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CONVERGENCE TO EQUILIBRIUM FOR GRANULAR MEDIA EQUATIONS AND THEIR EULER SCHEMES

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We introduce a new interacting particle system to investigate the behavior of the nonlinear, nonlocal diffusive equation already studied by Benachour et al. [3, 4]. We first prove an uniform (with respect to time) propagation of chaos. Then, we show that the solution of the nonlinear PDE converges exponentially fast to equilibrium recovering a result established by an other way by Carrillo, McCann and Vilanni [7]. At last we provide explicit and Gaussian confidence intervals for the convergence of an implicit Euler scheme to the stationary distribution of the nonlinear equation.

1. Introduction. In this paper, we study an interacting particle system and an implicit Euler scheme to describe and solve numerically, by a probabilistic way, the following nonlinear equation:

$$\frac{\partial u}{\partial t} = \nabla \cdot [\nabla u + u \nabla W * u], \tag{1.1}$$

where $u(t, \cdot)$ is a time-dependent probability measure on $\mathbb{R}^d$ and $W : \mathbb{R}^d \to \mathbb{R}$ is an interaction potential. The symbol ∇ stands for the gradient operator whereas $\nabla \cdot$ denotes the divergence operator. At last, $*$ stands for the convolution operator:

$$\nabla W * u(x) = \int \nabla W(x-y) u(dy).$$

The function W is supposed to be symmetric:

$$\forall x \in \mathbb{R}^d \qquad W(-x) = W(x), \tag{A1}$$

uniformly convex:

$$\exists \lambda > 0,\ \forall x, v \in \mathbb{R}^d \qquad \langle \text{Hess}\, W(x) v, v \rangle \geq \lambda \langle v, v \rangle, \tag{A2}$$

and such that its gradient is a locally Lipschitz function with polynomial growth: there exists a polynom P such that

$$\forall x, y \in \mathbb{R}^d \qquad |\nabla W(x) - \nabla W(y)| \leq |x-y||P(x) + P(y)|. \tag{A3}$$

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REMARK 1.1. This equation has several physical interpretations. In particular, equation (1.1), in the case when $W(x) = |x|^3$ and $d = 1$, arises in the modeling of granular media. Consider many infinitesimal particles colliding inelastically. Passing to the limit with the good renormalization between the frequency and the inelasticity of the collisions, the distribution of the velocity of "a particle among an infinity" turns to be solution of equation (1.1) ([5] provides a complete description of this framework).

The probabilistic interpretation of equation (1.1) is to consider a Markov process which law at time t is u. The nonlinear process $(\overline{X}_t)_{t\geq 0}$ is the solution of

$$\begin{cases} d\overline{X}_t = \sqrt{2}dB_t - \nabla W * u_t(\overline{X}_t)\,dt, \\ \mathcal{L}(\overline{X}_t) = u_t(dy), \qquad \text{for } t \geq 0, \end{cases} \tag{1.2}$$

where $\mathcal{L}(\overline{X}_t)$ stands for the law of $\overline{X}_t$.

REMARK 1.2. The process $(\overline{X}_t)_{t\geq 0}$ or the stochastic differential equation (1.2) (SDE) are said to be nonlinear since the coefficients of the SDE depend on the law of $\overline{X}_t$.

We want to investigate the long time behavior of the nonlinear process and provide a way to simulate its invariant law with Gaussian confidence intervals thanks to an implicit Euler scheme. To achieve this goal, we construct from $(\overline{X}_t)_{t\geq 0}$ an interacting particle system, that is, a diffusion on $(\mathbb{R}^d)^{\times N}$ such that:

1. The propagation of chaos holds uniformly in time: roughly speaking, the law of the first coordinate of the particle system at time t converges to u_t and a fixed number of particles are asymptotically independent when the size of the system converges to infinity.
2. The particle system converges to equilibrium, as $t \to \infty$, with an explicit and exponential rate.
3. The associated Euler scheme satisfies Gaussian concentration inequalities uniformly in time and in its size.

The natural way to associate a particle system to the nonlinear process is to replace the law u in the coefficients of equation (1.1) by the empirical measure of the system. Let $(B^i_\cdot)_{i\in\mathbb{N}}$ and $(X^i_0)_{i\in\mathbb{N}}$ be two independent collections of independent Brownian motions and independent random values with law u_0. One can introduce the process $(X^N_t)_{t\geq 0}$ solution of the following SDE:

$$\begin{cases} dX^{i,N}_t = \sqrt{2}dB^i_t - \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \nabla W(X^{i,N}_t - X^{j,N}_t)\,dt, & \text{for } i = 1, \dots, N, \\ X^{i,N}_0 = X^i_0, & \text{for } i = 1, \dots, N. \end{cases} \tag{1.3}$$

In Section 2, we explain why this particle system, which has been studied by Benachour et al. [3] cannot help us to fulfill the program: The propagation of chaos is not uniform in time and the particle system is ill-behaved when time grows. But Carrillo McCann and Vilanni [7] showed that the nonlinear process converges exponentially fast to equilibrium. The aim of this paper is to construct another particle system which shares this property.

We show, in Section 3, that the former particle system has a bad behavior in the direction $(v, \dots, v)$ and a good one in the other directions. Then we introduce a new particle system $(Y_t^N)_{t\geq 0}$ which is the projection of (1.3) on the orthogonal of $(v, \dots, v)$.

Section 4 is devoted to the study of this new particle system. We first establish that it satisfies the Bakry–Emery criterion with positive curvature λ, where λ is given by (A2). Then the invariant measure $u_\infty^{(N)}$ of the particle system satisfies a logarithmic Sobolev inequality with constant $2/\lambda$: For every smooth function f on $(\mathbb{R}^d)^{\times N}$,

$$\mathrm{Ent}_{u_\infty^{(N)}}(f^2) \leq \frac{2}{\lambda} \int |\nabla f|^2 \, du_\infty^{(N)},$$

where

$$\mathrm{Ent}_\mu(f^2) := \int f^2 \log f^2 \, d\mu - \int f^2 \, d\mu \log \int f^2 \, d\mu = \int f^2 \log \frac{f^2}{\int f^2 \, d\mu} \, d\mu.$$

This result implies that the relative entropy of $u_t^{(N)}$, the law of the particle system at time t, with respect to $u_\infty^{(N)}$ decreases exponentially fast,

$$\mathrm{Ent}\big(u_t^{(N)} \mid u_\infty^{(N)}\big) \leq \mathrm{Ent}\big(u_0^{(N)} \mid u_\infty^{(N)}\big) e^{-2\lambda t},$$

where

$$\mathrm{Ent}(\nu \mid \mu) = \begin{cases} \displaystyle\int \frac{d\nu}{d\mu} \log \frac{d\nu}{d\mu} \, d\mu, & \text{if } \nu \ll \mu, \\ +\infty, & \text{else.} \end{cases}$$

In Section 5 we establish that the uniform propagation of chaos holds for the convergence of the projected particle system.

THEOREM 1.3. *If W satisfies* (A1), (A2) *and* (A3), *then there exists a constant C such that, for every $N \in \mathbb{N}^*$,*

$$\sup_{t\geq 0} \mathbb{E}\Big(\big|Y_t^{i,N} - \overline{X}_t^i\big|^2\Big) \leq \frac{C}{N}.$$

At this point we are able to use the program. We recover in Section 6 the exponential convergence to equilibrium for the nonlinear PDE established in [7] in term of Wasserstein distance,

$$\mathrm{W}_2(\nu,\mu) := \sqrt{\inf \int\int \tfrac{1}{2}|x-y|^2\, d\pi(x,y)},$$

where the infimum is running over all probability measures π on $\mathbb{R}^d \times \mathbb{R}^d$ with respective marginals ν and μ.

THEOREM 1.4. *If W satisfies* (A1), (A2) *and* (A3), *then*, *there exists a constant K such that*

$$\mathrm{W}_2(u_t, u_\infty) \le K e^{-\lambda t}.$$

At last, in Section 7, we introduce the implicit Euler scheme that approximates the projected particle system. We show that it satisfies a logarithmic Sobolev inequality which is independent of its size and time. We deduce Gaussian concentration inequalities for the convergence of the empirical measure to its mean. This provides exacts confidence intervals for the simulation of the solution of (1.1) or its equilibrium measure u_∞.

THEOREM 1.5. *There exists a constant c such that*, *for every function f with Lipschitz seminorm less than* 1,

$$\mathbb{P}\left(\left|\frac{1}{N}\sum_{i=1}^{N} f(Y_n^{i,N,\gamma}) - \int f\, du_\infty\right| \ge r + c\gamma + \frac{c}{\sqrt{N}} + c e^{-\lambda\gamma n}\right) \le 2e^{-N\lambda r^2/2},$$

where $(Y_n^{N,\gamma})_{n\in\mathbb{N}}$ is the implicit Euler scheme with discretization step γ.

This result allows to choose the values of each parameter (number of particles, step and number of iterations of the Euler scheme): Suppose that c and λ are equal to 1, then to get a result with precision 4ε with probability $2e^{-a^2}$, one can take

$$r = \frac{2a\varepsilon}{a+1}, \qquad N = \frac{(a+1)^2}{4\varepsilon^2}, \qquad \gamma = \varepsilon \quad \text{and} \quad n = -\frac{\log\varepsilon}{\varepsilon^2}.$$

2. A simple particle system which is not enough precise. Benachour et al. [3, 4] study equation (1.1) from the probabilistic point of view. They establish a polynomial convergence to the equilibrium and they associate equation (1.1) with a nonlinear Markov process $(\overline{X}_t)_{t\ge 0}$ solution of (1.2) and an interacting particle system $(X_t^N)_{t\ge 0}$ solution of (1.3). In the sequel we introduce a family of independent nonlinear processes $(\overline{X}^i_\cdot)_{i\in\mathbb{N}^*}$ defined by

$$\begin{cases} d\overline{X}_t^i = \sqrt{2}\, dB_t^i - \nabla W * u_t(\overline{X}_t^i)\, dt, \\ \mathcal{L}(\overline{X}_t^i) = u_t(dy), \qquad \text{for } t \ge 0, \\ \overline{X}_0^i = X_0^i. \end{cases} \tag{2.1}$$

The process $(\overline{X}_t^i)_{t\geq 0}$ is driven by the same Brownian motion as the ith particle of the system and they are equal at time 0.

The nonlinear process is not simpler to study than equation (1.1). The fruitful idea, initiated in the general case by McKean and Kac, is to associate to $(\overline{X}_t)_{t\geq 0}$ an interacting particle system. This process is much more simpler to study or to simulate than $(\overline{X}_t)_{t\geq 0}$. Unfortunately, the three requirements of the program fail to be true for the particle system (1.3)!

2.1. *Propagation of chaos.* First, the propagation of chaos phenomenon holds but not uniformly in time.

THEOREM 2.1 (Benachour et al. [3]). *If* $\mathbb{E}(|X_0|^2)$ *is finite, then, there exists a constant* C *such that, for every* $T \geq 0$ *and* $N \in \mathbb{N}$,

$$\sup_{0\leq t\leq T} \mathbb{E}\Big(\big|X_t^{i,N} - \overline{X}_t^i\big|^2\Big) \leq \frac{CT^2}{N},$$

where the process $(\overline{X}_t^i)_{t\geq 0}$ *is defined in equation* (2.1).

2.2. *Long time behavior of the first particle system.* Secondly, it can be proved that the distribution of the particle system (1.3) does not converge when t grows to infinity to a probability measure. The simplest way to establish this point is to notice that, thanks to the assumption (A1), ∇W is odd and then, the mean of the empirical measure is equal to

$$\frac{1}{N}\sum_{i=1}^N X_t^{i,N} = \frac{1}{N}\sum_{i=1}^N B_t^i + \frac{1}{N}\sum_{i=1}^N X_0^i$$

which is a sum of a Gaussian random variable $\mathcal{N}(0, t/N)$ and a fixed (in time) random variable and then does not converge, as $t \to \infty$, to a probability measure. This fact suggests that the direction $(v, \dots, v)$ badly influences the long time behavior of the particle system.

Let us now establish that the situation is much better in the other directions. In the sequel, we deal with derivation operators on $\mathbb{R}^d$ and $(\mathbb{R}^d)^{\times N}$. The operators on the product space will be written with bold fonts. For example, the symbol $\boldsymbol{\nabla}$ stands for the gradient operator on $(\mathbb{R}^d)^{\times N}$.

Equation (1.3) which defines the particle system can be rewritten as:

$$dX_t^N = \sqrt{2}d\mathcal{B}_t - \boldsymbol{\nabla} U(X_t^N)\,dt,$$

where $\mathcal{B}_t = (B_t^1, \dots, B_t^N)$ and $U : \mathbb{R}^d \times \cdots \times \mathbb{R}^d \to \mathbb{R}$ is equal to

$$U(x_1, \dots, x_N) = \frac{1}{2N}\sum_{1\leq i,j\leq N} W(x_i - x_j).$$

Equivalently, the law of the particle system $v_t^{(N)}$ is solution of the linear Fokker–Planck equation:

$$\frac{\partial v^{(N)}}{\partial t} = \nabla \cdot [\nabla v^{(N)} + v^{(N)} \nabla U].$$

At this point, let us recall a result established by Bakry [2] which provides logarithmic Sobolev inequalities for the semigroup $(\mathbf{P}_t)_t$ associated to $(X_t^N)_{t\geq 0}$.

THEOREM 2.2 (Bakry [2]). *If* $(\mathbf{P}_t)_t$ *is the semigroup associated to the process* $(X_t^N)_{t\geq 0}$ *and* ρ *is a real number, then the following properties are equivalent*:

1. *For every* x *and* v *in* $(\mathbb{R}^d)^N$.

$$\langle \mathbf{Hess}\, U(x)v, v\rangle \geq \rho |v|^2;$$

2. *For every* $t \geq 0$, *and* x *in* $(\mathbb{R}^d)^N$, *the measure* $\mathbf{P}_t(\cdot)(x)$ *satisfies a logarithmic Sobolev inequality with constant*

$$C_t = \frac{2}{\rho}(1 - e^{-2\rho t}).$$

REMARK 2.3. If ρ is equal to 0, the constant C_t is equal to $4t$.

The next step is then the study of the Hessian matrix of U and in particular of its spectrum.

LEMMA 2.4. *If* W *satisfies* (A2), *then*, *the vectors* $\mathbf{v} = (v, \dots, v)$, *where* v *is a vector of* $\mathbb{R}^d$, *belong to the kernel of the matrix* $\mathbf{Hess}\, U(x)$.

PROOF. The matrix $\mathbf{Hess}\, U(x)$ has a special property: dividing $\mathbf{Hess}\, U(x)$ in blocks of size $d \times d$, one can see that each diagonal entry is equal to the sum of the other terms of its row. Then it is clear that a vector $(v, \dots, v)$ where v belongs to $\mathbb{R}^d$ is such that $(\mathbf{Hess}\, U(x))\mathbf{v} = 0$. $\square$

Proposition 3.1 perform a complete study of the eigenvalues and eigenspaces of $\mathbf{Hess}\, U(x)$.

As a conclusion, the law of the particle system at time t satisfies a logarithmic Sobolev inequality with constant $4t$. This result gives no information when time grows to infinity. Moreover, one can notice that the measure $\exp(-U)$ has an infinite mass.

2.3. *Long time behavior of the nonlinear process.* Nevertheless Carrillo, McCann and Villani [7] investigate the long time behavior of the solution of equation (1.1) studying the time evolution of the functional

$$\eta(u) = \int u(x)\log u(x)\,dx + \frac{1}{2}\int\int W(x-y)u(x)u(y)\,dx\,dy. \tag{2.2}$$

The method consists in an extension to the nonlinear framework of the Bakry and Emery strategy which is used in Section 4:

- show that the function β from $\mathbb{R}_+$ to $\mathbb{R}$ defined by $\beta(t) = \eta(u_t) - \eta(u_\infty)$ is decreasing;
- find a differential inequality between β' and β'' and then use Gronwall lemma.

This strategy is very attractive but the computation of β'' requires high attention and precision.

THEOREM 2.5 (Carrillo, McCann and Villani [7]). *If W satisfies* (A1) *and* (A2), *the solution of equation* (1.1) *converges exponentially fast to equilibrium. More precisely*, *there exists a constant K*, *depending on $\eta(u_0)$*, *such that*

$$\eta(u_t) - \eta(u_\infty) \le K\exp(-2\lambda t),$$

where u_∞ is the unique minimizer of η with the mean of u_0.

Thus, we face the paradoxical situation where the particle system does not behave as well as the nonlinear process it approximates.

3. Another particle system. In this section we carefully investigate the reason why the choice of the process $(X_t^N)_{t\ge 0}$ is not the best one.

3.1. *A remark on the nonlinear evolution.* First, the time evolution of the solution of equation (1.1) occurs in the class of the probability measures with mean equal to the one of the initial condition u_0. Indeed,

$$\begin{aligned}\mathbb{E}(\overline{X}_t) &= \mathbb{E}(\overline{X}_0) - \int_0^t \mathbb{E}[\nabla W * u_s(\overline{X}_s)]\,ds\\ &= \mathbb{E}(\overline{X}_0) - \int_0^t \mathbb{E}[\nabla W(\overline{X}_s - \overline{X}'_s)]\,ds,\end{aligned}$$

where $(\overline{X}'_t)_{t\ge 0}$ is an independent copy of $(\overline{X}_t)_{t\ge 0}$. According to assumption (A1) ∇W is an odd function. For every $s \in \mathbb{R}_+$ one has

$$\mathbb{E}[\nabla W(\overline{X}_s - \overline{X}'_s)] = 0,$$

and then $\mathbb{E}(\overline{X}_t) = \mathbb{E}(\overline{X}_0)$ for every $t \in \mathbb{R}_+$.

Since our aim is to construct a particle system which empirical measure is a good approximation of u_t, it seems reasonable to choose its mean near of the one of u_0. As we will see, this remark is the key of the problem.

3.2. *The projected particle system.* We have already stressed that the vectors $\mathbf{v} = (v, \dots, v)$ prevent the particle system from converging to equilibrium. The following result provides a complete description of spec $\mathbf{Hess}\, U$.

PROPOSITION 3.1. *If W satisfies* (A2), *then*:

1. *The matrix* $\mathbf{Hess}\, U(x)$ *admits* 0 *as an eigenvalue with multiplicity d and the associated eigenvectors are of the following form* $\mathbf{v} = (v, \dots, v)$ *where v belongs to* $\mathbb{R}^d$.
2. *The others eigenvalues are greater or equal than* λ.

PROOF. We have already noticed (see Lemma 2.4) that the vectors that can be written $(v, \dots, v)$ are in the kernel of $\mathbf{Hess}\, U(x)$. We have now to study the other directions. Let us decompose the Hessian matrix of U in blocks of size $d \times d$. The entries of $\mathbf{Hess}\, U$ are denoted by $(H_{ij})_{1 \le i, j \le N}$:

$$H_{ii} = \frac{1}{N} \sum_{j \neq i} \operatorname{Hess} W(x_i - x_j) \quad \text{and} \quad H_{ij} = -\frac{1}{N} \operatorname{Hess} W(x_i - x_j) \qquad \text{for } i \neq j.$$

The first point of the proposition is easy to check.

Let us study the other eigenvalues. In the sequel the matrix A stands for

$$A := \lambda \operatorname{Id} - \frac{\lambda}{N} \mathbb{1},$$

where $\mathbb{1}$ is the matrix whose all blocks of size $d \times d$ are equal to I the identity matrix on $\mathbb{R}^{d \times d}$:

$$\mathbb{1} = \begin{pmatrix} I & \dots & I \\ \vdots & \ddots & \vdots \\ I & \dots & I \end{pmatrix}.$$

The matrix H can be written $H(x) = A + H(x) - A$. The spectrum of the matrix A is $\{0, \lambda\}$ with respective multiplicity d and $d(N-1)$ and eigenvectors the sets $\{\mathbf{v} = (v, \dots, v)\}$ and

$$\mathcal{M} = \left\{ \mathbf{v} = (v_1, \dots, v_N),\ \sum_{i=1}^{N} v_i = 0 \right\}.$$

At last, let us show that $H - A$ is a nonnegative symmetric matrix. Denoting by

$$B_{ij} = \frac{1}{N} (\operatorname{Hess} W(x_i - x_j) - \lambda I) \qquad \text{for } i \neq j,$$

we have

$$(H - A)_{ii} = \sum_{j \neq i} B_{ij} \quad \text{and} \quad (H - A)_{ij} = -B_{ij} \qquad \text{for } i \neq j.$$

Moreover, $\langle B_{ij}v, v\rangle \geq \lambda|v|^2$ and B_{ij} is symmetric. Then, for every vector $\mathbf{v} = (v_1, \dots, v_N)$,

$$\begin{aligned}\langle (H - A)\mathbf{v}, \mathbf{v}\rangle &= \sum_{\substack{i \neq j\\ i,j=1}}^{N} \left(\langle B_{ij}v_i, v_i\rangle - \langle B_{ij}v_i, v_j\rangle\right)\\ &= \frac{1}{2}\sum_{\substack{i \neq j\\ i,j=1}}^{N} \langle B_{ij}(v_i - v_j), (v_i - v_j)\rangle \geq 0.\end{aligned}$$

This completes the proof. □

According to Proposition 3.1, the natural idea is to project the particle system on the set $\mathcal{M}$ which is orthogonal to $\{\mathbf{v} = (v, \dots, v)\}$:

$$\mathcal{M} = \left\{x \in \mathbb{R}^N,\ \sum_{i=1}^{N} x_i = 0\right\},$$

Project the vector X^N on $\mathcal{M}$ is nothing else than subtract to all its coordinates the mean of its coordinates. Let us introduce the process $(Y_t^N)_{t\geq 0}$ on $\mathbb{R}^{dN}$ defined by

$$Y_t^{i,N} = X_t^{i,N} - \frac{1}{N}\sum_{j=1}^{N} X_t^{j,N} \qquad \text{for all } i = 1, \dots, N.$$

By the definition of Y^N,

$$X_t^{i,N} - X_t^{j,N} = Y_t^{i,N} - Y_t^{j,N}.$$

Then, one has

$$Y_t^{i,N} = X_0^i - \frac{1}{N}\sum_{j=1}^{N} X_0^j + \sqrt{2}B_t^i - \frac{\sqrt{2}}{N}\sum_{j=1}^{N} B_t^j - \frac{1}{N}\sum_{j=1}^{N}\int_0^t \nabla W(Y_s^{i,N} - Y_s^{j,N})\,ds.$$

REMARK 3.2. Notice that the projected process $(Y_t^N)_{t\geq 0}$ is still a diffusion on $\mathcal{M}$. This is due to the invariance of the drift by translation of any vector in $\{\mathbf{v} = (v, \dots, v)\}$. One could rewrite Y_t^N with the help of $N - 1$ independent Brownian motions in a system of coordinates in $\mathcal{M}$ but this would destroy the symmetry in the writing of Y_t^N.

4. Long time behavior of the projected particle system. One of the most efficient tools to study the convergence to equilibrium for a diffusion process is to establish a logarithmic Sobolev inequality for its invariant probability measure. To get such a result one can use the following theorem.

THEOREM 4.1 (Bakry–Emery). *If for every* $x \in \mathbb{R}^{dN}$,

$$\mathbf{Hess}\, U(x) \geq \rho Id$$

then, the measure μ *with density*

$$\frac{1}{Z}e^{-U(x)}, \qquad \text{where } Z = \int e^{-U(x)}\, dx,$$

satisfies a logarithmic Sobolev inequality with constant $2/\rho$, *that is, for every smooth function* f,

$$\mathrm{Ent}_\mu(f^2) \leq \frac{2}{\rho}\int |\nabla f|^2\, d\mu,$$

where

$$\mathrm{Ent}_\mu(f^2) := \int f^2 \log f^2\, d\mu - \int f^2\, d\mu \log \int f^2\, d\mu.$$

REMARK 4.2. By an analogy with Riemannian geometry, we will call ρ the curvature of the diffusion process with infinitesimal generator $\Delta f - \nabla U \cdot \nabla f$.

The process $(Y_t^N)_{t\geq 0}$ on $\mathcal{M}$ has a curvature greater than λ. To establish this point, one can write the process $(X_t^N)_{t\geq 0}$ in an orthonormal basis with the first d vectors in $\mathcal{N}$ and the other in $\mathcal{M}$. The two processes are independent and the first one is a standard Brownian motion whereas the second one is diffusion with a curvature greater than λ. This is not surprising: This fact is contained in Proposition 3.1. As a consequence we get that its invariant measure satisfies a logarithmic Sobolev inequality with constant $2/\lambda$.

PROPOSITION 4.3. *If* W *satisfies* (A1) *and* (A2), *then, the invariant law of* (Y_t^N) *which is equal to*

$$u_\infty^{(N)} = \frac{1}{Z_N}\mathbb{1}_{\mathcal{M}}(y)\exp\left(\frac{1}{2N}\sum_{i,j=1}^{N} W(y_i - y_j)\right) dy,$$

with

$$Z_N = \int_{\mathcal{M}} \exp\left(\frac{1}{2N}\sum_{i,j=1}^{N} W(y_i - y_j)\right) dy,$$

satisfies a logarithmic Sobolev inequality with constant $2/\lambda$.

As is well known, this inequality implies an estimate for the convergence to equilibrium in terms of relative entropy.

COROLLARY 4.4. *If $u_t^{(N)}$ stands for the density of the law of Y_t^N, then, for every $t \geq 0$,*

$$\mathrm{Ent}\big(u_t^{(N)} \mid u_\infty^{(N)}\big) \leq \mathrm{Ent}\big(u_0^{(N)} \mid u_\infty^{(N)}\big) e^{-2\lambda t}.$$

REMARK 4.5. This result is quite classical but we reproduce its proof in order to give an idea of the strategy used in [7].

PROOF OF COROLLARY 4.4. We study the time evolution of the function α from $\mathbb{R}_+$ to $\mathbb{R}$ defined by

$$\alpha(t) := \mathrm{Ent}(u_t^{(N)} \mid u_\infty^{(N)}) = \int \log \frac{u_t^{(N)}}{u_\infty^{(N)}} \, du_t^{(N)}.$$

Using the fact that $u_t^{(N)}$ is solution of the Fokker–Planck equation

$$\frac{\partial u^{(N)}}{\partial t} = \nabla \cdot [\nabla u^{(N)} + u^{(N)} \nabla U],$$

one gets

$$\alpha'(t) = -\int \left| \frac{\nabla u_t^{(N)}}{u_t^{(N)}} - \frac{\nabla u_\infty^{(N)}}{u_\infty^{(N)}} \right|^2 du_t^{(N)}.$$

Moreover, the logarithmic Sobolev inequality for $u_\infty^{(N)}$ ensures that

$$\mathrm{Ent}(u_t^{(N)} \mid u_\infty^{(N)}) \leq \frac{1}{2\lambda} \int \left| \frac{\nabla u_t^{(N)}}{u_t^{(N)}} - \frac{\nabla u_\infty^{(N)}}{u_\infty^{(N)}} \right|^2 du_t^{(N)}.$$

So, α satisfies a differential inequality

$$\alpha'(t) \leq -2\lambda \alpha(t),$$

which completes the proof. □

5. Uniform propagation of chaos. We have now to establish the propagation of chaos phenomenon.

THEOREM 5.1. *If W satisfies* (A1) *and* (A2), *then there exists a constant C such that, for every $N \in \mathbb{N}^*$,*

$$\sup_{t \geq 0} \mathbb{E}\Big(\big|Y_t^{i,N} - \overline{X}_t^i\big|^2\Big) \leq \frac{C}{N}.$$

PROOF. The proof follows that of Theorem 2.1 established in [3] and the improvement comes from the fact that the new particle system is on $\mathcal{M}$.

Let us first define, for $i = 1, \dots, N$, the process $(\overline{Y}_t^{i,N})_{t\geq 0}$ by

$$\overline{Y}_t^{i,N} := \overline{X}_t^i - \frac{1}{N}\sum_{j=1}^N \overline{X}_t^j.$$

This is the ith coordinate of the projection on $\mathcal{M}$ of the vector $(\overline{X}_t^1, \dots, \overline{X}_t^N)$.

To control $\mathbb{E}(|Y_t^{i,N} - \overline{X}_t^i|^2)$, we write

$$\sqrt{\mathbb{E}\left(\left|Y_t^{i,N} - \overline{X}_t^i\right|^2\right)} \leq \sqrt{\mathbb{E}\left(\left|Y_t^{i,N} - \overline{Y}_t^{i,N}\right|^2\right)} + \sqrt{\mathbb{E}\left(\left|\overline{Y}_t^{i,N} - \overline{X}_t^i\right|^2\right)}. \tag{5.1}$$

The second term of the right-hand side is easy to deal with. By independence of the processes of $\overline{X}_t^i$ and $\overline{X}_t^j$,

$$\mathbb{E}\left(\left|\overline{Y}_t^{i,N} - \overline{X}_t^i\right|^2\right) = \frac{1}{N}\mathbb{E}\left(\left|\overline{X}_t^i\right|^2\right).$$

Moreover, the moments of the process $(\overline{X}_t)_{t\geq 0}$ are bounded, uniformly in time, as soon as they are finite at time 0.

LEMMA 5.2 (Benachour et al.). *If* $\mathbb{E}(|\overline{X}_0|^{2n})$ *is finite, then there exists a* K_n *such that*

$$\sup_{t\geq 0}\mathbb{E}\left(|\overline{X}_t|^{2n}\right) \leq K_n.$$

PROOF. Let us investigate the proof of the case $n = 1$ in order to show the role of assumptions (A1) and (A2). A complete proof of this result is performed by Proposition 3.8 in [3].

By the Itô formula, we get

$$\mathbb{E}\left(|\overline{X}_t|^2\right) - \mathbb{E}\left(|\overline{X}_0|^2\right) = -2\int_0^t \mathbb{E}[\overline{X}_s \cdot \nabla W * u_s(\overline{X}_s)]\,ds + 2t.$$

Let $(\overline{X}'_t)_{t\geq 0}$ be an independent copy of $(\overline{X}'_t)_{t\geq 0}$. Since ∇W is an odd function, one has

$$\begin{aligned}\mathbb{E}\left(|\overline{X}_t|^2\right) - \mathbb{E}\left(|\overline{X}_0|^2\right) &= -2\int_0^t \mathbb{E}[\overline{X}_s \cdot \nabla W(\overline{X}_s - \overline{X}'_s)]\,ds + 2t\\ &= -\int_0^t \mathbb{E}[(\overline{X}_s - \overline{X}'_s)\cdot \nabla W(\overline{X}_s - \overline{X}'_s)]\,ds + 2t.\end{aligned}$$

Assumption (A2) of strict convexity for W provides

$$\mathbb{E}\left(|\overline{X}_t|^2\right) - \mathbb{E}\left(|\overline{X}_0|^2\right) \leq -\lambda\int_0^t \mathbb{E}[(\overline{X}_s - \overline{X}'_s)^2]\,ds + 2t.$$

Then we use that $\mathbb{E}(\overline{X}_s) = \mathbb{E}(\overline{X}_0)$ for every $s \in \mathbb{R}_+$ to get

$$\mathbb{E}\Big(|\overline{X}_t|^2\Big) - \mathbb{E}\Big(|\overline{X}_0|^2\Big) \le -2\lambda \int_0^t \mathbb{E}\Big(|\overline{X}_s|^2\Big)\, ds + 2t\big(\lambda[\mathbb{E}(\overline{X}_0)]^2 + 1\big).$$

The Gronwall lemma completes the proof in the case $n = 1$:

$$\mathbb{E}\Big(|\overline{X}_t|^2\Big) \le \bigg[\big(\mathbb{E}(\overline{X}_0)\big)^2 + \frac{1}{2\lambda}\bigg](1 - e^{-2\lambda t}) + \mathbb{E}(\overline{X}_0^2) e^{-2\lambda t}.$$

Induction completes the proof of the Lemma 5.2. □

Let us turn to the first term of the right-hand side of (5.1).

$$\begin{aligned} Y_t^{i,N} - \overline{Y}_t^{i,N} &= -\frac{1}{N} \int_0^t \sum_{j=1}^N [\nabla W(Y_s^{i,N} - Y_s^{j,N}) - \nabla W * u_s(\overline{X}_s^i)]\, ds \\ &\quad - \frac{1}{N} \int_0^t \sum_{j=1}^N \nabla W * u_s(\overline{X}_s^j)\, ds. \end{aligned}$$

Using the Itô formula, one can get

$$\sum_{i=1}^N \big|Y_t^{i,N} - \overline{Y}_t^{i,N}\big|^2 = -2 \sum_{1 \le i,j \le N} \int_0^t [\rho_{ij}^{(1)}(s) + \rho_{ij}^{(2)}(s) + \rho_{ij}^{(3)}(s)]\, ds,$$

where

$$\begin{aligned} \rho_{ij}^{(1)}(s) &:= [\nabla W(Y_s^{i,N} - Y_s^{j,N}) - \nabla W(\overline{X}_s^i - \overline{X}_s^j)] \cdot [Y_s^{i,N} - \overline{Y}_s^{i,N}] \\ &= [\nabla W(Y_s^{i,N} - Y_s^{j,N}) - \nabla W(\overline{Y}_s^{i,N} - \overline{Y}_s^{j,N})] \cdot [Y_s^{i,N} - \overline{Y}_s^{i,N}], \\ \rho_{ij}^{(2)}(s) &:= [\nabla W(\overline{X}_s^i - \overline{X}_s^j) - \nabla W * u_s(\overline{X}_s^i)] \cdot [Y_s^{i,N} - \overline{Y}_s^{i,N}], \\ \rho_{ij}^{(3)}(s) &:= [\nabla W * u_s(\overline{X}_s^i)] \cdot [Y_s^{i,N} - \overline{Y}_s^{i,N}]. \end{aligned}$$

We treat the three terms one by one. To deal with $\rho_{ij}^{(1)}(s)$ one has to gather the crossing terms:

$$\sum_{1 \le i,j \le N} \rho_{ij}^{(1)}(s) = \frac{1}{2} \sum_{1 \le i,j \le N} [\rho_{ij}^{(1)}(s) + \rho_{ji}^{(1)}(s)].$$

The sum $\rho_{ij}^{(1)}(s) + \rho_{ji}^{(1)}(s)$ is equal to

$$[\nabla W(Y_s^{i,N} - Y_s^{j,N}) - \nabla W(\overline{Y}_s^{i,N} - \overline{Y}_s^{j,N})] \cdot [(Y_s^{i,N} - \overline{Y}_s^{i,N}) - (Y_s^{j,N} - \overline{Y}_s^{j,N})]$$

which is also equal to

$$[\nabla W(Y_s^{i,N} - Y_s^{j,N}) - \nabla W(\overline{Y}_s^{i,N} - \overline{Y}_s^{j,N})] \cdot [(Y_s^{i,N} - Y_s^{j,N}) - (\overline{Y}_s^{i,N} - \overline{Y}_s^{j,N})].$$

Then, by the convexity assumption, $\rho_{ij}^{(1)}(s) + \rho_{ji}^{(1)}(s)$ is bounded below by the quantity

$$\lambda\left|(Y_s^{i,N} - Y_s^{j,N}) - (\overline{Y}_s^{i,N} - \overline{Y}_s^{j,N})\right|^2 = \lambda\left|(Y_s^{i,N} - \overline{Y}_s^{i,N}) - (Y_s^{j,N} - \overline{Y}_s^{j,N})\right|^2.$$

Since the vectors Y^N and $\overline{Y}^N$ are on $\mathcal{M}$, the sum of their coordinates is equal to 0 and then, we get by a straightforward computation:

$$\begin{aligned}\sum_{1\le i,j\le N} \rho_{ij}^{(1)}(s) &\ge \frac{\lambda}{2} \sum_{1\le i,j\le N} \left|(Y_s^{i,N} - \overline{Y}_s^{i,N}) - (Y_s^{j,N} - \overline{Y}_s^{j,N})\right|^2 \\ &= \lambda N \sum_{i=1}^N \left|Y_s^{i,N} - \overline{Y}_s^{i,N}\right|^2 \\ &\quad -\lambda \sum_{1\le i,j\le N} (Y_s^{i,N} - \overline{Y}_s^{i,N}) \cdot (Y_s^{j,N} - \overline{Y}_s^{j,N}) \\ &= \lambda N \sum_{i=1}^N \left|Y_s^{i,N} - \overline{Y}_s^{i,N}\right|^2 - \lambda \left|\sum_{i=1}^N (Y_s^{i,N} - \overline{Y}_s^{i,N})\right|^2 \\ &= \lambda N \sum_{i=1}^N \left|Y_s^{i,N} - \overline{Y}_s^{i,N}\right|^2.\end{aligned}$$

Those lines contain the key point of the improvement of the result established in [3].

The second term on the right-hand side of (5.1) is controlled by the Cauchy–Schwarz inequality,

$$-\mathbb{E}\left[\sum_{j=1}^N \rho_{ij}^{(2)}(s)\right] \le \left[\mathbb{E}\left(\left|Y_s^{i,N} - \overline{Y}_s^{i,N}\right|^2\right)\right]^{1/2} \theta_i(s)^{1/2},$$

where

$$\theta_i(s) = \mathbb{E}\left[\left|\sum_{j=1}^N \left(\nabla W(\overline{X}_s^i - XJ_s) - \nabla W * u_s(\overline{X}_s^i)\right)\right|^2\right].$$

Using the independence of the copies of $\overline{X}_t$, the polynomial growth of ∇W and Lemma 5.2, one can prove that there exists a constant c such that

$$\theta_i(s) \le cN.$$

The key is to notice that

$$\mathbb{E}[\nabla W(\overline{X}_s^i - \overline{X}_s^j) - \nabla W * u_s(\overline{X}_s^i)] = 0.$$

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The third term of the right-hand side of (5.1) is controlled by the same way using Cauchy–Schwarz inequality and the fact that

$$\mathbb{E}[\nabla W(\overline{X}_s^i - \overline{X}_s^j)] = 0.$$

As a conclusion, we have proved that there exists a constant c such that

$$\sum_{i=1}^{N} \mathbb{E}\Big(\big|Y_t^{i,N} - \overline{Y}_t^{i,N}\big|^2\Big)$$

$$\leq -2\lambda N \int_0^t \sum_{i=1}^{N} \big|Y_s^{i,N} - \overline{Y}_s^{i,N}\big|^2 \, ds + c\sqrt{N} \int_0^t \Big[\mathbb{E}\Big(\big|Y_s^{i,N} - \overline{Y}_s^{i,N}\big|^2\Big)\Big]^{1/2} ds.$$

If $\alpha : \mathbb{R}_+ \to \mathbb{R}$ is defined by $\alpha(s) = \mathbb{E}(|Y_s^{i,N} - \overline{Y}_s^{i,N}|^2)$, then we have established that

$$\alpha'(s) \leq -2\lambda\alpha(s) + \frac{c}{\sqrt{N}}\sqrt{\alpha(s)}.$$

The Gronwall lemma completes the proof:

$$\sqrt{\alpha(t)} \leq \frac{c}{\lambda\sqrt{N}}(1 - e^{-\lambda t}). \qquad \square$$

6. Convergence to equilibrium for the nonlinear process. In this section we investigate the long time behavior of u_t thanks to the uniform propagation of chaos and the convergence to equilibrium for the projected particle system. Instead of proving convergence for the functional η [see equation (2.2)], we prove it in terms of Wasserstein distance.

DEFINITION 6.1. The Wasserstein metric between to probability measures μ and ν with finite second moments is defined by

$$\mathrm{W}_2(\nu, \mu) := \sqrt{\inf \int\int \tfrac{1}{2}|x - y|^2 \, d\pi(x, y)},$$

where the infimum is running over all probability measures π on $\mathbb{R}^d \times \mathbb{R}^d$ with respective marginals ν and μ.

THEOREM 6.2. *There exists a constant K such that*

$$\mathrm{W}_2(u_t, u_\infty) \leq K e^{-\lambda t}.$$

PROOF. By the triangular inequality, one has

$$\mathrm{W}_2(u_t, u_\infty) \leq \mathrm{W}_2(u_t, u_t^{(1,N)}) + \mathrm{W}_2(u_t^{(1,N)}, u_\infty^{(1,N)}) + \mathrm{W}_2(u_\infty^{(1,N)}, u_\infty).$$

By the definition of $\mathrm{W}_2(\cdot,\cdot)$ and the uniform propagation of chaos established in Theorem 5.1, one has

$$\mathrm{W}_2(u_t, u_t^{(1,N)}) + \mathrm{W}_2(u_\infty^{(1,N)}, u_\infty) \le 2\sqrt{\sup_{t\ge 0}\mathbb{E}\Big(\big|Y_t^{i,N} - \overline{X}_t^i\big|^2\Big)} \le \frac{2C}{\sqrt{N}}.$$

To deal with the quantity $\mathrm{W}_2(u_t^{(1,N)}, u_\infty^{(1,N)})$, we first make use of the following lemma.

LEMMA 6.3. *Let μ and ν be two probability measures on $\mathbb{R}^d$. Then, for every probability measures on $(\mathbb{R}^d)^{\times N}$, μ_N and ν_N with respective marginals $\mu,\dots,\mu$ and $\nu,\dots,\nu$,*

$$\mathrm{W}_2(\mu_N, \nu_N)^2 \ge N\mathrm{W}_2(\mu,\nu)^2.$$

PROOF. One has to use the definition of the Wasserstein metric for $\mathrm{W}_2(\mu_N,\nu_N)$ and choose the product measure $\mu\otimes\cdots\mu\otimes\nu\otimes\cdots\otimes\nu$. □

Then we get

$$\mathrm{W}_2(u_t^{(1,N)}, u_\infty^{(1,N)}) \le \frac{1}{\sqrt{N}}\mathrm{W}_2(u_t^{(N)}, u_\infty^{(N)}).$$

At this point we make use of a result established by Otto and Villani in [9] and reformulated by Bobkov, Gentil and Ledoux [6].

THEOREM 6.4 (Otto–Villani). *If μ is absolutely continuous and satisfies a logarithmic Sobolev inequality with constant C, then, for every probability measure ν absolutely continuous with respect to μ,*

$$\mathrm{W}_2(\mu,\nu)^2 \le \frac{C}{2}\,\mathrm{Ent}(\nu\,|\,\mu).$$

Thus, we have

$$\mathrm{W}_2(u_t^{(1,N)}, u_\infty^{(1,N)}) \le \sqrt{\frac{C}{2N}\,\mathrm{Ent}(u_t^{(N)}\,|\,u_\infty^{(N)})},$$

and as a consequence of the exponential decay of the relative entropy for the particle system,

$$\mathrm{W}_2(u_t^{(1,N)}, u_\infty^{(1,N)}) \le \sqrt{\frac{C}{2N}\,\mathrm{Ent}(u_0^{(N)}\,|\,u_\infty^{(N)})e^{-\lambda t}}.$$

At last, $u_0^{(N)}$ is the projection of $u_0^{\otimes N}$ on $\mathcal{M}$ and then $\mathrm{Ent}(u_0^{(N)}\,|\,u_\infty^{(N)})$ is of order N when u_0 is sufficiently kind, for example, if u_0 is absolutely continuous with a positive density.

As a conclusion, we have shown that for every N,

$$\mathrm{W}_2(u_t, u_\infty) \le \frac{K}{\sqrt{N}} + K e^{-\lambda t},$$

which completes the proof. □

7. Exact confidence intervals for the convergence of the implicit Euler scheme to the nonlinear process. To simulate the law of the solution of a stochastic differential equation,

$$dX_t = \sigma(X_t)\,dB_t + b(X_t)\,dt,$$

the most efficient probabilistic way is to use the Euler scheme with discretization step γ defined by:

$$X^\gamma_{n+1} = X^\gamma_n + \gamma b(X^\gamma_n) + \sigma(X^\gamma_n)(B_{\gamma(n+1)} - B_{\gamma n}).$$

Unfortunately, this algorithm does not necessarily converge when the coefficients σ and b are no longer uniformly Lipschitz.

An alternative way consists in using the implicit Euler scheme. Let us construct it in the case where one wants to simulate the law of the diffusion

$$dX_t = \sqrt{2}\,dB_t - \nabla U(X_t)\,dt,$$

where U satisfies (A2). The implicit Euler scheme is defined by:

$$X^\gamma_{n+1} = X^\gamma_n - \gamma \nabla U(X^\gamma_{n+1}) + \sqrt{2}(B_{\gamma(n+1)} - B_{\gamma n}).$$

The price to pay to get a convergent scheme is to invert, at each discretization step, the function

$$F(x) = x + \gamma \nabla U(x).$$

In our case this is not impossible: Since

$$\mathrm{Jac}\, F = \mathrm{Id} + \gamma\, \mathrm{Hess}\, U(x),$$

the classical gradient methods easily solve this problem even in large dimensions.

7.1. *Logarithmic Sobolev inequalities for the Euler scheme.* We prove here that, when the discretization step γ is smaller than $1/\lambda$, the iterated kernel K^n satisfies a logarithmic Sobolev inequality with an explicit constant depending on λ.

THEOREM 7.1. *If the discretization step γ is smaller that $1/\lambda$, then for every $n \in \mathbb{N}$, $x \in \mathbb{R}$ and (smooth) function f from $\mathbb{R}^d$ to $\mathbb{R}$, K^n satisfies a logarithmic Sobolev inequality with constant*

$$D_{\gamma,n} := \frac{4(1+\lambda\gamma)}{\lambda(2+\lambda\gamma)}\left(1 - \frac{1}{(1+\lambda\gamma)^{2n}}\right). \tag{7.1}$$

PROOF. If φ is the inverse function of F, let us denote by K the implicit Euler kernel defined by

$$Kf(x) = \mathbb{E}[(f \circ \varphi)(x + \sqrt{2\gamma}Y)],$$

where the law of Y a Gaussian law $\mathcal{N}(0, \mathrm{Id})$. One has to compute the entropy

$$\mathrm{Ent}_K(f^2) = \mathrm{Ent}_{\mu_{x,\gamma}}((f \circ \varphi)^2),$$

where $\mu_{x,\gamma}$ is the Gaussian law $\mathcal{N}(x, 2\gamma\,\mathrm{Id})$. In the sequel we omit the index of $\mu_{x,\gamma}$. The measure μ satisfies a logarithmic Sobolev inequality with constant 4γ (see [1]). Then,

$$\mathrm{Ent}_K(f^2) \le 4\gamma \int |\nabla(f \circ \varphi)(x)|^2 \, d\mu(x).$$

Now, by the definition of φ, we get

$$\mathrm{Jac}\,\varphi(x) = [\mathrm{Id} + \gamma\,\mathrm{Hess}\,U(x)]^{-1}.$$

Then, for every v in $\mathbb{R}^d$, one has

$$\langle \mathrm{Jac}\,\varphi(x)v, v\rangle \le (1 + \gamma\lambda)^{-1}|v|^2.$$

As a consequence, the kernels $(K(\cdot)(x))_x$ satisfy a logarithmic Sobolev inequality with the common constant $4\gamma/(1 + \lambda\gamma)$,

$$\mathrm{Ent}_K(f^2) \le \frac{4\gamma}{1 + \lambda\gamma} K(|\nabla f|^2). \tag{7.2}$$

On the other hand,

$$\nabla Kf(x) = \mathbb{E}\big[\mathrm{Jac}\,\varphi(Z)\nabla f\big(\varphi(Z)\big)\big].$$

Then K and ∇ satisfy the following commutation relation

$$|\nabla K(f)(x)| \le (1 + \gamma\lambda)^{-1} K(|\nabla f|)(x). \tag{7.3}$$

To prove the logarithmic Sobolev inequality for K^n, we mimic the proof performed in [2] for the diffusion processes

$$\mathrm{Ent}_K(f^2) = \sum_{i=1}^{n} K^{i-1}\,\mathrm{Ent}_K(g_{n-i}^2),$$

where g_{n-i} stands for $\sqrt{K^{n-i}(f^2)}$. The logarithmic Sobolev inequality (7.2) provides

$$\mathrm{Ent}_K(f^2) \le \frac{4\gamma}{1 + \lambda\gamma} \sum_{i=1}^{n} K^i(|\nabla g_{n-i}|^2).$$

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Now, we make use of the commutation relation (7.3) to get for $1 \le i \le n$,

$$|\nabla g_{n-i}|^2 = \frac{|\nabla K^{n-i}(f^2)|^2}{4K^{n-i}(f^2)} \le \frac{1}{(1+\lambda\gamma)^2} \frac{[K|\nabla K^{n-i-1}(f^2)|]^2}{4KK^{n-i-1}(f^2)}.$$

Then, since, from the Cauchy–Schwarz inequality,

$$\frac{(Kf)^2}{K(g)} \le K\left(\frac{f^2}{g}\right),$$

one has

$$\frac{[K|\nabla K^{n-i-1}(f^2)|]^2}{4KK^{n-i-1}(f^2)} \le K\left[\frac{|\nabla K^{n-i-1}(f^2)|^2}{4K^{n-i-1}(f^2)}\right] = K[|\nabla g_{n-i-1}|^2].$$

A straightforward induction shows that, for $1 \le i \le n$,

$$|\nabla g_{n-i}|^2 \le \frac{1}{(1+\lambda\gamma)^{2(n-i)}} K^{n-i}[|\nabla f|^2].$$

Then it follows that

$$\begin{aligned} \mathrm{Ent}_{K^n}(f^2) &\le 4\gamma\left[\sum_{i=0}^{n-1} \frac{1}{(1+\lambda\gamma)^{2i}}\right] K^n[|\nabla f|^2] \\ &= \frac{4\gamma}{1 - 1/(1+\lambda\gamma)^2}\left(1 - \frac{1}{(1+\lambda\gamma)^{2n}}\right) K^n[|\nabla f|^2] \\ &= \frac{4(1+\lambda\gamma)}{\lambda(2+\lambda\gamma)}\left(1 - \frac{1}{(1+\lambda\gamma)^{2n}}\right) K^n[|\nabla f|^2], \end{aligned}$$

and the proof is complete. □

7.2. *Concentration inequalities.* Logarithmic Sobolev inequalities provide Gaussian concentration properties *via* the Herbst's argument (see [8]). We will say that f is α-Lipschitz function if

$$\sup_{x \ne y} \frac{|f(x) - f(y)|}{|x - y|} = \alpha,$$

where $|\cdot|$ stands for the Euclidean norm.

THEOREM 7.2. *If a measure μ on $\mathbb{R}^d$ satisfies a logarithmic Sobolev inequality with constant C then for every α-Lipschitz* (*for the Euclidean norm*) *function g*,

$$\mathbb{P}\big(|g(X) - \mathbb{E}g(X)| \ge r\big) \le 2e^{-r^2/C\alpha^2}.$$

A straightforward application of this result is obtained choosing

$$g(x) = \frac{1}{N}\sum_{i=1}^{N} f(x_i),$$

where f is an 1-Lipschitz function on $\mathbb{R}^d$. The function g is $(1/\sqrt{N})$-Lipschitz and we get:

PROPOSITION 7.3. *If $(Z_n^{\gamma,N})_{n\in\mathbb{N}}$ is the implicit Euler scheme associated the projected particle system, then, for every 1-Lipschitz function f on $\mathbb{R}^d$ and every $r \geq 0$,*

$$\mathbb{P}\left(\left|\frac{1}{N}\sum_{i=1}^{N} f(Z_t^{\gamma,i,N}) - \mathbb{E}f(Z_t^{\gamma,i,N})\right| \leq r\right) \leq 2e^{-N\lambda r^2/2}.$$

7.3. *Weak convergence of the Euler scheme.* The previous result ensures that the empirical measure of the Euler scheme is concentrated around its mean as a Gaussian law with variance $1/\lambda$. To achieve the program, we need to use the following result that establishes the weak convergence of the Euler scheme, uniformly in time.

PROPOSITION 7.4 (Talay [10]). *There exists a constant c such that, for every $t \geq 0$, $N \in \mathbb{N}^*$ and every 1-Lipschitz function f,*

$$|\mathbb{E}f(Z_t^{\gamma,i,N}) - \mathbb{E}f(Y_t^{i,N})| \leq c\gamma.$$

This result is presented in [10].

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