam-Harris-Neveu tree $\mathbb{U}$. We assume that $L_{x,e}^u \sim \mathcal{L}_{x,e}$ and that the sub-arrays $(L_{x,e}^u)_{e\in\mathbb{E}}$, for $x \in \mathbb{X}$ and $u \in \mathcal{U}$ are independent from each other, as well as from

(ξ_n). For any (x, e, u), the $\mathfrak{X}$-valued random variable $L^u_{x,e}$ describes the offspring of an individual u of type x reproducing in the environment e. As we did previously, we note $N^u_{x,e} = \psi(L^u_{x,e})$ its measure valued counterpart. We endow $\mathbb{U}$ with the σ-algebra $\mathcal{U}$ of events that only depend on the n first generations of the tree for some n, and endow $\mathbb{X}^{\mathbb{U}}$ with the cylindrical σ-algebra. Then, let us build recursively a random labelled subtree $(\mathbb{T}, \mathbb{X})$, where $\mathbb{T} \subset \mathbb{U}$ and $X : \mathbb{T} \longrightarrow \mathbb{X}$ as follows. We note $x \in \mathbb{X}$ the type of the initial individual and set $X(\varnothing) = x$. Let us note $\mathbb{G}_n$ the random set of the individuals of $\mathbb{T}$ which are at height n. Then, conditionally on $\mathbb{G}_n$ and $X|_{\mathbb{G}_n}$, we set

$$\mathbb{G}_{n+1} = \left\{ui, u \in \mathbb{G}_n, 1 \leqslant i \leqslant N^u_{X(u),\xi_n}(\mathbb{1})\right\}.$$

Moreover, conditionally on $N^u_{X(u),\xi_n}(\mathbb{1}) = d$ we define $(X(u1), \cdots, X(ud)) = L^u_{X(u),\xi_n}$.

An induction argument shows that the process $(\tilde{Z}_n)$ defined by

$$\tilde{Z}_n = \sum_{u \in \mathbb{G}_n} \delta_{X(u)}$$

is distributed as (Z_n). Indeed,

$$\tilde{Z}_{n+1} = \sum_{v \in \mathbb{G}_{n+1}} \delta_{X(v)} = \sum_{u \in \mathbb{G}_n} \sum_{i=1}^{N^u_{X(u),\xi_n}(\mathbb{1})} \delta_{X(ui)} = \sum_{u \in \mathbb{G}_n} N^u_{X(u),\xi_n}$$

Then, using the map ψ defined in (V.1), we write

$$\tilde{Z}_{n+1} = \sum_{x \in \mathbb{X}} \sum_{\substack{u \in \mathbb{G}_n \\ X(u)=x}} \psi(L^u_{x,\xi_n}).$$

For any type $y \in \mathbb{X}$, conditionally on $\mathbb{G}_n$ and $\bar{\xi}$, the family $A_y = (\psi(L^u_{x,\xi_n}))_{u \in \mathbb{G}_n, X(u)=y}$ contains $\tilde{Z}_n(\mathbb{1}_y)$ independent variables, all of which are distributed according to the ψ-pushforward of the distribution $\mathcal{L}_{y,\xi_n}$. Moreover, the families $(A_y)_{y \in \mathbb{X}}$ are mutually independent conditionally on $\mathbb{G}_n$ and $\bar{\xi}$. Therefore, conditionally on $\tilde{Z}_n$ and $\bar{\xi}$, it holds

$$\tilde{Z}_{n+1} \stackrel{d}{=} \sum_{x \in \mathbb{X}} \sum_{k=1}^{\tilde{Z}_n(\mathbb{1}_x)} N^{k,n}_{x,\xi_n}.$$

Therefore the law of $\tilde{Z}_{n+1}$ conditionally on $\tilde{Z}_n$ is the same as the law of Z_{n+1} conditionally on Z_n, which proves that

$$(\tilde{Z}_n)_{n \geqslant 0} \stackrel{d}{=} (Z_n)_{n \geqslant 0}$$

since $Z_0 = \tilde{Z}_0$. In the sequel, by abuse of notation, we shall denote $Z_n = \tilde{Z}_n$.

V.3.2.2 Construction of the size biased tree

We introduce a slight modification in the above construction to create a biased law on the space of labelled trees. More precisely, we build a random labelled tree $(\mathbb{T}^*, X^*)$, equipped with a random sequence of marked individuals $(\mathbf{s}_n)_{n \geqslant 0}$, called the spine.

We recall that for almost any sequence of environments $\bar{e} = (e_n)_{n\geqslant 0} \in \mathcal{E}^{\mathbb{Z}_+}$, there exists a sequence of functions $(h_n) \in \mathcal{B}(\mathbb{X})^{\mathbb{Z}_+}$ and a sequence of positive real numbers (λ_n) satisfying Theorem 10. We fix such a sequence of environments and introduce, for each type x and almost any sequence $\bar{e}$ the biased probability distribution $\mathcal{L}^*_{x,\bar{e}}$ on $\mathfrak{X}$ such that

$$d\mathcal{L}^*_{x,\bar{e}}(Z) = \frac{Z(h_1)}{\lambda_0 h_0(x)} d\mathcal{L}_{x,e_0}(Z). \tag{V.18}$$

Remark V.8. In case of a finite type set $\mathbb{X} = \{1, \dots, d\}$, the functions h_0 and h_1 are respectively the limits of the right dominant eigenvectors of the matrices $M_{0,n}$ and $M_{1,n}$. It holds therefore $\mathbb{P}[N^*_{x,e} = u] = \frac{\langle u,h_1\rangle}{\lambda_0 h_0(x)} \mathbb{P}[N_{x,e_0} = u]$ for any vector $u \in \mathbb{Z}^d_+$ (with the abuse of notation $\mathcal{N} = \mathbb{Z}^d_+$). One recovers the biased distribution used in [GLP23]. This distribution is itself an extension of the biased laws introduced in [LPP95; KLPP97] for Galton Watson processes in constant environment respectively with one and a finite number of types.

In particular, for any $f \in \mathcal{B}(\mathbb{X})$, it holds

$$\int_{\mathfrak{X}} \psi(Z)(f) d\mathcal{L}^*_{x,e}(Z) = \int \psi(Z)(f) \frac{\psi(Z)(h_1)}{\lambda_0 h_0(x)} d\mathcal{L}_{x,e_0}(Z).$$

Let $\left(L^{u,*}_{x,\bar{e}}\right)_{x\in\mathbb{X}, \bar{e}\in\mathcal{E}^{\mathbb{Z}_+}, u\in\mathbb{U}}$ be an array of independent variables such that $L^{u,*}_{x,\bar{e}} \sim \mathcal{L}^*_{x,\bar{e}}$. We note once again $N^{u,*}_{x,\bar{e}} = \psi\left(L^{u,*}_{x,\bar{e}}\right)$.
If u is at height n in $\mathbb{U}$, conditionally on $u \in \mathbb{T}$, we note for short $L_u = L^u_{X(u),\xi_n}$ if $u \neq \mathbf{s}_n$, and $L_{\mathbf{s}_n} = L^{\mathbf{s}_n,*}_{X(\mathbf{s}_n),\xi_n}$ the tuple valued variables describing the offspring of the individuals. Similarly, we note $N_u = \psi(L_u)$ their measure valued counterparts. Then, conditionally on the environmental process (ξ_n), we build recursively the biased random tree $(\mathbb{T}^*, X^*)$, and the spine $(\mathbf{s}_n)_{n\geqslant 0}$ as follows.

We first set $X^*(\varnothing) = x$ and $\mathbf{s}_0 = \varnothing$. We define

$$\mathbb{G}^*_{n+1} = \{ui, u \in \mathbb{G}^*_n \backslash \{\mathbf{s}_n\}, 1 \leqslant i \leqslant N_u(\mathbb{1})\} \cup \{\mathbf{s}_n i, 1 \leqslant i \leqslant N_{\mathbf{s}_n}(\mathbb{1})\}.$$

If $u \in \mathbb{G}^*_n, u \neq \mathbf{s}_n$, conditionally on $N^u_{X(u),\xi_n}(\mathbb{1}) = d$ we set

$$(X^*(u1), \cdots, X^*(ud)) = L_u = L^u_{X(u),\xi_n}.$$

Moreover, conditionally on $N_{\mathbf{s}_n}(\mathbb{1}) = d$,

$$(X^*(\mathbf{s}_n 1), \cdots, X^*(\mathbf{s}_n d)) = L_{\mathbf{s}_n} = L^{\mathbf{s}_n,*}_{X(\mathbf{s}_n),\theta^n(\bar{\xi})}.$$

The next individual $\mathbf{s}_{n+1}$ of the spine is chosen at random among the children $\{\mathbf{s}_n 1, \dots \mathbf{s}_n d\}$ of $\mathbf{s}_n$ in such a way that $\mathbf{s}_{n+1} = \mathbf{s}_n i$ with probability proportional to $h_{n+1}(X^*(\mathbf{s}_n i))$. Similarly as in the regular construction, we note Z^*_n the occupation measure of the n-th generation of the labelled tree $(\mathbb{T}^*, X^*)$.

$$Z^*_n = \sum_{u\in\mathbb{G}^*_n} \delta_{X^*(u)}.$$

In this subsection $\mathbb{P}, \mathbb{P}_{\bar{\xi}}$ refer to the respective annealed and quenched law of this whole construction. Let us note $\mathcal{T}_n$ the σ-algebra on the set of labelled trees $\mathbb{U} \times \mathbb{X}^{\mathbb{U}}$ generated by the events that only depend on the first n generations of a tree, and $\mathcal{T} = \bigvee_{n\geqslant 0} \mathcal{T}_n$. The labelled trees $(\mathbb{T}, X)$ and $(\mathbb{T}^*, X^*)$ are random variables on $(\mathbb{U} \times \mathbb{U}^{\mathbb{X}}, \mathcal{T})$, we note $\mathcal{P}_{\bar{\xi}}$, $\mathcal{P}^*_{\bar{\xi}}$, $\mathcal{P}$ and $\mathcal{P}^*$ there respective quenched and annealed distributions. These are therefore probability measures on the space $(\mathbb{U} \times \mathbb{U}^{\mathbb{X}}, \mathcal{T})$. Note that W_n is $\sigma(\bar{\xi}, \mathcal{T}_n)$-measurable. Moreover, the following analog of equation (5.3) of [GLP14] holds

Proposition V.9. *For any $n \geqslant 0$, $\mathbb{P}(d\bar{\xi})$-almost surely,*

$$d\mathcal{P}^*_{\bar{\xi}}|_{\mathcal{T}_n} = W_n d\mathcal{P}_{\bar{\xi}}|_{\mathcal{T}_n}.$$

Proof. For two non-labelled trees t and $t \subset \mathbb{U}$, we note $t \overset{n}{=} t'$ if the two trees coincide up to height n and call $d_u(t)$ the number of offspring of the node $u \in t$. Let t be a tree of height n, $(B_u)_{u\in t} \in \mathcal{X}^t$ be an array of measurable subsets of $\mathbb{X}$ and $s \in \mathbb{G}_n(t)$. We note $s = (i_1 \dots i_n)$. Then the following equality of events holds :

$$\{\mathbb{T}^* \overset{n}{=} t\} \cap \bigcap_{u\in t} \{X(u) \in B_u\} \cap \{\mathbf{s}_n = s\} = \bigcap_{u\in t} \left\{ L_u \in \prod_{i=1}^{d_u(t)} B_u \right\} \cap \bigcap_{k=1}^{n} \{s_k = (i_1 \dots i_k)\}$$

On this event, if $l(u) = d$, then conditionally on $\mathcal{T}_d$, either $L_u \sim \mathcal{L}_{X(u),\xi_d}$ or $L_u \sim \mathcal{L}^*_{X(u),T^d(\bar{\xi})}$ depending on whether $u = \mathbf{s}_d$ or not. This yields

$$\begin{aligned} &\mathbb{P}\left[\{\mathbb{T}^* \overset{n}{=} t\} \cap \bigcap_{u\in t} \{X(u) \in B_u\} \cap \{\mathbf{s}_n = s\} \middle| \bar{\xi} \right] \\ &= \int \prod_{k=0}^{n-1} \prod_{\substack{u\in\mathbb{G}_k(t) \\ u\neq(i_1\dots i_k)}} \mathbb{1}_{L_u \in \prod_{i=1}^{d_t(u)} B_{ui}} d\mathcal{L}_{X(u),\xi_k}(L_u) \\ &\times \mathbb{1}_{L_k \in \prod_{i=1}^{d_t(i_1\dots i_k)} B_{(i_1\dots i_k i)}} \frac{h_n(X(i_1 \dots i_{k+1}))}{\sum_{i=1}^{d_{i_1\dots i_k}(t)} h_{k+1}(X(i_1 \dots i_k i))} d\mathcal{L}^*_{X(i_1\dots i_k),T^k(\bar{\xi})}(L_{(i_1\dots i_k)}) \end{aligned} \tag{V.19}$$

In this context, the definition (V.18) of the biased law $\mathcal{L}^*_{x,\bar{e}}$ yields, for all k,

$$d\mathcal{L}^*_{X(i_1\dots i_k),T^k(\bar{\xi})}(L_{(i_1\dots i_k)}) = \frac{\sum_{i=1}^{d_{i_1\dots i_k}(t)} h_{k+1}(X(i_1 \dots i_k i))}{\lambda_k h_k(X(i_1 \dots i_k))} d\mathcal{L}_{X(i_1\dots i_k),\xi_k}(L_{(i_1\dots i_k)}). \tag{V.20}$$

Therefore, plugging (V.20) into (V.19) yields

$$\begin{aligned} &\mathbb{P}\left[\{\mathbb{T}^* \overset{n}{=} t\} \cap \bigcap_{u\in t} \{X(u) \in B_u\} \cap \{\mathbf{s}_n = s\} \middle| \bar{\xi} \right] \\ &= \int \prod_{k=0}^{n-1} \prod_{u\in\mathbb{G}_k(t)} \mathbb{1}_{L_u \in \prod_{i=1}^{d_t(u)} B_{ui}} d\mathcal{L}_{X(u),\xi_k}(L_u) \times \frac{h_n(X(s))}{\lambda_{0,n} h_0(x)}, \end{aligned}$$

where we recall that x refers to the type of the initial individual $\varnothing = s_0$. Summing over all vertices s in the n-th generation of t, we finally get

$$\begin{aligned}\int \mathbb{1}_{\{\mathbb{T}\overset{n}{=}t\}\cap\bigcap_{u\in t}X(u)\in B_u} d\mathcal{P}^*_{\bar{\xi}}(\mathbb{T}^*) &= \mathbb{P}\left[\{\mathbb{T}^* \overset{n}{=} t\} \cap \bigcap_{u\in t}\{X(u)\in B_u\}\middle|\bar{\xi}\right] \\ &= \int \prod_{\substack{u\in t\\ l(u)\leqslant n-1}} \mathbb{1}_{L_u\in\prod_{i=1}^{d_t(u)} B_{ui}} d\mathcal{L}_{X(u),\xi_k}(L_u) \times \frac{Z_n(h_n)}{\lambda_{0,n}h_0(x)} \\ &= \int \prod_{\substack{u\in t\\ l(u)\leqslant n-1}} \mathbb{1}_{L_u\in\prod_{i=1}^{d_t(u)} B_{ui}} d\mathcal{L}_{X(u),\xi_k}(L_u) W_n \\ &= \int W_n \mathbb{1}_{\{\mathbb{T}\overset{n}{=}t\}\cap\bigcap_{u\in t}X(u)\in B_u} d\mathcal{P}_{\bar{\xi}}(\mathbb{T})\end{aligned}$$

The σ-algebra $\mathcal{T}_n$ is generated by events of the form $\{\mathbb{T} \overset{n}{=} t\} \cap \bigcap_{u\in t}\{X(u)\in B_u\}$. Therefore the proposition is proved. □

V.3.2.3 Nondegeneracy of the martingale

We define in this section $W = \limsup W_n \in [0,\infty]$. Since $\mathbb{P}(d\bar{\xi})$-almost surely, the process (W_n) is a $\mathcal{P}_{\bar{\xi}}$-nonnegative martingale, then $W = \lim W_n \in [0,\infty)$, $\mathcal{P}_{\bar{\xi}}$-almost surely, in $\mathbb{P}(d\bar{\xi})$-almost surely any environment sequence $\bar{\xi}$. Our main goal is to exhibit mild assumptions under which W is not degenerate, in the sense that $\mathcal{P}_{\bar{\xi}}(W > 0) > 0, \mathbb{P}(d\bar{\xi})$-almost surely. Let us fix a sequence $\bar{e} = (e_n)$. By Theorem 5.3.3 of [Dur10], if $\mathcal{P}^*_{\bar{e}}(W < \infty) = 1$ then $\int W d\mathcal{P}_{\bar{e}} = 1$, and thus $\mathcal{P}_{\bar{e}}(W > 0) > 0$. We focus now on the biased tree $\mathbb{T}^*$ and prove that $\mathcal{P}^*_{\bar{\xi}}(\limsup W_n < \infty), \mathbb{P}(d\bar{\xi})-$a.s.

To do so, following [GLP23], we introduce the quantity

$$A_n = \frac{(Z_n^* - \delta_{X^*(\mathbf{s}_n)})(h_n)}{\lambda_{0,n}Z_0^*(h_0)} - \sum_{k=0}^{n-1}\frac{(N_{\mathbf{s}_k} - \delta_{X^*(s_{k+1})})(h_{k+1})}{\lambda_{0,k+1}Z_0^*(h_0)}$$

We introduce $\mathcal{Y} = \sigma((\mathbf{s}_n, X^*(s_n), L_{\mathbf{s}_n})_{n\geqslant 0})$, the $\sigma-$algebra generated by the spine. Then the process (A_n) satisfies

$$\begin{aligned}\mathbb{E}[A_{n+1}|\mathcal{Y},\bar{\xi},\mathcal{T}_n] &= \mathbb{E}\left[\frac{(Z_{n+1}^* - \delta_{X^*(\mathbf{s}_{n+1})})(h_{n+1}) - (N_{\mathbf{s}_n} - \delta_{X^*(\mathbf{s}_{n+1})})}{\lambda_{0,n+1}Z_0^*(h_0)}\right. \\ &\qquad \left.- \sum_{k=0}^{n-1}\frac{(N_{\mathbf{s}_k} - \delta_{X^*(s_{k+1})})(h_{k+1})}{\lambda_{0,k+1}Z_0^*(h_0)}\middle|\mathcal{Y},\bar{\xi},\mathcal{T}_n\right] \\ &= \frac{\mathbb{E}[(Z_{n+1}^* - N_{\mathbf{s}_n})(h_{n+1})|\bar{\xi},\mathcal{Y},\mathcal{T}_n]}{\lambda_{0,n+1}Z_0^*(h_0)} - \sum_{k=0}^{n-1}\frac{(N_{\mathbf{s}_k} - \delta_{X^*(s_{k+1})})(h_{k+1})}{\lambda_{0,k+1}Z_0^*(h_0)}.\end{aligned} \tag{V.21}$$

Note that

$$Z_{n+1}^* = \sum_{u\in\mathbb{G}_n^*\setminus\{\mathbf{s}_n\}} L^u_{\xi_n,X(u)} + N_{\mathbf{s}_n}.$$

Thus

$$\begin{aligned}\mathbb{E}[(Z_{n+1}^* - N_{\mathbf{s}_n})(h_{n+1})|\bar{\xi}, \mathcal{Y}, \mathcal{T}_n] &= \sum_{u \in \mathbb{G}_n^* \setminus \{\mathbf{s}_n\}} \delta_{X(u)} M_n h_{n+1} \\ &= (Z_n^* - \delta_{X^*(\mathbf{s}_n)}) M_n h_{n+1} \\ &= \lambda_n (Z_n^* - \delta_{X^*(\mathbf{s}_n)}) h_n \end{aligned} \tag{V.22}$$

Plugging (V.22) into (V.21) proves that (A_n) is a $\mathbb{P}_{\bar{\xi}}$-martingale, conditionally on $\sigma(\bar{\xi}, \mathcal{Y})$. To ensure the convergence of this martingale, we need the following statement.

Lemma V.10. *We assume that there exists a nondecreasing and positive function f such that $\frac{1}{x\log(x)f(x)}$ is integrable at infinity and*

$$\mathbb{E}\left[\sup_{x \in \mathbb{X}} \frac{\mathbb{E}\left[Z_1(h_1)\log[Z_1(h_1)]f(Z_1(h_1))|\bar{\xi}, Z_0 = \delta_x\right]}{M_0 h_1(x)}\right] < \infty.$$

Then, $\mathbb{P}$-almost surely,

$$\limsup_{n\to\infty} \frac{1}{n} \log^+(N_{\mathbf{s}_n}(h_{n+1})) = 0.$$

Combining **Ass. V.4** with Proposition V.7 and Lemma V.10 proves that the series

$$\sum_{k \geqslant 0} \frac{N_{\mathbf{s}_k}(h_{k+1})}{\lambda_{0,k+1} Z_0^*(h_0)}$$

is $\mathbb{P}$-a.s. convergent. Moreover, noticing that $\delta_{X^*(s_{k+1})}(h_{k+1}) \leqslant 1$, the series

$$\sum_{k \geqslant 0} \frac{(N_{\mathbf{s}_k} - \delta_{X^*(s_{k+1})})(h_{k+1})}{\lambda_{0,k+1} Z_0^*(h_0)}$$

also converges $\mathbb{P}$-a.s. Its sum, which we denote S, is $\sigma(\bar{\xi}, \mathcal{Y})$ measurable, thus the process $(A_n + S)_{n \geqslant 0}$ is a nonnegative martingale with respect to the filtration $\sigma(\mathcal{Y}, \bar{\xi}, \mathcal{T}_n)$. Thus, the sequence (A_n) converges $\mathbb{P}$-a.s. This proves in turn the $\mathbb{P}$-almost sure convergence of

$$\frac{(Z_n^* - \delta_{X^*(\mathbf{s}_n)})(h_n)}{\lambda_{0,n} Z_0^*(h_0)}.$$

Since, once again, the quantity $\delta_{X^*(\mathbf{s}_n)}(h_n)$ is bounded by 1,

$$\frac{(Z_n^*)(h_n)}{\lambda_{0,n} Z_0^*(h_0)}$$

also converges $\mathbb{P}$-a.s. to a finite variable. As a consequence, for $\mathbb{P}$-almost any sequence $\bar{\xi} = (\xi_n)$, the sequence $(W_n)_{n \geqslant 0}$ converges $\mathcal{P}_{\bar{\xi}}^*$-almost surely to a finite limit and $W = \limsup W_n < \infty$, $\mathcal{P}_{\bar{\xi}}^*$-almost surely. This concludes the proof.

V.3.2.4 Proof of Lemma V.10

Proof. Let us control first the tail of the annealed distribution of $N_{\mathbf{s}_n}(h_{n+1})$:

$$\begin{aligned}
\mathbb{P}[N_{\mathbf{s}_n}(h_{n+1}) > t] &= \mathbb{E}[\mathbb{P}[N_{\mathbf{s}_n}(h_{n+1}) > t|\bar{\xi}, X^*(\mathbf{s}_n)]] \\
&= \mathbb{E}\left[\int \mathbb{1}_{[t,+\infty)}(Z(h_{n+1}))d\mathcal{L}^*_{X^*(\mathbf{s}_n),\theta^n(\bar{\xi})}(Z)\right] \\
&\leqslant \mathbb{E}\left[\sup_{x\in\mathbb{X}} \int \mathbb{1}_{[t,+\infty)}(Z(h_{n+1}))d\mathcal{L}^*_{x,\theta^n(\bar{\xi})}(Z)\right] \\
&\leqslant \mathbb{E}\left[\sup_{x\in\mathbb{X}} \int \mathbb{1}_{[t,+\infty)}(Z(h_1))d\mathcal{L}^*_{x,\bar{\xi}}(Z)\right] \\
&\leqslant \mathbb{E}\left[\sup_{x\in\mathbb{X}} \mathbb{P}[Z^*_1(h_1) > t|Z^*_0 = x, \bar{\xi}]\right]. \qquad \text{(V.23)}
\end{aligned}$$

By the Markov inequality, we derive for any nonnegative and nondecreasing function f,

$$\mathbb{P}[Z^*_1(h_1) > t|Z^*_0 = x, \bar{\xi}] \leqslant \frac{\sup_{x\in\mathbb{X}} \mathbb{E}\left[\log^+(Z^*_1(h_1))f(Z^*_1(h_1))|Z^*_0 = \delta_x, \bar{\xi} = (e_n)\right]}{f(t)\log^+(t)}. \qquad \text{(V.24)}$$

Plugging (V.24) in (V.23), we get

$$\begin{aligned}
\mathbb{P}[N_{\mathbf{s}_n}(h_{n+1}) > t] &\leqslant \mathbb{E}\left[\frac{\sup_{x\in\mathbb{X}} \mathbb{E}\left[\log^+(Z^*_1(h_1))f(Z^*_1(h_1))|Z^*_0 = \delta_x, \bar{\xi} = (e_n)\right]}{f(t)\log^+(t)}\right] \\
&\leqslant \frac{\mathbb{E}\left[\sup_{x\in\mathbb{X}} \mathbb{E}\left[\log^+(Z^*_1(h_1))f(Z^*_1(h_1))|Z^*_0 = \delta_x, \bar{\xi} = (e_n)\right]\right]}{f(t)\log^+(t)} \qquad \text{(V.25)}
\end{aligned}$$

For $\mathbb{P}$-almost any sequence $\bar{e} = (e_n)_{n\geqslant 0}$, setting $r(t) = t\log^+ t f(t)$, we may write

$$\begin{aligned}
\frac{\mathbb{E}\left[r(Z_1(h_1))|Z_0 = \delta_x, \bar{\xi} = \bar{e}\right]}{M_0 h_1(x)} &= \int \log^+(Z_1(h_1))f(Z_1(h_1))\frac{Z_1(h_1)}{\lambda_0 h_0(x)}d\mathcal{L}_{x,e_0}(Z_1) \\
&= \int \log^+(Z(h_1))f(Z(h_1))d\mathcal{L}^*_{x,\bar{e}}(Z) \\
&= \mathbb{E}\left[\log^+(Z^*_1(h_1))f(Z^*_1(h_1))|Z_0 = \delta_x, \bar{\xi} = (e_n)\right]. \qquad \text{(V.26)}
\end{aligned}$$

Assumption **Ass. V.5** can be rewritten

$$\mathbb{E}\left[\sup_{x\in\mathbb{X}} \frac{\mathbb{E}\left[r(Z_1(h_1))|Z_0 = \delta_x, \bar{\xi}\right]}{M_0 h_1(x)}\right] < \infty$$

which implies, by (V.26),

$$A := \mathbb{E}\left[\sup_{x\in\mathbb{X}} \mathbb{E}\left[\log^+(Z^*_1(h_1))f(Z^*_1(h_1))|Z^*_0 = \delta_x, \bar{\xi} = (e_n)\right]\right] < \infty.$$

Hence, by (V.25)

$$\mathbb{P}[N_{\mathbf{s}_n}(h_{n+1}) > t] \leqslant \frac{A}{f(t)\log^+(t)}.$$

Then, for any $c > 0$,

$$\begin{aligned}
\mathbb{E}\left[\sum_{n\geqslant 1} \mathbb{1}_{\frac{1}{n}\log(N_{\mathbf{s}_n}(h_{n+1}))>c}\right] &= \mathbb{E}\left[\sum_{n\in\mathbb{Z}_+} \mathbb{1}_{\frac{1}{c}\log(N_{\mathbf{s}_n}(h_{n+1}))>n}\right] \\
&= \sum_{n\geqslant 1} \mathbb{P}[\log^+(N_{\mathbf{s}_n}(h_{n+1})) > nc] \\
&= \sum_{n\geqslant 1} \mathbb{P}[N_{\mathbf{s}_n}(h_{n+1}) > \exp(nc)] \\
&\leqslant \sum_{n\geqslant 1} \frac{A}{ncf(\exp nc)}. \qquad (V.27)
\end{aligned}$$

A elementary change of variable shows that since $\frac{1}{t\log(t)f(t)}$ is integrable at $+\infty$, then so is $\frac{1}{uf(\exp(u))}$. Thus the sum (V.27) converges, which proves that $\limsup_{n\to\infty} \frac{\log^+(N_{\mathbf{s}_n}(h_{n+1}))}{n} = 0$. □

V.3.3 Almost sure convergence of the type distribution

Before proving Theorem 9, let us discuss its statement. As pointed out in the introduction, we are not able to guarantee that the error term $o\left(\left(\max(\tilde{\eta}e^{\lambda}, e^{\frac{\lambda}{2}}) + \varepsilon\right)^n\right)$ appearing in (V.6) is negligeable with respect to the quantity $Z_0M_{0,n}(f)$ for every $f \in \mathcal{B}(\mathbb{X})$. We highlight two specific subcases, which are neither exhaustive or exclusive of each other.

Case 1: If $\liminf_{n\to\infty}\left(\frac{Z_0M_{0,n}(f)}{\|Z_0M_{0,n}\|_{TV}}\right)^{\frac{1}{n}} > \max\left(\tilde{\eta}, e^{-\frac{\lambda}{2}}\right)$, then conditionally on $\{W > 0\}$ it holds

$$Z_n(f) \underset{n\to\infty}{\sim} WZ_0M_{0,n}(f) \quad \mathbb{P}\text{-a.s.} \qquad (V.28)$$

Case 2: If $A := \limsup_{n\to\infty}\left(\frac{Z_0M_{0,n}(f)}{\|Z_0M_{0,n}\|_{TV}}\right)^{\frac{1}{n}} < 1$, then it holds

$$\limsup_{n\to\infty}\left(Z_n(f)^{\frac{1}{n}}\right) \leqslant \max(\tilde{\eta}e^{\lambda}, e^{\frac{\lambda}{2}}, Ae^{\lambda}) \quad \mathbb{P}\text{-a.s.}$$

Any function f such that $c \leqslant f \leqslant C$ for some positive constants c, C (in particular $f = \mathbb{1}$) belongs to case 1 and satisfies (V.28). This yields the first claim of Corollary V.3. However for functions f such that $\inf_{x\in\mathbb{X}} f(x) = 0$, it seems possible that the typical distribution of types at time n is concentrated on types associated with small values of f, in which case we might have, with positive probability

$$\liminf_{n\to\infty}\left(\frac{Z_0M_{0,n}(f)}{\|Z_0M_{0,n}\|_{TV}}\right)^{\frac{1}{n}} \leqslant \max(\tilde{\eta}, e^{-\frac{\lambda}{2}}),$$

implying that along some subsequence (n_k), the main term in the asymptotic expansion (V.6) is in fact negligeable with respect ot the error term. Case 2 provides an upper bound on $Z_n(f)$ when one has additionally a sufficiently precise upper bound on the quenched mean of the distribution of types $\frac{Z_0M_{0,n}(f)}{\|Z_0M_{0,n}\|_{TV}}$.

The claim *ii)* of Theorem 9 provides a minorization condition relating a function f and the coupling measure, which guarantees that f belongs to Case 1.

Let us prove now Corollary V.3 and then Theorem 9

Proof of Corollary V.3. Plugging $f = \mathbb{1}$ in (V.6) and choosing ε small enough such that $\max(\tilde{\eta}e^{\lambda}, e^{\frac{\lambda}{2}}) + \varepsilon < e^{\lambda}$, we obtain

$$\|Z_n\|_{TV} = Z_n(\mathbb{1}) = W_n\|Z_0M_{0,n}\|_{TV} + \underset{n\to\infty}{o}\left(e^{n\lambda}\right) \quad \mathbb{P}\text{-a.s.}$$

By Theorem 10, *iii)*, conditionally on $\{W > 0\}$, we have

$$\lim_{n\to+\infty} \left(W_n\|Z_0M_{0,n}\|_{TV}\right)^{\frac{1}{n}} = e^{\lambda}$$

and thus, conditionally on $\{W > 0\}$,

$$Z_n(\mathbb{1}) \sim W\|Z_0M_{0,n}\|_{TV} \quad \mathbb{P}\text{-a.s.}$$

Using again Theorem 10, *iii)*, we derive $\lim_{n\to\infty} \left(Z_n(\mathbb{1})\right)^{\frac{1}{n}} = e^{\lambda}$. This proves (V.9).
Let us choose now $f \in \mathcal{B}(\mathbb{X})$. We apply (V.6) and derive, $\mathbb{P}$-almost surely, conditionally on $\{W > 0\}$, for any $\varepsilon > 0$,

$$\frac{Z_n(f)}{Z_n(\mathbb{1})} = \frac{Z_0M_{0,n}(f)}{\|Z_0M_{0,n}\|_{TV}} \times \frac{W_n\|Z_0M_{0,n}\|_{TV}}{Z_n(\mathbb{1})} + \underset{n\to\infty}{o}\left(\frac{\left(\max(\tilde{\eta}e^{\lambda}, e^{\frac{\lambda}{2}}) + \varepsilon\right)^n}{Z_n(\mathbb{1})}\right).$$

By (V.9), we know that, if $\varepsilon > 0$ is small enough, it holds $\mathbb{P}$-almost surely

$$\lim_{n\to\infty} \frac{\left(\max(\tilde{\eta}e^{\lambda}, e^{\frac{\lambda}{2}}) + \varepsilon\right)^n}{Z_n(\mathbb{1})} = 0$$

conditionally on $\{W > 0\}$, thus

$$\lim_{n\to+\infty} \frac{Z_n(f)}{Z_n(\mathbb{1})} - \frac{Z_0M_{0,n}(f)}{\|Z_0M_{0,n}\|_{TV}} \times \frac{W_n\|Z_0M_{0,n}\|_{TV}}{Z_n(\mathbb{1})} = 0,$$

where $\frac{Z_0M_{0,n}(f)}{\|Z_0M_{0,n}\|_{TV}}$ is bounded by 1. Hence, by (V.9), conditionally on $\{W > 0\}$, it holds $W_n\|Z_0M_{0,n}\|_{TV} \sim Z_n(\mathbb{1})$. From this, we finally derive

$$\lim_{n\to+\infty} \frac{Z_n(f)}{Z_n(\mathbb{1})} - \frac{Z_0M_{0,n}(f)}{\|Z_0M_{0,n}\|_{TV}} = 0.$$

□

Proof of Theorem 9. We note here $\pi_n = \frac{Z_0 M_{0,n}}{\|Z_0 M_{0,n}\|_{TV}} \in \mathcal{M}_1(\mathbb{X})$ the renormalized quenched mean of the population at time $n \geqslant 0$. Fix $f \in \mathcal{B}(\mathbb{X})$. We set

$$f_n = f - \frac{\pi_n(f) h_n}{\pi_n(h_n)} = f - \frac{Z_0 M_{0,n}(f)}{Z_0 M_{0,n}(h_n)} h_n = f - Z_0 M_{0,n}(f) \frac{h_n}{\lambda_{0,n} Z_0(h_0)}.$$

Note that $\pi_n(f_n) = 0$ and

$$\|f_n\|_\infty \leqslant \|f\|_\infty \left(1 + \frac{1}{\pi_n(h_n)}\right) \tag{V.29}$$

We also define

$$\Delta_n = \frac{Z_n(f_n)}{\lambda_{0,n}} = \frac{Z_n(f)}{\lambda_{0,n}} - W_n \frac{\pi_n(f)}{\pi_n(h_n)} Z_0(h_0) = \frac{Z_n(f) - W_n Z_0 M_{0,n}(f)}{\lambda_{0,n}}. \tag{V.30}$$

The main idea of the proof is to show that the random sequence $(\Delta_n)_{n \geqslant 0}$ converges to 0 exponentially fast in $\mathcal{L}^2(\mathbb{P}_{\bar{\xi}})$, $\mathbb{P}(d\bar{\xi})$-a.s. This will yield the desired result. Notice first that

$$\begin{aligned} Z_n(f_n)^2 &= \left(\sum_{u \in \mathbb{G}_n} f_n(X(u))\right)^2 \\ &= \sum_{u,v \in \mathbb{G}_n} f_n(X(u)) f_n(X(v)) \\ &= \sum_{u \in \mathbb{G}_n} f_n(X(u))^2 + \sum_{\substack{u,v \in \mathbb{G}_n \\ u \neq v}} f_n(X(u)) f_n(X(v)) \end{aligned} \tag{V.31}$$

where

$$\sum_{u \in \mathbb{G}_n} f_n(X(u))^2 = Z_n(f_n^2). \tag{V.32}$$

Moreover, each pair of distinct individuals $u \neq v \in \mathbb{G}_n$ has a unique last common ancestor w at some time $0 \leqslant p \leqslant n-1$, characterized by the property that w has two distinct children $u' \neq v' \in \mathbb{G}_{p+1}$ such that $u' \preceq u$ and $v' \preceq v$. This can be used to reorganize the right hand side term in (V.31) as

$$\sum_{\substack{u,v \in \mathbb{G}_n \\ u \neq v}} f_n(X(u)) f_n(X(v)) = \sum_{p=0}^{n-1} \underbrace{\sum_{w \in \mathbb{G}_p} \sum_{\substack{u',v' \in \mathbb{G}_{p+1} \\ w \preceq u',v' \\ u' \neq v'}} \sum_{\substack{u \in \mathbb{G}_n \\ u' \preceq u}} f_n(X(u)) \sum_{\substack{v \in \mathbb{G}_n \\ v' \preceq u}} f_n(X(v))}_{:= \Sigma_p(f_n)}. \tag{V.33}$$

Combining (V.31), (V.32) and (V.33) yields

$$Z_n(f_n)^2 = Z_n(f_n^2) + \sum_{p=0}^{n-1} \Sigma_p(f_n) \tag{V.34}$$

We want now to compute and estimate the quenched expectation of (V.34). The left hand side term is the easiest to deal with :

$$\mathbb{E}_{\bar{\xi}}\left[Z_n(f_n^2)\right] = Z_0 M_{0,n}(f_n^2) \leqslant \|f_n\|_\infty^2 \|Z_0 M_{0,n}\|_{TV}. \tag{V.35}$$

The right hand side term requires a more subtle handling. For each pair of distinct individuals u', v' at time $p+1$, the subtrees descending respectively form u', v' are independent conditionally on $\bar{\xi}$ and $\mathcal{F}_{p+1}$. This yields

$$\mathbb{E}_{\bar{\xi}}\left[\Sigma_p(f_n)\middle|\,\mathcal{F}_{p+1}\right] = \sum_{w \in \mathbb{G}_p} \sum_{\substack{u',v' \in \mathbb{G}_{p+1} \\ w \preceq u',v' \\ u' \neq v'}} M_{p+1,n}(f_n)(X(u'))M_{p+1,n}(f_n)(X(v')). \tag{V.36}$$

Therefore, the quantity $\mathbb{E}_{\bar{\xi}}\left[\Sigma_p(f_n)|\mathcal{F}_p\right]$ can be rewritten

$$\begin{aligned}
\mathbb{E}_{\bar{\xi}}\left[\Sigma_p(f_n)|\mathcal{F}_p\right] &= \mathbb{E}_{\bar{\xi}}\left[\mathbb{E}_{\bar{\xi}}\left[\Sigma_p(f_n)|\mathcal{F}_{p+1}\right]\middle|\,\mathcal{F}_p\right] \\
&= \sum_{w \in \mathbb{G}_p} \mathbb{E}_{\bar{\xi}}\left[\sum_{\substack{u',v' \in \mathbb{G}_{p+1} \\ w \preceq u',v' \\ u' \neq v'}} M_{p+1,n}(f_n)(X(u'))M_{p+1,n}(f_n)(X(v'))\middle|\,\mathcal{F}_p\right] \\
&= \sum_{w \in \mathbb{G}_p} V_p(M_{p+1,n}(f_n))(X(w)) \\
&= Z_p(V_p(M_{p+1,n}(f_n))),
\end{aligned}$$

where the quantity $V_p(g)(x)$ is defined for any $g \in \mathcal{B}(\mathbb{X})$ and $x \in \mathbb{X}$ as

$$V_p(g)(x) = \mathbb{E}_{\bar{\xi}}\left[\sum_{\substack{u,v \in \mathbb{G}_{p+1} \\ u \neq v}} g(X(u))g(X(v))\middle|\, Z_p = \delta_x\right].$$

Consequently,

$$\mathbb{E}_{\bar{\xi}}[\sigma_p(f_n)] = Z_0 M_{0,p}(V_p(M_{p+1,n}(f_n)))$$

and

$$\mathbb{E}_{\bar{\xi}}\left[\sum_{p=0}^{n-1} \Sigma_p(f_n)\right] = \sum_{p=0}^{n-1} Z_0 M_{0,p}(V_p(M_{p+1,n}(f_n))). \tag{V.37}$$

Notice that $|V_p(g)| \leqslant V_p(|g|)$ and $V_p(\alpha g)) = \alpha^2 V_p(g)$ for any $g \in \mathcal{B}(\mathbb{X})$ and any $\alpha \in \mathbb{R}$. Therefore, noting $|||V_p||| = \sup_{x \in \mathbb{X}} V_p(\mathbb{1})(x)$, it holds

$$|Z_0 M_{0,p}(V_p(M_{p+1,n}(f_n)))| \leqslant \left(\sup_{x \in \mathbb{X}} M_{p+1,n}(f_n)(x)\right)^2 \|Z_0 M_{0,p}\| |||V_p|||. \tag{V.38}$$

Using (V.14) and recalling that $\pi_n(f_n) = 0$, we derive

$$\begin{aligned}
|M_{p+1,n}(f_n)(x)| &= \left| M_{p+1,n}(f_n)(x) - \frac{h_k(x)}{\pi_0 M_{0,k} h_k} \|\pi_0 M_{0,n}\|_{TV} \pi_n(f_n) \right| \\
&\leqslant \|f_n\|_\infty \left\| \delta_x M_{p+1,n} - \frac{h_k(x)}{\pi_0 M_{0,k} h_k} \|\pi_0 M_{0,n}\|_{TV} \pi_n \right\|_{TV} \\
&\leqslant \frac{4}{\gamma_{n-1}} \|f_n\|_\infty \|\delta_x M_{p+1,n}\|_{TV} \prod_{i=p+1}^{n-1} (1-\gamma_i). \qquad \text{(V.39)}
\end{aligned}$$

Plugging (V.39) into (V.38) yields

$$|Z_0 M_{0,p}(V_p(M_{p+1,n}(f_n)))| \leqslant \left(\frac{4}{\gamma_{n-1}} \|f_n\|_\infty \|\!|M_{p+1,n}|\!\| \prod_{p+1}^{n-1} (1-\gamma_i) \right)^2 \|Z_0 M_{0,p}\| \, \|\!|V_p|\!\|. \quad \text{(V.40)}$$

with

$$\|\!|M_{p+1,n}|\!\| \leqslant \frac{1}{\gamma_p} \frac{\|Z_0 M_{0,n}\|_{TV}}{\|Z_0 M_{0,p+1}\|_{TV}},$$

by (V.16). Thus

$$\begin{aligned}
&|Z_0 M_{0,p}(V_p(M_{p+1,n}(f_n)))| \\
&\leqslant \left(\frac{4\|f_n\|_\infty}{\gamma_{n-1}\gamma_p} \frac{\|Z_0 M_{0,n}\|_{TV}}{\|Z_0 M_{0,p+1}\|_{TV}} \prod_{p+1}^{n-1} (1-\gamma_i) \right)^2 \|Z_0 M_{0,p}\|_{TV} \, \|\!|V_p|\!\| \\
&\leqslant \underbrace{\left(\frac{4\|f_n\|_\infty \|Z_0 M_{0,n}\|_{TV} \prod_{i=0}^{n-1}(1-\gamma_i)}{\gamma_{n-1}} \right)^2}_{\alpha_n} \times \underbrace{\frac{\|Z_0 M_{0,p}\|_{TV} \|\!|V_p|\!\|}{\gamma_p \|Z_0 M_{0,p+1}\|_{TV}^2 \prod_{i=0}^{p} (1-\gamma_i)^2}}_{\beta_p} \qquad \text{(V.41)}
\end{aligned}$$

Putting together (V.34), (V.35) with (V.37) and (V.41) yields, $\mathbb{P}(d\bar{\xi})$-a.s.

$$\mathbb{E}_{\bar{\xi}} \left[Z_n(f_n)^2 \right] \leqslant \|f_n\|_\infty^2 \|Z_0 M_{0,n}\|_{TV} + \alpha_n \sum_{p=0}^{n-1} \beta_p$$

and therefore

$$\mathbb{E}_{\bar{\xi}} \left[\Delta_n^2 \right] \leqslant \|f_n\|_\infty^2 \frac{\|Z_0 M_{0,n}\|}{\lambda_{0,n}^2} + \frac{\alpha_n}{\lambda_{0,n}^2} \sum_{p=0}^{n-1} \beta_p. \qquad \text{(V.42)}$$

To estimate the asymptotic behavior of the random sequences $(\|f_n\|_\infty)_{n\geqslant 0}$, $(\alpha_n)_{n\geqslant 0}$ and $(\beta_p)_{p\geqslant 0}$, we state the following lemma and postpone its proof to the end of the section.

Lemma V.11. *Under assumptions* ***Ass. V.1*** *to* ***Ass. V.6****, it holds* $\mathbb{P}(d\bar{\xi})$*-a.s.*

i) $\limsup_{n\to\infty} \|f_n\|_\infty^{\frac{1}{n}} \leqslant 1$

ii) $\limsup_{n\to\infty} (\alpha_n)^{\frac{1}{n}} \leqslant \left(\tilde{\eta} e^{\lambda}\right)^2$

iii) $\limsup_{p\to\infty} (\beta_p)^{\frac{1}{p}} \leqslant \frac{e^{-\lambda}}{\tilde{\eta}^2}$.

On the one hand, combining Lemma V.11, *i)* with Theorem 10, *iv)* and Theorem V.7, we derive

$$\lim_{n\to\infty} \left(\|f_n\| \frac{\|Z_0 M_{0,n}\|}{\lambda_{0,n}^2} \right)^{\frac{1}{n}} \leqslant e^{-\lambda} < 1 \quad \mathbb{P}(d\bar{\xi})\text{-a.s.}$$

which implies

$$\|f_n\| \frac{\|Z_0 M_{0,n}\|}{\lambda_{0,n}^2} = \underset{n\to\infty}{O} \left((e^{-\lambda} + \varepsilon)^n \right) \quad \mathbb{P}(d\bar{\xi})\text{-a.s.} \tag{V.43}$$

where $e^{-\lambda} + \varepsilon < 1$ for $\varepsilon > 0$ small enough. On the other hand, Lemma V.11,*iii)* clearly implies that $\mathbb{P}(d\bar{\xi})$-a.s., for any $\varepsilon > 0$ and for any p large enough, $\beta_p \leqslant \left(\frac{e^{-\lambda}}{\tilde{\eta}^2} + \varepsilon \right)^p$. This implies

$$\limsup_{n\to\infty} \left(\sum_{p=0}^{n-1} \beta_p \right)^{\frac{1}{n}} \leqslant \max\left(1, \frac{e^{-\lambda}}{\tilde{\eta}^2}\right) \quad \mathbb{P}(d\bar{\xi})\text{-a.s.} \tag{V.44}$$

Indeed,

- If $\frac{e^{-\lambda}}{\tilde{\eta}^2} < 1$ then for $\varepsilon > 0$ small enough $\frac{e^{-\lambda}}{\tilde{\eta}^2} + \varepsilon < 1$ and $\sum_{p=0}^{n-1} \beta_p = \underset{n\to\infty}{O}(1)$ $\mathbb{P}(d\bar{\xi})$-almost surely.

- If $\frac{e^{-\lambda}}{\tilde{\eta}^2} \geqslant 1$ then $\frac{e^{-\lambda}}{\tilde{\eta}^2} + \varepsilon > 1$ for all $\varepsilon > 0$, thus

$$\sum_{p=0}^{n-1} \beta_p = \underset{n\to\infty}{O} \left(\sum_{p=0}^{n-1} \left(\frac{1}{\tilde{\eta} e^{2\lambda}} + \varepsilon \right)^p \right) = \underset{n\to\infty}{O} \left(\left(\frac{1}{\tilde{\eta} e^{2\lambda}} + \varepsilon \right)^n \right) \quad \mathbb{P}(d\bar{\xi})\text{-a.s.}$$

  and taking an infimum over $\varepsilon > 0$ yields (V.44).

Using (V.44), Lemma V.11, *ii)* and Proposition V.7, we derive

$$\limsup_{n\to\infty} \left(\frac{\alpha_n}{\lambda_{0,n}^2} \sum_{p=0}^{n-1} \beta_p \right)^{\frac{1}{n}} \leqslant \max\left(e^{-\lambda}, \tilde{\eta}^2\right) < 1 \quad \mathbb{P}(d\bar{\xi})\text{-a.s.} \tag{V.45}$$

Thus for any $\varepsilon > 0$,

$$\frac{\alpha_n}{\lambda_{0,n}^2} \sum_{p=0}^{n-1} \beta_p = \underset{n\to\infty}{O} \left(\left(\max\left(e^{-\lambda}, \tilde{\eta}^2\right) + \varepsilon \right)^n \right) \quad \mathbb{P}(d\bar{\xi})\text{-a.s.} \tag{V.46}$$

Finally, plugging (V.43) and (V.46) into (V.42), we obtain that for any $\varepsilon > 0$, it holds

$$\mathbb{E}_{\bar{\xi}}[\Delta_n^2] = \underset{n\to\infty}{O} \left(\left(\max\left(e^{-\lambda}, \tilde{\eta}^2\right) + \varepsilon \right)^n \right) \underset{n\to\infty}{\longrightarrow} 0 \quad \mathbb{P}(d\bar{\xi})\text{-a.s.} \tag{V.47}$$

Hence, if $\delta^2 > \max\left(e^{-\lambda}, \tilde{\eta}^2\right)$, for any $\varepsilon > 0$, it holds $\mathbb{P}(d\bar{\xi})-$almost surely

$$\mathbb{P}_{\bar{\xi}}[|\Delta_n| > \delta^n] \leqslant \frac{\mathbb{E}_{\bar{\xi}}[\Delta_n^2]}{\delta^{2n}} = \underset{n\to\infty}{O}\left(\left(\frac{\max\left(e^{-\lambda}, \tilde{\eta}^2\right) + \varepsilon}{\delta^2}\right)^n\right)$$

Consequently, by Borel-Cantelli's lemma, it holds $|\Delta_n| \leqslant \delta^n$ for n large enough, $\mathbb{P}$-a.s. This proves that for all $\delta > \max\left(e^{-\frac{\lambda}{2}}, \tilde{\eta}\right)$,

$$\frac{Z_n(f)}{\lambda_{0,n}} = W_n \frac{Z_0 M_{0,n} f}{\lambda_{0,n}} + \underset{n\to\infty}{O}(\delta^n) \quad \mathbb{P}\text{-a.s.}$$

Noticing that $\lim_{n\to\infty} \lambda_{0,n}^{\frac{1}{n}} = e^{\lambda}$, we get finally, for any $\delta > \max(e^{\frac{\lambda}{2}}, \tilde{\eta}e^{\lambda})$, $\mathbb{P}$-a.s.,

$$Z_n(f) = W_n Z_0 M_{0,n} f + \underset{n\to+\infty}{O}(\delta^n).$$

Thus claim *i)* is proved. Let us prove now *ii)*. We assume now that the function $f \neq$ is nonnegative and satisfies $\mathbb{E}\left[\log^-\left(\nu M_{1,N}(f)\right)\right] < +\infty$ for some deterministic integer $N \geqslant 1$. Let us write

$$\mu M_{0,n} f \geqslant \mu M_{0,n-N} M_{n-N,n-N+1} M_{n-N+1,n} f$$

and notice that

$$M_{n-N,n-N+1} M_{n-N+1,n} f(x) \geqslant c_{n-N} m_{n-N,n-N+1}(x) \nu_{n-N}(M_{n-N+1,n} f).$$

Integrating the last inequality with respect to the measure $\mu M_{0,n-N}(dx)$, we obtain

$$\mu M_{0,n} f \geqslant c_{n-N} \|\mu M_{0,n-N+1}\|_{TV} \nu_{n-N}(M_{n-N+1,n}(f)). \tag{V.48}$$

As a consequence from Lemma IV.7, we know that $\lim_{n\to\infty} \gamma_{n-N}^{\frac{1}{n}} = 1$. Since $\gamma_{n-N} \leqslant c_{n-N} \leqslant 1$, we deduce

$$\lim_{n\to\infty} c_{n-N}^{\frac{1}{n}} = 1. \tag{V.49}$$

Finally, from the integrability property $\mathbb{E}\left[\log^-\left(\nu M_{1,n} f\right)\right] < +\infty$, and the stationarity of the sequence M_n, we deduce that for all $\alpha > 0$,

$$\sum_{n\geqslant 0} \mathbb{P}\left[\log^-\left(\nu_{n-N} M_{n-N+1,n}\right) \geqslant \alpha n\right] = \sum_{n\geqslant 0} \mathbb{P}\left[\log^-\left(\nu M_{1,N}\right) \geqslant \alpha n\right] < +\infty$$

which yields

$$\sum_{n\geqslant 0} \mathbb{P}\left[\left(\nu_{n-N} M_{n-N+1,n}\right)^{\frac{1}{n}} \leqslant e^{-\alpha}\right] < +\infty.$$

Using Borel-Cantelli's lemma and letting α go to 0, we deduce that

$$\liminf_{n\geqslant 0} \left(\nu_{n-N} M_{n-N+1,n}\right)^{\frac{1}{n}} \geqslant 1. \tag{V.50}$$

Plugging (V.49), (V.50) and (V.13) into (V.48) yields

$$\liminf_{n\to\infty} (\mu M_{0,n} f)^{\frac{1}{n}} \geqslant e^{\lambda} \quad \mathbb{P}(d\bar{\xi})\text{-a.s.}$$

which can be strengthen into $\lim_{n\to\infty} (\mu M_{0,n} f)^{\frac{1}{n}} = e^{\lambda}$, using once again (V.13) and the fact that f is bounded. Therefore, we have $\delta^n = o(Z_0 M_{0,n}(f))$ $\mathbb{P}$-a.s, for any $\delta \in (\max(e^{\frac{\lambda}{2}}, \tilde{\eta} e^{\lambda}), e^{\lambda})$. Plugging this into *i)* clearly yields both claims of *ii)*.

$\square$

Let us prove now Lemma V.11.

Proof of Lemma V.11. First notice that, by (V.29),

$$\limsup_{n\to\infty} (\|f_n\|_{\infty})^{\frac{1}{n}} \leqslant \limsup_{n\to\infty} \left(1 + \frac{1}{\pi_n(h_n)}\right)^{\frac{1}{n}}.$$

where, by definition of π_n and by Lemma V.6, it holds

$$\pi_n(h_n) = \frac{Z_0 M_{0,n} h_n}{\|Z_0 M_{0,n}\|_{TV}} = Z_0(h_0) \frac{\lambda_{0,n}}{\|Z_0 M_{0,n}\|_{TV}}.$$

Thus, applying Theorem 10, *iv)* and Lemma V.7 we derive $\lim_{n\to\infty} (\pi_n(h_n))^{\frac{1}{n}} = 1$, which in turns implies $\limsup_{n\to\infty} (\|f_n\|_{\infty})^{\frac{1}{n}} \leqslant 1, \mathbb{P}(d\bar{\xi})$-a.s. *i)* is therefore proved.
By Proposition IV.4, it holds

$$\lim_{n\to\infty} \left(\prod_{i=0}^{n-1} (1-\gamma_i)\right)^{\frac{1}{n}} = \tilde{\eta} \text{ and } \lim_{n\to\infty} (\gamma_{n-1})^{\frac{1}{n}} = 1 \quad \mathbb{P}(d\bar{\xi})\text{-a.s.} \tag{V.51}$$

The estimate *ii)* is now derived combining *i)* with (V.51) and Theorem 10, *iv)*.
Finally, notice that

$$\begin{aligned}
|V_p(g)(x)| &\leqslant \left| \mathbb{E}_{\bar{\xi}} \left[\sum_{\substack{u,v\in\mathbb{G}_{p+1} \\ u\neq v}} g(X(u))g(X(v)) \middle| Z_p = \delta_x \right] \right| \\
&\leqslant \|g|_{\infty}^2 \left| \mathbb{E}_{\bar{\xi}} \left[\sum_{\substack{u,v\in\mathbb{G}_{p+1} \\ u\neq v}} 1 \middle| Z_p = \delta_x \right] \right| \\
&\leqslant \|g|_{\infty}^2 \left| \mathbb{E}_{\bar{\xi}} \left[\sum_{u,v\in\mathbb{G}_{p+1}} 1 \middle| Z_p = \delta_x \right] \right| \\
&\leqslant |g\|_{\infty}^2 \mathbb{E}_{\bar{\xi}} \left[\|Z_{p+1}\|_{TV}^2 \middle| Z_p = \delta_x \right]
\end{aligned}$$

Thus $\|\!|V_p|\!\| \leqslant \sup_{x\in\mathbb{X}} \mathbb{E}_{\bar{\xi}} \left[\|Z_{p+1}\|_{TV}^2 \middle| Z_p = \delta_x \right]$. Moreover, assumption **Ass. V.6** ensures that the random quantity $\|\!|V_p|\!\|$ is $\mathbb{P}(d\bar{\xi})$-a.s. finite, and that $\mathbb{E}\left[|\log \|\!|V_p|\!\||\right] < \infty$. This log-integrability, combined with the stationarity of (V_p) implies that $\lim_{p\to\infty}(\|\|V_p\|\|)^{\frac{1}{p}} = 1$, by arguments similar to those of Proposition IV.4. Combining this with (V.51) and Theorem 10, *iii)* yields

$$\begin{aligned}
\limsup_{p\to\infty}(\beta_p)^{\frac{1}{p}} &= \limsup_{p\to\infty}\left(\frac{\|Z_0M_{0,p}\|_{TV}\|\|V_p\|\|}{\gamma_p\|Z_0M_{0,p+1}\|_{TV}^2\prod_{i=0}^p(1-\gamma_i)^2}\right)^{\frac{1}{p}} \\
&\leqslant \frac{\lim_{p\to\infty}\left(|Z_0M_{0,p}\|_{TV}^{\frac{1}{p}}\right)\lim_{p\to\infty}\left(\|\|V_p\|\|^{\frac{1}{p}}\right)}{\lim_{p\to\infty}\gamma_p^{\frac{1}{p}}\left(\lim_{p\to\infty}\|Z_0M_{0,p+1}\|^{\frac{1}{p}}\right)^2\left(\lim_{p\to\infty}\left(\prod_{i=0}^p(1-\gamma_i)\right)^{\frac{1}{p}}\right)^2} \\
&\leqslant \frac{e^{-\lambda}}{e^{2\lambda}}.
\end{aligned}$$

This concludes the proof of *iii)*. □

V.3.4 Extinction and explosion

Proof of Proposition V.4. Let us consider a MGWRE and assume that conditionally on survival, it holds almost surely $\limsup_{n\to\infty} Z_n(\{x_0\}) = +\infty$. In other words,

$$\mathbb{P}_{\bar{\xi}}[\mathbf{Ext}] + \mathbb{P}_{\bar{\xi}}[\limsup Z_n(x_0) = +\infty] = 1 \quad \mathbb{P}[d\bar{\xi}]\text{-a.s} \tag{V.52}$$

We additionally assume that assumptions **Ass. V.1** to **Ass. V.5** hold, therefore $\mathbb{P}_{\bar{\xi}}[W > 0|Z_0 = \delta_x] > 0, \mathbb{P}(d\bar{\xi})$-a.s., for any type $x \in \mathbb{X}$ and in particular for $x = x_0$.

Let us introduce the stopping time $T_K = \inf\{n \geqslant 1|Z_n(x_0) \geqslant K\}$ and note for each $(x, \bar{e}) \in \mathbb{X} \times \mathcal{E}^{\mathbb{Z}_+}$,

$$q_{\mathbf{NonExp}}(x, \bar{e}) = \mathbb{P}[\mathbf{NonExp}\,|Z_0 = \delta_x, \bar{\xi} = \bar{e}] = \mathbb{P}[W > 0|Z_0 = \delta_x, \bar{\xi} = \bar{e}] \leqslant 1.$$

Since $\mathbb{P}_{\bar{\xi}}[\mathbf{NonExp}\,|Z_n] = \prod_{x\in\mathbb{X}} q_{\mathbf{NonExp}}(x, \theta^n(\bar{\xi}))^{Z_n(x)}$, we have $\mathbb{P}_{\bar{\xi}}[\mathbf{NonExp}\,|T_K = n] \leqslant q_{\mathbf{NonExp}}(x_0, \theta^n(\bar{\xi}))^K$. Thus,

$$\mathbb{P}_{\bar{\xi}}[\mathbf{NonExp} \cap \{T_K < \infty\}] \leqslant \sum_{n\geqslant 1}\mathbb{P}_{\bar{\xi}}[T_K = n]q_{\mathbf{NonExp}}(x_0, \theta^n(\bar{\xi}))^K,$$

where $\mathbb{P}_{\bar{\xi}}[T_K = n]$ is $\sigma(\xi_0, \ldots \xi_{n-1})$-measurable and $q_{\mathbf{NonExp}}(x_0, \theta^n(\bar{\xi}))$ is $\sigma((\xi_k)_{k\geqslant n})$-measurable. Since the sequence (ξ_n) is here assumed to be i.i.d, the quantities $\mathbb{P}_{\bar{\xi}}[T_K = n]$ and $q_{\mathbf{NonExp}}(x_0, \theta^n(\bar{\xi}))$ are independent from each other. Hence, integrating with respect to the environment, we get

$$\begin{aligned}
\mathbb{P}[\mathbf{NonExp} \cap \{T_K < \infty\}] &\leqslant \sum_{n\geqslant 1}\mathbb{P}[T_K = n]\mathbb{E}\left[q_{\mathbf{NonExp}}(x_0, \theta^n(\bar{\xi}))^K\right] \\
&= \sum_{n\geqslant 1}\mathbb{P}[T_K = n]\mathbb{E}\left[q_{\mathbf{NonExp}}(x_0, \bar{\xi})^K\right] \\
&= \mathbb{P}[T_K < \infty]\mathbb{E}\left[q_{\mathbf{NonExp}}(x_0, \bar{\xi})^K\right].
\end{aligned} \tag{V.53}$$

By Theorem 8, $q_{\mathbf{NonExp}}(x_0, \bar{\xi}) < 1\, \mathbb{P}(d\bar{\xi})$-a.s. This yields $\lim_{K\to\infty} q_{\mathbf{NonExp}}(x_0, \bar{\xi})^K = 0, \mathbb{P}(d\bar{\xi})$-a.s. and thus $\lim_{K\to\infty} \mathbb{P}[\mathbf{NonExp} \cap \{T_K < \infty\}] = 0$. We derive therefore

$$\begin{aligned}
\mathbb{P}\left[\mathbf{NonExp} \cap \{\limsup Z_n(0) = +\infty\}\right] &= \mathbb{P}\left[\mathbf{NonExp} \cap \bigcap_{K \geqslant 1} \{T_K = \infty\}\right] \\
&= \lim_{K\to\infty} \mathbb{P}[\mathbf{NonExp} \cap \{T_K < \infty\}] \\
&= 0,
\end{aligned}$$

hence

$$\mathbb{P}_{\bar{\xi}}\left[\mathbf{NonExp} \cap \{\limsup Z_n(0) = +\infty\}\right] = 0 \quad \mathbb{P}(d\bar{\xi})\text{-a.s.}$$

Finally, the equality

$$\mathbb{P}\left[\mathbf{NonExp} \cap \{\limsup Z_n(0) = +\infty\}\right] = \mathbb{E}\left[\mathbb{P}_{\bar{\xi}}\left[\mathbf{NonExp} \cap \{\limsup Z_n(0) = +\infty\}\right]\right]$$

yields

$$\mathbb{P}_{\bar{\xi}}\left[\mathbf{NonExp} \cap \{\limsup Z_n(0) = +\infty\}\right] = 0 \quad \mathbb{P}(d\bar{\xi})\text{-a.s.}$$

From this and (V.52), we derive $\mathbb{P}(d\bar{\xi})$-a.s.,

$$\mathbb{P}_{\bar{\xi}}\left[\mathbf{NonExp}\right] + \mathbb{P}_{\bar{\xi}}\left[\{\limsup Z_n(0) = +\infty\}\right] \leqslant 1 = \mathbb{P}_{\bar{\xi}}\left[\mathbf{Ext}\right] + \mathbb{P}_{\bar{\xi}}\left[\{\limsup Z_n(0) = +\infty\}\right].$$

Hence $\mathbb{P}_{\bar{\xi}}\left[\mathbf{NonExp}\right] \leqslant \mathbb{P}_{\bar{\xi}}\left[\mathbf{Ext}\right], \mathbb{P}(d\bar{\xi})$-a.s. Combining this with the clear inclusion $\mathbf{Ext} \subset \mathbf{NonExp}$ proves that $\mathbf{Ext}$ and $\mathbf{NonExp}$ only differ by a $\mathbb{P}$-negligible event. □

V.4 Discrete time model for an age structured population

V.4.1 Presentation of the model

In this subsection, we focus on an example of a population model where the type accounts for the age of the individuals. We choose to model this notion of the age of an individual by the integer number of generations since its birth. Therefore, we take $\mathbb{X} = \mathbb{Z}_+$, and $\mathcal{X} = \mathcal{P}(\mathbb{Z}_+)$. At each time step, an individual of time x may create a random number of individuals of age 0, which we refer to as *newborns*. Moreover, at each time step, after giving birth to newborns, each individual may die or survive to the next time step, in which case its offspring will additionally contain an individual of age $x+1$. For each $x \in \mathbb{X}$, $e \in \mathcal{E}$, let $F_{x,e}$ be an integer valued variable of mean $f_{x,e}$ (encoding the number of newborns), and $S_{x,e}$ be a Bernoulli variable of parameter $s_{x,e}$ (encoding the survival or death of the individual), independent of $F_{x,e}$. We define $L_{x,e} = (0, \dots, 0) \in \mathbb{X}^k$ if $(S_{x,e}, L_{x,e}) = (0, k)$ and $L_{x,e} = (x+1, 0, \dots, 0) \in \mathbb{X}^{k+1}$ if $(S_{x,e}, L_{x,e}) = (1, k)$. Choosing $\mathcal{L}_{x,e}$ as the distribution of $L_{x,e}$, we introduce a particular class of GWRE which is adapted to model an age structured population. In particular, we write $N^{k,n}_{x,e} = \psi(L^{k,n}_{x,e}) = F_{x,e}\delta_0 + S_{x,e}\delta_{x+1}$. Noting $f_{x,e} = \mathbb{E}[F_{x,e}]$ and $s_{x,e} = \mathbb{E}[F_{x,e}]$, the mean matrix of such a process is of the form

$$M(e)=\begin{pmatrix} f_{0,} & s_{0,e} & 0 & 0 & \dots \\ f_{1,e} & 0 & s_{1,e} & 0 & \dots \\ f_{2,e} & 0 & 0 & s_{2,e} & \ddots \\ f_{3,e} & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This is clearly an infinite dimensional generalization of the Leslie matrices presented for example in [Cas10]. For this reason we call a MGWRE with such offspring distribution a Leslie-GWRE or (L-GWRE for short). In this context, the random variables Z_n are point measures on the discrete type space $\mathbb{X}=\mathbb{Z}_+$. Therefore Z_n is characterized by the vector $(Z_n(k))_{k\geqslant 0}$, where we note for short $Z_n(k)=Z_n(\{k\})$.

V.4.2 Preliminary assumption

We recall that if X_1 and X_2 are two real valued random variables we say that X_2 dominates stochastically X_1 (and note $X_1 \preceq X_2$) if

$$\forall t \in \mathbb{R},\, \mathbb{P}[X_1 \geqslant t] \leqslant \mathbb{P}[X_2 \geqslant t].$$

In the whole section, we assume that the following assumption holds.

Ass. V.7. The random variables $(F_{x,e}, S_{x,e})_{x\in\mathbb{X}, e\in\mathcal{E}}$ satisfy

i) $s_{x,e}, f_{x,e} > 0$ for any $x \in \mathbb{X}$, $e \in \mathbb{E}$.

ii) $F_{x+1,e} \preceq F_{x,e}$ and $S_{x+1,e} \preceq S_{x,e}$ for any $x \in \mathbb{X}, e \in \mathbb{E}$.

iii) $\mathbb{E}[\log f_{0,\xi_0}] < \infty$

iv) $\mathbb{E}\left[\left|\log\left(\sup_{x\in\mathbb{X}} \frac{s_{x,\xi_0}}{f_{x,\xi_0}}\right)\right|\right] < \infty$

By Proposition IV.16, we have

Proposition V.12. *Consider a L-GWRE $(Z_n)_{n\geqslant 0}$, with a stationary and ergodic environmental sequence $\bar{\xi}=(\xi_n)_{n\geqslant 0}$, such that the offspring random variables $(F_{x,e}, S_{x,e})_{x\in\mathbb{X}, e\in\mathcal{E}}$ satisfy **Ass. V.7**. Then the associated quenched first moment semigroup (M_n) satisfies Assumptions **Ass. V.1** to **Ass. V.3**, and therefore the conclusions of Theorems 10 and 7 hold.*

In **Ass. V.7**, we restrict ourselves to offspring distributions $(F_{x,e})_{x\geqslant 0}, (S_{x,e})_{x\geqslant 0}$ which are nonincreasing as a function of x, with respect to the stochastic domination relation. This is consistent with the intuition that older individuals have both a higher chance of dying and tend to make less children than younger ones.

Remark V.13. In some biological contexts, one might expect that the peak age in terms of fertility or survival is some age $x_0 > 0$. This is not compatible with Assumption **Ass. V.7**-*ii)*. To allow this situation to happen, one can weaken *ii)* and assume instead that the variables $(F_{x,e})$ and $(S_{x,e})$ are decreasing only after some rank $x=x_0$. Then some additional controls on the variables $F_{x,e}$ and $S_{x,e}$ for $x \leqslant x_0$ allow to deduce **Ass. V.1**-**Ass. V.3** using Proposition IV.16 and Remark IV.17.

V.4.3 Non degeneneracy of (W_n) for L-GWRE

Let us now focus on assumption **Ass. V.5** and see how to check it when working with L-GWREs.

Lemma V.14. *Let (Z_n) be a L-GWRE satisfying **Ass. V.7** and assume that there exists some $\varepsilon > 0$ such that at least one of the following assertions holds :*

$$\mathbb{E}\left[\sup_{x\in\mathbb{X}} \frac{\sum_{k\geqslant 0}(k+1)\log^+(k+1)^{1+\varepsilon}\mathbb{P}\left[F_{x,\xi_0}=k|\bar{\xi}\right]}{f_{x,\xi_0}}\right] < \infty, \tag{V.54}$$

$$\mathbb{E}\left[\sup_{x\in\mathbb{X}} \frac{\sum_{k\geqslant 0}\log^+(k+1)^{1+\varepsilon}\mathbb{P}\left[F_{x,\xi_0}\geqslant k|\bar{\xi}\right]}{f_{x,\xi_0}}\right] < \infty. \tag{V.55}$$

*Then **Ass. V.5** holds. If additionally $\lambda > 0$, then the conclusions of Theorem 8 hold.*

We postpone the proof of this lemma at the end of Subsection V.4.3. This statement proves that for L-GWREs, assumption **Ass. V.5** is a consequence of some uniform integrability assumption on the variables $(F_{x,e})_{x,e}$. Unfortunately, the argument of the supremum both in (V.54) and (V.55) is not monotonous in $F_{x,e}$, thus the monotonicity assumption **Ass. V.7**-*ii)* does not allow to simply remove the supremum in criterions (V.54) and (V.55). However, these criterions can be pretty easily checked assuming some control on the tails of the variables $(F_{x,e})_{x,e}$. For any real valued sequences (u_k) and (v_k) and any $\beta > 0$, we set

$$u_k \overset{\beta}{\asymp} v_k \Leftrightarrow \beta^{-1}v_k \leqslant u_k \leqslant \beta v_k \text{ for all } k \geqslant 0.$$

Let us use this notation to introduce three classes of distributions for $(F_{x,e})_{x\in\mathbb{X},e\in\mathcal{E}}$.

Definition V.15. *Let $(F_{x,e})_{x\in\mathbb{X},e\in\mathcal{E}}$ be a familiy of integer-valued random variables. We say that $(F_{x,e})_{x\in\mathbb{X},e\in\mathcal{E}}$ belongs to*

- ***BS** if and only if for each $(x,e)\in\mathbb{X}\times\mathcal{E}$, there exists a deterministic integer $A_{x,e}$ and a real number $\varepsilon > 0$ such that*

$$\mathbb{P}[F_{x,e}\geqslant A_{x,e}] = 0 \text{ and } \mathbb{E}\left[\sup_{x\in\mathbb{X}} \frac{A_{x,\xi_0}\left(\log^+(A_{x,\xi_0})\right)^{1+\varepsilon}}{f_{x,\xi_0}}\right] < \infty,$$

- ***ExpTail** if and only if for each $(x,e)\in\mathbb{X}\times\mathcal{E}$, there exists $q(x,e) < 1, \alpha(x,e) > 0$ and $\beta(e) > 0$ such that*

$$\mathbb{P}[F_{x,e}\geqslant k] \overset{\beta(e)}{\asymp} \alpha(x,e)q(x,e)^k$$

and

$$\mathbb{E}\left[\frac{\beta(\xi_0)^2}{1-\sup_{x\in\mathbb{X}} q(x,\xi_0)}\right] < \infty.$$

- ***PolyTail*** *if and only if for each* $(x,e) \in \mathbb{X} \times \mathcal{E}$, *there exists* $\delta(x,e) > 1, \alpha(x,e) > 0$ *and* $\beta(e) > 0$, *it holds*

$$\mathbb{P}[F_{x,e} \geqslant k] \overset{\beta(e)}{\asymp} \alpha(x,e) k^{-\delta(x,e)},$$

where

$$\mathbb{E}\left[\beta(\xi_0)^2 \inf_{x \in \mathbb{X}} \delta(x,\xi_0)\right] < \infty \text{ and } \mathbb{E}\left[\frac{\beta(\xi_0)^2}{(\inf_{x\in\mathbb{X}} \delta(x,\xi_0) - 1)^{1+\varepsilon}}\right] < \infty \text{ for some } \varepsilon > 0.$$

When the random variables $(F_{x,e})_{x\in\mathbb{X}, e\in\mathcal{E}}$ are in one of the three previous classes, our assumption **Ass. V.5** can be verified, allowing to apply Theorem 8.

Proposition V.16. *Consider a supercritical L-GWRE satisfying* ***Ass. V.7***. *We assume that the family of variables* $(F_{x,e})_{x,e}$ *belongs to one of the classes* ***BS***, ***ExpTail***, ***PolyTail***. *Then, the process* (W_n) *is well defined, it is a nonnegative martingale and its limit* W *satisfies*

$$\mathbb{P}_{\bar{\xi}}[W > 0] > 0 \quad \mathbb{P}(d\bar{\xi})\text{-}a.s.$$

Proof. by Proposition V.12, assumptions **Ass. V.1** to **Ass. V.3** hold when **Ass. V.7** is satisfied. Since we additionally assume that $\lambda > 0$, it only remains to check **Ass. V.5** to apply Theorem 8. Let us check it in each of the three cases.

- If $(F_{x,e})_{x\in\mathbb{X}, e\in\mathcal{E}}$ belongs to **BS**, then

$$\sum_{k\geqslant 0} (k+1)\log(k+1)^{1+\varepsilon} \frac{\mathbb{P}[F_{x,e} = k]}{f_{x,e}} \leqslant \frac{(A_{x,e})\log(A_{x,e})^{1+\varepsilon}}{f_{x,e}}.$$

  Therefore, applying criterion (V.54), we can show that **Ass. V.5** is satisfied as soon as

$$\mathbb{E}\left[\sup_{x\in\mathbb{X}} \frac{A_{x,\xi_0}\log(A_{x,\xi_0})^{1+\varepsilon}}{f_{x,\xi_0}}\right] < \infty.$$

- Assume now that $(F_{x,e})_{x\in\mathbb{X}, e\in\mathcal{E}}$ belongs to **ExpTail**. In this case, we apply criterion (V.55). On the one hand, since $\mathbb{P}[F_{x,e} \geqslant k] \geqslant \beta(e)^{-1}\alpha(x,e)q(x,e)^k$, we can derive

$$\begin{aligned} f_{x,e} &= \mathbb{E}[F_{x,e}] \\ &= \sum_{k\geqslant 1} \mathbb{P}[F_{x,e} \geqslant k] \\ &\geqslant \beta(e)^{-1}\alpha(x,e) \sum_{k\geqslant 1} q(x,e)^k \\ &\geqslant \beta(e)^{-1}\alpha(x,e) \frac{q(x,e)}{1-q(x,e)}. \end{aligned}$$

On the other hand, for any $\varepsilon > 0$,

$$\sum_{k\geqslant 0} \log^+(k+1)^{1+\varepsilon}\mathbb{P}\left[F_{x,e} \geqslant k\right] \leqslant \beta(e)\alpha(x,e)\sum_{k\geqslant 1}\log^+(k+1)^{1+\varepsilon}q(x,e)^k.$$

Consequently, noting $C = \sup_{k\geqslant 1}\frac{\log^+(k+1)^{1+\varepsilon}}{k}$,

$$\begin{aligned}\frac{\sum_{k\geqslant 0}\log^+(k+1)^{1+\varepsilon}\mathbb{P}\left[F_{x,e}\geqslant k\right]}{f_{x,e}} &\leqslant \beta(e)^2(1-q(x,e))\sum_{k\geqslant 1}\log^+(k+1)^{1+\varepsilon}q(x,e)^{k-1}\\ &\leqslant C\beta(e)^2(1-q(x,e))\sum_{k\geqslant 1}kq(x,e)^{k-1}\\ &\leqslant C\beta(e)^2(1-q(x,e))\frac{1}{(1-q(x,e))^2}\\ &\leqslant C\frac{\beta(e)^2}{1-q(x,e)}.\end{aligned}$$

Taking a supremum over $x\in\mathbb{X}$ and integrating over the environmental space, we get indeed

$$\mathbb{E}\left[\sup_{x\in\mathbb{X}}\frac{\sum_{k\geqslant 0}\log^+(k+1)^{1+\varepsilon}\mathbb{P}\left[F_{x,\xi_0}\geqslant k|\xi\right]}{f_{x,e}}\right] \leqslant C(\varepsilon)\mathbb{E}\left[\frac{\beta(\xi_0)^2}{1-\sup_{x\in\mathbb{X}}q(x,\xi_0)}\right] < \infty,$$

which proves **Ass. V.5**.

- We suppose finally that $(F_{x,e})_{x\in\mathbb{X},e\in\mathcal{E}}$ belongs to **Polytail**. On the one hand, we derive similarly as in the previous case

$$\begin{aligned}f_{x,e} &= \sum_{k\geqslant 1}\mathbb{P}[F_{x,e}\geqslant k]\\ &\geqslant \beta(e)^{-1}\alpha(x,e)\sum_{k\geqslant 1}k^{-\delta(x,e)}\\ &\geqslant \beta(e)^{-1}\alpha(x,e)\int_1^{+\infty}t^{-\delta(x,e)}dt\\ &\geqslant \frac{\alpha(x,e)}{(\delta(x,e)-1)\beta(e)}\end{aligned}$$

and for any $\varepsilon > 0$,

$$\sum_{k\geqslant 0}\log^+(k+1)^{1+\varepsilon}\mathbb{P}\left[F_{x,e}\geqslant k\right] \leqslant \beta(e)\alpha(x,e)\sum_{k\geqslant 1}\log^+(k+1)^{1+\varepsilon}k^{-\delta(x,e)}.$$

Hence, we get

$$\frac{\sum_{k\geqslant 0}\log^+(k+1)^{1+\varepsilon}\mathbb{P}\left[F_{x,e}\geqslant k\right]}{f_{x,e}} \leqslant \beta(e)^2(\delta(x,e)-1)\sum_{k\geqslant 1}\log^+(k+1)^{1+\varepsilon}k^{-\delta(x,e)}$$
$$\leqslant \beta(e)^2(\delta(x,e)-1)u(1+\varepsilon,\delta(x,e))$$

where $u:(s,a)\in(1,\infty)^2\mapsto\sum_{k\geqslant 1}\log(k+1)^s k^{-a}$. A careful study of u (see Appendix, Lemma A) allows to show that

$$u(1+\varepsilon,\delta(x,e))\leqslant C_1+\frac{C_2}{(\delta(x,e)-1)^{2+\varepsilon}},$$

where C_1 and C_2 are two explicit numbers depending only on ε. This yields

$$\frac{\sum_{k\geqslant 0}\log^+(k+1)^{1+\varepsilon}\mathbb{P}\left[F_{x,e}\geqslant k\right]}{f_{x,e}} \leqslant C_1\beta(e)^2(\delta(x,e)-1)$$
$$+\,C_2\frac{\beta(e)^2}{(\delta(x,e)-1)^{1+\varepsilon}}$$

and finally

$$\mathbb{E}\left[\sup_{x\in\mathbb{X}}\frac{\sum_{k\geqslant 0}\log^+(k+1)^{1+\varepsilon}\mathbb{P}\left[F_{x,\xi_0}\geqslant k|\bar{\xi}\right]}{f_{x,e}}\right]\leqslant C_1\mathbb{E}\left[\beta(\xi_0)^2(\inf_{x\in\mathbb{X}}\delta(x,\xi_0)-1)\right]$$
$$+\,C_2\mathbb{E}\left[\frac{\beta(\xi_0)^2}{(\inf_{x\in\mathbb{X}}\delta(x,\xi_0)-1)^{1+\varepsilon}}\right]<\infty.$$

As a consequence, **Ass. V.5** also holds on the class **PolyTail**.

□

Proof of Lemma V.14. We assume that the distributions of the variables $(F_{x,e},S_{x,e})$ satisfy **Ass. V.7**. Let us check that (V.54) and (V.55) both imply **Ass. V.5**. Conditionally on $Z_0=\delta_x$, it holds

$$h_1(0)F_{x,\xi_0}\leqslant Z_1(h_1)=S_{x,\xi_0}h_1(x+1)+F_{x,\xi_0}h_1(0)\leqslant(1+F_{x,\xi_0})$$

since $S_{x,\xi_0}\in\{0,1\}$ and h_1 is positive and bounded by 1. Therefore

$$M_0h_1(x)=\mathbb{E}[Z_1(h_1)|\bar{\xi},Z_0=\delta_x]\geqslant f_{x,\xi_0}h_1(0). \tag{V.56}$$

Moreover, notice that $h_1(0)=\lim_{n\to\infty}\frac{\|\delta_0M_{1,n}\|_{TV}}{\|\!|M_{1,n}\|\!|}\geqslant d_1$ by definition of d_1. By Lemma IV.20 and Remark IV.17, we can choose d_1 in such a way that

$$1\geqslant d_1\geqslant\inf_{x,n}\frac{s_{0,\xi_0}\ldots s_{n,\xi_n}f_{n+1,\xi_{n+1}}}{s_{x+0,\xi_0}\ldots s_{x+n,\xi_n}f_{x+n+1,\xi_{n+1}}}$$

Moreover, **Ass. V.7**-*ii)* implies that the sequences $(f_{x,e})_{x\geqslant 0}$ and $(s_{x,e})_{x\geqslant 0}$ are nonincreasing, thus

$$\inf_{x,n} \frac{s_{0,\xi_0} \dots s_{n,\xi_n} f_{n+1,\xi_{n+1}}}{s_{x+0,\xi_0} \dots s_{x+n,\xi_n} f_{x+n+1,\xi_{n+1}}} \geqslant 1$$

and $d_1 = 1$. Therefore, under **Ass. V.7**, it holds $h_1(0) \geqslant 1$ $\mathbb{P}(d\xi)$-a.s., hence $h_1(0) = 0$ since $\|h_1\|_\infty = 1$. Plugging this in (V.56) yields

$$M_0 h_1(x) \geqslant f_{x,\xi_0}. \tag{V.57}$$

We choose to verify **Ass. V.5** with $f(t) = \log^+(t)^\varepsilon$, which satisfies the integrability assumption $\int^{+\infty}(t\log^+(t)f(t))^{-1}dt < \infty$. Therefore, we set $g(t) = t(\log^+(t))^{1+\varepsilon}$ and derive

$$\begin{aligned}
\mathbb{E}\left[g(Z_1(h_1))|Z_0 = \delta_x, \bar{\xi}\right] &\leqslant \mathbb{E}\left[g(F_{x,\xi_0}+1)|\bar{\xi}\right] \\
&\leqslant \sum_{k\geqslant 0} g(k+1)\mathbb{P}\left[F_{x,\xi_0} = k|\bar{\xi}\right] \\
&\leqslant \sum_{k\geqslant 0} (k+1)\log^+(k+1)^{1+\varepsilon}\mathbb{P}\left[F_{x,\xi_0} = k|\bar{\xi}\right].
\end{aligned} \tag{V.58}$$

Putting together (V.58) and (V.57), taking a supremum in x and integrating with respect to $\bar{\xi}$, we show that

$$\mathbb{E}\left[\sup_{x\in\mathbb{X}} \frac{\mathbb{E}\left[g(Z_1(h_1))|\bar{\xi}, Z_0 = \delta_x\right]}{M_0h_1(x)}\right] \leqslant \mathbb{E}\left[\sup_{x\in\mathbb{X}} \frac{\sum_{k\geqslant 0}(k+1)\log^+(k+1)^{1+\varepsilon}\mathbb{P}\left[F_{x,\xi_0} = k|\bar{\xi}\right]}{f_{x,\xi_0}}\right].$$

As a consequence, (V.54) is indeed sufficient for **Ass. V.5**. Let us check now that (V.55) also implies **Ass. V.5**. We perform an integration by parts in (V.58) and derive

$$\mathbb{E}\left[g(Z_1(h_1))|Z_0 = \delta_x, \bar{\xi}\right] \leqslant \sum_{k\geqslant 1}(g(k+1) - g(k))\mathbb{P}\left[F_{x,\xi_0} \geqslant k|\bar{\xi}\right].$$

An elementary function study (see Appendix, Lemma B) allows to show that there exists some constant A depending only on $\varepsilon > 0$, such that $g(k+1) - g(k) \leqslant A\log(1+k)^{1+\varepsilon}$ for all $k \geqslant 1$. This implies

$$\mathbb{E}\left[g(Z_1(h_1))|Z_0 = \delta_x, \bar{\xi}\right] \leqslant A\sum_{k\geqslant 1}\log(1+k)^{1+\varepsilon}\mathbb{P}\left[F_{x,\xi_0} \geqslant k|\bar{\xi}\right]. \tag{V.59}$$

Similarly, combining (V.57) and (V.59), one proves that

$$\mathbb{E}\left[\sup_{x\in\mathbb{X}} \frac{\mathbb{E}\left[g(Z_1(h_1))|\bar{\xi}, Z_0 = \delta_x\right]}{M_0h_1(x)}\right] \leqslant A\mathbb{E}\left[\sup_{x\in\mathbb{X}} \frac{\sum_{k\geqslant 1}\log(1+k)^{1+\varepsilon}\mathbb{P}\left[F_{x,\xi_0} \geqslant k|\bar{\xi}\right]}{f_{x,\xi_0}}\right],$$

therefore (V.55) indeed implies **Ass. V.5**. □

V.4.4 Asymptotic type distribution in the L-GWRE

In the case of a L-GWRE, the condition **Ass. V.6** which is needed to apply Theorem 9 is satisfied as soon as the offpring distributions $(F_{x,e})_{x,e}$ satisfy the moment condition

$$\mathbb{E}\left[\log^+\left(\sup_{x\in\mathbb{X}}\mathbb{E}[(F_{x,\xi_0}+1)^2]\right)\right] < \infty.$$

Since checking this last condition on a particular model is pretty straightforward, we do not discuss it further and focus now on the additionnal condition

$$\mathbb{E}[\log^+(\nu M_{1,N}(f))] < +\infty$$

which guarantees that the assertion *ii)* of Theorem 9 holds. Namely, we state and prove the following proposition.

Proposition V.17. *Consider a L-GWRE such that Assumption* ***Ass. V.7*** *holds. We assume moreover that* $\mathbb{E}[-\log(s_{x,\xi_0})] < +\infty$ *for all* $x \in \mathbb{X}$*. Then for any non-negative and nonzero function* f*, there exists an deterministic integer* $N \geqslant 0$ *such that*

$$\mathbb{E}[\log^+(\delta_0 M_{1,N}(f))] < +\infty$$

.

When the conditions of this Proposition hold, as well as the moment conditions required for Theorem 9, we can apply Theorem 9,*ii)* to any function f, and in particular to $f = \mathbb{1}_y$ for any type y. We obtain hence

$$\lim_{n\to\infty}\frac{Z_n(y)}{\mathbb{E}[Z_n(y)|Z_0,\bar{\xi}]} = W \quad \mathbb{P}\text{-a.s.}$$

as well as $\lim\,(Z_n(y))^{\frac{1}{n}} = e^{\lambda}$ almost surely on $\{W > 0\}$, for any initial population Z_0 and any type $y \in \mathbb{X}$.

Proof of Proposition V.17. Let f be non-negative and non-zero. We can choose $y \in \mathbb{X}$ such that $f(y) > 0$ and write $f \geqslant f(y)\mathbb{1}_y$. Thus, to prove $\mathbb{E}[\log^-(\delta_0 M_{1,N} f)] < +\infty$ we only need to prove that $\mathbb{E}[\log^-(M_{1,N}(0,y))] < +\infty$, where $M_{1,N}(0,y)$ refers to the entry on the 0-th row and y-th column of the matrix $M_{0,N}$. We set $N = y+1$, and write $M_{1,y+1}(0,y) \geqslant M_1(0,1)\ldots M_y(y-1,y) = \prod_{x=1}^{y} s_{x-1,\xi_x}$. Hence,

$$\mathbb{E}[\log^-(M_{1,N}(0,y))] \leqslant \sum_{x=1}^{y}\mathbb{E}[\log^-(s_{x-1,\xi_x})] < +\infty,$$

if $\mathbb{E}[-\log(s_{x,\xi_0})] < +\infty$ for all x. ☐

V.4.5 Extinction and explosion

As explained in the introduction, we do not expect a dichotomy between the events **Ext** and $\{W > 0\}$ in general when $\mathbb{X}$ is infinte. However, Proposition V.4 states that this dichotomy holds when there exists a type x_0 such that $\limsup_{n\to\infty} Z_n(x_0) = +\infty$ a.s. on the survival event. In the next statement, we prove that under mild assumptions, this situation happens in the particular case of an L-GWRE.

Proposition V.18. *Let (Z_n) be a L-GWRE, satisfying the assumptions of Proposition V.12, as well as **Ass. V.5**. We suppose additionally that*

i) $\inf_{x,e} \mathbb{P}[F_{x,e} = 0] = a > 0$

ii) $s = \sup_{x,e} s_{x,e} < 1$.

Then (V.11) holds a.s. on survival. If additionally $\bar{\xi}$ is i.i.d then Proposition V.4 applies and

$$\mathbb{P}\left[\mathbf{NonExp} \triangle \mathbf{Ext}\right] = 0.$$

Notice that under the stochastic monotonicity assumption **Ass. V.7**,*ii)*, it holds

$$a = \inf_{e\in\mathbb{E}} \mathbb{P}[F_{0,e} = 0] \text{ and } s = \sup_{e\in\mathbb{E}} s_{0,e}.$$

Our proof relies on two lemmas, in which we prove results of independent interest.

Lemma V.19. *Let (Z_n) be a L-GWRE such that $\inf_{x,e} \mathbb{P}[F_{x,e} = 0] = a > 0$ and $s = \sup_{x,e} s_{x,e} < 1$, then*

$$\mathbb{P}_{\bar{\xi}}[\mathbf{Ext}] + \mathbb{P}_{\bar{\xi}}[\lim_{n\to+\infty} \|Z_n\| = +\infty] = 1 \quad \mathbb{P}(d\bar{\xi})\text{-}a.s.$$

Lemma V.20. *Let (Z_n) be a L-GWRE such that $s = \sup_{x,e} s_{x,e} < 1$, then it holds*

$$\left\{\lim_{n\to\infty} \|Z_n\| = +\infty\right\} \subset \left\{\limsup_{n\to\infty} Z_n(0) = +\infty\right\}$$

up to a $\mathbb{P}$-negligeable event.

Once again, we postpone the proofs of the lemmas and focus first on the proof of Proposition V.18, which is rather short.

Proof of Proposition V.18. Combining Lemmas V.19 and V.20, we derive

$$\mathbb{P}_{\bar{\xi}}[\mathbf{Ext}] + \mathbb{P}_{\bar{\xi}}[\limsup Z_n(0) = +\infty] \geqslant 1 \quad \mathbb{P}(d\bar{\xi})\text{-a.s.}$$

Since the events **Ext** and $\{\limsup Z_n(0) = +\infty\}$ are clearly incompatible, this proves

$$\mathbb{P}_{\bar{\xi}}[\mathbf{Ext}] + \mathbb{P}_{\bar{\xi}}[\limsup Z_n(0) = +\infty] = 1 \quad \mathbb{P}(d\bar{\xi})\text{-a.s,}$$

in other words, $\mathbb{P}$-almost surely, conditionally on survival, it holds $\limsup_{n\to\infty} Z_n(0) = +\infty$. Thus Proposition V.4 applies if $\bar{\xi}$ is i.i.d. $\square$

Let us move now to the proof of the lemmas.

Proof of Lemma V.19. We adapt here arguments from [Har68, Theorem 11.2]. We notice that

$$\mathbb{P}[Z_1 = 0|Z_0, \bar{\xi}] = \prod_{x \in \mathbb{X}} \mathbb{P}[(F_{x,\xi_0}, S_{x,\xi_0}) = 0|\bar{\xi}]^{Z_0(x)} \geqslant (as)^{\|Z_0\|}.$$

Let us note

$$\begin{aligned} R_K(\bar{e}, Z) &= \mathbb{P}[0 < \|Z_n\| \leqslant K \text{ infinitely often } |\bar{\xi} = \bar{e}, Z_0 = Z], \\ \Omega_K &= \{Z \in \mathcal{N} | 0 < \|Z\| \leqslant K\}, \\ \bar{R}_K &= \sup_{\substack{\bar{e} \in \mathcal{E}^{\mathbb{Z}_+} \\ Z \in \Omega_K}} R_K(\bar{e}, Z) \\ \text{and } T_K &= \inf\{n \geqslant 1, Z_n \in \Omega_K\} \in \mathbb{Z}_+ \cup \{+\infty\}, \end{aligned}$$

for each $K \geqslant 1$, $Z \in \mathcal{N}$ and $\bar{e} \in \mathcal{E}^{\mathbb{Z}_+}$. It clearly holds

$$\begin{aligned} R_K(\bar{e}, Z) &= \mathbb{E}\left[R_K(\theta^{T_K}(\bar{e}), Z_{T_K}) \mathbb{1}_{T_K < \infty} | \bar{\xi} = \bar{e}, Z_0 = Z\right] \\ &\leqslant \bar{R}_K \mathbb{P}[T_K < \infty | \bar{\xi} = \bar{e}, Z_0 = Z]. \end{aligned} \tag{V.60}$$

However, $\{Z_1 = 0\} \subset \{T_K = \infty\}$, thus

$$\mathbb{P}[T_K < \infty | \bar{\xi} = \bar{e}, Z_0 = Z] \leqslant 1 - \mathbb{P}[Z_1 = 0 | \bar{\xi} = \bar{e}, Z_0 = Z] \leqslant 1 - (as)^{\|Z\|}.$$

In particular, if $Z \in \Omega_K$ then $\|Z\| \leqslant K$ thus

$$\mathbb{P}[T_K < \infty | \bar{\xi} = \bar{e}, Z_0 = Z] \leqslant 1 - (as)^K.$$

Plugging this into (V.60), and taking a supremum over $Z \in \Omega_K$, we get

$$\bar{R}_K \leqslant \bar{R}_K \sup_{\substack{\bar{e} \in \mathcal{E}^{\mathbb{Z}_+} \\ Z \in \Omega_K}} \mathbb{P}[T_K < \infty | \bar{\xi} = \bar{e}, Z_0 = Z] \leqslant \bar{R}_K \left(1 - (as)^K\right),$$

where $1 - (as)^K < 1$. Thus $\bar{R}_K = 0$ for any $K \geqslant 1$. Hence for any $Z \in \mathcal{N}$ and $\bar{e} \in \mathcal{E}^{\mathbb{Z}_+}$ it holds

$$\mathbb{P}\left[0 < \liminf \|Z_n\| < \infty | \bar{\xi} = \bar{e}, Z_0 = Z\right] = \lim_{K \to \infty} R_K(\bar{e}, Z) = 0,$$

which concludes. □

Proof of Lemma V.20. By Definition V.2, the L-GWRE (Z_n) satisfies the recurrence relation

$$Z_{n+1}(x+1) = \sum_{k=1}^{Z_n(x)} S_{x,\xi_n}^{k,n} \text{ and } Z_{n+1}(0) = \sum_{x \in \mathbb{X}} F_{x,\xi_n}^{k,n},$$

the $(S_{x,e}^{k,n} : e \in \mathcal{E}, k, n, x \geqslant 0)$ are mutually independent Bernoulli variables such that $\mathbb{E}[S_{x,e}^{k,n}] = s_{x,e} \leqslant s$. Thus, we may consider a coupling of the array $(S_{x,e}^{k,n})_{k,n,x,e}$ with an array $(\tilde{S}_{x,e}^{k,n})_{x,k,n,e}$ of mutually independent Bernoulli variables of mean s in such a way that almost surely, for any $k, n \in \mathbb{Z}_+, x \in \mathbb{X}, e \in \mathcal{E}$

$$S^{k,n}_{x,e} \leqslant \tilde{S}^{k,n}_{x,e}.$$

For each integer K, we define a process $(\tilde{Z}^K_n)$ with values in $\mathcal{N}$ by setting $\tilde{Z}_0 = Z_0$ and for any $n \geqslant 0$,

$$\tilde{Z}^K_{n+1}(x+1) = \sum_{i=1}^{\tilde{Z}^K_n(x)} \tilde{S}^{i,n}_{x,\xi_n} \text{ for } x \geqslant 0, \text{ and } \tilde{Z}^K_{n+1}(0) = K.$$

Recalling the notation $\Omega_K = \{\sup_{n\geqslant 0} Z_n(0) \leqslant K\}$, a mere induction allows to show that for all $n \in \mathbb{Z}_+$, we have $Z_n \mathbb{1}_{\Omega_K} \leqslant \tilde{Z}^K_n$. Moreover, it holds

$$\|\tilde{Z}^K_{n+1}\| = K + \sum_{x\in\mathbb{X}} \sum_{i=1}^{Z_n(x)} S^{i,n}_{x,\xi_n},$$

and conditionally on $\bar{\xi}$ and $\tilde{Z}^K_n$

$$\|\tilde{Z}^K_{n+1}\| \overset{d}{=} K + \sum_{i=1}^{\|\tilde{Z}^K_n\|} S^{i,n},$$

where the $S^{i,n}$ are independent Bernoulli variables of parameter s. Therefore, the process $(\|Z^K_n\|)_{n\geqslant 0}$ is distributed as a Galton-Watson process with immigration, where the offspring distribution is a Bernoulli law of parameter s and the immigration is deterministic with value K. This process describes a monotype particle system where each particle present at time n survives independently from the others with probability s (and dies with probability $1-s$), and K new particles arrive at each time step. As a consequence, noting $S'^n_{i,k}$ the Bernoulli variable describing the survival from time n to time $n+1$ of the i-th particle arrived at time k, it is clear that the law of the process $(\tilde{Z}^K_n)_{0\leqslant n}$ satisfies

$$(\tilde{Z}^K_n)_{n\geqslant 0} \overset{d}{=} \left(\sum_{k=1}^{n} \sum_{i=1}^{K} S'^k_{i,k} \dots S'^{n-1}_{i,k} + \sum_{i=1}^{\|\tilde{Z}^K_0\|} S'^0_{i,0} \dots S'^{n-1}_{i,0}. \right)_{n\geqslant 0},$$

where the $(S'^n_{i,k})_{i,k,n\geqslant 0}$ are mutually independent. At a fixed time n, it holds therefore

$$\tilde{Z}^K_n \overset{d}{=} \sum_{k=1}^{n} \sum_{i=1}^{K} S'^k_{i,k} \dots S'^{n-1}_{i,k} + \sum_{i=1}^{\|\tilde{Z}^K_0\|} S'^0_{i,0} \dots S'^{n-1}_{i,0} \overset{d}{=} \sum_{k=0}^{n-1} \sum_{i=1}^{K} S'^1_{i,k} \dots S'^k_{i,k} + \sum_{i=1}^{\|\tilde{Z}^K_0\|} S'^0_{i,0} \dots S'^{n-1}_{i,0}.$$

And as $n \to \infty$, it holds $\mathbb{P}$-almost surely

$$\sum_{k=0}^{n-1} \sum_{i=1}^{K} S'^1_{i,k} \dots S'^k_{i,k} + \sum_{i=1}^{\|\tilde{Z}^K_0\|} S'^0_{i,0} \dots S'^{n-1}_{i,0} \xrightarrow[n\to\infty]{} \sum_{k=0}^{+\infty} \sum_{i=1}^{K} S'^1_{i,k} \dots S'^k_{i,k} =: S^K.$$

Therefore,
$$\|\tilde{Z}_n^K\| \xrightarrow[n\to\infty]{d} S^K.$$
Additionally, note that $\mathbb{E}[S^K] = \sum_{i=1}^K \sum_{k=0}^{+\infty} \mathbb{E}[S'^1_{i,k} \dots S'^k_{i,k}] \leqslant K/(1-s) < \infty$ thus S^K is $\mathbb{P}$-a.s. finite.

In particular, since $\|\tilde{Z}_n^K\|$ and S^K are integer-valued, this convergence implies
$$\begin{aligned}
\mathbb{P}\left[\liminf_{n\to\infty} \|\tilde{Z}_n^K\| = +\infty\right] &= \mathbb{P}\left[\bigcap_{A\in\mathbb{Z}_+^*} \bigcup_{N\geqslant 0} \bigcap_{n\geqslant N} \left\{\|\tilde{Z}_n^K\| > A\right\}\right] \\
&= \liminf_{A\to\infty} \limsup_{N\to\infty} \mathbb{P}\left[\bigcap_{n\geqslant N} \left\{\|\tilde{Z}_n^K\| > A\right\}\right] \\
&\leqslant \liminf_{A\to\infty} \limsup_{N\to\infty} \mathbb{P}\left[\|\tilde{Z}_N^K\| > A\right] \\
&\leqslant \liminf_{A\to\infty} \mathbb{P}\left[S^K > A\right] \\
&= 0.
\end{aligned}$$
Since once again $Z_n \mathbb{1}^{\Omega^K} \leqslant \tilde{Z}_n^K$, it holds therefore
$$\mathbb{P}[\liminf_{n\to+\infty} \mathbb{1}_{\Omega_K} \|Z_n\| = +\infty] = 0.$$
Therefore, $\mathbb{P}$-almost surely, either
$$\forall K \geqslant 1,\ \sup_{n\geqslant 1} Z_n(0) > K, \text{ i.e. } \limsup_{n\to\infty} Z_n(0) = +\infty$$
or
$$\liminf_{n\to\infty} \|Z_n\| < +\infty.$$
Therefore, up to a $\mathbb{P}$-negligible event, it holds
$$\{\lim \|Z_n\| = +\infty\} = \{\liminf \|Z_n\| = +\infty\} \subset \{\limsup Z_n(0) = +\infty\}.$$
$\square$

V.5 Appendix

Lemma A. Let $u : (a, s) \in (1+\infty)^2 \mapsto \sum_{k\geqslant 1} (\log k)^s k^{-a}$. For all $a, s > 1$, it holds
$$u(a,s) = \frac{\Gamma(s+1)}{(a-1)^{s+1}} + 2\left(\frac{s}{e}\right)^s.$$

Proof. We note $\psi_{s,a}(t) = \log(t) t^{-a/s}$ and notice that $u(a,s) = \sum_{k\geqslant 1} \psi_{s,a}(k)^s$. The derivative of $\psi_{s,a}$ is
$$\psi'_{s,a}(t) = t^{-\frac{a}{s}-1}\left(1 - \frac{a}{s}\log(t)\right),$$
Thus $\psi_{s,a}$ and $\psi_{s,a}^s$ are increasing on $(1, e^{s/a}]$ and decreasing on $(e^{s/a}, +\infty)$. Therefore, we

write

$$u(s,a) \leqslant \sum_{1\leqslant k\leqslant e^{s/a}-1} \psi_{s,a}(k)^s + 2\psi_{s,a}(e^{s/a})^s + \sum_{e^{s/a}+1\leqslant k} \psi_{s,a}(k)^s.$$

Where

$$\sum_{1\leqslant k\leqslant e^{s/a}-1} \psi_{s,a}(k)^s \leqslant \int_1^{e^{s/a}} \psi_{s,a}(t)^s dt$$

and

$$\sum_{1+e^{s/a}\leqslant k} \psi_{s,a}(k)^s \leqslant \int_{e^{s/a}}^{+\infty} \psi_{s,a}(t)^s dt.$$

This yields

$$u(s,a) \leqslant \int_1^{+\infty} (\log t)^s\, t^{-a} dt + 2\left(\frac{s}{a}\right)^s \exp(s/a)^{-a}.$$

On the one hand, setting $z = (a-1)\log(t)$, one gets

$$\begin{aligned}
\int_1^{+\infty} \log(t)^s t^{-a} dt &= \int_1^{+\infty} \log(t)^s e^{-(a-1)\log(t)} \frac{dt}{t} \\
&= \frac{1}{(a-1)^{s+1}} \int_0^{+\infty} z^s e^{-z} dz \\
&= \frac{\Gamma(s+1)}{(a-1)^{s+1}}
\end{aligned}$$

On the other hand

$$\left(\frac{s}{a}\right)^s \exp(s/a)^{-a} = \left(\frac{s}{ae}\right)^s \leqslant \left(\frac{s}{e}\right)^s$$

since $a > 1$. Therefore

$$u(s,a) \leqslant \frac{\Gamma(s+1)}{(a-1)^{s+1}} + 2\left(\frac{s}{e}\right)^s$$

which is indeed the desired result. □

Lemma B. Let $\varepsilon > 0$ and $g(t) = t\,(\log t)^{1+\varepsilon}$ for $t \geqslant 1$. Then there exists $A > 0$ such that $g(k+1) - g(k) \leqslant A\,(\log(k+1))^{1+\varepsilon}$ for any $k \geqslant 1$.

Proof. We compute the derivative

$$g'(t) = (\log t)^{1+\varepsilon} + (1+\varepsilon)\,(\log t)^{\varepsilon}\,, \quad t \geqslant 1.$$

By the mean value theorem, for $t \geqslant 1$, it holds therefore

$$\begin{aligned}
g(t+1) - g(t) &\leqslant \sup_{t\leqslant s\leqslant t+1} g'(s) \\
&\leqslant g'(t+1) \\
&\leqslant (\log(t+1))^{1+\varepsilon} + (1+\varepsilon)\,(\log(t+1))^{\varepsilon} \\
&\leqslant \left(1 + \frac{1+\varepsilon}{\log(2)}\right) (\log(1+t))^{1+\varepsilon}\,.
\end{aligned}$$

This holds therefore in particular for $t = k \in \mathbb{Z}_+$. □

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Maxime LIGONNIÈRE

Produit de matrices aléatoires et branchement multitype en dimension infinie

Résumé

Les processus de Galton-Watson forment une large classe de modèles de populations en temps discret, où des individus se reproduisent aléatoirement, de manière asexuée et sans interaction. L'objectif de cette thèse est d'en étudier une variante où chaque individu a un type et la reproduction des individus est influencée par un environnement, représenté par un processus aléatoire stationnaire et ergodique. Plus précisément, la loi de probabilité de la descendance d'un individu dépend à la fois du type de cet individu et de l'état de l'environnement à l'instant où il vit. L'étude de ce processus requiert la compréhension de sa moyenne dite *quenched*, c'est à dire conditionnelle à la suite des environnements rencontrés par le système. Cette moyenne est alors liée à un produit d'opérateurs linéaires positifs aléatoires, qui agissent sur des espaces de mesures et de fonctions. Dans le cas où les types des individus prennent leurs valeurs dans un ensemble fini, ces opérateurs sont en fait simplement des matrices positives. La théorie des produits de matrices aléatoires, initiée dans les années 1960, est très riche et a permis d'aboutir récemment à une compréhension fine des processus de Galton-Watson associés. On se concentre ici sur le cas moins exploré où l'ensemble des types possibles est infini. Nous établissons tout d'abord un résultat d'ergodicité pour des produits d'opérateurs aléatoires en dimension finie. Il nécessite l'obtention d'une propriété de contraction projective, qu'on obtient grâce à l'application de minorations de Doeblin sur une chaîne de Markov inhomogène sous-jacente. Pour les processus de Galton-Watson associés, ces résultats d'ergodicité impliquent la croissance exponentielle de la population à un taux déterministe, indépendant de la population initiale et donné par l'exposant de Lyapounov du produit d'opérateurs. De plus, la répartition des types dans la population moyenne ne dépend plus de la population initiale en temps long. De ces résultats en moyenne quenched, nous déduisons tout d'abord l'extinction presque sûre de la population quand l'exposant de Lyapunov est négatif. Ensuite, nous obtenons un théorème de Kesten-Stigum, qui, lorsque l'exposant de Lyapunov est positif, sous de bonnes conditions d'intégrabilité, garantit que sur un événement de probabilité non nulle, la population survit et sa taille croit à une vitesse géométrique déterminée par l'exposant de Lyapounov. De plus, la distribution des types en son sein est celle prévue par la moyenne quenched du processus, qui asymptotiquement ne dépend pas de la population initiale. Nous appliquons ces résultats à une classe de processus modélisant une population structurée en âge (sans notion d'âge maximal) dont les matrices moyennes sont des variantes infinies des matrices de Leslie, utilisées depuis plusieurs décennies en démographie et écologie.

Mots clés : Opérateurs aléatoires, Processus de branchement, Modèles de dynamique des populations, Probabilités

Abstract

The Galton-Watson processes form a large class of discrete time population models, in which individuals engage in an asexual and random reproduction, without interacting with each other. In this thesis, we study processes of this class in which each individual has a type and the reproduction of individuals is affected by a notion of environment represented by a stationary and ergodic process. Namely, the probability distribution of the offpsring of an individual depends both on its type and the state of the environment at the time he lives. The study of such a process relies on the understanding of its quenched mean, that is, the mean of the population conditionally on the environmental process. This mean is linked to a product of random positive linear operators which act on some measures and functions space. In the case where the types of the individuals take their values in a finite set, these operators are merely positive matrices. The rich theory of products of random matrices, intiated in the 1960s, has allowed to obtain precise results on the associated Galton-Watson processes in the last decades. We focus here in the case of an infinite set of possible types. We start by obtaining a ergodicity result for the associated products of operators. It requires some projective contraction property, which we derive from a Doeblin minoration on some underlying time inhomogeneous Markov Chain. In terms of the associated Galton-Watson processes, this ergodicity result yields the exponential growth of the quenched mean of the population at a rate given by the Lyapunov exponent of the operator product. Moreover, the distribution of types after a long time does not depend on the initial population. From the study of the quenched mean, we first deduce that the population goes extinct almost surely when the Lyapunov exponent is negative. Then, we prove a Kesten-Stigum type result, which assumes that the Lyapunov exponent is positive and that some integrability conditions holds and guarantees that with postive probability, the population survives and grows exponentially fast at a rate given by the Lyapunov exponent. Moreover, the distribution of the types coincides asymptotically with its quenched mean counterpart and thus does not depend on the initial population. We apply those results to a class of processes modelling an age-structured population. The mean matrices associated with this example are infinite dimensional variants of the Leslie matrices, which have been used in demographics and ecology for several decades.

Keywords : Random operators, Branching Processes, Population models, Probability