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Mathematical Models for Charged Particle Diffusion and Transport

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Considerate la vostra semenza:
Fatti non foste a viver come bruti,
Ma per seguir virtute e canoscenza.

Dante, Inferno XXVI, vv.118-120

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Florence, December 1999. Simona Mancini

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Introduction

1.1 What is concerned with?

In this thesis we are concerned with the mathematical derivation of the solution of evolution problems arising from the study of the motion of charged or uncharged particles. In particular, the thesis is divided in two main subjects: the proofs of the existence and uniqueness proofs of the solution of evolutions problems by means of the theory of semigroups; and the justification of a drift-diffusion type model, derived from a kinetic equation.

In this chapter, after a small introduction to the arguments of this thesis, we give a physical description of the partial differential equations (Vlasov and Boltzmann equations) which govern the evolution of a system of particles (electrons) subject to a force field (i.e. electric and/or magnetic field).

In chapter 2 we recall some basic definitions and theorems of functional analysis. Then, we give an overview of the theory of linear and affine semigroups. Finally, we describe the different techniques applied in asymptotic analysis.

Chapter 3 concerns different type of models for the boundary conditions. Starting from the Vlasov equation equipped with nonhomogeneous no-reentry boundary conditions (see section 3.1), passing to nonhomogeneous dissipative and conservative boundary conditions (see section 3.2), and ending by multiplicative boundary conditions (see section 3.3). In particular, the results of sections 3.1 and 3.2 respectively gave rise to two publications: [46] and [47]. While, section 3.3 is a work in progress. This kind of models have already been studied by various mathematicians, see [3], [8], [37]. Also, in some previous papers (see [44] and [45]) it was proved the existence and uniqueness of the solution of evolution problems defined on one and three dimensional bounded region and equipped by a *general* kind of boundary conditions.

Chapters 4 and 5 are devoted to the derivation of a drift-diffusion model starting from a three dimensional kinetic equation. Chapter 4 concerns the collision-less case, i.e. collisions of particles against molecules of the host medium are neglected, and leaded to the article (see [28]). Chapter 5 is the development to the isotropic collision case and is a work in progress. The derivation of drift-diffusion models is interesting from a numerical point of view. In fact, the number of unknown variables in the distribution function describing the system of particles is reduced from seven (three positions, three velocities and time) to four (two positions, energy and time). This fact simplifying the writing codes for numerical simulations.

1.2 The Kinetic Equations

In this section, we introduce the kinetic equations which will be studied in this thesis. Kinetic problems arise when modeling and studying the motion of an ensemble of particles. In particular, we shall consider charged particles (electrons) moving in a prescribed region under the action of a driving force and obeying the laws of classical mechanics. As the treatment of semi-classical or quantum kinetic equation is beyond the purposes of this thesis, we shall not consider quantum mechanics laws. From a probabilistic point of view, kinetic equations give the equation of the probability density of the ensemble.

1.2.1 The Vlasov Equation

We begin considering a single electron moving in the vacuum under the influence of a force field F. The vector $x = x(t)$ belonging to $\mathbb{R}^3$ describe the position of the electron and may depends on time t, and the velocity vector is given by $v = v(t) \in \mathbb{R}^3$. If for instance the force governing the motion of the electron is due to an electric field E, then, denoting by $-q < 0$ the electron charge, the force field acting on the electron is given by $F = -qE$. Finally, if the electron is submitted also to a magnetic field B, then the force field is given by $F = -q(E + v \times B)$.
Consider an initial data $f_I = f_I(x, v)$ which describes the probability density of the electron to be in a position x with velocity v at the initial time $t = 0$. Our goal is to derive a continuum equation for the probability density $f = f(x, v, t)$ which evolves

from f_I. We shall assume that $f_I \geq 0$ and that:

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_I(x, v)\, dx\, dv = 1.$$

We shall assume that f does not change along trajectories, i.e. for every $x \in \mathbb{R}^3$, $v \in \mathbb{R}^3$ and for every $t \geq 0$:

$$f(w(t; x(t; x, v), v(t; x, v)), t) = f_I(x, v), \tag{1.2.0}$$

where $w(t; x, v)$ is the trajectory of a particle, $x(0; x, v) = x$ and $v(0; x, v)$. Differentiating (1.2.1) with respect to t, we obtain, for $t > 0$:

$$\partial_t f + v \cdot \nabla_x f - F/m \cdot \nabla_v f = 0, \tag{1.2.0}$$

where we denote by ∂_t the derivative with respect to t and by $\nabla_x f$ and $\nabla_v f$ the gradients of f with respect to x and v. Equation (1.2.1) is the *Liouville* equation and it governs the evolution of the position-velocity probability density f of the electron in the force field F under the assumption that the electron moves according to the laws of classical mechanics. The motion is assumed to take place without any interference from the environment or equivalently that the electron moves in a vacuum.

We now consider the case of an ensemble consisting of M particles all having the same mass m. The position vector x and the velocity vector v of the ensemble are $3M$-dimensional vectors, i.e. $x = (x_1, ..., x_M)$ and $v = (v_1, ..., v_M)$ where x_i, $v_i \in \mathbb{R}^3$ represents the position and velocity vector of the i-th particle of the ensemble. The force filed F is $3M$-dimensional too, $F = (F_1, ..., F_M)$, and in general depends on all $6M$ position and velocity coordinates and on the time t. By $F_i = F_i(x, v, t)$, $i = 1, ..., M$ we denote the force acting on the i-th particle. The joint position-velocity probability density of the M-particle ensemble f, satisfies the classical Liouville equation, for $t > 0$:

$$\partial_t f + v \cdot \nabla_x f - F/m \cdot \nabla_v f = 0, \tag{1.2.0}$$

with $x \in \mathbb{R}^{3M}$, $v \in \mathbb{R}^{3M}$. Equation (1.2.1) is linear and hyperbolic and it is usually supplemented by the initial condition $f(x, v, t = 0) = f_I(x, v)$.
We remark that the non-negativity of f is preserved by the evolution process by the Liouville equation: if $f_I \geq 0$, then the solution f of (1.2.1) is such that $f \geq 0$ too

for any $x \in \mathbb{R}^3$, $v \in \mathbb{R}^3$ and $t \geq 0$. Moreover, if the force field $F = -qE$ is assumed to be divergence free with respect to the velocity, $div_v F = 0$, then the conservation of the whole space integral holds:

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(x, v, t)\, dx\, dv = 1.$$

We note that a mathematical analysis of the Liouville equation from a semigroup point of view in L^2 spaces is done in [55].

Two main problems arise in the study of the Liouville equation for an ensemble of many particle interacting:

- Models for the driving force F are not readily available.
- The dimension of the M-particle ensemble phase space is $6M$, which is very large for numerical simulations.

The *Vlasov* equation is then introduced to solve these problems (in particular to reduce the dimension of the Liouville equation). In order to derive it, we assume that:

- The charged particles (electrons) move in the vacuum, or equivalently the effects of the host medium are negligible.
- The force field is independent on the velocity vector, thus $div_v F = 0$.
- The motion is governed by an external electric field E_{ext} and by a two-particle internal force E_{int}.

We do not explain all the steps needed to derive the Vlasov equation, we just recall that, first it is derived a system of equations for the position-velocity densities of subensembles consisting of d electrons ($d = 1, ...M$) called the BBGKY-hierarchy (Bogoliubov [13], Born and Green [16], Kirkwood [40]-[41] and Yvon [59]). Then, once the formal limit for M going to ∞ is carried out, a solution of the hierarchy, determined by a single function of three position, three velocity coordinates and time, is constructed.

This solution, based on the assumption that the particles of a small subensemble move independently of each other, is the electron number density in the physical phase space $\mathbb{R}^3 \times \mathbb{R}^3$. Moreover, it is the solution of the Vlasov equation which can be considered as an *aggregated* one-particle Liouville equation supplemented by a self-consistent field relation:

$$\partial_t f + v \cdot \nabla_x f - q/m\, E_{eff} \cdot \nabla_v f = 0,\ x \in \mathbb{R}^3,\ v \in \mathbb{R}^3,\ t > 0$$
$$E_{eff}(x,t) = E_{ext}(x,t) + \int_{\mathbb{R}^3} n(x',t) E_{int}(x,x')\, dx',\ x \in \mathbb{R}^3,\ t > 0 \tag{1.2.0}$$

where $f = f(x,v,t)$ and $n = n(x,t)$ represent the expected electron number densities in the phase space and in the position space respectively. In other words f is the number of electrons per unit volume in an infinitesimal neighborhood of (x,v) at time t, and n is the number of electrons per unit volume in an infinitesimal neighborhood of x at time t:

$$n = \int_{\mathbb{R}^3} f(x,v,t)\, dv.$$

We stress out that the Vlasov equation is useful to describe the short time behavior of a system of weakly interacting mass points, this is the case for example of a rarefied gas whose particles interact with relatively weak, long range forces, such as the electrons of a ionized gas (Coulomb force) or the stars of a stellar system (gravitational force), [20], [21].

Furthermore, the Vlasov equation is usually employed to model the Coulomb interaction caused by a typical weak long range force, for example in astrophysics. Whereas, we shall assume that the effective force field F is given data of our models. Hence we shall not consider the coupling of the Vlasov equation with the *Poisson* or the *Maxwell* equations. For a mathematical overview of results on the existence and uniqueness of solution of the Vlasov-Poisson equation we refer the readers to [31], [24], [4].

1.2.2 The Boltzmann equation

In the Vlasov equation the short range interactions are not taken into account. Nevertheless, these interactions, usually called *scattering* or *collisions* of particles, heavily influence the motion of particles if the study is done on a sufficiently large time scale. In particular, for an accurate description of charge transport the short range interactions of the particles with the environment are significantly important.

It was *Boltzmann* who in 1872 derived for the first time an operator in which also the short range forces are considered, [13]. From this fact the extended Vlasov equation is called Boltzmann equation. The most relevant fact of the Boltzmann equation is the appearance of a nonlinear and nonlocal collision operator which cause many mathematical difficulties in the analytical and numerical study. We shall consider a linearization of this collision operator, which anyhow leads to a sufficiently correct description of the motion of the charge particles.

We now give a formal derivation of the collision operator based on phenomenological considerations. In plasma physics or in semiconductors physics the obstacles on which the electrons collide are ions or neutral molecules, which are much more heavier than electrons.

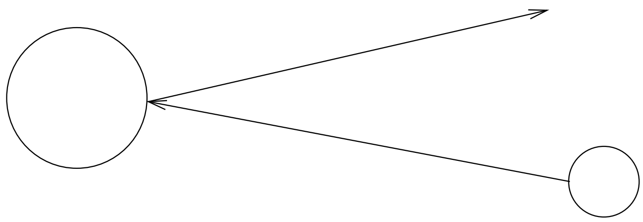

FIGURE 1.1. Collision between an electron and a ion

We assume that a collision yields to an instantaneously scattering of the particle from one state to another in such a way that its velocity vector change extremely fast, while the change of the position vector take place slowly. In other words, we may say that a particle which at time t is in the state (x, v') and collides at the same time t will be in the state (x, v).

The collision operator, Q, will be derived as the sum of two terms (L and G) describing respectively the loss and gain of particles in the volume $dx\, dv$.
Let $s(v \to v')$ denote the probability density of particles with position x at time t, to shift a velocity v into v' because of a scattering event. We remark that $s(v \to v')$ may also depend on x. Then $s(v \to v')\, dv'$ is the probability of a particle with velocity v to undergo a collision in a volume dv' of v'. Being $f(x, v, t)\, dx\, dv$ the number of particles present at time t in the unit volume $dx\, dv$, the number of particles sent in dv' by a collision event in the time dt is equal to:

$$s(v \to v')\, dv'\, dt\, f(x, v, t)\, dx\, dv.$$

Therefore, the total number of particles leaving $dx\,dv$ in the time dt is given by:

$$\left(\int_{\mathbb{R}^3} s(v \to v')\,dv'\right) dt\,f(x,v,t)\,dx\,dv,$$

and this by definition is equal to the loss term L of the collision operator:

$$L(f)(x,v,t) = \int_{\mathbb{R}^3} s(v \to v')f(x,v,t)\,dv'.$$

The gain term G is then computed analogously, and it is given by:

$$G(f)(x,v,t) = \int_{\mathbb{R}^3} s(v' \to v)f(x,v',t)\,dv'.$$

Finally, the collision operator Q, given by the sum of $-L$ and G, reads:

$$Q(f)(x,v,t) = \int_{\mathbb{R}^3} [s(v' \to v)f' - s(v \to v')f]\,dv',$$

where we have used the standard notation $f = f(x,v,t)$ and $f' = f(x,v',t)$. As a consequence, the linear Boltzmann equation reads, for $x \in \mathbb{R}^3,\ v \in \mathbb{R}^3,\ t > 0$:

$$\partial_t f + v \cdot \nabla_x f - F/m \cdot \nabla_v f = Q(f), \tag{1.2.0}$$

where F denotes the force field, and thus may contains an electric force and/or a magnetic force which may be a data of the transport problem or may have to be determined by means of the Poisson or Maxwell equation. We remark, that a more rigorous approach is given in [20], and that the existence, globally in time, of solution of the Boltzmann equation in the field-free case ($F = 0$) is proved in [32].

We note that the integration of $Q(f)$ with respect to v gives 0, this implying that the total number of electrons is conserved. In other words, collisions neither destroy nor generates particles.

Another important property of Q is related to the relaxation of the state of the ensemble towards local thermodynamic equilibrium. The quantity:

$$\lambda(x,v) = \int_{\mathbb{R}^3} s(v \to v')\,dv'$$

is called collision frequency. It measure the strength of the interaction at the state (x,v) corresponding to the transition rate s. Its reciprocal $\tau(x,v) = \lambda^{-1}$ is the relaxation time describing the average time between two consecutive collisions at the state (x,v). The relaxation time τ represents the time scale on which the density f

relaxes towards an equilibrium state. Furthermore, it is possible to prove that the relaxation time τ is the scale on which f returns to the equilibrium density from the perturbed state f_I along the characteristics.
We remark that, this is not the case if collisions are not considered (Vlasov equation), in fact there is no mechanism which forces the particle ensemble to relax towards the thermodynamic equilibrium in the large time limit. This fact, is mathematically represented by the well known H-Theorem (see [20]).

We shall assume that the obstacles are at the thermodynamic equilibrium at the temperature T, in order words they are animated by a random velocity of zero mean and of square mean $v_0 = (k_B T/m_0)^{1/2}$, where m_0 is the mass of the obstacles and k_B is the Boltzmann constant. In the collision the random velocity is transmitted to the electrons, that is to say that after the collision the electrons will have a random velocity of zero mean and of square mean $v = (k_B T/m)^{1/2}$. Such a statistical distribution is given by the *Maxwellian*:

$$M(v) = \frac{1}{(2\pi m k_B T)^{d/2}} \exp\left(\frac{-|v|^2 m}{2T}\right),$$

where d is the dimension of the space.
When the thermodynamic equilibrium between the electrons and ions is locally reached we have that $f(x,v,t) = n(x,t)M(v)$ where n is the density of electrons:

$$n(x,t) = \int_{\mathbb{R}^3} f(x,v,t)\, dv.$$

If so, the collision operator no more affect the electrons distribution, this implying that $Q(M)(x,v,t) = 0$ for every $v \in \mathbb{R}^3$. Thus, for every $v \in \mathbb{R}^3$:

$$\int_{\mathbb{R}^3} s(v' \to v)M(v') - s(v \to v')M(v)\, dv' = 0.$$

Usually we ask more, for every $v, v' \in \mathbb{R}^3$:

$$s(v' \to v)M(v') - s(v \to v')M(v) = 0.$$

This assumption, together with the following definition of the scattering cross section $\phi(v,v')$:

$$\phi(v,v') = s(v \to v')M(v')^{-1} = s(v' \to v)M(v)^{-1} = \phi(v',v),$$

yields to:

$$Q(f)(x,v,t) = \int_{\mathbb{R}^3} \phi(v',v)[M(v)f' - M(v')f]\, dv'.$$

We remark now that there exists different types of collisions: elastic, inelastic and superelastic. In order to fix ideas, we consider a weakly ionized plasma where the interactions of the electrons are essentially done with the neutral molecules. We shall assume that the neutral molecules are at rest, which for instance is a reasonable assumption as the electrons mass is much smaller than the mass of the neutral molecules.

In the elastic collision there is no exchange of energy between electrons and neutral molecules, thus, denoting by v' the electron velocity before the collision and by v the one after the collision, we have that $|v'| = |v|$ (remark that the electron velocity direction change in the collision). The collision operator Q simplifies to:

$$Q_e(f)(x,v,t) = \int_{\mathbb{R}^3} \phi_e(v',v)(f' - f)\delta\left(\frac{m|v'|^2}{2} - \frac{m|v|^2}{2}\right) dv',$$

where δ is a Dirac mass.

On the other hand, in the inelastic collision an internal degree of energy ε_k $(k = 1, ..., k)$ of the neutral molecule is excited, then:

$$\varepsilon_k + \frac{m|v'|^2}{2} = \frac{m|v|^2}{2}.$$

Being the superelastic collision is the inverse process, we have:

$$\frac{m|v'|^2}{2} = \varepsilon_k + \frac{m|v|^2}{2}.$$

Therefore the inelastic collision operator reads:

$$Q_k(f)(x,v,t) = \int_{\mathbb{R}^3} (s(v',v)f' - s(v,v')f)\, dv',$$

and the Boltzmann equation is given by, for $x \in \mathbb{R}^3$, $v \in \mathbb{R}^3$ and $t > 0$:

$$\partial_t f + v \cdot \nabla_x f - F/m \cdot \nabla_v f = Q_e(f) + \sum_k Q_k(f), \tag{1.2.0}$$

where the sum is to be considered extended to all the inelastic collision processes. In the sequel we will be concerned only by elastic collision operators.

2

Mathematical Preliminaries

2.1 Notations

In this section we recall some definitions and results of the theory of linear operators. Let X and Y denote two Banach spaces, and let A be a linear operator from X to Y, with domain $D(A) \subset X$ and range $R(A) \subset Y$.

We shall denote by $\mathcal{B}(X,Y)$ the family of linear and bounded operators A acting from X to Y and with $D(A) = X$ and $R(A) = Y$. If $Y = X$, then we shall write $\mathcal{B}(X) = \mathcal{B}(X,X)$.

Let $G(A)$ denote the graph of the operator A:

$$G(A) = \{(f,g) \in X \times Y : g = Af \ , \ f \in D(A)\} \subset X \times Y.$$

We remark that if A is a linear operator then $G(A)$ is linear.

Definition 2.1.1 *The operator A is closed if its graph, $G(A)$, is a closed set of $X \times Y$: $G(A) = \overline{G}(A)$.*

We shall denote by $\mathcal{C}(X,Y)$ the family of closed operators acting from X to Y. We recall just some properties of bounded and closed operators.

Proposition 2.1.1 *Let A, B be two linear operators and X, Y, Z three Banach spaces. Then, we have:*
(i) A is closed if and only if $D(A)$ is closed.
(ii) A closed, B bounded, and $D(A) \subset D(B)$, then $A + B$ is closed.
(iii) Let A be invertible, then A is closed if and only if A^{-1} is closed.
(iv) If $A \in \mathcal{C}(X,Y)$, $B \in \mathcal{B}(Y,Z)$ and $B^{-1} \in \mathcal{B}(Z,Y)$, then $AB \in \mathcal{C}(X,Y)$.

We remark that from Proposition 2.1.1 (i) it follows that $\mathcal{B}(X,Y) \subset \mathcal{C}(X,Y)$. In fact, as $A \in \mathcal{B}(X)$ then it is bounded and $D(A) = X$ which is closed in X.

Usually, in order to verify that a given operator L is closed, it is more convenient to apply the following characterization:

Proposition 2.1.2 *The operator $A \in \mathcal{C}(X,Y)$ if and only if for every sequence $\{f_n\} \in D(A)$ such that $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} Af_n = g$ we have $f \in D(A)$ and $g = Af$.*

Let $A \in \mathcal{C}(X,Y)$, and consider, for $f \in D(A)$, the graph norm:

$$\|f\|_A = \|Af\|_Y + \|f\|_X \tag{2.1.0}$$

We recall that the space $D(A)$ equipped by the graph norm $\|\cdot\|_A$ is a Banach space and that $A \in \mathcal{B}(D(A), Y)$.

Remark that, if $A \in \mathcal{C}(X,Y)$ and if we consider the sequence of functions $\{f_n\} \in D(A)$, then from Proposition 2.1.2 it follows that

$$A(\lim_{n\to\infty} f_n) = \lim_{n\to\infty}(Af_n).$$

This fact is useful in applications; it is therefore natural to study weather a given operator A is closed or not, and if not to ask weather it has a closable extension or not.

Definition 2.1.2 *An operator $A \notin \mathcal{C}(X,Y)$ is closable if it has a closed extension, i.e. if there exists a closed operator $\overline{A}$ such that $D(A) \subset D(\overline{A})$ and $\overline{A}f = Af$ for every $f \in D(A)$.*

One way to prove that an operator A is closable is given by the following proposition:

Proposition 2.1.3 *If for every $\{f_n\} \subset D(A)$ such that $f_n \to 0$ and $Af_n \to g$ we have $g = 0$, then there exists an operator $\overline{A}$ such that:*
(i) $\overline{A}$ is an extension of A;
(ii) $\overline{A} \in \mathcal{C}(X,Y)$ and $G(\overline{A}) = \overline{G}(A)$;
(iii) every closed extension of A, is an extension of $\overline{A}$.

We recall that $\overline{A}$ is called the *closure* of A, and that it is usually defined as follows. Let $\{f_n\} \subset D(A)$ such that to the limit $n \to \infty$, f_n converges to f and Af_n tends to $g \in Y$, then the operator $\overline{A}$ acts like A and its domain is given by:

$$D(\overline{A}) = \{f \,:\, f = \lim_{n\to\infty} f_n,\; f_n \in D(A)\,,\; \overline{A}f = g\}.$$

Given an Hilbert space H and an operator A with domain $D(A)$ dense in H and range $R(A) \subset H$, the *adjoint* operator of A, denoted by A^*, is the operator defined by:

$$(Ax, y) = (x, A^*y)$$

for $x \in D(A)$ and $y \in D(A^*)$, where:

$$D(A^*) = \{y \in H \,:\, (Ax, y) = (x, A^*y) \;\forall\; x \in D(A)\},$$

and we have denoted by $(\cdot, \cdot)$ the inner product in H. We remark that A^* is a closed operator even if A is not closed.

Definition 2.1.3 *If the operator A with domain $D(A)$ dense in H is symmetric (i.e. $(Ax, y) = (x, Ay)$ for every $x, y \in D(A)$) and $D(A) = D(A^*)$, then A is a self-adjoint operator.*

Example: Consider the integral operator $K : D(K) = L^2[0, 1] \to L^2[0, 1]$ defined by:

$$Kf(x) = \int_0^1 k(x, y)\, f(y)\, dy \tag{2.1.0}$$

where $k(x, y)$ is a function defined for $(x, y) \in [0, 1] \times [0, 1]$ and such that:

$$\int_0^1 \int_0^1 |k|^2\, dx\, dy < +\infty.$$

Applying the Hölder inequality, it follows $K \in \mathcal{B}(L^2([0, 1]))$. Moreover, K is a self-adjoint operator, provided that $k(y, x) = \overline{k(x, y)}$, where $\overline{k(x, y)}$ is the coniugate of $k(x, y)$. ■

Let L be a linear operator, not necessarily bounded, with $D(L) \subset X$ and $R(L) \subset X$ and consider the operator $L_z = zI - L$ where $z \in \mathbb{C}$ (I is the identity operator on X). Note that, $D(L_z) = D(L)$. Considering the values of z for which L_z admit an inverse, we have:

Definition 2.1.4 *Let L_z^{-1} exist, $D(L_z^{-1})$ be dense in X and L_z^{-1} be bounded. For such values of z we define the resolvent set, $\rho(L)$, as:*

$$\rho(L) = \{z \in \mathbb{C} \,:\, L_z^{-1} \text{ exists, } D(L_z^{-1}) \text{ is dense in } X \text{ and } L_z^{-1} \text{ is bounded}\}.$$

We recall that the $z \in \mathbb{C}$ such that L_z^{-1} does not exists (i.e. $Lf = zf$ for $f \neq 0$) are the eigenvalues of the operator L and that the function f is the eigenfunction associated to the eigenvalue z. The *spectra* of L is by definition the set:

$$\sigma(L) = \rho(L)^c,$$

where with $\rho(L)^c$ we have denoted the complement set of $\rho(L)$. Note that the eigenvalues of L belong to $\sigma(L)$. There exist more detailed classifications of the spectra of an operator L (i.e. discrete spectra, continuous spectra, residual spectra,...), but we shall not need this classification in the remainder.

We remark that $\sigma(L)$ is a closed set, since $\rho(L)$ is an open one. Moreover, if $L \in \mathcal{C}(X)$ and $z \in \rho(L)$, then $(zI - L)^{-1} \in \mathcal{B}(X)$ and it is usually denoted by $R(z, L)$ and called *resolvent operator* or *resolvent* of L.

Example: As examples of the resolvent set and of the spectra, we consider the free streaming operator A given by:

$$Af = -\frac{df}{dx} \ , \quad D(A) \subset C[0,1] \ , \quad R(A) \subset C[0,1]. \tag{2.1.0}$$

We note that A it is not bounded. Moreover, if the domain of A is defined by:

$$D(A) = \{f \in C([0,1]),\ Af \in C([0,1]),\ f(0) = 0\}$$

then we have that $\rho(A) = \mathbb{C}$ and therefore $\sigma(A) = \emptyset$.
On the other hand, if the domain of A is defined by:

$$D(A) = \{f \in C([0,1]),\ Af \in C([0,1])\}$$

then we have that $\rho(A) = \emptyset$ and therefore $\sigma(A) = \mathbb{C}$.
In this example is underlined the fact that the choice of the domain of the operator L heavily affects the study of the resolvent operator $R(z, L)$. ■

For completeness, we list some of the properties of the resolvent of a closed operator:

Proposition 2.1.4 *Let $L \in \mathcal{C}(X)$. Then,*
(i) $LR(z,L) = zR(z,L) - I \ \in \mathcal{B}(X) \quad \forall\, z \in \rho(L)$.
(ii) $LR(z,L)f = R(z,L)Lf, \quad \forall\, f \in D(L), \quad z \in \rho(L)$.
(iii) $R(z,L) - R(z',L) = (z - z')R(z,L)R(z',L), \quad \forall\, z, z' \in \rho(L)$.
(iv) $R(z,L)R(z',L) = R(z',L)R(z,L), \quad \forall\, z, z' \in \rho(L)$.

Finally, we quote a result about the composition of the spectra of self-adjoint operators.

Proposition 2.1.5 *If L is a self-adjoint operator, then $\sigma(L)$ consists only of real eigenvalue. Moreover, every $z = a + ib$ with $b \neq 0$ belongs to $\rho(L)$ and we have $\|R(z, L)\| \leq 1/b$.*

We recall now that:

Definition 2.1.5 *An operator $A : X \to Y$ is compact if the image of the unitary ball of the space X, $A(B_X)$, is relatively compact in Y.*

We shall denote the class of compact operators from X to Y by $\mathcal{K}(X, Y)$. An important class of compact operator is the one composed by the Hilbert-Schmidt operators. We give a characterization of these operators:

Proposition 2.1.6 *Let $H = L^2(\Omega)$, and $k(x, y) \in L^2(\Omega \times \Omega)$. Then the operator:*

$$Kf(x) = \int_{\Omega} k(x, y) f(y)\, dy$$

is a Hilbert-Schmidt operator.
Conversely, every Hilbert-Schmidt operator on L^2 can be uniquely represented by means of a function $k(x, y) \in L^2(\Omega \times \Omega)$.

Two theorems about compact operator will be needed in what follows. The Krein-Rutman theorem concerning the eigenvalues of a compact operator, and the Fredholm alternative concerning the existence of solution of equation of this form: $(I - A)f = g$.

Theorem 2.1.1 (Krein-Rutman) *Let X be a Banach space and let C be a convex cone with vertex 0. Assume that C is closed, with Int $C \neq \emptyset$ and $C \cap (-C) = \{0\}$. If $A \in \mathcal{K}(X)$ is such that $A(C \setminus \{0\}) \subset Int\, C$, then there exist a function $f \in Int\, C$ and a $z > 0$ such that $Af = zf$. Moreover z is the only eigenvalue associated to an eigenfunction of A in C. Finally, $z = \max\{|\zeta|\,,\, \zeta \in \sigma(T)\}$ and it as molteplicity (geometric and algebraic) equal to 1.*

Theorem 2.1.2 (Fredholm alternative) *If $A \in \mathcal{K}(X)$. Then:*
(i) dim$(N(I - A)) < \infty$;
(ii) $R(I - A) \in \mathcal{C}(X)$, and $R(I - A) = N(I - A^*)^{\perp}$;
(iii) $N(I - A) = \{0\} \leftrightarrow R(I - A) = X$;
(iv) dim$(NI - A))$ =dim$(N(I - A^*))$.

The Fredholm alternative shows that either for every $g \in X$ the equation $(I-A)f = g$ admits a unique solution f, or the homogeneous equation $(I-A)f = 0$ has n linearly independent solutions and then the equation $(I-A)f = g$ is solvable if and only if f verifies n orthogonality conditions, i.e. $f \in N(I-A^*)^{\perp}$.

We also recall the Lax-Milgram theorem on *continuous* and *coercive* bilinear forms.

Definition 2.1.6 *Let H be an Hilbert space. The bilinear form $a(u,v) : H \times H \to \mathbb{R}$ is:*

- *continuous if there exists C such that $|a(u,v)| \leq C|u|\,|v|$ for every $u, v \in H$;*
- *coercive if there exists $\alpha > 0$ such that $a(v,v) \geq \alpha |v|^2$ for every $v \in H$.*

Theorem 2.1.3 (Lax-Milgram) *Let $a(u,v)$ be a continuous and coercive bilinear form on H. Then for every $\varphi \in H^*$ there exists a unique $u \in H$ such that*

$$a(u,v) = \langle \varphi, v \rangle$$

for all $v \in H$.

We conclude this section recalling the definitions of Sobolev spaces. Let $\Omega \subset \mathbb{R}^N$ be open and $1 \leq p \leq \infty$. The Sobolev space $W^{1,p}$ is defined by (in the sense of distribution):

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) \,:\, \frac{\partial u}{\partial x_i} \in L^p\,,\; i = 1...N \right\}$$

It is clear that if $u \in C^1(\Omega) \cap L^p(\Omega)$ and $\dfrac{\partial u}{\partial x_i} \in L^p$ for all $i = 1...N$, then $u \in W^{1,p}(\Omega)$ and the partial derivative in the usual sense coincide with those in the $W^{1,p}$ sense. In the sequel we shall work in the space $H^1 = W^{1,2}$. This space, equipped by the norm

$$\|u\|_{H^1} = \left(\|u\|_{L^2}^2 + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^2}^2 \right)^{1/2},$$

is a Hilbert space.
The space $W_0^{1,2}(\Omega) = H_0^1(\Omega)$ denotes the closure of C_0^1 in $W^{1,2}(\Omega)$. $W_0^{1,2}(\Omega)$ is *in practice* the space of those functions u becoming 0 on the boundary $\partial\Omega$.
By H^{-1} we denote the dual space of $H_0^1(\Omega)$, and by $H^{1/2}$ the fractional Sobolev

space $W^{1/2,2}$ defined by, for $0 < s < 1$:

$$W^{s,2} = \left\{ u \in L^2(\Omega) \, : \, \frac{|u(x) - u(y)|}{|x-y|^{s+N/2}} \in L^2(\Omega \times \Omega) \right\}.$$

We remark that, given a suitably regular open set $\Omega \subset \mathbb{R}^N$, the trace operator is defined from $W^{1,p}(\Omega)$ in $L^p(\Gamma)$, where $\Gamma = \partial\Omega$ is the boundary of Ω. Moreover, the functions belonging to $L^p(\Gamma)$ and do not have traces, while the one belonging to $W^{1,p}(\Gamma)$ do. The most important properties of the trace operator are:

- If $u \in W^{1,p}(\Omega)$, then $u|_\Gamma \in W^{1-1/p,p}(\Gamma)$ and for every $u \in W^{1,p}(\Omega)$:

$$\|u|_\Gamma\|_{W^{1-1/p,p}(\Gamma)} \leq C\|u\|_{W^{1,p}(\Omega)};$$

- The trace operator $u \to u|_\Gamma$ is surjective;

- For every $u, v \in H^1(\Omega)$ the Green formula holds true.

2.2 The Theory of Semigroups

In literature, there exists several results about the existence and uniqueness of the solution of Vlasov or Boltzmann equations. Nevertheless, the region in which the motion of particles takes place is usually considered as unbounded, i.e. $x \in \mathbb{R}^3$. On the other hand, the physics of several applications, suggests to relay the particles in bounded domains (see [20], [30], [48], [46], [47]). This is the reason way we shall consider domains with bounded positions and unbounded velocities.

In particular, we shall study the Vlasov equation defined in *bounded* domains from a semigroup point of view. We shall see that the assumption of being closed, is essential for an operator in order to generate a semigroup. Moreover, we shall also see how the fact of considering unbounded velocities heavily affects the above generation.

The study of the generation of a semigroup by an operator is not only interesting in itself, but it also is one of the step of the Hilbert expansion or the moments methods, as we shall see in the next section.

In the present section we recall the main definitions and results of the theory of linear (see [8], [39], [53]), and 'affine' semigroups (see [6], [9]).

2.2.1 Semigroups of linear operators

Let X be a Banach space and L a linear closed (not necessarily bounded) operator; we shall denote such a class of operator by $\mathcal{C}(X)$. There are different types of semigroups. We are concerned with the class of *strongly continuous* semigroup of *contractions* and with the more general class of *strongly continuous* semigroups.

Definition 2.2.1 *A semigroup of operators $Z(t)$, with $t > 0$, is a strongly continuous semigroup if:*
(i) $Z(0) = I$;
(ii) $Z(t+s) = Z(t)Z(s)$ *for every* $t, s \geq 0$ *(semigroup property)*
(iii) $\lim_{t \downarrow 0} Z(t)x = x \ , \quad \forall\, x \in X.$

It is possible to prove that:

Proposition 2.2.1 *If $Z(t)$ is a strongly continuous semigroup, then there exist two constants $M \geq 1$ and $\omega \geq 0$ such that, for $t \geq 0$:*

$$\|Z(t)\| \leq M e^{\omega t}. \tag{2.2.0}$$

If $M = 1$ and $\omega = 0$, i.e. $\|Z(t)\| \leq 1$ for every $t \geq 0$, then $Z(t)$ is called strongly continuous semigroup of *contractions*.
A well known characterization of the strongly continuous semigroups is given by the Hille-Yiosida theorem:

Theorem 2.2.1 (Hille-Yosida) *The linear operator L is such that:*
(i) $L \in \mathcal{C}(X)$, *and* $D(L)$ *dense in* X;
(ii) The resolvent set $\rho(L)$ *contains* $(\beta, +\infty)$ *and for every* $z > \beta$:

$$\|R(z, L)^k\| \leq \frac{M}{(z-\beta)^k} \ , \quad k = 1, 2, \dots \ . \tag{2.2.0}$$

if and only if L is the generator of a strongly continuous semigroup $\{Z(t)\}$.

We remark that, if $\beta = 0$, $M = 1$ and condition (ii) is replaced by:
(ii)' The resolvent set $\rho(L)$ *contains* $(0, +\infty)$ *and for every* $z > 0$:

$$\|R(z, L)\| \leq \frac{1}{z},$$

then, the Hille-Yosida theorem becomes simpler and $Z(t)$ is a strongly continuous semigroup of contractions.

To verify conditions (i) and (ii) of the Hille-Yosida theorem is not always the best way of proving that a linear operator L generates a strongly continuous semigroup of contractions (for example when the functional space is a Hilbert space). Another characterization of the strongly continuous semigroups of contractions is the Lumer-Phillips theorem. In order to state this result we need some preliminaries.

Let X a Banach space, and let X^* be its dual.

Definition 2.2.2 *A linear operator L is dissipative if for every $x \in D(L)$ there is a $x^* \in F(x) = \{x^* \ : x^* \in X^* \text{ and } (x^*, x) = \|x\|^2 = \|x^*\|^2\}$ such that:*

$$\mathrm{Re}\,(Lx, x^*) \leq 0.$$

By $\mathrm{Re}\, x$ we denote the real part of x. We have the Lumer-Phillips theorem:

Theorem 2.2.2 (Lumer-Phillips) *Let L be a linear operator with dense domain $D(L)$ in X.*
If L is dissipative and there is a $z_0 > 0$ such that the range, $R(z_0 I - L)$, of $z_0 I - L$ is the whole space X, then L is the generator of a strongly continuous semigroup of contraction on X.
Conversely, if L is the generator of a strongly continuous semigroup of contractions on X, then $R(zI - L) = X$ for all $z > 0$ and L is dissipative.

We recall that from the Lumer-Phillips theorem we have (see [53]):

Corollary 2.2.1 *Let X be a Banach space and L a closed operator with dense domain in L. If both L and L^* are dissipative operators, then L is the generator of a strongly continuous semigroup of contractions on X.*

There also exists a characterization for positive semigroups. We assume that the Banach space X has a *generating and normal* cone X^+, i.e. X satisfies:

$$X = X^+ - X^- \text{ and } X^* = X^{*+} - X^{*-},$$

where we have denoted by X^+ and X^- the cones in X and by X^{*+} and X^{*-} the cones in X^*. Moreover, assume that the operator L has dense domain in X and a positive resolvent, i.e. for every $x \in X^+$ and for some z_0, $z > z_0$: $R(z, L)x \subset X^+$.

Definition 2.2.3 *Let L be a linear operator; we define the value:*

$$s(L) = \inf\{\omega \in \mathbb{R} \,:\, (\omega, +\infty) \subset \rho(L)\,,\ R(z, L) \geq 0,\ \forall\, z > \omega\},$$

and the type of the semigroup generated by L:

$$\omega(L) = \inf\{\omega \in \mathbb{R} \,:\, \exists\, M \geq 1\,,\; \|Z(t)\| \leq Me^{\omega t},\; \forall\, t \geq 0\}.$$

From the Hille-Yosida theorem follows that if an operator has a positive resolvent then the semigroup that it generates is positive, and vice versa. Thus, if L generates a semigroup and has a positive resolvent, then $s(L) \leq \omega(L)$.
Let now $\overline{s}(L)$ be the spectral bound of the operator L, defined by:

$$\overline{s}(L) = \sup\{\mathrm{Re}\, z \,:\, z \in \sigma(L)\}.$$

If L has a positive resolvent, then $\overline{s}(L) \leq s(L) \leq \omega(L)$. Moreover, we have the following characterization for the positive semigroups:

Theorem 2.2.3 (Arendt1) *Let the Banach space X be ordered as above, and let the operator L such that $D(L)$ is dense in X, and $R(z, L) > 0$. If there exists $z_0 > s(L)$ such that $\|R(z, L)x\| \geq c\|x\|$ for every $x \in X^+$, then L generates a positive semigroup and $s(L) = \omega(L)$.*

The Arendt theorem is useful in applications. In fact, in its hypothesis it has an inverse estimate with respect to the Hille-Yosida theorem. In turn, the converse does not hold, and even if Theorem (2.2.3) says that L generates a strongly continuous semigroup, it gives not the value of M of the estimate (2.2.1), which has to be computed by means of other techniques (for example, using equivalent norms).

However, as we shall see in section 3.3, not always it is possible to prove the Hille-Yiosida estimate or the Arendt estimate. Nevertheless, when the studied operator is proved to have a positive resolvent, the generation of an integrated semigroup is then possible, as it is stated in the next theorem. We begin giving the definition of n-times integrated semigroup:

Definition 2.2.4 *Let $n \in N$. A strongly continuous family $S(t)$, for $t \geq 0$ is called n-times integrated semigroup if $S(0) = 0$ and given $s, t \geq 0$:*

$$\begin{aligned} S(t)S(s) = \frac{1}{(n-1)!}\bigg(&\int_t^{s+t} (s+t-r)^{n-1} S(r)\, dr \\ &- \int_0^s (s+t-r)^{n-1} S(r)\, dr\bigg). \end{aligned} \tag{2.2.0}$$

Moreover, $S(t)$ is called non-degenerate if $S(t)x = 0$ for all $t \geq 0$ implies $x = 0$. Finally, $S(t)$ is called exponentially bounded if there exist $M, w \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{wt}$ for all $t \geq 0$.

We remark that equation (2.2.4) yields to:

$$S(t)S(s) = S(s)S(t) \ , \quad \forall \, s,t \geq 0$$

and

$$S(t)S(0) = 0 \ , \quad \forall \, t \geq 0.$$

Theorem 2.2.4 (Arendt2) *Let X be a Banach space with generating and normal cone X^+. Let the operator L has dense domain in X. If the resolvent set of L contains the interval $(a, +\infty)$ for some positive a and the resolvent $R(\lambda, L)$ is positive for $\lambda > a$, then L is the generator of an integrated semigroup $S(t)$ for every $t \geq 0$.*

We now quote a classical result about bounded perturbations of the generator of a strongly continuous semigroup:

Proposition 2.2.2 (Perturbation) *Let X be a Banach space and let L be the generator of a strongly continuous semigroup $Z(t)$ (i.e. $\|Z(t)\| \leq Me^{\omega t}$). If B is a bounded linear operator on X (i.e. $B \in \mathcal{B}(X)$) then $L + B$ is the generator of a strongly continuous semigroup $S(t)$ on X, satisfying:*

$$\|S(t)\| \leq Me^{(\omega + \|B\|)t}. \tag{2.2.0}$$

We recall also a theorem by Pazy which may be seen as a perturbation theorem:

Theorem 2.2.5 (Pazy) *Let L_n be a sequence of generators of strongly continuous semigroups and assume that:*
(i) As $n \to \infty$, $L_n x \to Lx$ for every $x \in D$ where D is a dense subset of X.
(ii) There exists a z_0 with $Re\, z_0 > \omega$ for which $(z_0 I - L)D$ is dense in X.
Then the closure $\overline{L}$ of L, is the generator of a strongly continuous semigroup.
Moreover, if $Z_n(t)$ and $Z(t)$ are the strongly continuous semigroups generated by L_n and $\overline{L}$ respectively, then:

$$\lim_{n \to \infty} Z_n(t)x = Z(t)x \ , \quad \forall \, x \in X \ , \ t \geq 0$$

and the above limit is uniform in t for t in bounded intervals.

We finally recall a theorem by Chernoff (see [23], [57]) about the generation of a semigroup by the sum of two generators of semigroups.

Theorem 2.2.6 (Chernoff) *Let $T(t)$ and $S(t)$ be strongly continuous semigroups on X satisfying the stability condition:*

$$\|[T(t/n)S(t/n)]^n\| \leq Me^{\omega t},$$

for every $t \geq 0, n \in \mathbb{N}$ and for some constants $M \geq 1, \omega \in \mathbb{R}$. Consider the sum $A + B$ on $D = D(A) \cap D(B)$ of the generators A of $T(t)$ and B of $S(t)$ and assume that D and $(z_0 - A - B)D$ are dense in X for some z_0. Then, the closure $\overline{A + B}$, generates a strongly continuous semigroup $U(t)$ given by the Trotter product formula:

$$U(t)f = \lim_{n\to\infty}[T(t/n)S(t/n)]^n f \; , \; f \in X, \tag{2.2.0}$$

with uniform convergence for t in bounded intervals.

2.2.2 The evolution problem

Let X be a Banach space and let L be a linear operator from $D(L) \subset X$ into X. Given $u_0 \in X$ the abstract Cauchy problem for L with initial data u_0 is to find a (strong) solution $u(t)$ to the initial value problem:

$$\begin{cases} \dfrac{d}{dt}u(t) = Lu(t) \; , \quad t > 0 \\ u(0) = u_0 \, . \end{cases} \tag{2.2.0}$$

By a (strong) solution we mean an X valued function $u(t)$ such that $u(t)$ is continuous for $t \geq 0$, continuously differentiable, $u(t) \in D(L)$ for $t > 0$ and (2.2.2) is satisfied. We remark that, since $u(t) \in D(L)$ for $t > 0$ and u is continuous at $t = 0$, if $u_0 \notin \overline{D}(L)$ then (2.2.2) has not a solution.

It is well known that if the operator L generates a strongly continuous semigroup $Z(t)$, then the abstract Cauchy problem for L, with $u_0 \in X$, has a unique solution $u(t) = Z(t)u_0$, and this solution is such that $\|u(t)\| \leq Me^{\omega t}\|u_0\|$.

As said in the introduction of this section, we are concerned with bounded regions, therefore the transport equation will be equipped by some kind of *boundary* conditions. When studying kinetic models it is reasonable to define boundary conditions giving the relation between the *outgoing* density of particles and the *incoming*

one. Correspondingly, $D(L)$ is characterized by these boundary conditions.
In order to write such type of boundary conditions we first need to define the *incoming* and *outgoing* boundaries. To fix ideas, we consider a bounded and convex set in $\mathbb{R}^3 \times \mathbb{R}^3 \supset \Omega \times \mathbb{R}^3$, then the incoming and outgoing boundaries are respectively given by:

$$\Gamma^- = \{(x,v) : \, x \in \partial\Omega \, , \; v \in \mathbb{R}^3 \, , \; \nu(x) \cdot v < 0\},$$

$$\Gamma^+ = \{(x,v) : \, x \in \partial\Omega \, , \; v \in \mathbb{R}^3 \, , \; \nu(x) \cdot v > 0\},$$

where $\nu(x)$ is the unit outward normal vector to $\partial\Omega$ at x. We are now,in the position to define the *traces* of the probability density function f as the restriction of f on $\Gamma^\pm$, $f_\pm = f|_{\Gamma^\pm}$. The traces $f_\pm$ can be used to represents the outgoing and incoming flux of particles through the boundary $\partial\Omega$.

We stress that the operator which associates to f its traces may not be bounded. If so the transport operator of the evolution problem may not be closed (see [22]).

There are three main types of boundary condition correlating f_+ and f_-. Namely, the reflection, the diffusive and the Maxwell boundary conditions. We just write the last type of boundary condition, as the first two types may be obtained from it. Indeed, the Maxwell boundary conditions reads:

$$\begin{aligned} f_-(x,v) &= \alpha f_+(x,v_*) \\ &+(1-\alpha)\int_{(x,v)\in\Gamma^+} K(x,v' \to v) f(x,v') |\nu(x)\cdot v'| \, dv', \end{aligned} \tag{2.2.0}$$

where $x \in \partial\Omega$, $0 \leq \alpha \leq 1$, $v_* = -v$ is the specular reflected velocity and $K(x, v' \to v)$ is the rate of particle which before colliding the boundary has a velocity v' and are reflected with a velocity v. It is easily seen then that, if $\alpha = 0$ then we have pure diffusion, while if $\alpha = 1$ we only have the specular reflection.

All the above type of boundary conditions may also be described as follows by a bounded and linear operator Λ,

$$f_- = \Lambda f_+. \tag{2.2.0}$$

If the norm of Λ is strictly smaller than one, then we say that we have *dissipative* boundary conditions. While if Λ has norm equal to one, then the boundary conditions are called *conservative*. We shall also study a third type of boundary conditions called *multiplicative*. Indeed, if $\|\Lambda f_+\| \geq \alpha \|f_-\|$ with $\alpha > 1$, then Λ has norm larger than one and we refer to *multiplicative* boundary conditions. This kind of boundary conditions has already been studied in the contest of evolution problems by S.Totaro (see [58]). Moreover, results about the generation of semigroup in kinetic equation (with the null force field, $F = 0$) equipped by multiplicative boundary conditions may be found in [44] and [45].

We finally remark that in applications there may also exist a source of particles on the boundaries usually described by means of a given function q defined on Γ^-. If so, we shall call the boundary conditions *nonhomogeneous*. The evolution problem equipped by nonhomogeneous boundary conditions will be called *affine* evolution problem.

Indeed, it is possible to write the explicit form of the solution of the nonhomogeneous problem in terms of the semigroup generated by the associated (homogeneous) linear evolution problem ([6], [8]). This fact will become clear after having introduced some more notations. We shall distinguish two cases, the time independent source case, and the time dependent source case.

Definition 2.2.5 *Let X be a Banach space and let A and L be two operators with domain $D(A)$ and $D(L)$ both with values in X. The operator A is said to be an affine operator associated to L if $D(A)$ is an affine subspace associated to $D(L)$:*

$$f_1 - f_2 \in D(L), \quad \forall f_1, f_2 \in D(A),$$

$$f + g \in D(A), \quad \forall f \in D(A), g \in D(L),$$

and

$$A(f+g) = Af + Lg \ , \quad \forall f \in D(A)\, , g \in D(L).$$

We remark that is possible to apply this definition even if the nonhomogeneity appears in the evolution equation, and not only if it appears in the boundary conditions.

Let A be an affine operator, with time independent source term q, associated to the linear operator L and consider the affine Cauchy problem:

$$\begin{cases} \dfrac{d}{dt}u(t) = Au(t) \ , \quad t > 0 \\ \\ u(0) = u_0 \in D(A), \end{cases} \tag{2.2.0}$$

where A is time independent.

If L is the generator of the semigroup $Z(t)$, then the unique solution of (2.2.2) reads:

$$u(t) = u_0 + \int_0^t Z(t) A u_0 \, ds \,, \quad t \geq 0, \tag{2.2.0}$$

and is such that $u(t) \in C[0, +\infty] \cap C^1(0, +\infty)$.

Moreover, if there exists a $p \in D(A)$ such that $Ap = 0$, then, replacing u_0 in (2.2.2) by $(u_0 - p) + p$, it follows that $Au_0 = L(u_0 - p)$ and so:

$$u(t) = p + Z(t)(u_0 - p), \tag{2.2.0}$$

Finally, if we define the family of operators $U(t)$, for every $x \in D(A)$ and $t \geq 0$:

$$U(t)x = x + \int_0^t Z(t) A x \, ds,$$

then $U(t)$ is an *affine* strongly continuous semigroup associated to $Z(t)$, as it shown in the following proposition.

Proposition 2.2.3 *The family of operators $U(t)$ satisfies:*
(i) For every $t \geq 0$, $U(t) : D(A) \to D(A)$. Moreover, it is an affine operator associated with $Z(t)|_{D(L)}$.
(ii) $U(0) = I, \quad U(t+s) = U(t)\,U(s) \; \forall \, t, s \geq 0$.
(iii) $\lim_{t \downarrow 0} U(t)x = x \,, \quad \forall \, x \in X$.
(iv) $|U(t)x_1 - U(t)x_2| \leq M e^{\omega t} |x_1 - x_2|, \; \forall \, x_1, x_2 \in D(A)$.

On the other hand, if the source term q depends explicitly on time, denoting by A_t the affine operator associated to L with $0 \leq t < t_0$, the solution of problem:

$$\begin{cases} \dfrac{d}{dt} u(t) = A_t u(t) \,, \quad t > 0 \\ \\ u(0) = u_0 \in D(A_0), \end{cases} \tag{2.2.0}$$

reads as follows, for $0 \leq t < t_0$:

$$u(t) = p(t) + Z(t)[u_0 - p(0)] + \int_0^t Z(t-s)[A_s p(s) - p'(s)] ds, \tag{2.2.0}$$

where $p(t) = p(\cdot, \cdot, t)$ is a suitable function from $[0, t_0)$ in X such that $p(t) \in D(A_t)$. We remark that, for every $0 \leq t < t_0$, $D(A_t)$ is an affine subspace associated to $D(L)$, and it may be represented by:

$$D(A_t) = p(t) + D(L).$$

This representation is not unique, because it may be $D(A_t) = p(t) + D(L)$ as well as $D(A_t) = p_1(t) + D(L)$ for some other function $p_1 \in D(A_t)$.
If $p(t)$ has the additional property $A_t p(t) - p'(t) = 0$, then (2.2.2) becomes, for $0 \leq t < t_0$:

$$u(t) = p(t) + Z(t)[u_0 - p(0)]. \tag{2.2.0}$$

Finally, we remark that the main point in solving affine problems, once the semigroup of the linear associated problem is known, is to find a regular representation $p(t)$ for $D(A_t)$.

We end this section, stating a theorem on the existence and uniqueness of the solution of a Cauchy problem when the operator we consider is the generator of an integrated semigroup.

Theorem 2.2.7 (Arendt3) *Assume that $f_0 \in D(L^2)$, and that the operator L is the generator of an integrated semigroup, then there exists a unique solution of the Cauchy problem*

$$\begin{cases} \dfrac{df(t)}{dt} = Lf \\ \\ f(0) = f_0 \in D(L^2) \end{cases} \tag{2.2.0}$$

given by:

$$f(t) = S(t)Lf_0 + f_0 \,, \quad \forall\, t \geq 0, \tag{2.2.0}$$

where $S(t)$ is the integrated semigroup generated by L.
Moreover, if $f_0 \geq 0$ then $f(t) \geq 0$ for every $t \geq 0$.

2.3 Asymptotic Analysis

Many approaches are possible in order to study a transport problems. In this thesis, we shall consider also the derivation of a *drift-diffusion* type model from a kinetic one. The interest of such an approach relies mainly on the fact that the number of independent variables is reduced from seven (three position coordinates, three velocity coordinates and time) to four variables. From an numerical point of view, this simplification of the model leads to the possibility of more efficient numerical simulations than the integration of the kinetic equation via classical methods such as Monte Carlo method. Moreover, fluid dynamical models seem to be a good compromise between the contradictory requirements of physical accuracy and computational efficiency.

There are two different main approaches to the drift-diffusion equations: the *Hilbert expansion*, and the *moments* methods.

The Hilbert expansion method, is based on a perturbation argument. Depending on the physics of the model, it is possible to define a small adimensional parameter α, and then a scaling of the variables. Consequently, a scaled version of the kinetic equation is obtained, where now the unknown function depends on adimensional variables and is denoted by f^α. This function is then expanded in powers of α:

$$f^\alpha = f_0 + \alpha f_1 + \alpha^2 f_2 + ...,$$

This expansion is called the Hilbert expansion, and has been introduced for the first time by Hilbert in 1912 ([38]).
The next step in the derivation of the drift-diffusion model, is to replace f^α by its expansion and then to solve the systems written for the coefficients of the powers 0, 1 and 2 of α. It is easy to see that to solve the first order system we must solve the zero order one, and to solve the second order system we need to know the solution of the zero and first order systems.
The fact that the Hilbert expansion is usually truncated at the second order, become clear when writing the solvability condition for the second order system. In effect, the drift-diffusion equation is given by the solvability condition, and so there is no need to study the successive orders.
The drift-diffusion equations are of the type:

$$\begin{aligned} J &= -D(\nabla_x n - nF) \\ \partial_t n - \nabla_x J &= 0, \end{aligned} \tag{2.3.0}$$

where J is the current density, n the density of the particles, F is the electric field and D is a suitably defined diffusion tensor. They where derived for the first time by van Roosbroek ([60]). The name drift-diffusion originates from the type of dependence of the current densities J on the densities n and on the force field F.
We end recalling that there exist other type of fluid dynamical models as the *hydrodynamic* equations, founded by means of a *Chapman-Enskog expansion*.

In recent years, however, another approach to the derivation of the drift-diffusion equations has been studied. This method is known in semiconductor physics as the *spherical harmonics expansions* method (leading to the SHE-models) or as the *energy transport* method. The main reasons for choosing this approach is that on intermediate scaling it is possible to establish macroscopic models more complex than the drift-diffusion. In turns, these models better describe the physics of the intermediate regime than the drift-diffusion model and still are less expensive than kinetic models for applications.
The main steps are the following:

- Write the scaled version of the kinetic equation.
- Prove existence and uniqueness of the solution f^α of the scaled kinetic equation for every α.
- Define the current J^α and prove that its limit is a function J of the form:

$$J = -D(\nabla_x + F\partial_\varepsilon)f,$$

  where f is the limit function of f^α, ∂_ε is the partial derivative with respect to the kinetic energy and D is a suitable defined diffusion tensor.

- Pass to the limit $\alpha \to 0$ in the scaled kinetic equation and prove that the limit function f satisfies a continuity equation of the type:

$$\partial_t f - (\nabla_x - F\partial_\varepsilon)J = 0.$$

Compared with the Hilbert expansion, the moments method requires a good deal of physical intuition or a priori knowledge about the solution of the kinetic equation. The main ingredient is an ansatz for the phase space density which prescribes the dependence on the velocity and which contains several parameters depending on position and time. After inserting the ansatz, the kinetic equation is multiplied by a number of linearly independent functions of velocity and integrated over the velocity space. The result are differential equations for the time and space dependent parameters. We remark that, in some cases, not all integrations can be carried out explicitly.

3

Semigroup Techniques and the Vlasov equation

The Vlasov equation is useful in modeling particle transport problems on a time scale much shorter than the mean time between two consecutive scattering collisions, ([48]). For instance, in plasma or in semiconductor physics (see [26] and [48] respectively) it is used to model the flux of electrons.

When collision effects between particles have to be considered, then a collision term $Q(f, f)$ is added on the right hand side of the Vlasov equation. This additional term may lead to more difficult procedures, for example in asymptotic analysis, but as far as the generation of a semigroup is concerned, it may be treated as a bounded perturbation ([39], [53]).

In recent years the Vlasov equation has been considered under many different aspects. For example it has been studied to deal with the runaway phenomenon ([19]), or in connection with the asymptotic analysis of transport problems in order to derive diffusion, hydrodynamic and SHE models ([26], [48]) and to get numerical simulations. However, the physics of the problems suggested by several applications leads to the study of the Vlasov equation in *bounded* domains. In fact, the boundary conditions can heavily affect the form and the behavior of the solution.

In this chapter we study the evolution of a system of particles with mass m and charge q_0, not subject to scattering events, moving in a one dimensional bounded region under the influence of a constant electric field E and interacting with the boundaries. This situation is modeled by the onedimensional Vlasov equation equipped with some kind boundary conditions. We shall assume that the boundary conditions, describing the relation between the incoming flux of particles and the outgoing one, are defined by means of a linear and bounded operator Λ. This operator my represent the Maxwell boundary conditions as well as the reflection boundary conditions or a linear combination of them. Moreover, on the boundaries of the region a source of particles q is also taken into account. This fact leads to nonhomogeneous evolution problems.

It is worth observing that this kind of problem has already been the subject of several papers. For instance, Bardos in [3] assumed bounded coefficients for general linear differential equations equipped by Dirichlet boundary condition. In other words, both the outgoing and incoming densities of particles were supposed to be zero. The study and the generation of a semigroup done by Bardos is based on trajectories methods. He also established some results on numerical approximations. We stress out that, Bardos studied the problem only for a bounded range of velocity, whereas in this chapter the velocity v varies over the whole space $\mathbb{R}$.

Moreover, Beals, Greenberg, Protopopescu and Van der Mee ([37], [7]) have studied this kind of evolution problem, too. In fact, they consider the whole Boltzmann equation and their statements are given under the essential assumption that all the considered operators are closed. They prove the generation of a semigroup only in the case of dissipative conditions on the boundary and the explicit form of the solution is *not* given in the case of sources terms on the boundaries. Finally, we remark that the solution we find is different and in some sense more general than the one found by the mentioned authors by means of the trajectories methods because we use abstract techniques.

In section 3.1 we give the prove of the existence and uniqueness of the solution of the Vlasov equation defined in a slab and with source terms on the boundaries. The proof is based on techniques of analysis for elliptic operators and on the theory of linear and affine semigroups. By means of this kind of approach we are also able to write the explicit form of the solution and to consider its approximations.

In section 3.2 we study the one-dimensional Vlasov equation in a rod with nonhomogeneous dissipative and conservative boundary conditions. The study is carried out by means of the theory of semigroups and affine operators. Existence, uniqueness and positivity of the solution are proved. The explicit form of the solution is derived and an approximating sequence is given.

Finally in section 3.3 it is proved the generation of an integrated semigroup by a transport equation equipped with multiplicative boundary conditions. Hence, the existence and uniqueness of the solution of the related evolution problem are also proved and the explicit form of this solution is given.

3.1 The Vlasov equation in a slab with source terms on the boundaries

We consider the behavior of a system of particles in a one-dimensional bounded region and moving under the influence of a constant electric field E. The boundary conditions are given by (2.2.2) with $\Lambda = 0$ and with an incoming source of particles, (non re-entry nonhomogeneous boundary conditions).

In section 3.1.1 we describe the model and define the functional space and the operator A_t that we need to write the abstract form of our problem. We also recall the general abstract form of the solutions both in the time dependent and time independent boundary source cases ([6] and [9]). Successively, we introduce a sequence of operators which approximate the operator A_t and we prove convergence theorems for the sequence of operators that we have defined.

In section 3.1.2 we prove the main theorem: the generation of a strongly continuous contraction semigroup by using the sequence of operators defined in section 3.1.1. The proof of the theorem is based on a paper by Lunardi and Vespri ([43]) about the generation of strongly continuous contraction semigroups by elliptic operators with unbounded coefficients. Following [43] and by means of duality arguments we prove also that this theorem holds in L^p spaces with $1 \leq p < \infty$. The fact that the generation of a semigroup holds also in the L^1 space is relevant in physics, because in this space the norm of the density function f gives the total number of particles present at time t in the considered region.

In section 3.1.3 we show that the assumptions of the Trotter approximation theorem ([57]) are fulfilled by the sequence of operators previously defined. Hence, we are able to state the convergence of the semigroups generated by the sequence of operators to the semigroup which gives the solution of the abstract form of our problem.

Finally, in section 3.1.4 we give the explicit form of the solution of the problem and we show that it is possible to consider also its approximations, as done by Bardos in [3].

3.1.1 The problem

We study the evolution of a system of charged particles, with mass m and charge q_0, moving in a slab under the influence of a constant electric field E. Moreover, we assume that there is some kind of particle injection at the boundaries of the slab. As we disregard scattering events, the problem is modeled by the following Vlasov equation:

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} + a\frac{\partial u}{\partial v} = 0, \tag{3.1.0}$$

where $a = q_0 E/m$ is constant and represents the force due to the given electric field acting on the particles. The unknown function $u = u(x, v, t)$ represents the density of the particles which at time t are in a position $x \in [-b, +b]$ and have velocity $v \in \mathbb{R}$.

The initial and boundary conditions needed to solve this problem read as follows:

$$u(x, v, 0) = u_0(x, v), \qquad x \in [-b, +b], \quad v \in \mathbb{R}, \tag{3.1.0}$$

and

$$\begin{cases} u(-b, v, t) = q_1(v, t), & v > 0, \quad t \geq 0, \\ \\ u(+b, v, t) = q_2(v, t), & v < 0, \quad t \geq 0, \end{cases} \tag{3.1.0}$$

where the non negative functions $q_1(v, t)$ and $q_2(v, t)$ represent two sources of particles at the boundaries $x = -b$ and $x = +b$, respectively.

In order to write the problem in an abstract form, we introduce the set $\Omega = [-b, +b] \times \mathbb{R}$ and we consider the Banach space $X = L^2(\Omega)$, endowed with the usual L^2 norm, $\|f\|_2 = \|f\|$ for $f \in X$.

Moreover, let us define the operator:

$$A_t f = -v\frac{\partial f}{\partial x} - a\frac{\partial f}{\partial v}, \tag{3.1.0}$$

with domain and range:

$$D(A_t) = \left\{ f \in X, v\frac{\partial f}{\partial x} \in X, \frac{\partial f}{\partial v} \in X, f \text{ satisfies } (3.1.1) \right\}, \tag{3.1.0}$$

$$R(A_t) \subset X.$$

We remark that A_t is a nonlinear time dependent operator because of the nonhomogeneous boundary conditions which appear in the definition of its domain. Furthermore, we assume that the source $q_1(\cdot,t)$ belongs to $L^2(0,+\infty)$ for every $t \geq 0$ and that $q_2(\cdot,t)$ belongs to $L^2(-\infty,0)$ for every $t \geq 0$.

The abstract form of problem (3.1.1), (3.1.1) and (3.1.1) then reads as follows:

$$\begin{cases} \dfrac{du(t)}{dt} = A_t u(t)\,,\ t > 0 \\ \\ u(0) = u_0 \,\in D(A_0), \end{cases} \tag{3.1.0}$$

where $u(\cdot,\cdot,t)$ is now a function defined on $[0,+\infty)$ with values in X.

In order to prove that (3.1.1) has a unique strongly continuous solution, we shall first consider the following auxiliary linear operator L:

$$Lf = -v\frac{\partial f}{\partial x} - a\frac{\partial f}{\partial v}, \tag{3.1.0}$$

with domain and range:

$$\begin{gathered} D(L) = \left\{ f \in X, v\frac{\partial f}{\partial x} \in X, \frac{\partial f}{\partial v} \in X, f(x,v)|_{\partial\Omega} = 0 \right\}, \\ R(L) \subset X. \end{gathered} \tag{3.1.0}$$

It is easy to prove that, for each given $t > 0$, A_t is an affine operator associated to L. In fact,

$$\begin{gathered} f_1 - f_2 \in D(L),\ \forall\, f_1, f_2 \in D(A_t), \\ A_t(f+g) = A_t f + Lg,\ \forall\, f \in D(A_t),\ g \in D(L). \end{gathered} \tag{3.1.0}$$

We remark that the physical meaning of the operator L has no relevance. As a matter of fact, we need it in order to apply the theory of affine operators, to prove that the problem (3.1.1) has a unique solution and to derive its explicit form.
In fact, if we prove that the operator L is the generator of a semigroup $T(t)$, then the solution of the Cauchy problem (3.1.1), in the case of time independent sources q_1 and q_2, can be written as follows:

$$u(t) = u_0 + \int_0^t T(s) A_t u_0 ds\,, \tag{3.1.0}$$

where we recall that $u_0 \in D(A_t)$.
Moreover, if there exists a function $p = p(x, v) \in D(A_t)$ such that $A_t p = 0$, then (3.1.1) simplifies to:

$$u(t) = p + T(t)(u_0 - p). \tag{3.1.0}$$

On the other hand, if the source terms $q_1(t)$ and $q_2(t)$ are time dependent, then the solution of problem (3.1.1) can be written as follows:

$$u(t) = p(t) + T(t)[u_0 - p(0)] + \int_0^t T(t-s)[A_t p(s) - p'(s)]ds, \tag{3.1.0}$$

where $p(t) = p(\cdot, \cdot, t)$ is a function from $\Omega \times [0, t_0)$ in X, ($t_0 \leq +\infty$) such that $p(t) \in D(A_t)$. Furthermore, if $p(t)$ is also such that $A_t p(s) - p'(s) = 0$, where p' is a strong derivative, then (3.1.1) becomes:

$$u(t) = p(t) + T(t)[u_0 - p(0)]. \tag{3.1.0}$$

We shall give an example of the function p in the case of time independent sources q_1 and q_2 in section 3.1.4.

By using (3.1.1) and the choice axiom, it possible to prove that every affine operator A_t associated to a linear operator L has the representation

$$D(A_t) = p(t) + D(L), \tag{3.1.1}$$

where p(t) is a suitable function belonging to $D(A_t)$.
The representation (3.1.1) is not unique because we might have $D(A_t) = p(t) + D(L)$ as well as $D(A_t) = p_1(t) + D(L)$ for some other function $p_1(t)$. However, it can be proved that this fact does not affect all the results we have quoted,(see [6]).

Moreover, if the operator L, which A_t is affine to, is such that its closure $\overline{L}$ generates a strongly continuous semigroup, we can define $\tilde{A}_t$, an extension of A_t, in the following way:

$$D(\tilde{A}_t) = p(t) + D(\overline{L}) = \{f \in X, f = p(t) + f_0, f_0 \in D(\overline{L})\}$$

$$\tilde{A}_t f = A_t p(t) + \overline{L} f_0 \tag{3.1.0}$$

where $p(t)$ is the function used to represent A_t by means of (3.1.1).

Thus, we can apply formulas (3.1.1), (3.1.1), (3.1.1), (3.1.1), where now $T(t)$ is the semigroup generated by $\overline{L}$ and the abstract evolution problem is (3.1.1) with A_t replaced by $\tilde{A}_t$.

We now consider the following sequence of linear operators L_n which approximate L. For every $n \in \mathbb{N}$:

$$L_n f = \frac{k}{n}\Delta f + Lf = \frac{1}{n}\Delta f + Lf, \tag{3.1.0}$$

with domain and range:

$$D(L_n) = D(L) \cap \{f \in X, \Delta f \in X\},$$
$$R(L_n) \subset X, \tag{3.1.0}$$

where Δ is to be considered with respect to x and v, and where k is a dimensional constant which for simplicity we consider equal to 1.

Let us now define also the operators $A_{t,n}$ which are the affine operators associated to L_n and which approximate the operator A_t. For every $n \in \mathbb{N}$:

$$A_{t,n} f = \frac{1}{n}\Delta f + A_t f, \tag{3.1.0}$$

with domain and range:

$$D(A_{t,n}) = D(A_t) \cap \{f \in X, \Delta f \in X\},$$
$$R(A_{t,n}) \subset X. \tag{3.1.0}$$

It is worth to remark that $D = D(L_n)$ and $D(A_{t,n})$ are *independent* of n. The following two lemmas hold:

Lemma 3.1.1 *Let L and L_n be defined by (3.1.1) and (3.1.1), then we have:*

$$\lim_{n\to\infty} \|L_n f - Lf\| = 0 \quad \forall f \in D = D(L_n) \subset D(L).$$

Proof:

For every $f \in D$, we have:

$$\|L_n f - Lf\| = \left\| \frac{1}{n}\Delta f + Lf - Lf \right\| = \frac{1}{n}\|\Delta f\|,$$

which tends to 0 as n goes to ∞. ∎

Lemma 3.1.2 *Let A_t and $A_{t,n}$ be defined by (3.1.1) and (3.1.1), then for every $t \geq 0$:*

$$\lim_{n\to\infty} \|A_{t,n}f - A_t f\| = 0 \quad \forall f \in D(A_{t,n}) \subset D(A_t).$$

Proof:

As in the previous lemma, the proof follows from the definition of $D(A_{t,n})$. ■

3.1.2 Generation of the semigroup

In this section we prove the generation of a strongly continuous semigroup of contractions by the operator L_n by using the techniques of [43]. By this result, it will follow that the closure $\overline{L}$ of the linear operator L is the generator of a strongly continuous semigroup of contractions (i.e. $\overline{L} \in \mathcal{G}(1,0;X)$, [39] and [53]).

Theorem 3.1.1 *The operator L_n defined by (3.1.1), (3.1.1) is the generator of a strongly continuous semigroup of contractions, that is $L_n \in \mathcal{G}(1,0;X)$.*

Proof:

The proof is a rather simplified version of the proofs of [43] with $\Omega = \mathbb{R}^n$. We give it here for sake of completeness. Let us fix $n \in \mathbb{N}$ and consider the following bilinear form associated to the operator L_n:

$$\begin{aligned} \hat{a}(f,\varphi) = \langle L_n f, \varphi\rangle = &-\frac{1}{n}\int_{-b}^{+b}\int_{\mathbb{R}} \left(\frac{\partial f}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial \varphi}{\partial v}\right) dxdv \\ &+ \int_{-b}^{+b}\int_{\mathbb{R}} \left(-v\frac{\partial f}{\partial x} - a\frac{\partial f}{\partial v}\right)\varphi\, dxdv \end{aligned} \tag{3.1.0}$$

for every $f = f(x,v) \in D(L_n)$ and $\varphi = \varphi(x,v) \in W_0^{1,2}(\Omega) \subset D(L_n)$.

As $W_0^{1,2}(\Omega)$ is dense in X, for every $f \in D(L_n)$ the map $\varphi \mapsto \hat{a}(f,\varphi)$ can be extended with continuity to X in such a way that there exists one and only one $h \in X$ such that $\hat{a}(f,\varphi) = \langle h, \varphi\rangle$; thus $\hat{a}(f,\varphi) = 0\ \forall\varphi$ if and only if $f = 0$. It follows that, for every $\lambda > 0$ and for every $g \in X$, $f \in D(L_n)$ is a solution of the resolvent equation:

$$\lambda f - L_n f = g, \tag{3.1.0}$$

if and only if for every $\varphi \in W_0^{1,2}(\Omega)$ we have:

$$\begin{aligned} &\frac{1}{n}\int_{-b}^{+b}\int_{\mathbb{R}}\left(\frac{\partial f}{\partial x}\frac{\partial \varphi}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial \varphi}{\partial v}\right)dxdv \\ &+\int_{-b}^{+b}\int_{\mathbb{R}}\left(v\frac{\partial f}{\partial x}+a\frac{\partial f}{\partial v}+\lambda f\right)\varphi\,dxdv=\int_{-b}^{+b}\int_{\mathbb{R}}g\varphi\,dxdv. \end{aligned} \tag{3.1.0}$$

Following [43], we now approximate $v \in \mathbb{R}$ by means of some bounded v_m as follows; given $m \in \mathbb{N}$, we put:

$$v_m = \begin{cases} v & \text{if } |v| \le m, \\ \\ \dfrac{mv}{|v|} & \text{otherwise.} \end{cases} \tag{3.1.0}$$

If we define the bilinear form $\hat{a}_m(f_m, \varphi)$ as the bilinear form $\hat{a}(f, \varphi)$ (3.1.2) in which v is replaced by v_m, then $\hat{a}_m$ is continue and coercive on $H^1(\Omega) = W^{1,2}(\Omega)$.

Moreover, replacing in equation (3.1.2) v by v_m and L_n by $L_{n,m}$, where $L_{n,m}$ is defined as L_n with v replaced by v_m and $D(L_{n,m}) = D(L_n)$, we have that the resolvent equation:

$$\lambda f_m - L_{n,m} f_m = g,$$

thanks to the Lax-Millgram theorem, has a unique solution $f_m \in D(L_{n,m})$ for every $g \in X$, where f_m are the solutions corresponding to the problem with the truncated velocities v_m. Therefore, replacing φ with the solution f_m in (3.1.2), we obtain:

$$\begin{aligned} &\frac{1}{n}\int_{-b}^{+b}\int_{\mathbb{R}}\left(\frac{\partial f_m}{\partial x}\frac{\partial f_m}{\partial x}+\frac{\partial f_m}{\partial v}\frac{\partial f_m}{\partial v}\right)dxdv \\ &+\int_{-b}^{+b}\int_{\mathbb{R}}\left(v_m\frac{\partial f_m}{\partial x}+a\frac{\partial f_m}{\partial v}+\lambda f_m\right)f_m\,dxdv=\int_{-b}^{+b}\int_{\mathbb{R}}g f_m\,dxdv. \end{aligned} \tag{3.1.0}$$

Since f_m must belong to $D(L_{n,m})$, we have from (3.1.2):

$$\frac{1}{n}\|\nabla f_m\|^2 + \lambda\|f_m\|^2 \le \|g\|\,\|f_m\|. \tag{3.1.0}$$

Following the same arguments of [43] it is possible to prove that, being f_m equi-bounded functions in $H^1(\Omega)$, they converge weakly as $m \to \infty$ to f, which is the solution of (3.1.2). Therefore we also have:

$$\frac{1}{n}\|\nabla f\|^2 + \lambda\|f\|^2 \le \|g\|\,\|f\|, \tag{3.1.0}$$

from which we get:

$$\|f\| \le \frac{1}{\lambda}\|g\|, \tag{3.1.0}$$

which is the well known Hille-Yosida estimate. Thus, $L_n \in \mathcal{G}(1, 0; X)$. ■

We remark that by the same techniques of [43] it is possible to prove the generation of a strongly continuous contraction semigroup also when the coefficients of the first order derivatives appearing in the definition of L_n are unbounded functions of x and v with not more than linear growth at ∞.

It is also possible to prove that L_n generates a strongly continuous contraction semigroup in $L^p(\Omega)$ with $p > 2$. On the other hand, in $L^\infty(\Omega)$ there is not generation of semigroup because the domain is not dense in X, even if estimate (3.1.2) still holds (see [43]).
Nevertheless, by using estimate (3.1.2) in $L^\infty(\Omega)$ we are able to state that L_n generate a strongly continuous semigroup also in $L^1(\Omega)$. This fact has a precise physical meaning because in $L^1(\Omega)$ the norm of the density function $u(x, v, t)$ gives the total number of particles present at time t in the region Ω. Hence, from the generation of a strongly continuous contraction semigroup, follows that it is possible to have bounds of the total number of particles which are in Ω at time t by means of the total number of particles which are in Ω at time 0, $\|u_0\|$.

Define the linear operator $L_{n;1}$:

$$L_{n;1}f = \frac{1}{n}\Delta f + Lf \quad \forall f \in L^1(\Omega)$$

$$D(L_{n;1}) = D(L) \cap \{f \in L^1(\Omega), \Delta f \in L^1(\Omega),\ f(x, v)|_{\partial\Omega} = 0\}, \tag{3.1.0}$$

$$R(L_{n;1}) \subset L^1(\Omega).$$

we remark that $L_{n;1}f = L_n f$, where now in the definition of $D(L_n)$ the space X is $L^1(\Omega)$.

Theorem 3.1.2 *The operator $L_{n;1}$ defined as in (3.1.2) is the generator of a strongly continuous semigroup of contractions, i.e. $L_{n;1} \in \mathcal{G}(1, 0; L^1(\Omega))$.*

Proof:

Define the dual operator of $L_{n;1}$, L_n^*, as follows:

$$L_n^* f = \frac{1}{n}\Delta f - Lf, \tag{3.1.0}$$

with domain and range:

$$D(L_n^*) = D(L) \cap \{f \in L^\infty(\Omega), \Delta f \in L^\infty(\Omega)\}, \tag{3.1.0}$$

$$R(L_n^*) \subset L^\infty(\Omega).$$

We remark that in definition (3.1.2) the space X in $D(L)$ is $L^\infty(\Omega)$.

By the same procedure used to show that equation (3.1.2) has a solution in $L^p(\Omega)$ we can prove that the equation $\lambda f - L_n^* f = g$ has a solution f in $L^\infty(\Omega)$ for every $g \in L^\infty(\Omega)$ and that estimate (3.1.2) holds (see [43]).

Let $h \in L^1(\Omega)$ and, for every $\lambda > 0$, consider the following resolvent equation for the operator $L_{n;1}$:

$$\lambda w - L_{n;1} w = h. \tag{3.1.0}$$

If $h \in C_0^\infty(\Omega)$, by using again the results of [43], there exists a unique solution $w \in C_0^\infty(\Omega)$ of the resolvent equation (3.1.2). Thus, we have the existence of the solution of (3.1.2) in $L^1(\Omega)$, because $C_0^\infty(\Omega)$ is dense in $L^1(\Omega)$. If f is the solution of $\lambda f - L_n^* f = g$, by means of duality arguments we have:

$$\begin{aligned}
\|w\|_1 &= \sup_{g \in L^\infty(\Omega),\, \|g\|_\infty \le 1} \int_{-b}^{+b} \int_{\mathbb{R}} wg \, dxdv \\
&= \int_{-b}^{+b} \int_{\mathbb{R}} w(\lambda - L_n^*) f \, dxdv \\
&= \int_{-b}^{+b} \int_{\mathbb{R}} (\lambda - L_{n,1}) w f \, dxdv \\
&= \int_{-b}^{+b} \int_{\mathbb{R}} hf \, dxdv \\
&\le \|h\|_1 \, \|f\|_\infty \\
&\le \|h\|_1 \frac{1}{\lambda} \|g\|_\infty \le \frac{1}{\lambda} \|h\|_1,
\end{aligned}$$

which is the Hille-Yosida estimate in $L^1(\Omega)$. ■

We remark that every statement that we shall prove in the remainder of the present section holds in $L^p(\Omega)$ spaces for every p such that $1 \le p < \infty$.

3.1.3 Convergence of the semigroup generated by L_n

In order to prove the convergence of the semigroup generated by L_n to the semigroup generated by the closure $\overline{L}$ of L, we first prove the lemmas below.

Lemma 3.1.3 *The resolvent sequence $\{R(\lambda, L_n)\}$ strongly converges as n goes to ∞ for any given $\lambda > 0$.*

Proof:
Let $\varepsilon > 0$ be fixed, $f \in D$ and n, $m \in \mathbb{N}$ such that $n > m$. Given any $\lambda > 0$, since $D(L_n) = D$ does not depend on n, we have:

$$\|(\lambda I - L_n)^{-1} f - (\lambda I - L_m)^{-1} f\|$$

$$\begin{aligned}
&= \|(\lambda I - L_n)^{-1}[f - (\lambda I - L_n)(\lambda I - L_m)^{-1} f]\| \\
&\leq \frac{1}{\lambda}\| [(\lambda I - L_m) - (\lambda I - L_n)](\lambda I - L_m)^{-1} f\| \\
&= \frac{1}{\lambda}\|(\lambda I - L_m - \lambda I + L_n)(\lambda I - L_m)^{-1} f\| \\
&= \frac{1}{\lambda}\|(\lambda I - L_m)^{-1}(L_n - L_m) f\| \\
&\leq \frac{1}{\lambda}\frac{1}{\lambda}\|L_n f - L_m f\| < \frac{\varepsilon}{\lambda^2},
\end{aligned}$$

where the last inequality holds because $\{L_n f\}$ is convergent (as proved in Lemma 3.1.1), and so it is a Cauchy sequence. Hence, it follows that $\{R(\lambda, L_n) f\}$ is a Cauchy sequence, and therefore it converges if $f \in D$. Since D is dense in X and the resolvent operator $R(\lambda, L_n)$ are uniformly bounded, this result holds for any $f \in X$. ■

Lemma 3.1.4 *The operator $(I - \alpha L_n)^{-1}$, where $\alpha > 0$, strongly converges to the identity operator I as $\alpha \to 0^+$, uniformly with respect to n.*

Proof:
By taking into account estimate (3.1.2), we have for every $f \in D = D(L_n)$:

$$\|(I - \alpha L_n)^{-1} f - f\|$$

$$\begin{aligned}
&= \|(I - \alpha L_n)^{-1}[f - (I - \alpha L_n) f]\,\| \\
&\leq \|(I - \alpha L_n)^{-1}\|\, \|f - f + \alpha L_n f\|
\end{aligned}$$

$$= \left\| \left[\alpha \left(\frac{I}{\alpha} - L_n \right) \right]^{-1} \right\| \|\alpha L_n f\|$$

$$= \left\| \frac{1}{\alpha} \left(\frac{1}{\alpha} I - L_n \right)^{-1} \right\| \|\alpha L_n f\|$$

$$\leq \frac{1}{\alpha} \alpha \|\alpha L_n f\| = \alpha \left(\left\| \frac{1}{n} \Delta f \right\| + \|Lf\| \right) \leq \alpha \left(\|\Delta f\| + \|Lf\| \right),$$

which approaches 0 as $\alpha \to 0^+$, uniformly in $n \in \mathbb{N}$. Since D is dense in X, the above result holds for every $f \in X$. ■

Theorem 3.1.3 *The closure $\overline{L}$ of the operator L generates a strongly continuous semigroup of contractions in X. Moreover, for every $t \geq 0$, we have:*

$$\lim_{n\to\infty} \|exp(tL_n)f - exp(t\overline{L})f\| = 0 \quad \forall f \in X,$$

and the above limit is uniform in t for t in bounded intervals.

Proof:
From Lemma 3.1.3 and 3.1.4, it follows that there exists an operator $\hat{L}$ such that $\hat{L} \in \mathcal{G}(1, 0; X)$ and the semigroup generated by L_n, $exp(tL_n)$, strongly converges to the semigroup $exp(t\hat{L})$ generated by $\hat{L}$, (see [39] and [57]).

If we prove that $(\lambda I - L)D$ is dense in X for a fixed $\lambda > 0$, we can apply a theorem of [53], (see the Pazy theorem 2.2.5) and have that $\hat{L} = \overline{L}$.

Thus, let $\lambda > 0$ be fixed, let $\overline{g} = \overline{g}(x, v) \in C_0^\infty(\Omega)$ and define the following function

$$\overline{f}(x,v) = \begin{cases} \dfrac{1}{a} \exp\left(\dfrac{-\lambda v}{a}\right) \displaystyle\int_{-\gamma}^{v} \overline{g}\left(x - \frac{v^2}{2a} + \frac{v'^2}{2a}, v'\right) \exp\left(\frac{\lambda v'}{a}\right) dv' \,, v < 0 \\[2ex] -\dfrac{1}{a} \exp\left(\dfrac{-\lambda v}{a}\right) \displaystyle\int_{v}^{\gamma} \overline{g}\left(x - \frac{v^2}{2a} + \frac{v'^2}{2a}, v'\right) \exp\left(\frac{\lambda v'}{a}\right) dv' \,, v > 0, \end{cases} \tag{3.1.0}$$

where $\gamma = \sqrt{2a\left|x + b - \frac{v^2}{2a}\right|}$. It is easy to see that $\overline{f} \in D$ and

$$(\lambda I - L)\overline{f} = \overline{g}\,. \tag{3.1.1}$$

Thus, $(\lambda I - L)D \supset C_0^\infty$ and $(\lambda I - L)D$ is dense in X because C_0^∞ has this property. ■

3.1.4 Approximation and Solution

In section 3.1.3 we have proved that the linear operator $\overline{L}$ generates a strongly continuous semigroup of contractions.

On the other hand, the operator A_t is an affine operator associated to the operator L, so it is possible to define an extension $\tilde{A}_t$ of A_t, such that $\tilde{A}_t$ is affine to $\overline{L}$, see (3.1.2). Then, relations (3.1.1) holds for the operator $\overline{L}$ and $\tilde{A}_t$ and we can use (3.1.1) and (3.1.1).

For example, in the time independent source case, the solution of problem (3.1.1) with $\tilde{A}_t$ in place of A_t is given by

$$u(t) = p + \exp(t\overline{L})(u_0 - p). \tag{3.1.1}$$

It follows, from Theorem 3.1.3, that $exp(tL_n)$ approximates $exp(t\overline{L})$. Hence, given p such that $A_t p = 0$, if we find a sequence $\{p_n\} \subset D(A_{t,n})$ converging to p, we have that the following sequence

$$u_n(t) = p_n + \exp(tL_n)(u_0 - p_n) \tag{3.1.1}$$

converges to the solution (3.1.4). A similar result can be proved in the time dependent source case.

As regards the explicit form of the function p appearing in (3.1.1) we have the following proposition:

Proposition 3.1.1 *In the time independent sources case, if $q_1 = q_1(\cdot) \in C^1(0,\infty) \cap L^2(0,\infty)$ and if $q_2 = q_2(\cdot) \in C^1(-\infty,0) \cap L^2(-\infty,0)$, then the solution of (3.1.1) is written explicitly as in (3.1.4), where p is given by*

$$p(x,v) = \begin{cases} q_1(\sqrt{2a| - x - b + v^2/2a|}) & \text{if } v > 0 \\ \\ q_2(-\sqrt{2a(-x + b + v^2/2a)}) & \text{if } v < 0 \end{cases} . \tag{3.1.1}$$

Proof:

It is easy to check that the function $p(x,v)$ given by (3.1.1) satisfies the equation $A_t p = 0$. This can be done by using the following transformations:

$$\begin{cases} \hat{x} = |x + b - v^2/2a| \\ \hat{v} = v \end{cases} \tag{3.1.1}$$

for $v > 0$, and

$$\begin{cases} \hat{x} = x - b - v^2/2a \\ \hat{v} = v \end{cases} \tag{3.1.1}$$

for $v < 0$.

We consider only the case $v > 0$ with $x + b - v^2/2a \geq 0$, the cases $v > 0$ with $x + b - v^2/2a \leq 0$ and $v < 0$ are analogous. With the above change of variables, defining $\Phi(\hat{x}, \hat{v}) = p(\hat{x} - b + \hat{v}^2/2a, \hat{v})$, equation $A_t p = 0$ reads as follows:

$$a \frac{\partial \Phi}{\partial \hat{v}} = 0, \tag{3.1.1}$$

with boundary condition:

$$\Phi(-v^2/2a, v) = q_1(v) \,, \quad v > 0.$$

By integrating (3.1.4) with respect to $\hat{v}$ and considering the boundary condition and the change of variable (3.1.4) we have that the function $p(x, v)$ solution of $A_t p = 0$ is given by (3.1.1). We remark that $p(x, v) \in D(A_t)$ owing to the regularity assumptions made on q_1 and q_2. ■

We end this section remarking that it is not possible to apply the techniques of the present section to prove the generation of a semigroup if the boundary conditions are not of no-reentry (i.e. $\Lambda \neq 0$). In fact,the space

$$\left\{ f \in X, v\frac{\partial f}{\partial x} \in X, \frac{\partial f}{\partial v} \in X, f_- = \Lambda f_+ \right\}$$

is not dense in $W_0^{1,2}$.

3.2 Vlasov equation with generalized nonhomogeneous boundary conditions

In this section, we study the one-dimensional Vlasov equation which models a system of particles of charge q_0 and mass m moving in a rod under the influence of a known constant electric field E. We describe the boundary conditions giving the *incoming* density of particles at the boundaries of the slab by means of the sum of a linear bounded and positive operator Λ applied to the *outgoing* particle density and of a particle source given by a known function $q(\pm b, v, t)$. If $\|\Lambda\| < 1$, we call this type of conditions *nonhomogeneous dissipative* boundary conditions. Whereas, if $\|\Lambda\| = 1$, the boundary condition are called *nonhomogeneous conservative*.

Note that the boundary sources $q(\pm b, v, t)$ represent some kind of particle injection into the rod. We also remark that non re-entry boundary conditions are such that $0 = \|\Lambda\| < 1$, whereas specular reflection boundary conditions are such that$\|\Lambda\| = 1$ (see section 3.2.3).

In the previous section, see also [46], we studied this problem by means of the theory of elliptic operators with unbounded coefficients and *affine* semigroup ([6], [9]). The result was shown only in the case of nonhomogeneous boundary conditions with $\Lambda = 0$ and led to the possibility of approximating the solution.

In this section, the study of the problem is based mainly on the theory of semigroups. In particular we apply a theorem by Chernoff ([23], [39], [53]), a theorem by Pazy ([53]), and the theory of affine semigroups ([6], [9]). This kind of approach enables us to prove existence, uniqueness and positivity of the solution and to derive its explicit form in terms of the boundary sources. The approximation of the solution follows from the Trotter product formula ([23], [57]).

In section 3.2.1 we introduce the functional spaces needed to study the problem, we define the affine operator V which describes the Vlasov equation with the nonhomogeneous boundary conditions and the linear operator L which describes the corresponding Vlasov equation with homogeneous boundary conditions.

In section 3.2.2 we assume that $\|\Lambda\| < 1$, we recall the Chernoff theorem and prove that the operator L, with $\|\Lambda\| < 1$, satisfies its assumptions. Hence, the semigroup generated by the closure $\overline{L}$ of L can be written by means of the Trotter product formula.

In section 3.2.3 we prove the generation of a continuous semigroup of contractions by the closure of the operator L with $\|\Lambda\| = 1$. This can be done in two ways: applying the Pazy theorem ([53]), and applying again the Chernoff theorem and using the results of section 3.2.2. Moreover, we shall prove that the generators of the semigroup are equal.

In section 3.2.4 we conclude giving the explicit form of the solution of the nonhomogeneous problem by means of the theory of affine operators and semigroups.

3.2.1 The problem

We consider a rod of length $2b$, the set $\Omega = [-b, +b] \times \mathbb{R}$ and introduce the *incoming* and *outgoing* sets:

$$\Omega^{in} = (\{-b\} \times (0, +\infty)) \cup (\{+b\} \times (-\infty, 0)),$$

and

$$\Omega^{out} = (\{-b\} \times (-\infty, 0)) \cup (\{+b\} \times (0, +\infty)).$$

Further, we consider the functional space

$$X = L^1(\Omega) \quad , \quad \|f\| = \int_\Omega |f(x, v)|\, dx\, dv,$$

and denote by X^+ the positive cone of X. Let us also define the *incoming* and *outgoing* functional spaces:

$$X^{in} = L^1(\Omega^{in}; |v|dv), \quad X^{out} = L^1(\Omega^{out}; |v|dv), \tag{3.2.0}$$

with norms respectively given by:

$$\|f^{in}\|_{in} = \int_0^{+\infty} v|f(-b, v)|\, dv + \int_{-\infty}^0 |v|\, |f(b, v)|\, dv, \tag{3.2.0}$$

and

$$\|f^{out}\|_{out} = \int_{-\infty}^0 |v|\, |f(-b, v)|\, dv + \int_0^{+\infty} v|f(b, v)|\, dv. \tag{3.2.0}$$

We shall denote by f_i^{in} and f_i^{out}, where $i = 1, 2$, the i-th 'component' of the functions f^{in} and f^{out} belonging to the component spaces X_i^{out} and X_i^{in}, and by $\| \cdot \|_{i,in}$ and $\| \cdot \|_{i,out}$, $i = 1, 2$, the associated norms. For instance:

$$\|f_1^{in}\|_{1,in} = \int_0^{+\infty} v|f(-b, v)|\, dv$$

denotes the first term on the right hand side of the norm $\| \cdot \|_{in}$ given by (3.2.1). In other words, we may interpret X^{in} as $X_1^{in} \times X_2^{in}$ and X^{out} as $X_1^{out} \times X_2^{out}$ and $f^{in} = \begin{pmatrix} f_1^{in} \\ f_2^{in} \end{pmatrix}$. We remark that, if f is a particle density, then $f \in X^+$ and $\|f\|$ gives the total number of particles in the rod. Moreover, $\|f^{in}\|_{in}$ and $\|f^{out}\|_{out}$ respectively are the total ingoing and outgoing fluxes at the boundaries, [37].

The evolution of a system of charged particles, with mass m and charge q_0, moving under the influence of a constant electric field E, during the time interval between two successive scattering events, is modeled by the Vlasov equation:

$$\frac{\partial n}{\partial t}(x,v,t) + v\frac{\partial n}{\partial x}(x,v,t) + a\frac{\partial n}{\partial v}(x,v,t) = 0 \qquad (3.2.0)$$

where $(x,v) \in \Omega$ and $t > 0$. Moreover, $a = q_0 E/m$ is a given constant and $n(x,v,t)$ represents the density of particles which, at time t, are in a position $x \in [-b,+b]$ and have velocity $v \in \mathbb{R}$. Equation (3.2.1) is equipped with the initial condition:

$$n(x,v,0) = n_0(x,v), \quad (x,v) \in \Omega. \qquad (3.2.0)$$

We write the boundary conditions describing the relation between the incoming and the outgoing particle densities at the boundaries $x = \pm b$ of the rod by means of a linear bounded and positive operator Λ, with domain $D(\Lambda) = X^{out}$ and range $R(\Lambda) \subset X^{in}$ and with $\|\Lambda\| \leq 1$, and by means of a given positive function

$$q = \begin{pmatrix} q_1(-b,v) \\ q_2(b,v) \end{pmatrix} \in X^{in}$$

which represents an incoming flux of particles:

$$n^{in} = \Lambda n^{out} + q. \qquad (3.2.0)$$

In (3.2.1), n^{in} and n^{out} are the traces of the density function n:

$$n^{in} = n|_{\Omega^{in}}, \qquad n^{out} = n|_{\Omega^{out}},$$

where we recall that for instance

$$n^{in} = n|_{\Omega^{in}} = \begin{pmatrix} n_1^{in} \\ n_2^{in} \end{pmatrix}, \quad n_1^{in} = n(-b,v), v > 0, \quad n_2^{in} = n(b,v), v < 0.$$

In order to prove existence and uniqueness of the solution of system (3.2.1)-(3.2.1) and to write its explicit form, we transform it into the following Cauchy problem in the space X:

$$\begin{cases} \dfrac{dn(t)}{dt} = Vn(t)\,, \quad t > 0 \\ \\ n(0) = n_0 \in D(V). \end{cases} \qquad (3.2.0)$$

In (3.2.1), $n(t) = n(\cdot,\cdot,t)$ is a function from $[0,+\infty)$ with values in X, and the operator V is defined as follows:

$$Vf = -v\frac{\partial f}{\partial x} - a\frac{\partial f}{\partial v},$$

$$D(V) = \left\{ f \in X : -v\frac{\partial f}{\partial x} \in X, -a\frac{\partial f}{\partial v} \in X \;,\; f^{in} = \Lambda f^{out} + q \right\} \tag{3.2.0}$$

$$R(V) \subset X,$$

where $D(V)$ and $R(V)$ are the domain and range of V, respectively. It can be proved that $f^{in} \in X^{in}$, $f^{out} \in X^{out}$, provided that $f \in X$ and $-v\dfrac{\partial f}{\partial x} \in X$, ([37]).

It is easy to show that the operator V is an affine operator associated to the following linear operator L, ([6], [9]):

$$Lf = -v\frac{\partial f}{\partial x} - a\frac{\partial f}{\partial v},$$

$$D(L) = \left\{ f \in X : -v\frac{\partial f}{\partial x} \in X, -a\frac{\partial f}{\partial v} \in X \;,\; f^{in} = \Lambda f^{out} \right\} \;, \quad R(L) \subset X. \tag{3.2.0}$$

We remark that in (3.2.1) we have asked the function $f \in D(L)$ to satisfy the homogeneous boundary condition

$$f^{in} = \Lambda f^{out}. \tag{3.2.0}$$

3.2.2 The semigroup generated by $\overline{L}$

In this section we shall assume that $\|\Lambda\| < 1$. By using the Chernoff theorem, we shall prove that the operator $\overline{L}$, defined as the closure of L given by (3.2.1), generates a strongly continuous semigroup of contractions: $\overline{L} \in \mathcal{G}(1,0;X)$ ([39]). We shall also show that such a semigroup is positive. In order to apply the Chernoff theorem 2.2.6, we define the two operators:

$$L_1 f = -a\frac{\partial f}{\partial v} \;, \quad D(L_1) = \{f \in X : L_1 f \in X\} \;, \quad R(L_1) \subset X \tag{3.2.0}$$

and

$$L_2 f = -v\frac{\partial f}{\partial x} \;,\; D(L_2) = \{f \in X : L_2 f \in X \;,\; f^{in} = \Lambda f^{out}\} \;,\; R(L_2) \subset X. \tag{3.2.0}$$

Lemma 3.2.1 *The linear operator L_1 generates a strongly continuous semigroup of contractions, $[T(t)]_{t\geq 0}$, i.e. $L_1 \in \mathcal{G}(1,0;X)$. Moreover, $[T(t)]_{t\geq 0}$ maps X^+ into itself.*

Proof:
It is easy to see that the semigroup generated by L_1 has the form

$$exp(tL_1)f = f(x, v - at)$$

for all $t \geq 0$. ■

Consider the linear operator

$$A = \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix} \quad , \quad D(A) = X^{in} \quad , \quad R(A) \subset X^{out}, \tag{3.2.0}$$

where the operators A_{12} and A_{21} are defined by

$$\begin{aligned} A_{12} f_2^{in} = f_2^{in} e^{2bz/v} \quad &, \quad D(A_{12}) = X_2^{in} \quad , \quad R(A_{12}) \subset X_1^{out}, \\ A_{21} f_1^{in} = f_1^{in} e^{-2bz/v} \quad &, \quad D(A_{21}) = X_1^{in} \quad , \quad R(A_{21}) \subset X_2^{out}, \end{aligned} \tag{3.2.0}$$

and where we recall that

$$X^{in} = L^1((\{-b\} \times (0, +\infty)); vdv) \times L^1(\{+b\} \times (-\infty, 0)); |v|dv) = X_1^{in} \times X_2^{in},$$

$$X^{out} = L^1((\{-b\} \times (-\infty, 0)); |v|dv) \times L^1(\{+b\} \times (0, +\infty)); |v|dv) = X_1^{out} \times X_2^{out}.$$

Lemma 3.2.2 *The operator A given by (3.2.2) is such that $\|A\| \leq 1$.*

Proof:
We begin by computing the norm of A_{12} and A_{21}. Since $\|\cdot\|_{i,in}$ and $\|\cdot\|_{i,out}$, for $i = 1, 2$, are the i-th components of the norms $\|\cdot\|_{in}$ and $\|\cdot\|_{out}$, defined by (3.2.1) and (3.2.1), we have from (3.2.2):

$$\|A_{12} f_2^{in}\|_{1,out} = \int_{-\infty}^{0} |v|\, |f_2^{in}|\, e^{2bz/v}\, dv$$

$$\leq \int_{-\infty}^{0} |v|\, |f_2^{in}|\, dv = \int_{-\infty}^{0} |v|\, |f(b,v)|\, dv = \|f_2^{in}\|_{2,in}\, ;$$

therefore $\|A_{12}\| \leq 1$. Analogously, we have that $\|A_{21}\| \leq 1$. Thus, we obtain

$$\|Af^{in}\|_{out} = \|A_{12}f_2^{in}\|_{1,out} + \|A_{21}f_1^{in}\|_{2,out} \leq \|f_2^{in}\|_{2,in} + \|f_1^{in}\|_{1,in} = \|f^{in}\|_{in}$$

and so $\|A\| \leq 1$. ∎

Lemma 3.2.3 *Under the assumption that* $\|\Lambda\| < 1$*, the linear operator* L_2 *generates a strongly continuous semigroup of contractions,* $[S(t)]_{t\geq 0}$*, i.e.* $L_2 \in \mathcal{G}(1,0;X)$*. Moreover S(t) maps* X^+ *into itself.*

Proof:
Consider the resolvent equation of the linear operator L_2:

$$zf + v\frac{\partial f}{\partial x} = g, \tag{3.2.0}$$

where $z > 0$ and g is a given element of X.
The solution $f(x,v)$ of (3.2.2) has the form

$$f(x,v) = \begin{cases} C_1(v)e^{-z(x+b)/v} + \dfrac{1}{v}\displaystyle\int_{-b}^{x} g(x',v)e^{-z(x-x')/v}\,dx'\,, & v > 0 \\ \\ C_2(v)e^{-z(x-b)/v} + \dfrac{1}{|v|}\displaystyle\int_{x}^{+b} g(x',v)e^{-z(x-x')/v}\,dx'\,, & v < 0 \end{cases}, \tag{3.2.0}$$

where the functions $C_1(v)$ and $C_2(v)$ must be chosen so that $f \in D(L_2)$. Hence, the boundary condition (3.2.1) has to be satisfied. We have from (3.2.2):

$$f_1^{in} = f(-b,v) = C_1(v) \quad v > 0,$$

$$f_2^{in} = f(+b,v) = C_2(v) \quad v < 0,$$

$$f_2^{out} = f(+b,v) = C_1(v)e^{-2bz/v} + \frac{1}{v}\int_{-b}^{b} g(x',v)e^{-z(b-x')/v}\,dx' \quad v > 0,$$

$$f_1^{out} = f(-b,v) = C_2(v)e^{2bz/v} + \frac{1}{|v|}\int_{-b}^{b} g(x',v)e^{z(b+x')/v}\,dx' \quad v < 0.$$

On the other hand, from the boundary condition (3.2.1) we obtain:

$$C = \Lambda G + \Lambda A C, \tag{3.2.0}$$

where:

$$C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \in X^{in}, \qquad G = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in X^{out}, \tag{3.2.0}$$

with $g_1 = g_1(v)$ and $g_2 = g_2(v)$ respectively given by

$$g_1(v) = \frac{1}{|v|} \int_{-b}^{b} g(x', v) e^{z(b+x')/v} \, dx', \ v < 0$$

$$g_2(v) = \frac{1}{v} \int_{-b}^{b} g(x', v) e^{-z(b-x')/v} \, dx', \ v > 0$$

and where the linear operator A is defined by (3.2.2). The unique solution of (3.2.2) has the form:

$$C = (I - \Lambda A)^{-1} \Lambda G, \tag{3.2.0}$$

because we assumed that $\|\Lambda\| < 1$ and so $\|\Lambda A\| \le \|\Lambda\| \, \|A\| < 1$ by Lemma 3.2.2. This means that $C_1(v)$ and $C_2(v)$ are uniquely determined and so (3.2.2) gives the unique solution of the resolvent equation (3.2.2).

Moreover, from relation (3.2.2), it easy to see that $f(x,v) \in X^+$ if $g(x,v) \in X^+$. Hence, the resolvent operator $R(z, L_2)$ is a positive operator with $D(R(z, L_2)) = X$ for every $z > 0$. Therefore $R(z, L_2)$ is bounded , ([56]), and L_2 is a closed operator. We can obtain the Hille-Yosida estimate for $R(z, L_2)$ directly from the resolvent equation. In fact, if $g \in X^+$, by integrating (3.2.2) with respect to x and v, we have:

$$z\|f\| - \|f^{in}\|_{in} + \|f^{out}\|_{out} = \|g\|. \tag{3.2.0}$$

By using the boundary conditions (3.2.1) and the fact that $\|\Lambda\| < 1$, it follows from (3.2.2) that:

$$z\|f\| \le \|g\|. \tag{3.2.0}$$

By the positivity of $R(z, L_2)$, we have that (3.2.2) holds not only if $g \in X^+$ but for each $g \in X$. Hence (3.2.2) gives the Hille-Yosida estimate for every $z > 0$. Thus, L_2 generates a strongly continuous semigroup of contractions, which is positive because the resolvent of L_2 is positive. ■

We remark that $D(L) = D(L_1 + L_2) = D(L_1) \cap D(L_2)$ is dense in X because $C_0^\infty(\Omega) \subset D(L)$. In order to apply the Chernoff theorem (2.2.6) we have to prove that there exists a λ_0 such that $(\lambda_0 I - L_1 - L_2) D(L)$ is dense in X. In fact, we shall prove the following

Lemma 3.2.4 *Under the assumption that* $\|\Lambda\| < 1$, *the equation* $(zI - L_1 - L_2)f = g$ *has a unique solution for every* $g \in C_0^1(\Omega)$, *and* $z > 0$.

Proof:

Consider the equation:

$$zf + v\frac{\partial f}{\partial x} + a\frac{\partial f}{\partial v} = g \tag{3.2.0}$$

coupled with the boundary conditions (3.2.1). It is not difficult to see that the function $f(x,v)$ given by:

$$f(x,v) = \begin{cases} +\dfrac{1}{a}\exp\left(\dfrac{-zv}{a}\right)\displaystyle\int_{-\gamma}^{v} g\left(x - \frac{v^2}{2a} + \frac{v'^2}{2a}, v'\right)\exp\left(\frac{zv'}{a}\right)dv'\,, v<0 \\ \\ -\dfrac{1}{a}\exp\left(\dfrac{-zv}{a}\right)\displaystyle\int_{v}^{\gamma} g\left(x - \frac{v^2}{2a} + \frac{v'^2}{2a}, v'\right)\exp\left(\frac{zv'}{a}\right)dv'\,, v>0 \end{cases}, \tag{3.2.0}$$

where $\gamma = \gamma(x,v) = \sqrt{2a\left|b + x - \dfrac{v^2}{2a}\right|}$, satisfies equation (3.2.2) and belongs to the set:

$$D' = \left\{ f \in X : -v\frac{\partial f}{\partial x} \in X, -a\frac{\partial f}{\partial v} \in X\,,\; f|_{\partial\Omega} = 0 \right\} \tag{3.2.0}$$

Note that D' is dense in X and also that $D' \subset D(L)$. Moreover, from (3.2.2) and (3.2.2) it follows that $(zI - L_1 - L_2)D' \supset C_0^1(\Omega)$, which implies that $(zI - L_1 - L_2)D(L)$ is dense in X. ∎

Theorem 3.2.1 *The operator* $\overline{L} = \overline{L_1 + L_2}$ *generates a strongly continuous semigroup of contractions,* $[Z(t)]_{t\geq 0}$ *given by*

$$Z(t)f = \lim_{n\to\infty}[T(t/n)S(t/n)]^n f\,,\; f \in X, \tag{3.2.0}$$

where $[T(t)]_{t\geq 0}$ *and* $[S(t)]_{t\geq 0}$ *are the semigroups generated by the operators* L_1 *and* L_2*, respectively. Moreover,* $[Z(t)]_{t\geq 0}$ *maps the positive cone of* X *into itself.*

Proof:

By the Chernoff theorem 2.2.6, we obtain that the operator $\overline{L} = \overline{L_1 + L_2}$ generates a strongly continuous semigroup of contractions. If we denote by $[Z(t)]_{\geq 0}$ the semigroup generated by $\overline{L}$, then we have that it can be written by means of the Trotter product formula (3.2.1) The positivity of $Z(t)$ follows from the positivity of $T(t)$ and $S(t)$. ∎

3.2.3 Conservative boundary conditions

In the previous section we proved that the closure $\overline{L}$ of the operator L generates a strongly continuous positive semigroup of contractions under the assumptions that $\|\Lambda\| < 1$. Here we investigate the generation of a semigroup in the case $\|\Lambda\| = 1$. We call this kind of boundary conditions *conservative* because a particular case of $\|\Lambda\| = 1$ is when the operator Λ is such that $\|\Lambda f^{out}\|_{in} = \|f^{out}\|_{out}$ for every $f^{out} \in X^{out}$; it then follows from (3.2.1) that $\|f^{in}\|_{in} = \|f^{out}\|_{out}$., i.e. the total ingoing flux equals the total outgoing flux.

As an example, we may consider specular reflection boundary conditions:

$$f(-b, v) = f(-b, -v), v > 0, \quad f(b, v) = f(b, -v), v < 0.$$

In fact, we have in this case:

$$\|\Lambda f^{out}\|_{in} = \int_0^{+\infty} v\, f(-b, v)\, dv + \int_{-\infty}^0 |v| f(b, v)\, dv =$$

$$= \int_{-\infty}^0 |v'|\, f(-b, v')\, dv' + \int_0^{+\infty} v' f(b, v')\, dv' = \|f^{out}\|_{out}\, .$$

Let us now define the operator $\mathcal{L}$

$$\mathcal{L}f = -v\frac{\partial f}{\partial x} - a\frac{\partial f}{\partial v},$$

$$D(\mathcal{L}) = \left\{ f \in X,\ -v\frac{\partial f}{\partial x} \in X,\ -a\frac{\partial f}{\partial v} \in X,\ f^{in} = \Lambda f^{out} \right\} \qquad (3.2.0)$$

$$R(\mathcal{L}) \subset X,$$

where $\|\Lambda\| = 1$ and $D(\mathcal{L})$ and $R(\mathcal{L})$ are the domain and range of $\mathcal{L}$, respectively. We remark that the difference between the operator $\mathcal{L}$ given by (3.2.3) and L given by (3.2.1) is that the boundary condition is conservative for $\mathcal{L}$, whereas it is dissipative for L. Moreover, let us introduce the sequence of operators L_n defined by

$$L_n f = Lf,$$
$$D(L_n) = \{f \in X, L_n f \in X,\ f^{in} = (1 - 1/n)\Lambda f^{out}\},\ R(L_n) \subset X. \tag{3.2.0}$$

In order to prove that assumptions (i) and (ii) of the Pazy theorem 2.2.5 are satisfied, we consider the set D' defined by (3.2.2). Since D' is dense in X, assumption (i) of Theorem 2.2.5 holds as it is stated in the following Lemma.

Lemma 3.2.5 *The closure $\overline{L}_n$ of the operator L_n, defined in (3.2.3), generates a strongly continuous semigroup of contractions, for every $n \in \mathbb{N}$, i.e. $\overline{L}_n \in \mathcal{G}(1, 0; X)$. Moreover, if $f \in D'$, then the sequence $\overline{L}_n f$ converges to $\mathcal{L}f$, i.e.:*

$$\lim_{n \to 0} \|\overline{L}_n f - \mathcal{L}f\| = 0, \quad \forall f \in D'.$$

Proof:
The generation of the semigroup by $\overline{L}_n$ follows directly from Theorem 3.2.1 because L_n is a Vlasov operator with dissipative boundary conditions.
Moreover, $\overline{L}_n f = L_n f = \mathcal{L}f$, for every $f \in D' \subset D(L_n) \subset D(\overline{L}_n)$ and for every $n \in \mathbb{N}$. ■

We prove in the next Lemma that assumption (ii) of Theorem 2.2.5 holds too.

Lemma 3.2.6 *Let D' be defined by (3.2.2); then there exists a $\lambda_0 > 0$ such that $(\lambda_0 I - \mathcal{L})D'$ is dense in X.*

Proof:
See the proof of Lemma 3.2.4. ■

We obtain from Theorem 2.2.5 the following

Theorem 3.2.2 *The closure $\overline{\mathcal{L}}$ of $\mathcal{L}$, generates a strongly continuous semigroup of contractions, $[U(t)]_{t \geq 0}$ and, for every $f \in X$, we have:*

$$U(t)f = \lim_{n \to \infty} U_n(t)f, \tag{3.2.0}$$

where $[U_n(t)]_{t \geq 0}$ is the semigroups generated by $\overline{L}_n$.

The generation of a semigroup can be proved also by means of the Chernoff theorem (2.2.6). In fact, $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ where $\mathcal{L}_1$ is defined by (3.2.2) and $\mathcal{L}_2$ by (3.2.2). We use a different notation because $\|\Lambda\| = 1$.

From Lemma 3.2.1 we have that the semigroup generated by $\mathcal{L}_1$ has the form $exp(t\mathcal{L}_1)f = f(x, v - at)$ for all $t \geq 0$.

As far as the operator $\mathcal{L}_2$ is concerned, we already know from the previous section that the sequence of the closures $\overline{\mathcal{L}}_{2,n}$ of the operators $\mathcal{L}_{2,n}$:

$$\mathcal{L}_{2,n}f = \mathcal{L}_2 f\ ,\ D(\mathcal{L}_{2,n}) = \{f \in X : \mathcal{L}_{2,n}f \in X\ ,\ f^{in} = (1 - 1/n)\Lambda f^{out}\},$$
$$R(\mathcal{L}_{2,n}) \subset X, \tag{3.2.0}$$

generate strongly continuous semigroup of contractions $[S_n(t)]_{t\geq 0}$ for every $n \in \mathbb{N}$. As done in the Lemma 3.2.5 and 3.2.6, we can show that the assumptions of the Pazy theorem 2.2.5 hold for the operators $\mathcal{L}_{2,n}$ and $\mathcal{L}_2$ with $D = D'$. Thus, we can conclude that the closure $\overline{\mathcal{L}}_2$ of the operator $\mathcal{L}_2$ is the generator of a strongly continuous semigroup of contractions $[\mathcal{S}(t)]_{t\geq 0}$.

By applying again the Chernoff theorem 2.2.6 the following lemma holds.

Lemma 3.2.7 *The closure $\overline{\mathfrak{L}}$ of the operator $\mathfrak{L}$ defined by $\mathfrak{L}f = \mathcal{L}_1 f + \overline{\mathcal{L}}_2 f$, , is the generator of a strongly continuous semigroup of contractions, $[\mathfrak{U}(t)]_{t\geq 0}$.*

We remark that the operator $\mathcal{L}$ defined by (3.2.3) is equal to $\mathcal{L}_1 + \mathcal{L}_2$, whereas the operator $\mathfrak{L}$ is defined as $\mathcal{L}_1 + \overline{\mathcal{L}}_2$. Moreover, the closures of both $\mathcal{L}$ and $\mathfrak{L}$ generate a semigroup and are extensions of the operator $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$. Now we prove that these semigroups are equal and that $\overline{\mathcal{L}} = \overline{\mathfrak{L}}$.

Theorem 3.2.3 *The semigroup $U(t)$ generated by $\overline{\mathcal{L}}$ and the semigroup $\mathfrak{U}(t)$ generated by $\overline{\mathfrak{L}}$ are equal:*

$$U(t) = \mathfrak{U}(t)\ , \quad \forall\, t \geq 0.$$

Moreover,

$$\overline{\mathcal{L}} = \overline{\mathcal{L}_1 + \mathcal{L}_2} = \overline{\mathfrak{L}} = \overline{\mathcal{L}_1 + \overline{\mathcal{L}}_2}$$

Proof:

Let us consider the following Cauchy problem:

$$\begin{cases} \dfrac{dn(t)}{dt} = \overline{\mathcal{L}}u(t)\,, \quad t > 0, \\ \\ u(0) = u_0 \in D(\mathcal{L}) \subset D(\overline{\mathcal{L}}), \end{cases} \tag{3.2.0}$$

where $\mathcal{L}$ is defined by (3.2.3). From (3.2.2) we have that $u(t) = U(t)u_o \in D(\overline{\mathcal{L}})$,$\forall t \geq 0$. Since $\overline{\mathfrak{L}}$ defined in Lemma 3.2.7 is an extension of $\overline{\mathcal{L}}$, we have that $u(t) = U(t)u_o \in D(\overline{\mathfrak{L}})$,$\forall t \geq 0$. Hence, $u(t)$ satisfies also the problem

$$\begin{cases} \dfrac{dn(t)}{dt} = \overline{\mathfrak{L}}u(t)\,, \quad t > 0, \\ \\ u(0) = u_0 \in D(\mathcal{L}) \subset D(\overline{\mathcal{L}}) \subset D(\overline{\mathfrak{L}}). \end{cases} \tag{3.2.0}$$

Then $u(t) = \mathfrak{U}(t)u_0$. From the uniqueness of the solution of (3.2.3), we obtain that $U(t)u_0 = \mathfrak{U}(t)u_0$, for every $u_o \in D(\mathcal{L})$ and $t \geq 0$. Since $D(\mathcal{L})$ is dense in X it follows that $U(t)f = \mathfrak{U}(t)f$, for every $f \in X$ and every $t \geq 0$. Moreover, we can conclude that $\overline{\mathcal{L}} = \overline{\mathfrak{L}}$ and so $\overline{\mathcal{L}_1 + \mathcal{L}_2} = \overline{\mathcal{L}_1 + \overline{\mathcal{L}}_2}$. ∎

Furthermore, by using Theorem 3.2.3 it is possible to write explicitly the form of the semigroup $U(t)$ (and also $\mathfrak{U}(t)$):

$$\mathfrak{U}(t) = \lim_{m\to\infty} [T(t/m)\mathcal{S}(t/m)]^m f =$$

$$= \lim_{m\to\infty} [T(t/m) \lim_{n\to\infty} S_n(t/m)]^m f = U(t) =$$

$$= \lim_{n\to\infty} U_n(t) = \lim_{n\to\infty} \left(\lim_{m\to\infty} [T(t/m)S_n(t/m)]^m f \right).$$

3.2.4 The solution

In this section we shall give the form of the solution of the problem (3.2.1) in the case $\|\Lambda\| < 1$. The case $\|\Lambda\| = 1$ can be treated analogously.

In order to write the explicit form of the solution of the Cauchy problem (3.2.1) by means of the strongly continuous semigroup of contractions $Z(t)$ generated by $\overline{L}$ the closure of L, we need to prove that the operator $\overline{V}$ is an affine operator associated to $\overline{L}$.

From the definition of affine operators it is easy to see that the operator V is an affine operator associated to L. It is not difficult to prove, by using the definition of the closure of an operator and definition of affine operator, that also $\overline{V}$ is affine to $\overline{L}$.

Assume, first that the source term q in (3.2.1) is time independent, then, from the theory of affine operators ([6], [9]), we have that the solution of the Cauchy problem (3.2.1) with $\overline{V}$ instead of V reads as follows:

$$n(t) = p + Z(t)(n_0 - p), \tag{3.2.0}$$

where p is a suitable element of $D(V) \subset D(\overline{V})$, such that $Vp = \overline{V}p = 0$. On the other hand, if the source term q depends explicitly on time then the solution of problem (3.2.1) can be written as follows:

$$n(t) = p(t) + Z(t)[n_0 - p(0)] + \int_0^t Z(t-s)[Vp(s) - p'(s)]ds, \tag{3.2.0}$$

where $p(t) = p(\cdot, \cdot, t)$ is a suitable function from $[0, t_0)$ in X such that $p(t) \in D(V)$. If $p(t)$ is such that $Vp(s) - p'(s) = 0$, then (3.2.4) becomes:

$$n(t) = p(t) + Z(t)[n_0 - p(0)]. \tag{3.2.0}$$

As regards the explicit form of the function p appearing in (3.2.4) we have the following proposition:

Proposition 3.2.1 *In the time independent sources case, if* $q_1 = q_1(\cdot) \in C^1(0, \infty) \cap L^1(0, \infty)$ *and if* $q_2 = q_2(\cdot) \in C^1(-\infty, 0) \cap L^1(-\infty, 0)$, *then the solution of (3.2.1) is written explicitly as in (3.2.4), where p is given by*

$$p(x, v) = \begin{cases} q_1(\sqrt{2a| - x - b + v^2/2a|}) & \text{if } v > 0, \\ \\ q_2(-\sqrt{2a(-x + b + v^2/2a)}) & \text{if } v < 0. \end{cases} \tag{3.2.0}$$

Proof:

It is easy to check that the function $p(x, v)$ given by (3.2.1) satisfies the equation $Vp = 0$. This can be done by using the following transformations:

$$\begin{cases} \hat{x} = |x + b - v^2/2a| \\ \\ \hat{v} = v \end{cases} \tag{3.2.0}$$

for $v > 0$, and

$$\begin{cases} \hat{x} = x - b - v^2/2a \\ \hat{v} = v \end{cases} \tag{3.2.0}$$

for $v < 0$.

We consider only the case $v > 0$ with $x + b - v^2/2a \geq 0$, the cases $v > 0$ with $x + b - v^2/2a \leq 0$ and $v < 0$ are analogous. With the above change of variables, defining $\Phi(\hat{x}, \hat{v}) = p(\hat{x} - b + \hat{v}^2/2a, \hat{v})$, equation $Vp = 0$ reads as follows:

$$a\frac{\partial \Phi}{\partial \hat{v}} = 0, \tag{3.2.0}$$

with boundary condition:

$$\Phi(-v^2/2a, v) = q_1(v)\,, \quad v > 0.$$

By integrating (3.2.4) with respect to $\hat{v}$ and considering the boundary condition and the change of variable (3.2.4) we have that the function $p(x, v)$ solution of $Vp = 0$ is given by (3.2.1). We remark that $p(x, v) \in D''$ where D'' is given by:

$$D'' = \left\{ f \in X : -v\frac{\partial f}{\partial x} \in X, -a\frac{\partial f}{\partial v} \in X\,,\ f|_{\partial\Omega} = q \right\} \tag{3.2.0}$$

and $D'' \subset D(V)$ owing to the regularity assumptions made on q_1 and q_2. ■

3.3 The Generation of a Semigroup with Multiplicative Boundary Condition

The proofs of the previous section are not applicable to the case of multiplicative boundary conditions. In fact, the assumption $\|\Lambda < 1$ is necessary in order to prove the generation of a semigroup (see lemma 3.2.3). If multiplicative boundary conditions are taken into account, that is to say if $\|\Lambda\| > 1$, then it is clear that an *absorption* term must be added to the Vlasov equation. In particular, in this section we study the transport equation (i.e. Vlasov equation with force field $F = 0$) summed with an absorption term proportional to the modulus of the velocity. Moreover, we consider multiplicative boundary conditions, but we stress that the same results we are going to prove hold also for dissipative or conservative boundary conditions. This kind of problem as already been investigated,, but only in the case of bounded velocities. For instance, in [44] it was proved the existence and uniqueness of the solution of the Boltzmann equation equipped with multiplicative boundary conditions; this was done proving the generation of a semigroup. While in [45] a three dimensional problem was studied, proving the generation of a semigroup and writing the explicit form of the solution for the nonhomogeneous model.

In this section, under a specific assumption on the initial distribution function (i.e. the density function at time $t = 0$), we shall prove that the transport equation has a unique solution also when it is equipped by multiplicative boundary conditions and unbounded velocities are considered. This result is justified applying the theory of integrated and positive semigroups.

3.3.1 The model

We consider a system of charged or uncharged particles moving in a rod of length $2b$ and not subject to an external force filed. Moreover, the particles may be absorbed by the host medium, the absorption rate being proportional to the modulus of the velocity. The interaction of the particles with the boundaries of the rod is described by means of a linear and bounded operator Λ. We shall assume the norm of the operator Λ to be smaller, equal or even bigger than 1. Before writing explicitly the evolution problem we are concerned with, we define the set and functional space we shall need in order to study the problem.

Consider the set $\Omega = [-b, +b] \times \mathbb{R}$ and introduce the *incoming* and *outgoing* sets:

$$\Omega^{in} = (\{-b\} \times (0, +\infty)) \cup (\{+b\} \times (-\infty, 0)),$$

and

$$\Omega^{out} = (\{-b\} \times (-\infty, 0)) \cup (\{+b\} \times (0, +\infty)).$$

Further, consider the functional space

$$X = L^1(\Omega) \quad , \quad \|f\| = \int_\Omega |f(x, v)| \, dx \, dv,$$

and denote by X^+ the positive cone of X. Let us also define the *incoming* and *outgoing* functional spaces:

$$X^{in} = L^1(\Omega^{in}; |v| dv), \quad X^{out} = L^1(\Omega^{out}; |v| dv), \tag{3.3.0}$$

with norms respectively given by:

$$\|f^{in}\|_{in} = \int_0^{+\infty} v|f(-b,v)|\,dv + \int_{-\infty}^{0} |v|\,|f(b,v)|\,dv, \qquad (3.3.0)$$

and

$$\|f^{out}\|_{out} = \int_{-\infty}^{0} |v|\,|f(-b,v)|\,dv + \int_0^{+\infty} v|f(b,v)|\,dv. \qquad (3.3.0)$$

We shall denote by f_i^{in} and f_i^{out}, where $i = 1, 2$, the i-th 'component' of the functions f^{in} and f^{out} belonging to the component spaces X_i^{out} and X_i^{in}, and by $\|\cdot\|_{i,in}$ and $\|\cdot\|_{i,out}$, $i = 1, 2$, the associated norms. For instance:

$$\|f_1^{in}\|_{1,in} = \int_0^{+\infty} v|f(-b,v)|\,dv$$

denotes the first term on the right hand side of the norm $\|\cdot\|_{in}$ given by (3.3.1). In other words, we may interpret X^{in} as $X_1^{in} \times X_2^{in}$ and X^{out} as $X_1^{out} \times X_2^{out}$ and $f^{in} = \begin{pmatrix} f_1^{in} \\ f_2^{in} \end{pmatrix}$. We remark that, if f is a particle density, then $f \in X^+$ and $\|f\|$ gives the total number of particles in the rod. Moreover, $\|f^{in}\|_{in}$ and $\|f^{out}\|_{out}$ respectively are the total ingoing and outgoing fluxes at the boundaries, [37].

The function $n = n(x, v, t)$, representing the density of the particles which at time t are in $x \in [-b, +b]$ with velocity $v \in \mathbb{R}$, must then satisfy the following equation:

$$\frac{\partial n}{\partial t} + v\frac{\partial n}{\partial x} + |v|\sigma n = 0 \qquad (3.3.0)$$

where $\sigma > 0$ is the absorption cross section and it represents the probability of a particle to be absorbed by the host medium. Equation (3.3.1) is equipped by the initial condition

$$n(x, v, 0) = n_0(x, v), \qquad (3.3.0)$$

and the boundary conditions:

$$n^{in} = \Lambda n^{out} \qquad (3.3.0)$$

where $\|\Lambda\|$ may be smaller, equal or bigger than 1.

Introducing the linear operator:

$$Lf = -v\frac{\partial f}{\partial x} - |v|\sigma f$$

$$D(L) = \left\{ f \in X,\ v\frac{\partial f}{\partial x} \in X,\ |v|\sigma f \in X,\ f_{in} = \Lambda f_{out} \right\}, \quad R(L) \subset X, \qquad (3.3.0)$$

the evolution problem (3.3.1)-(3.3.1) reads:

$$\begin{cases} \dfrac{dn(t)}{dt} = Ln(t) \ , \quad t > 0 \\ \\ n(0) = n_0 \ \in D(L^2). \end{cases} \tag{3.3.0}$$

where now $n(\cdot, \cdot, t)$ is a function from $[0, +\infty)$ with values in X, and

$$D(L^2) = \{f \in D(L), \ Lf \in D(L)\}.$$

3.3.2 The integrated semigroup

We want to prove that the assumptions of theorem 2.2.4 hold. Let us define the linear operator A by:

$$A = \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix} \quad , \quad D(A) = X^{in} \quad , \quad R(A) \subset X^{out}, \tag{3.3.0}$$

where the operators A_{12} and A_{21} are defined by

$$\begin{aligned} A_{12} f_2^{in} &= f_2^{in} e^{2b\lambda/v} \, e^{-2b\sigma} \quad , \quad D(A_{12}) = X_2^{in} \quad , \quad R(A_{12}) \subset X_1^{out}, \\ A_{21} f_1^{in} &= f_1^{in} e^{-2b\lambda/v} \, e^{-2b\sigma} \quad , \quad D(A_{21}) = X_1^{in} \quad , \quad R(A_{21}) \subset X_2^{out}, \end{aligned} \tag{3.3.0}$$

and where we recall that

$$X^{in} = L^1((\{-b\} \times (0, +\infty)); vdv) \times L^1(\{+b\} \times (-\infty, 0)); |v|dv) = X_1^{in} \times X_2^{in},$$

$$X^{out} = L^1((\{-b\} \times (-\infty, 0)); |v|dv) \times L^1(\{+b\} \times (0, +\infty)); |v|dv) = X_1^{out} \times X_2^{out}.$$

Lemma 3.3.1 *The norm of the operator A is smaller than 1. In particular,*

$$\|A\| < \exp(-2b\sigma).$$

Proof:
From the definition of the spaces X^{in} and X^{out} we obtain:

$$\|Af\|_{out} = \int_{-\infty}^{0} |f_2(v)| \exp(2b\lambda/v) \, \exp(-2b\sigma) \, dv$$

$$+ \int_{0}^{+\infty} |f_1(v)| \exp(-2b\lambda/v) \exp(-2b\sigma) \, dv$$

$$\leq \exp(-2b\sigma)\left(\int_{-\infty}^{0}|f_2(v)|\,dv+\int_{0}^{\infty}|f_1(v)|\,dv\right)=\exp(-2b\sigma)\|f\|_{in}.$$

■

We shall prove that the linear operator L is the generator of an integrated semigroup.

Theorem 3.3.1 *If one of the following*
(i) the norm of Λ *is smaller or equal to* 1*:* $\|\Lambda\| \leq 1$*;*
(ii) the norm of Λ *is bigger than* 1*,* $\|\Lambda\| > 1$*, and*

$$\sigma > \frac{\ln\|\Lambda\|}{2b}; \tag{3.3.0}$$

holds, then L *generates an integrated semigroup, which is positive.*

Proof:
Consider the resolvent equation

$$v\frac{\partial f}{\partial x}+(\lambda+|v|\sigma)f=g \tag{3.3.0}$$

where g is a given element of X. Integrating with respect to x, we obtain:

$$f(x,v)=\begin{cases}\exp(\delta(x+b))C_1+\dfrac{1}{v}\displaystyle\int_{-b}^{x}\exp(\delta(x-x'))g(x',v)\,dx'\,,v>0\\[2ex] \exp(\delta(x-b))C_2-\dfrac{1}{v}\displaystyle\int_{x}^{+b}\exp(\delta(x-x'))g(x',v)\,dx'\,,v<0\end{cases}, \tag{3.3.0}$$

where $\delta = -(\lambda + |v|\sigma)/v$ and $C_1 = C_1(v)$ and $C_2 = C_2(v)$ are to be determined by means of the boundary conditions. Evaluating $f(x,v)$ for $x = \pm b$ we get:

$$f^{out} = Af^{in} + G$$

where A is defined in (3.3.2) and G is given by:

$$G=\begin{pmatrix} g_1\\ g_2\end{pmatrix}\in X^{out} \tag{3.3.0}$$

with :

$$g_1=g_1(v)=\int_{-b}^{+b}\exp(-\delta(b+x'))g(x',v)\,dx' \text{ for}\,,v<0$$

and

$$g_2 = g_2(v) = \int_{-b}^{+b} \exp(\delta(b - x'))g(x', v)\,dx' \ \text{for}\,, v > 0.$$

Taking into account the boundary conditions (3.3.1), we write:

$$f^{in} = \Lambda A f^{in} + \Lambda G$$

and so:

$$f^{in} = (I - \Lambda A)^{-1}\Lambda G \tag{3.3.0}$$

provided that $\|\Lambda A\| < 1$. The above condition is satisfied when $\|\Lambda\| \leq 1$, because $\|A\| < 1$. Moreover, if the absorption cross section is such that

$$\sigma > \frac{\ln \|\Lambda\|}{2b}, \tag{3.3.0}$$

then $\|\Lambda A\| < 1$. Relation (3.3.2) may be seen as a condition for avoiding the blow up of the system of particles.

We now remark that $C_0^\infty([-b, +b] \times \mathbb{R})$ is contained in $D(L)$ and is dense in X, thus $D(L)$ is dense in X. Moreover, X has a generating and normal cone.

It is clear from relations (3.3.2) and (3.3.2) that, if $g \in X^+$, then C_1 and C_2 both belong to X^+, and $G \in X^+$ too. Thus, the resolvent $R(\lambda, L)$ is positive and the linear operator L generates an integrated semigroup $S(t)$. ■

Therefore, from the theory of integrated semigroups follows that: there exists a unique solution of the Cauchy problem 3.3.1 given by:

$$n(t) = S(t)Ln_0 + n_0\ , \quad \forall\, t \geq 0, \tag{3.3.0}$$

where $S(t)$ is the integrated semigroup generated by L. Moreover, if $n_0 \geq 0$ then $n(t) \geq 0$ for every $t \geq 0$.

Define the operator $\mathcal{A}$ as follows

$$\mathcal{A} = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \quad , \quad D(\mathcal{A}) = X^{in} \quad , \quad R(\mathcal{A}) \subset X, \tag{3.3.0}$$

where the operators A_{11} and A_{22} are defined by

$$\begin{aligned} A_{11} f_2^{in} &= f_1^{in} e^{\delta(x+b)}\,, \quad , \quad D(A_{22}) = X_2^{in} \quad , \quad R(A_{11}) \subset X, \\ A_{21} f_1^{in} &= f_2^{in} e^{\delta(x-b)} \quad , \quad D(A_{21}) = X_1^{in} \quad , \quad R(A_{22}) \subset X. \end{aligned} \tag{3.3.0}$$

We have:

Lemma 3.3.2 *The norm of the operator $\mathcal{A}$ given by (3.3.2) is smaller than $1/\lambda$.*

Proof:

We first remark that if $v > 0$, then:

$$\int_{-b}^{+b} \exp\left(-(x+b)\frac{\lambda + |v|\sigma}{v}\right) dx$$

$$= \frac{-v}{\lambda + |v|\sigma}\left(\exp\left(\frac{-2b}{v}(\lambda + |v|\sigma)\right) - 1\right) \leq \frac{v}{\lambda + |v|\sigma}.$$

Analogously, if $v < 0$, then:

$$\int_{-b}^{+b} \exp\left(-(x-b)\frac{\lambda + |v|\sigma}{v}\right) dx \leq -\frac{v}{\lambda + |v|\sigma}.$$

Hence, as f_1 and f_2 do not depend on x, we have that:

$$\|\mathcal{A}f\| = \int_{-b}^{+b}\int_{-\infty}^{0} f_2 \exp\delta(x-b)\, dx\, dv + \int_{-b}^{+b}\int_{0}^{+\infty} f_1 \exp\delta(x+b)\, dx\, dv$$

$$\leq \int_{-\infty}^{0} -\frac{v}{\lambda + |v|\sigma} f_2\, dv + \int_{0}^{+\infty} \frac{v}{\lambda + |v|\sigma} f_1\, dv \leq \frac{1}{\lambda}\|f\|_{in}.$$

■

By means of the operator $\mathcal{A}$ and of relation (3.3.2), the solution f of the resolvent equation reads

$$f = R(\lambda, L)g = \mathcal{A}(I - \Lambda A)^{-1}\Lambda\mathcal{G} + R(\lambda, L_0)g \tag{3.3.0}$$

where L_0 is the following operator:

$$L_0 f = Lf\ , \quad D(L_0) = \left\{f \in X,\ v\frac{\partial f}{\partial x} \in X,\ |v|\sigma f \in X,\ f^{in} = 0\right\} \tag{3.3.0}$$

Finally, from relation 3.3.2 it easily obtained that:

$$\|R(\lambda, L)\| \leq \frac{M}{\lambda}\ , \quad M = \frac{1}{1 - \|\Lambda\|}.$$

4

Diffusion Driven by Collisions I

As said in chapter 2 the asymptotic limits of kinetic equations towards a fluid dynamic equation are of interest in numerical analysis. In fact, the number of unknowns diminish from seven to four, this fact leading to a more efficient numerical simulation. In this chapter we shall see the derivation of a drift-diffusion type equation starting from a three dimensional Vlasov equation.

The model on which is based the Vlasov equation is suggested by an application to ionic thrusters. This kind of thrusters are applied on satellites, the main aim being to avoid the transport of too heavy quantities of propellers. A ionic thruster is composed of two coaxial cylinder containing in the gap between them a gas (namely the Xenon). Electrons are injected in the cavity, ionize the gas and are accelerated parallely to the cylinders by means of an imposed electric field. In order to have a better efficiency, a magnetic field orthogonal to the axis of the cylinders is also applied, this forcing the electrons to spend a longer time in between the cylinders, thus ionizing a bigger quantity of gas (for more details about the device see [17], [33], for numerical applications [27], [42] and for related physical approaches [49], [50]). The electrons are subject also to a convex combination of specular reflection and of diffusion by collision against the boundaries.

Our goal is a mathematically rigorous derivation of a diffusion model. We shall study a simplified model of the ionic thruster, in fact we shall consider two parallel plates instead of the coaxial cylinders. This fact, only simplifies the writing of the equations but does not affect the mathematical derivation of the diffusion model which we shall prove. Moreover, in a first approach, collisions of electrons against the neutral molecules of the gas are neglected. This restriction will be waived in the next chapter.

It has first been demonstrated in [1] (by probabilistic arguments) and in [2] (by functional theoretic arguments) that collisions with the boundary can drive a particle system towards a diffusion regime. To some extent, the present work is a follow-up of [2] in a different physical framework but it also departs from [2] in important mathematical aspects.

In [2], the authors consider a collision-less neutral gas flowing in a thin domain (e.g. the gap between two plane parallel plates) and subject to a combination of specular reflection and pure diffusion at the solid plates. If the ratio (denoted by α) of the distance between the two plates to the typical longitudinal length scale (i.e. along the planes) is small, they show that the large time behavior (on time scales of order α^{-2}) of the particle distribution function is, to leading order, given by $n(\underline{\xi}, t)M(x, v)$, where $\underline{\xi}$ denotes the longitudinal position variable while x is the transversal coordinate, v is the velocity and t the time. M is the normalized distribution function in transversal sections of the domain (typically, M is the Maxwellian at the plates temperature) and n is a solution of a diffusion equation, which describes how the flow evolves between the plates.

The associated diffusion constant is the most important characteristic parameter of the flow at this scale. It is related to the typical distance that a particle travels between two encounters with the plates. Particles with grazing velocities (i.e. almost tangent to the plates) travel a very large distance before hitting them, giving rise to a large diffusivity. In fact, for the present plane parallel geometry, the diffusivity of [2] is infinite. It has been shown in [15] (by probabilistic tools) and in [35] (by functional theoretic arguments) that a logarithmic time rescaling restores a finite diffusivity.

In our case, motivated by the physical context [26], we consider elastically diffusive collisions at the plates: the particles are re-emitted with the same energy as their incident one and with a random velocity direction. As a consequence, the large time behavior of the distribution function is given by an 'energy distribution function' $F(\underline{\xi}, \varepsilon, t)$, where $\varepsilon = |v|^2/2$ is the particle kinetic energy and F satisfies a diffusion equation in both position and energy. Energy diffusion is caused by the combined effects of the electric field and of the collisions with the plates. Position and energy diffusions are not independent (in other words, the diffusion is degenerate) because total energy is preserved during both free flight and collisions. The resulting diffusion model is known in semiconductor physics as the SHE model (for Spherical Harmonics Expansion, a terminology originating from its early derivation, see [10] and references therein) and is also used in plasma physics [30] and gas discharge physics [54].

Another very important fact in our model is the presence of a strong magnetic field directed transversally to the plates. Particle motion then does not occur along straight lines, but rather, along helices whose axis are parallel to the magnetic field lines. The distance that a particle travels between two encounters with the plates cannot be larger than the radii of these helices (the so-called 'Larmor' radius). This limitation results in the finiteness of the diffusivity without time rescaling.

Mathematically, the present problem belongs to the class of diffusion approximation problems for kinetic equations (see e.g. [12], [5] in the context of neutron transport, [51], [36] for semiconductors and [29] for plasma). Two methods are usually developed: the Hilbert expansion method [5] and the moment method [36]. The latter, although providing only weak convergence results without explicit rates, is more flexible as it requires only mild regularity assumptions on the solution. It proceeds in three steps:

(i) prove that the time asymptotic profile of the distribution function is of the form $F(\underline{\xi}, \varepsilon, t)$, with the macroscopic variable F still to be determined at this stage.

(ii) by using the conservation properties of the collision operator, show that the macroscopic variable obeys a continuity equation, i.e. that its time derivative is balanced by the divergence of a current to be determined.

(iii) find an expression for the current. This is the delicate point of the proof, as the current depends on the first (order α) correction to the asymptotic profile. It is determined through the so-called 'auxiliary function', which essentially provides the response of the microscopic system to gradients of the macroscopic variable. The equation for the current is found as a moment of the kinetic equation through the auxiliary function (hence justifying the terminology 'moment method').

Point (ii) of the proof is actually straightforward. For (i), we first derive trace estimates (that to our knowledge are original) of the distribution function at the boundary (the plates). These estimates allow us to show that, in the limit $\alpha \to 0$, the traces converge in an L^2 sense to a function which is independent of the velocity direction $\omega = v/|v|$. It is then easy to prove that this property 'propagates' inside the domain and that the limiting distribution function is independent of the transversal coordinate x and of ω, the result to be proved. The trace estimate directly follows from the dissipative character of the boundary operator measured in terms of the maximal eigenvalue of its projection onto the orthogonal space to the equilibrium states. This technique has also proved its usefulness in other applications as well (see [11] for an application to semiconductor super-lattices).

Concerning point (iii), the auxiliary equation takes the form $\mathcal{A}_0^* f = g$, where $\mathcal{A}_0^*$ is the adjoint of the leading order operator in the kinetic equation and g are special functions which measure the microscopic effects of gradients in the macroscopic variable F. In the present case, the auxiliary equation is not solvable in f for arbitrary data g. Instead, we need to use the fact that, for our specific data, the action of the magnetic field balances the singularity which appears at grazing velocities. Note that we cannot apply the theory of [2]: indeed, in our case, hypothesis (52) of [2], which would mean the integrability of the function $\omega \in \mathbb{S}^2 \to |\omega_x|^{-1}$ on the sphere, breaks down due to grazing velocities.

This chapter given rise to the publication [28] and is organized as follows. First, an introduction to the kinetic model which serves as starting point to the present work is given in section 4.1 together with the scaling of the kinetic equation due to the geometry of the model and with the main theorem to be proved. Then, in section 4.2, we focus on the study of the boundary collision operator. In section 4.3, we deduce both an existence result and fundamental trace estimates for the scaled kinetic model. Section 4.4 is the core of the chapter: it develops the convergence proof of the kinetic model towards the diffusion model. The proof is divided in several steps corresponding to points (i) to (iii) mentioned above. In section 4.5, we state and prove some properties of the diffusive tensor.

4.1 The model, the scaling and the main theorem

We study the transport properties of electrons in a plasma thruster. For simplicity we consider a domain which consists of the gap between two parallel plates separated by a distance much smaller than their transverse dimensions. We respectively indicate by $\hat{X} = (\hat{x}, \hat{y}, \hat{z})$ and by $\hat{v} = (\hat{v}_x, \hat{v}_y, \hat{v}_z)$ the position and the velocity vectors of an electron in between the plates, where the $\hat{x}$ direction is perpendicular to the plates, and the $\hat{y}$ and $\hat{z}$ directions are the parallel ones. Electrons moving between the two planes are subject to a given electric field along the $\hat{y}$ and $\hat{z}$ directions, $\hat{E} = \hat{E}(\hat{y}, \hat{z}) = (0, \hat{E}_y, \hat{E}_z)$, and to a given magnetic field along the $\hat{x}$ direction $\hat{B} = \hat{B}(\hat{y}, \hat{z}) = (\hat{B}_x, 0, 0)$ both independent on time.

Collisions of electrons against the neutral molecules of the gas contained in the region $0 < \hat{x} < l$ are neglected in a first approach. Therefore, the electrons are supposed to move in between the planes according to a collision-less transport equation, and we can write the following Vlasov equation for the electron distribution function $\hat{f}(\hat{X}, \hat{v}, \hat{t})$:

$$\frac{\partial \hat{f}}{\partial \hat{t}} + \hat{v}_x \frac{\partial \hat{f}}{\partial \hat{x}} + \hat{v}_y \frac{\partial \hat{f}}{\partial \hat{y}} + \hat{v}_z \frac{\partial \hat{f}}{\partial \hat{z}} - \frac{q}{m}\left(\hat{E}_y \frac{\partial \hat{f}}{\partial \hat{v}_y} + \hat{E}_z \frac{\partial \hat{f}}{\partial \hat{v}_z}\right)$$
$$-\frac{q}{m}\hat{B}_x\left(\hat{v}_z \frac{\partial \hat{f}}{\partial \hat{v}_y} - \hat{v}_y \frac{\partial \hat{f}}{\partial \hat{v}_z}\right) = 0, \tag{4.1.0}$$

where $\hat{X} \in \hat{\Omega} = \{[0, l] \times \mathbb{R}^2\}$, $\hat{v} \in \mathbb{R}^3$ and $-q < 0$ and m are the electron charge and mass.

The collision of the electrons against the planes are modeled by a combination of specular and diffusive reflection. In order to write them, let us introduce the set $\hat{\Theta} = \hat{\Omega} \times \mathbb{R}^3$, consider its boundary $\hat{\Gamma} = \hat{\gamma} \times \mathbb{R}^3$, where $\hat{\gamma} = \{0, l\} \times \mathbb{R}^2$, and the following *incoming* and *outgoing* sets:

$$\hat{\Gamma}^- = \left(\{0\} \times \mathbb{R}^2 \times \{\hat{v} \in \mathbb{R}^3, \hat{v}_x > 0\}\right) \cup \left(\{l\} \times \mathbb{R}^2 \times \{\hat{v} \in \mathbb{R}^3, \hat{v}_x < 0\}\right),$$

$$\hat{\Gamma}^+ = \left(\{0\} \times \mathbb{R}^2 \times \{\hat{v} \in \mathbb{R}^3, \hat{v}_x < 0\}\right) \cup \left(\{l\} \times \mathbb{R}^2 \times \{\hat{v} \in \mathbb{R}^3, \hat{v}_x > 0\}\right),$$

where for instance $(\{0\} \times \mathbb{R}^2 \times \{\hat{v} \in \mathbb{R}^3, \hat{v}_x > 0\})$ will be used to represent electrons entering the region $0 < \hat{x} < l$ through the boundary plane $\hat{x} = 0$.
Indicating by $\hat{f}_-$ and $\hat{f}_+$ the traces of the distribution function $\hat{f}$ on the sets $\hat{\Gamma}^-$ and $\hat{\Gamma}^+$:

$$\hat{f}_- = \hat{f}|_{\hat{\Gamma}^-} \quad , \quad \hat{f}_+ = \hat{f}|_{\hat{\Gamma}^+},$$

the boundary conditions at $\hat{x} = 0$ and $\hat{x} = l$ read as follows:

$$\hat{f}_-(\hat{X}, \hat{v}) = \beta \hat{f}_+(\hat{X}, \hat{v}_*) + (1 - \beta)\mathcal{K}(\hat{f}_+)(\hat{X}, \hat{v}), \quad (\hat{X}, \hat{v}) \in \hat{\Gamma}^-, \tag{4.1.0}$$

where the accommodation coefficient $\beta = \beta(\hat{x}, \hat{y}, \hat{z}, |\hat{v}|)$ may depend upon $\hat{x}(= 0, l)$, $\hat{y}$, $\hat{z}$, $|\hat{v}|$, and is such that $0 \leq \beta < 1$. The velocity $\hat{v}_*$ is the specularly reflected

velocity given by $\hat{v}_* = (-\hat{v}_x, \hat{v}_y, \hat{v}_z)$.
The operator $\mathcal{K}(\hat{f}_+)(\hat{X}, \hat{v})$ is the diffusive reflection operator acting on the set of functions defined on $\hat{\Gamma}^+$ with values on functions defined on $\hat{\Gamma}^-$ and it is given by:

$$\mathcal{K}(\hat{f}_+)(\hat{X}, \hat{v}) = \int_{\{\omega' \in S^2, (\hat{X}, \hat{v}') \in \hat{\Gamma}^+\}} K(\hat{X}, |\hat{v}|; \omega' \to \omega)\, \hat{f}_+(\hat{X}, |v|\omega')\, |\omega'_x|\, d\omega', \quad (4.1.0)$$

where the velocity direction ω belongs to the unit sphere S^2 and is such that $\hat{v} = |\hat{v}|\omega = \hat{u}\omega$, with $\hat{u} = |\hat{v}|$. The dependence of the kernel K with respect to $\hat{X} = (\hat{x}, \hat{y}, \hat{z})$, $\hat{x} = \{0, l\}$, $(\hat{y}, \hat{z}) \in \mathbb{R}^2$ and $|\hat{v}|$ will be omitted, otherwise specified. The quantity $K(\omega' \to \omega)|\omega_x| d\omega$ is the probability of an electron impinging on a plane at a position $(\hat{y}, \hat{z})$ with velocity modulus $|\hat{v}|$ and velocity direction ω' to be reflected with new velocity direction ω belonging to the solid angle $d\omega$ (and the same velocity modulus).

Our model is based on the assumption that the distance between the planes is small compared with their dimension. Thus, we introduce a small parameter α defined by:

$$\alpha = \frac{l}{L} \ll 1. \quad (4.1.0)$$

The typical scale of variation of the transverse coordinate $\hat{x}$ is thus αL. On the other hand, the typical scale for the parallel coordinate $\hat{\underline{\xi}} = (\hat{y}, \hat{z})$ is L. Therefore, dimensionless position coordinates $(x, \underline{\xi}) = (x, y, z) \in [0, 1] \times \mathbb{R}^2$ are defined by:

$$\hat{x} = \alpha L x\,, \quad \hat{\underline{\xi}} = L\underline{\xi}.$$

Let E_0 be a typical scale of the externally imposed electric field. A typical velocity scale V_0 is the velocity reached by a particle in this electric field over a distance L and given by $V_0 = \sqrt{qE_0L/m}$. Dimensionless electric fields $E(\underline{\xi}) = (0, E_y(\underline{\xi}), E_z(\underline{\xi}))$ and velocity $v = (v_x, v_y, v_z)$ are then defined by:

$$\hat{E}(\hat{\underline{\xi}}) = E_0 E(\underline{\xi})\,, \quad \hat{v} = V_0 v.$$

Let B_0 be the typical scale of magnetic field, and $r_L = mV_0/qB_0$ the associated Larmor radius, that is, the radius of the circular trajectory of an electron of velocity V_0 in a uniform magnetic field of magnitude B_0 directed perpendicularly to the velocity. For the magnetic confinement to be efficient, r_L must be of the order of the distance αL between the boundary planes, which imposes $B_0 = mV_0/qL\alpha =$

$E_0/\alpha V_0$. The time needed by a particle to cross the distance L is L/V_0. Since we are looking for a diffusion regime, we set the typical time scale to $t_0 = L/\alpha V_0$. The dimensionless magnetic field is of the form $B(\underline{\xi}) = (B_x(\underline{\xi}), 0, 0)$ and the time t are then defined by:

$$\hat{t} = t_0 t\,, \quad \hat{B}(\hat{\underline{\xi}}) = B_0\, B(\underline{\xi}).$$

In what follows, when it will not lead to confusion, we will write $B = B(\underline{\xi})$ for $B_x = B_x(\underline{\xi})$.

With the above change of coordinates, equation (4.1) is written in dimensionless form as follows:

$$\alpha^2 \frac{\partial f^\alpha}{\partial t} + \alpha \left(v_y \frac{\partial f^\alpha}{\partial y} + v_z \frac{\partial f^\alpha}{\partial z} - E_y \frac{\partial f^\alpha}{\partial v_y} - E_z \frac{\partial f^\alpha}{\partial v_z} \right) + \\ + v_x \frac{\partial f^\alpha}{\partial x} - B \left(v_z \frac{\partial f^\alpha}{\partial v_y} - v_y \frac{\partial f^\alpha}{\partial v_z} \right) = 0, \qquad (4.1.0)$$

which may also be written compactly as:

$$\alpha^2 \frac{\partial f^\alpha}{\partial t} + \alpha \left(\underline{v} \cdot \nabla_{\underline{\xi}} f^\alpha - \underline{E} \cdot \nabla_{\underline{v}} f^\alpha \right) + v_x \frac{\partial f^\alpha}{\partial x} - (\underline{v} \times B) \cdot \nabla_{\underline{v}} f^\alpha = 0, \qquad (4.1.0)$$

where $f^\alpha = f^\alpha(\alpha L x, L\underline{\xi}; V_0 v; t_0 t)$ is the distribution function depending on the scaled position vector $X = (x, \underline{\xi}) = (x, y, z) \in \Omega = [0,1] \times \mathbb{R}^2$, the scaled velocity vector $v = (v_x, \underline{v}) = (v_x, v_y, v_z) \in \mathbb{R}^3$ and the scaled time t; the scaled electric field $\underline{E}$ is given by $\underline{E} = (E_y, E_z)$ and the cross product $\underline{v} \times B$ is equal to $(0, v_z B, -v_y B)$. The scaled boundary conditions are then given by:

$$f^\alpha_-(X, v) = \beta f^\alpha_+(X, v_*) + (1-\beta)\mathcal{K}(f^\alpha_+)(X, v), \quad (X, v) \in \Gamma^-, \qquad (4.1.0)$$

and where f^α_- and f^α_+ are the traces of f^α on the sets Γ^- and Γ^+:

$$f^\alpha_- = f^\alpha|_{\Gamma^-} \quad , \quad f^\alpha_+ = f^\alpha|_{\Gamma^+}. \qquad (4.1.0)$$

Now $\Gamma = \gamma \times \mathbb{R}^3$, with $\gamma = \{0, 1\} \times \mathbb{R}^2$. The accommodation coefficient $\beta = \beta(x, y, z, |v|)$ may depend upon the scaled coordinates $x = 0, 1$ (on the boundary), y, z, $|v|$, and $0 \le \beta < 1$ and the velocity v_* is the specularly reflected velocity $v_* = (-v_x, v_y, v_z)$.

Our goal is to show that the limit of f^α as $\alpha \to 0$ is a function $F(\underline{\xi}, \varepsilon, t)$ of the longitudinal coordinate $\underline{\xi}$, of the energy $\varepsilon = |v|^2/2$ and of the time which obeys a diffusion equation in the position-energy space. This equation is often referred to in the literature as the SHE model (see introduction for references). More precisely, we shall prove the following theorem:

Theorem 4.1.1 *Under assumptions (4.2.1) to (4.4.1) and (4.4.2), f^α converges to f^0 as $\alpha \to 0$ in the weak star topology of $L^\infty([0,T], L^2(\Theta))$ for any $T > 0$, where $f^0(X, v, t) = F(\underline{\xi}, |v|^2/2, t)$ and $F(\underline{\xi}, \varepsilon, t)$ is a distributional solution of the problem:*

$$4\pi\sqrt{2\varepsilon}\frac{\partial F}{\partial t} + \left(\nabla_{\underline{\xi}} - \underline{E}\frac{\partial}{\partial \varepsilon}\right)\cdot \underline{J} = 0, \tag{4.1.1}$$

$$\underline{J}(\underline{\xi}, \varepsilon, t) = -\mathbb{D}\left(\nabla_{\underline{\xi}} - \underline{E}\frac{\partial}{\partial \varepsilon}\right) F(\underline{\xi}, \varepsilon, t), \tag{4.1.2}$$

$$F|_{t=0} = F_I, \tag{4.1.3}$$

where F_I is a suitable initial condition, in the domain $(\underline{\xi}, \varepsilon) \in \mathbb{R}^2 \times (0, \infty)$. The diffusion tensor $\mathbb{D} = \mathbb{D}(\underline{\xi}, \varepsilon)$ is given by

$$\mathbb{D}(\underline{\xi}, \varepsilon) = (2\varepsilon)^{3/2}\int_0^1 \int_{\mathbb{S}^2} \underline{D}(x, \omega; \underline{\xi}, \varepsilon) \otimes \underline{\omega}\, dx\, d\omega, \tag{4.1.3}$$

where $\underline{\omega} = (\omega_y, \omega_z)$, $\underline{D} = (D_y, D_z)$, $\underline{D} \otimes \underline{\omega}$ is the tensor product $(D_i\omega_j)_{i,j\in\{y,z\}}$ and $D_i(x, \omega; \underline{\xi}, |v|^2/2)$, $(i = y, z)$ is a solution of the problem

$$\begin{cases} -v_x \dfrac{\partial D_i}{\partial x} + (\underline{v} \times B) \cdot \nabla_{\underline{v}} D_i = \omega_i\,, \text{ in } \Theta\,, \\ \\ (D_i)_+ = \mathcal{B}^*(D_i)_-\,, \text{ on } \Gamma\,, \end{cases} \tag{4.1.3}$$

unique, up to an additive function of $\underline{\xi}$ and ε.

Remark 4.1.1 By hypothesis 4.4.2, the diffusion tensor vanishes for vanishing energy $\mathbb{D}(\underline{\xi}, \varepsilon = 0) = 0$, $\forall \underline{\xi} \in \mathbb{R}^2$. Therefore, the diffusion problem in ε is degenerate and does not require additional boundary conditions at $\varepsilon = 0$. This hypothesis is satisfied in practice [26].

We introduce the functional setting and prove the existence of solutions of (4.1) in the following sections.

4.2 The boundary operator: assumptions and properties

We begin with some notations. We denote $\mathcal{S}_\pm(x)$, $x = 0, 1$ the following half-spheres:

$$\mathcal{S}_+(0) = \mathcal{S}_-(1) = \{\omega \in \mathbb{S}^2, \omega_x < 0\} \ , \ \mathcal{S}_-(0) = \mathcal{S}_+(1) = \{\omega \in \mathbb{S}^2, \omega_x > 0\}, \quad (4.2.0)$$

where we recall that $\omega = (\omega_x, \omega_y, \omega_z) = \dfrac{v}{|v|}$. We introduce the domain $\mathcal{S} = [0,1] \times \mathbb{S}^2$, with its associated *incoming* and *outgoing* boundaries defined by:

$$\mathcal{S}_- = (\{0\} \times \mathcal{S}_-(0)) \cup (\{1\} \times \mathcal{S}_-(1)) \, , \, \mathcal{S}_+ = (\{0\} \times \mathcal{S}_+(0)) \cup (\{1\} \times \mathcal{S}_+(1)).$$

We define the inner products on $L^2(\Theta)$ and on $L^2(\Gamma_\pm)$ respectively by:

$$(f,g)_\Theta = \int_\Theta f g \, d\theta \ , \quad (f,g)_{\Gamma_\pm} = \int_{\Gamma_\pm} f g \, |v_x| \, d\Gamma.$$

where $d\theta = dx d\underline{\xi} dv$ is the volume element in phase space, and $d\Gamma = \sum_{x=0,1} d\underline{\xi} dv$ is the surface element. The inner products on $L^2(\mathcal{S})$, $L^2(\mathcal{S}_\pm)$ are defined analogously:

$$(f,g)_\mathcal{S} = \int_0^1 \int_{\mathbb{S}^2} (fg) \, dx \, d\omega, \quad (4.2.0)$$

$$(f,g)_{\mathcal{S}_\pm} = \int_{\mathcal{S}_\pm(0)} |\omega_x| \, (fg) \, d\omega + \int_{\mathcal{S}_\pm(1)} |\omega_x| \, (fg) \, d\omega, \quad (4.2.0)$$

We now introduce the orthogonal projection $Q_\pm$ of $L^2(\mathcal{S}_\pm)$ on the space $\mathcal{C}^\pm$ of constant functions on each connected component of $\mathcal{S}_\pm$, i.e.:

$$Q_\pm f(x,\omega) = \frac{1}{\pi} \int_{\mathcal{S}_\pm(x)} |\omega'_x| \, f(x,\omega') \, d\omega' \ , \ \omega \in \mathcal{S}_\pm(x) \, , x = 0, 1 \, , \quad (4.2.0)$$

and its orthogonal complement $P_\pm = \mathrm{I}_{\mathcal{S}_\pm} - Q_\pm$, where $\mathrm{I}_{\mathcal{S}_\pm}$ is the identity.

We define the boundary operator $\mathcal{B}$ in the following way: for $\phi \in L^2(\mathcal{S}_+)$, $\mathcal{B}\phi \in L^2(\mathcal{S}_-)$ and

$$\mathcal{B}(\phi) = \beta J \phi + (1-\beta) \mathcal{K}(\phi), \quad (4.2.0)$$

where the mirror reflection operator J from $L^2(\mathcal{S}_+)$ to $L^2(\mathcal{S}_-)$ is defined by:

$$J\phi(x,\omega) = \phi(x,\omega_*), \quad (4.2.0)$$

with $\omega_* = (-\omega_x, \omega_y, \omega_z)$. Moreover, its adjoint J^* from $L^2(\mathcal{S}_-)$ to $L^2(\mathcal{S}_+)$ is also the mirror reflection operator, and so J and J^* satisfy $J^*J = I_{\mathcal{S}_+}$, $JJ^* = I_{\mathcal{S}_-}$.

Further, we list the required assumptions on the operator $\mathcal{K}$. We recall that we omit the dependence of $\mathcal{K}$ on $(x, \underline{\xi}, |v|)$ when the context is clear.

Hypothesis 4.2.1 *We assume that the kernel K satisfies the following properties:*
(i) Positivity:

$$K(\omega' \to \omega) > 0, \tag{4.2.0}$$

for almost all $(\omega, \omega') \in \mathcal{S}_-(x) \times \mathcal{S}_+(x)$, $x = 0, 1$.
(ii) Flux conservation:

$$\int_{\mathcal{S}_-(x)} K(\omega' \to \omega)\, |\omega_x|\, d\omega = 1\,,\; x = 0, 1\,. \tag{4.2.0}$$

(iii) Reciprocity relation:

$$K(\omega' \to \omega) = K(-\omega \to -\omega'), \quad \forall\, (\omega, \omega') \in \mathcal{S}_-(x) \times \mathcal{S}_+(x)\,,\; x = 0, 1\,. \tag{4.2.0}$$

Positivity and flux conservation are natural physical assumptions. Reciprocity is due to the time reversibility of the microscopic interaction process which occurs at the boundary. Its relevance is discussed in [20]. From relations (4.2.1) and (4.2.1), the *normalization identity* easily follows:

$$\int_{\mathcal{S}_+(x)} K(\omega' \to \omega)\, |\omega'_x|\, d\omega' = 1\,, \quad x = 0, 1. \tag{4.2.0}$$

Moreover, from Hypothesis 4.2.1, we derive the *Darrozes-Guiraud inequality*:

Lemma 4.2.1 *(i) Let $f_+ \in L^2(\mathcal{S}_+)$ and $f_- = \mathcal{B} f_+$. Then,*

$$\int_{\mathcal{S}_-(x)} |f_-(x,\omega)|^2\, |\omega_x|\, d\omega \leq \int_{\mathcal{S}_+(x)} |f_+(x,\omega)|^2\, |\omega_x|\, d\omega\,, \quad x = 0, 1. \tag{4.2.0}$$

(ii) $\mathcal{B}$, as an operator from $L^2(\mathcal{S}_+)$ to $L^2(\mathcal{S}_-)$, is of norm 1.

Proof:
(i) We have, for $x = 0, 1$:

$$\int_{\mathcal{S}_-(x)} |\omega_x|\, |f_-(x,\omega)|^2\, d\omega \leq \int_{\mathcal{S}_-(x)} \beta\; |f_+(x,\omega_*)|^2\, |\omega_x|\, d\omega$$

$$+ \int_{\mathcal{S}_-(x)} (1-\beta) \left| \int_{\mathcal{S}_+(x)} K(x, \omega' \to \omega)\, |(f_+(x,\omega')|\, |\omega'_x|\, d\omega' \right|^2 |\omega_x|\, d\omega$$

$$\leq \int_{\mathcal{S}_+(x)} \beta\; |f_+(x,\omega)|^2\, |\omega_x|\, d\omega + (1-\beta) \int_{\mathcal{S}_+(x)} |f_+(x,\omega')|^2 |\omega'_x|\, d\omega',$$

where we have used the Cauchy-Schwartz inequality and the normalization identity (4.2).

(ii) From (4.2.1), we deduce that $|\mathcal{B}f|^2_{L^2(\mathcal{S}_-)} \leq |f|^2_{L^2(\mathcal{S}_+)}$, and therefore that $\|\mathcal{B}\| \leq 1$. Now, from (4.2), any f in $\mathcal{C}^+$ is such that $\mathcal{B}f = Jf$. Then, $|\mathcal{B}f|^2_{L^2(\mathcal{S}_-)} = |f|_{L^2(\mathcal{S}_+)}$, showing that $\|\mathcal{B}\| = 1$. ■

We remark that the dual operator $\mathcal{B}^*$ of $\mathcal{B}$ maps $L^2(\mathcal{S}_-)$ to $L^2(\mathcal{S}_+)$ according to:

$$\mathcal{B}^*\phi = \beta J^*\phi + (1-\beta)\mathcal{K}^*(\phi) \tag{4.2.0}$$

with

$$\mathcal{K}^*(\phi)(x,\omega) = \int_{\mathcal{S}_-(x)} K(x,\omega \to \omega')\,|\omega'_x|\,\phi(x,\omega')\,d\omega',\ \omega \in \mathcal{S}_+(x)\,,\ x = 0,1. \tag{4.2.0}$$

Hypothesis 4.2.2 *The operator $\mathcal{K}$ is compact.*

We note that this implies that $\mathcal{K}^*$ is compact. We now prove:

Lemma 4.2.2 *Under Hypothesis 4.2.1, 4.2.2, we have:*

$$N(I - J\mathcal{B}^*) = \mathcal{C}^-,$$

$$N(I - J^*\mathcal{B}) = \mathcal{C}^+,$$

where, for instance, $N(I - J\mathcal{B}^*)$ denotes the Null-Space of the operator $I - J\mathcal{B}^*$.

Proof:

We first remark from (4.2) that a function φ of $\mathcal{C}^+$ satisfies $(I - J^*\mathcal{B})\varphi = 0$. Conversely, let $\varphi \in L^2(\mathcal{S}_+)$ be a solution of $(I - J^*\mathcal{B})\varphi = 0$, then:

$$\varphi - \beta J^*J\varphi - (1-\beta)J^*\mathcal{K}(\varphi) = 0,$$

which implies (because $\beta < 1$) that $(I - J^*\mathcal{K})\varphi = 0$. This equation can be decomposed in:

$$(I - J^*\mathcal{K}(0))\varphi_0 = 0, \quad (I - J^*\mathcal{K}(1))\varphi_1 = 0,$$

where $\varphi_x = \varphi|_{\mathcal{S}_+(x)}$ and $\mathcal{K}(x) = \mathcal{K}|_{L^2(\mathcal{S}_+(x))}$, $x = 0,1$.

The operators $J^*\mathcal{K}(x)$, $x = 0,1$ satisfy the following properties which are deduced from hypotheses 4.2.1 and 4.2.2:

(i) $J^*\mathcal{K}(x)$ is a compact operator on $L^2(\mathcal{S}_+(x))$.
(ii) $J^*\mathcal{K}(x)$ is positive: $\varphi \geq 0$ implies $J^*\mathcal{K}(x)(\varphi) > 0$.
(iii) The constant function 1 is an eigenfunction of $J^*\mathcal{K}(x)$ associated with the eigenvalue 1, i.e. $J^*\mathcal{K}(x)(1) = 1$.

Thanks to the Krein-Rutman theorem, it follows that 1 is an eigenvalue of multiplicity 1. Therefore, $N(I - J^*\mathcal{K}(x)) = \mathrm{Span}\{1\}$, $x = 0, 1$ and thus $N(I - J^*\mathcal{K}) = \mathcal{C}^+$, which proves the result. The proof is clearly similar for $I - J\mathcal{B}^*$. ■

Lemma 4.2.3 *The following equalities hold:*

$$\mathcal{B}Q_+ = Q_-\mathcal{B} = JQ_+ = Q_-J\,, \quad \mathcal{B}P_+ = P_-\mathcal{B}. \tag{4.2.0}$$

and similarly for $\mathcal{B}^*$.

Proof:
First, it is clear that $JQ_+ = Q_-J$. Let $\varphi = P_+\varphi + Q_+\varphi$ be the decomposition of $\varphi \in L^2(\mathcal{S}_+)$. Then:

$$\mathcal{B}\varphi = \mathcal{B}P_+\varphi + \mathcal{B}Q_+\varphi. \tag{4.2.0}$$

But $Q_+\varphi \in \mathcal{C}^+$ and, by Lemma 4.2.2, $\mathcal{B}Q_+\varphi = JQ_+\varphi = Q_-J\varphi$. Therefore,

$$\mathcal{B}\varphi = \mathcal{B}P_+\varphi + Q_-J\varphi. \tag{4.2.0}$$

We shall prove that (4.2) is the decomposition of $\mathcal{B}\varphi$ on P_- and Q_-. Indeed, let $q \in \mathcal{C}^-$, then, by duality, $(\mathcal{B}P_+\varphi, q)_{\mathcal{S}_-} = (P_+\varphi, \mathcal{B}^*q)_{\mathcal{S}_+}$, and $\mathcal{B}^*q = J^*q \in \mathcal{C}^+$. Hence, from the definition $P_+\varphi$, $(P_+\varphi, J^*q)_{\mathcal{S}_+} = 0$. Therefore, $\mathcal{B}P_+$ is orthogonal to $\mathcal{C}^-$ which proves the desired property. We deduce $\mathcal{B}P_+\varphi = P_-\mathcal{B}\varphi$ and $\mathcal{B}Q_+\varphi = Q_-\mathcal{B}\varphi$, which ends the proof of the Lemma. ■

By elementary operator theory, we remark that, for every $\underline{\xi} \in \mathbb{R}^2$ and for every $|v| \in \mathbb{R}^+$, there exists $k(\underline{\xi}, |v|)$ such that $\|\mathcal{K}P_+\| \leq k(\underline{\xi}, |v|) < 1$. In the remainder, we shall assume that the constant $k(\underline{\xi}, |v|)$ is bounded away from 1, as $\underline{\xi}$ and $|v|$ vary. More precisely, we assume the following:

Hypothesis 4.2.3 *(i) There exists a constant $k < 1$ such that:*

$$\|\mathcal{K}P_+\|_{\mathcal{L}(L^2(\mathcal{S}_+), L^2(\mathcal{S}_-))} \leq k < 1\,, |v| \in \mathbb{R}^+\,, \underline{\xi} \in \mathbb{R}^2. \tag{4.2.0}$$

(ii) There exists $\beta_0 < 1$ *such that* $0 \le \beta \le \beta_0 < 1$, $\underline{\xi} \in \mathbb{R}^2$, $|v| > 0$, $x = 0, 1$.

It follows that:

$$\|\mathcal{B}P_+\|_{\mathcal{L}(L^2(\mathcal{S}_+),L^2(\mathcal{S}_-))} \le \sqrt{\beta_0 + (1-\beta_0)k^2} = k_0. \tag{4.2.0}$$

We note that, if the kernel K is constant, then $\mathcal{K} = JQ_+$, and so $\mathcal{K}P_+ = JQ_+P_+ = 0$, and $\mathcal{K}$ satisfies the assumption (4.2.3). More generally, we have:

Lemma 4.2.4 *If* $K(\omega' \to \omega) \ge C > 0$, *where* C *is a constant, then assumption (4.2.3) holds.*

Proof:
It is enough to prove the result for $x = 1$, the proof being analogous for $x = 0$ (with the sign of ω_x reversed). We prove equivalently that there exists k, $0 < k < 1$, only depending on C such that:

$$|\mathcal{K}(1)\varphi|^2_{L^2(S_-(1))} \le k^2 |\varphi|^2_{L^2(S_+(1))}, \tag{4.2.0}$$

for all $\varphi \in L^2(S_+(1))$ such that:

$$\int_{\omega_x > 0} \varphi\, |\omega_x|\, d\omega = 0. \tag{4.2.0}$$

Omitting the dependence of the kernel $K(\omega' \to \omega)$ upon $\underline{\xi}$ and $|v|$, we have:

$$\begin{aligned} \mathcal{K}\varphi(\omega) &= \int_{\omega'_x > 0} K(\omega' \to \omega)\, \varphi(\omega')\, |\omega'_x|\, d\omega' \\ &= \int_{\omega'_x > 0} (K(\omega' \to \omega) - C/2)\, \varphi(\omega')\, |\omega'_x|\, d\omega' \end{aligned}$$

because of (4.2). But $K(\omega' \to \omega) - C/2 \ge C/2 > 0$ and

$$\int_{\omega'_x > 0} (K(\omega' \to \omega) - C/2)\, |\omega'_x|\, d\omega' = 1 - C\pi/2 \ge C\pi/2 > 0.$$

Therefore, noting $k^2 = 1 - C\pi/2$, we have $0 < k^2 < 1$ and by the Cauchy-Schwartz inequality:

$$\begin{aligned} |\mathcal{K}\varphi(\omega)|^2 &\le k^2 \int_{\omega'_x > 0} (K(\omega' \to \omega) - C/2)\, |\varphi(\omega')|^2\, |\omega'_x|\, d\omega' \\ &\le k^2 \int_{\omega'_x > 0} K(\omega' \to \omega) |\varphi(\omega')|^2\, |\omega'_x|\, d\omega'. \end{aligned}$$

Then, using the normalization property, we deduce that:

$$|\mathcal{K}\varphi(\omega)|^2_{L^2(\mathcal{S}_-(1))} = \int_{\omega_x<0} |K\varphi(\omega)|^2 \, |\omega_x| \, d\omega$$

$$\leq k^2 \int_{\omega'_x>0} |\varphi(\omega')|^2 \, |\omega'_x| \, d\omega' = k^2 |\varphi|^2_{L^2(\mathcal{S}_+(1))}.$$

■

As a final remark,we note that the following lemma holds.

Lemma 4.2.5 *Let $\phi = \phi(|v|^2)$ bounded and $f \in L^2(\mathcal{S}_+)$. Then $\mathcal{B}(\phi f) = \phi\mathcal{B}(f)$. Similarly, for $f \in L^2(\mathcal{S}_-)$, $\mathcal{B}^*(\phi f) = \phi\mathcal{B}^*(f)$.*

4.3 The transport operator

We define the following operator on $L^2(\Theta)$:

$$\mathcal{A}^\alpha f = \underline{v} \cdot \nabla_{\underline{\xi}} f - \underline{E} \cdot \nabla_{\underline{v}} f + \frac{1}{\alpha} \left(v_x \frac{\partial f}{\partial x} - (\underline{v} \times B) \cdot \nabla_{\underline{v}} \right), \tag{4.3.0}$$

with domain $D(\mathcal{A}^\alpha)$ defined by:

$$D(\mathcal{A}^\alpha) = \{ f \in L^2(\Theta),\, \mathcal{A}^\alpha f \in L^2(\Theta), P_+ f_+ \in L^2(\Gamma_+), \\ Q_+ f_+ \in L^2_{loc}(\Gamma_+), f_- = \mathcal{B} f_+ \}.$$

We need such regularity on the two projections of the trace f_+ in order to prove that the operator $\mathcal{A}^\alpha$ is closed (see lemmas 4.3.2, 4.3.3 and 4.3.4).

$$L^2_{loc}(\Gamma_\pm) = \{ f \text{ such that } \phi(|v|) f \in L^2(\Gamma_\pm), \forall \phi \in C^\infty, \phi \text{ bounded} \}. \tag{4.3.0}$$

$L^2_{loc}(\Gamma_\pm)$ is equipped with the family of semi-norms $|\cdot|_{\Gamma_\pm, R}$:

$$|f|^2_{\Gamma_\pm, R} = \int_{\Gamma_\pm, |v| \leq R} |v_x| \, |f|^2 \, d\Gamma. \tag{4.3.0}$$

$L^2_{loc}(\Gamma)$ is defined in a similar way with the family of semi-norms $|\cdot|_{\Gamma,R}$.

The goal of the present section is to establish trace estimates for functions of $D(\mathcal{A}^\alpha)$ which will show that $D(\mathcal{A}^\alpha)$ is closed for the graph norm

$$|f|^2_{\mathcal{A}^\alpha} = |f|^2_{L^2(\Theta)} + |\mathcal{A}^\alpha f|^2_{L^2(\Theta)}.$$

We then prove that $\mathcal{A}^\alpha$ generates a strongly continuous semigroup of contractions, thus providing an existence setting for the kinetic problem (4.1). As showed in the previous chapter, the fact of proving that $\mathcal{A}^\alpha$ is a closed operator, turns out to be crucial for the generation of a semigroup. We assume:

Hypothesis 4.3.1 *(i)* $\underline{E} = \underline{E}(\underline{\xi}) \in (W^{1,\infty}(\mathbb{R}^2))^2$.
(ii) $B = B(\underline{\xi}) \in C^1 \cap W^{1,\infty}(\mathbb{R}^2)$.
(iii) There exists a constant $B_0 > 0$ *such that* $|B(\underline{\xi})| \geq B_0 > 0$, *for every* $\underline{\xi} \in \mathbb{R}^2$.

We first establish a Green's Formula for functions in $D(\mathcal{A}^\alpha)$. Let us notice that for two functions $f, g \in C_0^1(\Theta)$ we have :

$$(\mathcal{A}^\alpha f, g)_\Theta = -(f, \mathcal{A}^\alpha g)_\Theta + \frac{1}{\alpha}\left(\int_{\mathbb{R}^2\times\mathbb{R}^3} v_x\,(fg)|_{x=1}\,d\Gamma - \int_{\mathbb{R}^2\times\mathbb{R}^3} v_x\,(fg)|_{x=0}\,d\Gamma\right)$$
$$= -(f, \mathcal{A}^\alpha g)_\Theta + \frac{1}{\alpha}\left(\int_{\Gamma_+} |v_x|\,f_+ g_+\,d\Gamma - \int_{\Gamma_-} |v_x|\,f_- g_-\,d\Gamma\right). \quad (4.3.0)$$

Define the space :

$$H(\mathcal{A}^\alpha) = \{f \in L^2(\Theta), \mathcal{A}^\alpha f \in L^2(\Theta)\} \quad (4.3.0)$$

By making $f = g$ in (4.3), we see that the assumption $f \in H(\mathcal{A}^\alpha)$ is not sufficient to guarantee the integrability of $|f_+|^2$ over the boundary because of the minus sign at the right-hand-side of (4.3). Following [3], [52], we define:

$$H_0(\mathcal{A}^\alpha) = \{f \in H(\mathcal{A}^\alpha),\ f_- \in L^2(\Gamma_-)\} = \{f \in H(\mathcal{A}^\alpha),\ f_+ \in L^2(\Gamma_+)\}. \quad (4.3.0)$$

Then, from [3], [52], we deduce:

Lemma 4.3.1 (Green's Formula) *Under Hypothesis 4.3.1, for* f, g *in* $H_0(\mathcal{A}^\alpha)$ *with compact support with respect to* v, *we have:*

$$(\mathcal{A}^\alpha f, g)_\Theta + (f, \mathcal{A}^\alpha g)_\Theta = \frac{1}{\alpha}\left((f_+, g_+)_{\Gamma_+} - (f_-, g_-)_{\Gamma_-}\right). \quad (4.3.0)$$

We first prove that the "non constant" part of the trace at the boundary of a function of $D(\mathcal{A}^\alpha)$ is controlled by the graph norm.

Lemma 4.3.2 *If* $f \in D(\mathcal{A}^\alpha)$, *then there exists a constant* $C > 0$ *such that:*

$$|P_- f_-|^2_{L^2(\Gamma_-)} \leq |P_+ f_+|^2_{L^2(\Gamma_+)} \leq \frac{2\alpha}{1-k_0}(\mathcal{A}^\alpha f, f)_\Theta \leq C\alpha |f|^2_{\mathcal{A}^\alpha}. \quad (4.3.0)$$

Proof:
We apply Green's Formula (4.3.1) with a cut off function $\chi_R(|v|^2)$ such that:

$$\chi_R(|v|^2) = \chi(|v|^2/R^2),$$

where $\chi \in C^\infty(\mathbb{R}^+)$, $0 \le \chi \le 1$, $\chi(u) = 1$ for $u < 1$ and $\chi(u) = 0$ for $u > 2$ and obtain, thanks to Lemma 4.2.3 and Hypothesis 4.2.3:

$$\begin{aligned}
2(\mathcal{A}^\alpha \chi_R f, \chi_R f)_\Theta &= \frac{1}{\alpha}\left(\int_{\Gamma_+} |v_x|\,|\chi_R f_+|^2\,d\Gamma - \int_{\Gamma_-} |v_x|\,|\chi_R f_-|^2\,d\Gamma\right) \\
&= \frac{1}{\alpha}\left(\int_{\Gamma_+} |v_x|\,|\chi_R|^2\,|f_+|^2\,d\Gamma - \int_{\Gamma_-} |v_x|\,|\mathcal{B} f_+|^2\,|\chi_R|^2\,d\Gamma\right) \\
&= \frac{1}{\alpha}\int_{\Gamma_+} |v_x|\,\left(|P_+ f_+|^2 + |Q_+ f_+|^2\right)|\chi_R|^2\,d\Gamma \\
&\quad - \frac{1}{\alpha}\int_{\Gamma_-} |v_x|\,\left(|P_- \mathcal{B} f_+|^2 + |Q_- \mathcal{B} f_+|^2\right)|\chi_R|^2\,d\Gamma \\
&= \frac{1}{\alpha}\left(\int_{\Gamma_+} |v_x|\,|P_+ f_+|^2\,|\chi_R|^2\,d\Gamma - \int_{\Gamma_-} |v_x|\,|\mathcal{B} P_+ f_+|^2\,|\chi_R|^2\,d\Gamma\right) \\
&\ge \frac{1-k_0^2}{\alpha}\int_{\Gamma_+} |v_x|\,|P_+ f_+|^2 |\chi_R|^2\,d\Gamma,
\end{aligned}$$

where we have used that $P_- f_- = P_- \mathcal{B} f_+ = \mathcal{B} P_+ f_+$. But, we have:

$$\mathcal{A}^\alpha(\chi_R f) = \chi_R \mathcal{A}^\alpha f + \frac{2\underline{E}\cdot\underline{v}}{R^2} f \chi'\left(\frac{|v|^2}{R^2}\right). \tag{4.3.0}$$

Therefore,

$$2(\mathcal{A}^\alpha \chi_R f, \chi_R f)_\Theta \le 2|\mathcal{A}^\alpha f|_{L^2(\Theta)}\,|f|_{L^2(\Theta)} + (C/R)|f|^2_{L^2(\Theta)}.$$

The result follows by letting $R \to \infty$. ■

We now notice that, if $f \in D(\mathcal{A}^\alpha)$, then $Q_- f_- = J Q_+ f_+$ (thanks to Lemma 4.2.3). Thus, there exists a single function $q(f) = q(x, \xi, |v|)$, $x = 0, 1, \xi \in \mathbb{R}^2, |v| > 0$, such that

$$q = Q_- f_- \text{ , on}\,\Gamma_-, \quad q = Q_+ f_+ \text{ , on}\,\Gamma_+ \tag{4.3.0}$$

We have:

Lemma 4.3.3 *Let $f \in D(\mathcal{A}^\alpha)$, then:*

$$|q(f)|^2_{\Gamma,R} \leq C\left(\alpha|f|^2_{\mathcal{A}^\alpha} + R|f|^2\right) \tag{4.3.0}$$

Proof:

We multiply $\mathcal{A}^\alpha f$ by $\text{sgn}(v_x)\phi(x)f$, where $\phi(x) = 2x - 1$ and $\text{sgn}(v_x)$ is the sign of v_x. Note that $\phi(1) = 1$, $\phi(0) = -1$. Hence, applying Green's Formula (4.3.1) with the cut off function χ_R, we obtain:

$$\begin{aligned}
&(\,\mathcal{A}^\alpha \chi_R f, \text{sgn}(v_x)\phi(x)\chi_R f\,)_{L^2(\Theta)} + \frac{2}{\alpha}|\sqrt{v_x}\chi_R f|^2_\Theta \\
&= \frac{1}{\alpha}\left(\int_{\mathbb{R}^2\times\mathbb{R}^3} \frac{|v_x|}{2}\,|\chi_R f(1)|^2\,d\Gamma + \int_{\mathbb{R}^2\times\mathbb{R}^3} \frac{|v_x|}{2}\,|\chi_R f(0)|^2\,d\Gamma\right) \\
&= \frac{1}{\alpha}\int_{\Gamma_+} \frac{|v_x|}{2}\left(|P_+ f_+|^2 + |Q_+ f_+|^2\right)|\chi_R|^2\,d\Gamma \\
&+ \frac{1}{\alpha}\int_{\Gamma_-} \frac{|v_x|}{2}\left(|P_- f_-|^2 + |Q_- f_-|^2\right)|\chi_R|^2\,d\gamma \\
&= \frac{1}{\alpha}\left(\int_{\Gamma_+} \frac{|v_x|}{2}\,|P_+ f_+|^2\,|\chi_R|^2\,d\Gamma + \int_{\Gamma_-} \frac{|v_x|}{2}\,|P_- f_-|^2\,|\chi_R|^2\,d\Gamma\right) \\
&+ \frac{1}{\alpha}\left(\int_{\Gamma_+} \frac{|v_x|}{2}\,|Q_+ f_+|^2\,|\chi_R|^2\,d\Gamma + \int_{\Gamma_-} \frac{|v_x|}{2}\,|Q_- f_-|^2\,|\chi_R|^2\,d\Gamma\right).
\end{aligned}$$

The second term on the right hand side is estimated from below by:

$$\frac{1}{\alpha}\left(\int_{\Gamma_+} \frac{|v_x|}{2}\,|q(f)|^2|\chi_R|^2\,d\Gamma + \int_{\Gamma_-} \frac{|v_x|}{2}\,|q(f)|^2|\chi_R|^2\,d\Gamma\right) > \frac{1}{\alpha}|q(f)|^2_{\Gamma,R},$$

whereas, the first term on the left hand side is smaller than $|f|^2_{\mathcal{A}^\alpha}$, and the second term on the left hand side is bounded by $(C/\alpha)|\sqrt{v_x}\chi_R f|^2_{L^2(\Theta)} = C(R/\alpha)|f|^2_{L^2(\Theta)}$. Hence, (4.3.3) follows. ■

Thanks to lemmas 4.3.2, 4.3.3, we can prove:

Lemma 4.3.4 *The operator $\mathcal{A}^\alpha$ given by (4.3) is a closed operator.*

Proof:

Assume that a sequence of function $f_n \in D(\mathcal{A}^\alpha)$ converges to a function f, and that $\mathcal{A}^\alpha f_n$ converges to g. It is clear that $g = \mathcal{A}^\alpha f$ in the distributional sense. There is still to prove that $P_+ f_+ \in L^2(\Gamma_+)$, $Q_+ f_+ \in L^2_{loc}(\Gamma_+)$ and $f_- = \mathcal{B} f_+$. From Lemmas

4.3.2 and 4.3.3, we have that $P_+ f_{n+}$ and $Q_+ f_{n+}$ are Cauchy sequence in $L^2(\Gamma_+)$ and $L^2_{loc}(\Gamma_+)$ respectively, thus convergent to $P_+ f_+$ and $Q_+ f_+$. (This follows from the fact that the traces converge in a weak sense, like e.g. $H^{-1/2}$). Then, the continuity of $\mathcal{B}$ allows to find $f_- = \mathcal{B} f_+$. ■

In order to prove that the operator $-\mathcal{A}^\alpha$ given by (4.3) generates a strongly continuous semigroup of contractions it is sufficient to prove that its dual operator, $\mathcal{A}^{\alpha *}$, is accretive.

Lemma 4.3.5 *(i) We have $\mathcal{A}^{\alpha *} f = -\mathcal{A}^\alpha f$, with domain:*

$$\begin{aligned} D(\mathcal{A}^{\alpha *}) &= \{f \in L^2(\Theta)\,, \mathcal{A}^{\alpha *} f \in L^2(\Theta),\ P_- f_- \in L^2(\Gamma_-), \\ &\qquad Q_- f_- \in L^2_{loc}(\Gamma_-), f_+ = \mathcal{B}^* f_-\}, \end{aligned} \tag{4.3.0}$$

where $\mathcal{B}^*$ is the adjoint of $\mathcal{B}$ given by (4.2).
(ii) $\mathcal{A}^{\alpha *}$ is accretive, i.e. $(\mathcal{A}^{\alpha *} f, f)_\Theta \geq 0$, $\forall f \in D(\mathcal{A}^{\alpha *})$.

Proof:
(i) First, define $\widehat{\mathcal{A}^{\alpha *}}$ by $\widehat{\mathcal{A}^{\alpha *}} f = -\mathcal{A}^\alpha f$ with the domain defined by (4.3.0). Let $f \in D(\mathcal{A}^\alpha)$ and $f^* \in D(\widehat{\mathcal{A}^{\alpha *}})$. By Green's Formula (4.3.1), we have:

$$(\mathcal{A}^\alpha f\,, f^*)_\Theta = (f\,, -\mathcal{A}^\alpha f^*)_\Theta + (f_+, f^*_+)_{\Gamma_+} - (f_-, f^*_-)_{\Gamma_-}$$

But,

$$\begin{aligned} (f_+, f^*_+)_{\Gamma_+} - (f_-, f^*_-)_{\Gamma_-} &= (f_+, f^*_+)_{\Gamma_+} - (\mathcal{B} f_+, f^*_-)_{\Gamma_-} \\ (f_+, f^*_+)_{\Gamma_+} &- (f_+, \mathcal{B}^* f^*_-)_{\Gamma_-} = 0. \end{aligned}$$

Therefore, $(\mathcal{A}^\alpha f, f^*)_\Theta \leq C(f^*)\,|f|_{L^2(\Theta)}$ proving that $f^* \in D(\mathcal{A}^{\alpha *})$. (The truncation argument, which must be used as above, is omitted for brevity), and that $D(\widehat{\mathcal{A}^{\alpha *}}) \subseteq D(\mathcal{A}^{\alpha *})$. Now, $D(\widehat{\mathcal{A}^{\alpha *}})$ is closed for the graph norm from the same arguments as for $\mathcal{A}^\alpha$. Therefore, $D(\widehat{\mathcal{A}^{\alpha *}})$ and $D(\mathcal{A}^{\alpha *})$ are two closed spaces for the graph norm, which contain the same dense subspace (for instance $\mathcal{D}(\Theta)$). Thus they are equal.
(ii) Let $f \in D(\mathcal{A}^{\alpha *})$. We have, thanks to Green's Formula (via a truncation argument which is omitted):

$$2(\mathcal{A}^{\alpha *} f, f)_\Theta = -\frac{1}{\alpha}\left(\int_{\Gamma_+} |v_x|\,|f_+|^2\,d\Gamma - \int_{\Gamma_-} |v_x|\,|f_-|^2\,d\Gamma\right)$$

$$= \frac{1}{\alpha}\left(\int_{\Gamma_-} |v_x|\,|f_-|^2\,d\Gamma - \int_{\Gamma_+} |v_x|\,|\mathcal{B}^* f_-|^2\,d\Gamma\right) \geq 0.$$

The inequality follows from the Darrozes-Guiraud formula for $\mathcal{B}^*$, the proof of which is omitted. ■

Collecting all the previous Lemmas and using the Lumer-Phillips theorem (see [53]) gives the following:

Theorem 4.3.1 *The operator $-\mathcal{A}^\alpha$ given by (5.4) generates a strongly continuous semigroup of contractions on $L^2(\Theta)$.*

4.4 Convergence towards the macroscopic model

In this section, we study the limit as $\alpha \to 0$ of the solution of the following problem:

$$\begin{cases} \alpha \dfrac{\partial}{\partial t} f^\alpha + \mathcal{A}^\alpha f^\alpha = 0 \quad \text{in } \Theta \\ f_-^\alpha = \mathcal{B} f_+^\alpha \quad \text{on } \Gamma \\ f^\alpha|_{t=0} = f_I\,. \end{cases} \tag{4.4.0}$$

To avoid the treatment of initial layers, we assume that the initial data are *well prepared*, as specified below:

Hypothesis 4.4.1 *We suppose that there exists a function F_I such that*

$$f_I(x, \underline{\xi}, v) = F_I(\underline{\xi}, |v|^2/2)$$

and that f_I satisfies

$$f_I \in L^2(\Theta),\ (\underline{v}\cdot\nabla_{\underline{\xi}} - \underline{E}\cdot\nabla_{\underline{v}}) f_I \in L^2(\Theta).$$

We note that hypothesis 4.4.1 implies that $f_I \in D(\mathcal{A}^\alpha)$ for all $\alpha > 0$. By Theorem 4.3.1, for any $T > 0$, problem (4.4) has a unique solution f^α belonging to $C^0([0,T], D(\mathcal{A}^\alpha)) \cap C^1([0,T], L^2(\Theta))$. In this section we are concerned with the proof of the main theorem 4.1.1. The proof will be divided in the following steps. First, we establish estimates on f^α showing the existence of a weak limit f^0 which does not depend on x and ω. After studying the auxiliary problem (4.1.1), we show that the current converges weakly and we establish equation (4.1.2). Finally, we derive the continuity equation (4.1.1), which concludes the proof.

4.4.1 Weak limit of f^α

Lemma 4.4.1 *There exists a constant C, only depending on the data, such that:*

$$|f^\alpha|_{L^\infty(0,T;L^2(\Theta))} \le C\,, \tag{4.4.1}$$

$$\int_0^T |P_+ f_+^\alpha|^2_{L^2(\Gamma_+)}\,dt \le \alpha^2 C\,. \tag{4.4.2}$$

Moreover, for any $R > 0$, there exists C_R only depending on R and on the data, such that:

$$\int_0^T |q(f^\alpha)|^2_{\Gamma,R}\,dt \le C_R. \tag{4.4.2}$$

where the definition of $q(f^\alpha)$ is given by (4.3).

Proof:

Multiplying the first equation of (4.4) by f^α, we have:

$$\alpha\left(|f^\alpha(t)|^2_{L^2(\Theta)} - |f_0|^2_{L^2(\Theta)}\right) + \int_0^t (\mathcal{A}^\alpha f^\alpha, f^\alpha)_\Theta\,ds = 0, \tag{4.4.2}$$

and using (4.3.2):

$$|f^\alpha(t)|^2_{L^2(\Theta)} + \frac{1-k_0^2}{2\alpha^2}\int_0^t |P_+ f_+^\alpha|^2_{L^2(\Gamma_+)}\,ds \le |f_I|^2_{L^2(\Theta)}, \tag{4.4.2}$$

which immediately gives (4.4.1) and (4.4.2). Moreover, from the proof of Lemma 4.3.3, we deduce:

$$|q(f^\alpha)|^2_{\Gamma,R} \le C\left(\alpha(\mathcal{A}^\alpha \chi_R f^\alpha, f^\alpha \chi_R \mathrm{sgn}(v_x)\phi)_\Theta + CR|f^\alpha|^2_{L^2(\Theta)}\right).$$

But, using (4.3) to evaluate $\mathcal{A}^\alpha(\chi_R f^\alpha)$ and the fact that $\mathcal{A}^\alpha f^\alpha = -\alpha\frac{\partial}{\partial t}f^\alpha$, we obtain:

$$\int_0^T |q(f^\alpha)|^2_{\Gamma,R}\,ds$$

$$\le -\alpha^2\int_0^T \left(\chi_R\frac{\partial}{\partial t}f^\alpha, \chi_R f^\alpha \mathrm{sgn}(v_x)\phi\right)_\Theta ds + CR\int_0^T |f^\alpha|^2_{L^2(\Theta)}\,ds$$

$$\le -\alpha^2\left[\int_\Theta |f^\alpha|^2 \mathrm{sgn}(v_x)\phi\chi_R^2\,dx\,d\underline{\xi}\,dv\right]_0^T + C\left(RT|f_0|^2_{L^2(\Theta)}\right) \le C_R.$$

Thus, (4.4.1)is proved. ■

As a consequence of Lemma 4.4.1, as α tends to 0, there exists a subsequence, still denoted by f^α, of solutions of problem (4.4), which converges in $L^\infty(0,T;L^2(\Theta))$ weak star to a function f^0. Furthermore, using the diagonal extraction process, the subsequence of $q(f^\alpha)$ converges to a function $q(x,\underline{\xi},|v|,t)$ with $x = 0,1$ in $L^2(0,T,L^2(\gamma \times B_R))$ weak star for any R, where B_R is the ball centered at 0 and of radius R in the velocity space. Also, from (4.4.2), the traces $P_+ f^\alpha_+$ and $P_- f^\alpha_-$ converge in $L^2(0,T;L^2(\Gamma_+))$ and $L^2(0,T;L^2(\Gamma_-))$ (respectively) strongly towards zero.

Finally, we note that, since f^α is bounded in $L^\infty(0,T;L^2(\Theta))$, then, by equation (4.4), $\mathcal{A}^\alpha f^\alpha$ is bounded (and even tends to zero) in $H^{-1}(0,T;L^2(\Theta))$. This implies that $(v_x f^\alpha)|_\Gamma$ is bounded in $H^{-1}(0,T;H^{-1/2}(\gamma \times B_R))$ for any ball B_R in the velocity space, by standard properties of $H(div)$ spaces (see [52], [34]). Therefore, the traces of f^α on Γ have limits in the distributional sense that are the traces of f^0 on Γ. By the preceding considerations, we deduce that the traces $f^0_\pm$ of f^0 on $\Gamma_\pm$ satisfy:

$$P_- f^0_- = P_+ f^0_+ = 0\,,\;\; Q_- f^0_- = Q_+ f^0_+ = q,$$

and so:

$$f^0|_\Gamma = q, \tag{4.4.2}$$

where $q = q(x,\underline{\xi},|v|,t)$, with $x = 0,1$, is independent of ω.

We now introduce the weak formulation:

Lemma 4.4.2 *Let f^α be the solution of problem (4.4). Then, f^α is a weak solution, i.e. for any test function $\phi \in C^1([0,T] \times \Theta)$, with a compact support in Θ and such that $\phi(\cdot,\cdot,T) = 0$, we have:*

$$\int_0^T \int_\Theta f^\alpha \left(\alpha \frac{\partial}{\partial t}\phi + (\underline{v} \cdot \nabla_{\underline{\xi}}\phi - \underline{E} \cdot \nabla_{\underline{v}}\phi) \right) dt\, d\theta + \alpha \int_\Theta f_I \phi|_{t=0}\, d\theta$$

$$+\frac{1}{\alpha} \int_0^T \int_\Theta f^\alpha \left(v_x \frac{\partial}{\partial x}\phi - (\underline{v} \times B) \cdot \nabla_{\underline{v}}\phi \right) dt\, d\theta \tag{4.4.2}$$

$$= \frac{1}{\alpha} \left(\int_0^T \int_{\Gamma_+} |v_x|\, f^\alpha_+ (\phi_+ - \mathcal{B}^* \phi_-)\, dt\, d\Gamma \right).$$

Proof:

Multiply equation (4.4) by ϕ, use Green's Formula (4.3.1) and the boundary conditions. ■

Lemma 4.4.3 *The limit function f^0 is a function of $(\underline{\xi}, |v|, t)$ only, i.e.*

$$f^0 = f^0(\underline{\xi}, |v|, t).$$

Proof:

Using (4.4.2) with ϕ with a compact support in Θ, we get:

$$\alpha^2 \int_0^T \int_\Theta f^\alpha \frac{\partial}{\partial t}\phi \, dt\, d\theta + \alpha \int_0^T \int_\Theta f^\alpha (\underline{v} \cdot \nabla_{\underline{\xi}} \phi - \underline{E} \cdot \nabla_{\underline{v}} \phi)\, dt\, d\theta$$
$$+ \int_0^T \int_\Theta f^\alpha \left(v_x \frac{\partial}{\partial x}\phi - (\underline{v} \times B) \cdot \nabla_{\underline{v}} \phi \right) dt\, d\theta + \alpha^2 \int_\Theta f_I \phi|_{t=0}\, d\theta = 0. \tag{4.4.2}$$

Hence, when $\alpha \to 0$ in (4.4.1), using the fact that f^α is bounded in $L^\infty(0, T, L^2(\Theta))$, we get:

$$\int_0^T \int_\Theta f^0 \left(v_x \frac{\partial}{\partial x}\phi - (\underline{v} \times B) \cdot \nabla_{\underline{v}} \phi \right) dt\, d\theta = 0. \tag{4.4.2}$$

This is equivalent to saying that f^0 is a distributional solution of the problem

$$\mathcal{A}^0 f^0 = 0\,, \quad f^0|_\Gamma = q, \tag{4.4.2}$$

where $\mathcal{A}^0$ is defined by:

$$\mathcal{A}^0 f = v_x \frac{\partial f}{\partial x} - (\underline{v} \times B) \cdot \nabla_{\underline{v}} f, \tag{4.4.2}$$

and $q = q(x, \xi, |v|, t)$ is independent of ω, as a consequence of (4.4.1).

To solve problem $\mathcal{A}^0 f = 0$, we first note that $\mathcal{A}^0$ operates only on the variables $(x, \omega) \in [0,1] \times \mathbb{S}^2$, and that $\underline{\xi} \in \mathbb{R}^2$ and $|v| \geq 0$ are mere parameters. Indeed, we can write:

$$\mathcal{A}^0 f = |v| \omega_x \frac{\partial f}{\partial x} + B(\underline{\xi}) \frac{\partial f}{\partial \omega}(e_x \times \omega) \tag{4.4.2}$$

where $\dfrac{\partial f}{\partial \omega}(e_x \times \omega)$ is the differential of f with respect to $\omega \in \mathbb{S}^2$ acting on the tangent vector to $\mathbb{S}^2$, $e_x \times \omega$.

We therefore only consider the dependence of f on $(x, \omega) \in [0,1] \times \mathbb{S}^2$. Note that $\mathbb{S}^2$ can be parameterized by the map $\omega(\sigma, \underline{\omega})$, where $\sigma = \omega_x/|\omega_x| \in \{-1, 1\}$ and $\underline{\omega} = (\omega_y, \omega_z)$. The fact that σ is equal to ± 1 underlines that we need two maps to parameterize the sphere in this way.

Next, we note $R^+_{(x,\sigma)}(\underline{\omega})$, the rotation of $\underline{\omega}$ about the x-axis of an angle bx, where

$$b = \frac{B(\underline{\xi})}{|v|\omega_x}\ , \quad \omega_x = \sigma\sqrt{1 - \omega_y^2 - \omega_z^2}.$$

In other words, $\underline{\omega}^\dagger = R^+_{(x,\sigma)}(\underline{\omega}) = (\omega_y^\dagger, \omega_z^\dagger)$ is given by:

$$\begin{cases} \omega_y^\dagger = \omega_y \cos bx - \omega_z \sin bx \\ \\ \omega_z^\dagger = \omega_y \sin bx + \omega_z \cos bx \end{cases} \tag{4.4.2}$$

We note that $\underline{\omega}^\dagger$ also depends on $|v|$ and $\underline{\xi}$, but we shall not stress this dependence otherwise needed.

Similarly, $R^-_{(x,\sigma)}(\underline{\omega})$ is the rotation of an angle $-bx$. Obviously, $\underline{\omega}^\dagger = R^+_{(x,\sigma)}(\underline{\omega})$ if and only if $\underline{\omega} = R^-_{(x,\sigma)}(\underline{\omega}^\dagger)$. Also, $R^-_{(x',\sigma)} R^+_{(x,\sigma)} = R^+_{(x-x',\sigma)} = R^-_{(x'-x,\sigma)}$. Then, with the change of unknowns:

$$f^\dagger(x, \sigma, \underline{\omega}) = f(x, \sigma, R^+_{(x,\sigma)}(\underline{\omega})), \tag{4.4.2}$$

a simple application of the chain rule gives:

$$|v|\omega_x \frac{\partial f^\dagger}{\partial x} = v_x \frac{\partial f}{\partial x} - (\underline{v} \times B) \cdot \nabla_{\underline{v}} f = \mathcal{A}^0 f = 0, \tag{4.4.2}$$

which means that $f^\dagger$ is constant with respect to x. Note that in obtaining (4.4.1), we have used that $\omega_x = \sigma|\omega_x|$ and $|\omega_x| = \sqrt{1 - |\underline{\omega}|^2} = \sqrt{1 - |\underline{\omega}^\dagger|^2}$ with $|\underline{\omega}|^2 = |\omega_y|^2 + |\omega_z|^2$. Therefore,

$$f^\dagger(x, \sigma, \underline{\omega}) = f^\dagger(0, \sigma, \underline{\omega}) = f^\dagger(1, \sigma, \underline{\omega}) \tag{4.4.2}$$

which, back to f, gives:

$$f(x, \sigma, \underline{\omega}) = f(0, \sigma, R^-_{(x,\sigma)}(\underline{\omega})) = f(1, \sigma, R^+_{(1-x,\sigma)}(\underline{\omega})). \tag{4.4.2}$$

Now, let f^0 be a solution of (4.4.1). Then, there exists $q(x)$, $x = 0, 1$, such that,

$$f^0(0, \sigma, \underline{\omega}) = q(0),$$

$$f^0(1, \sigma, \underline{\omega}) = q(1).$$

From (4.4.1), this implies that

$$f^0(x, \sigma, \underline{\omega}) = q(0) = q(1) = q.$$

Writing again the full set of variables, we get:

$$f^0(x, v, t) = q(\underline{\xi}, |v|, t) \tag{4.4.2}$$

which was the result to be proved. ■

Remark 4.4.1 The trace estimates (4.4.2) and (4.4.1) were essential to establish that the traces of the limit function f^0 converge in an L^2 sense. In [2], the convergence of the traces is in a weaker sense and it is not clear why the limit trace should satisfy the boundary operator.

Formula (4.4.1) implies that:

$$\begin{aligned} f_+(1, \sigma, \underline{\omega}) &= f_-(0, \sigma, R^-_{(1,\sigma)}(\underline{\omega})), \quad \sigma = 1, \\ f_+(0, \sigma, \underline{\omega}) &= f_-(1, \sigma, R^+_{(1,\sigma)}(\underline{\omega})), \quad \sigma = -1, \end{aligned} \tag{4.4.2}$$

which can be written compactly $f_+ = M_- f_-$, thus defining a map M_- from $L^2(\mathcal{S}_-)$ to $L^2(\mathcal{S}_+)$. The inverse map M_+ from $L^2(\mathcal{S}_+)$ to $L^2(\mathcal{S}_-)$ is defined by $f_- = M_+ f_+$ with:

$$\begin{aligned} f_-(0, \sigma, \underline{\omega}) &= f_+(1, \sigma, R^+_{(1,\sigma)}(\underline{\omega})), \quad \sigma = 1, \\ f_-(1, \sigma, \underline{\omega}) &= f_+(0, \sigma, R^-_{(1,\sigma)}(\underline{\omega})), \quad \sigma = -1, \end{aligned} \tag{4.4.2}$$

Remark that M_- and M_+ are isometrics between the spaces $L^2(\mathcal{S}_-)$ and $L^2(\mathcal{S}_+)$. Formulas (4.4.1) and (4.4.1) are illustrated on Figure 4.1. The mappings M_- and M_+ will be used below. We also define the kinetic energy $\varepsilon = |v|^2/2$ and the energy distribution function $F(\underline{\xi}, \varepsilon, t) = f^0(\underline{\xi}, |v|, t)$.

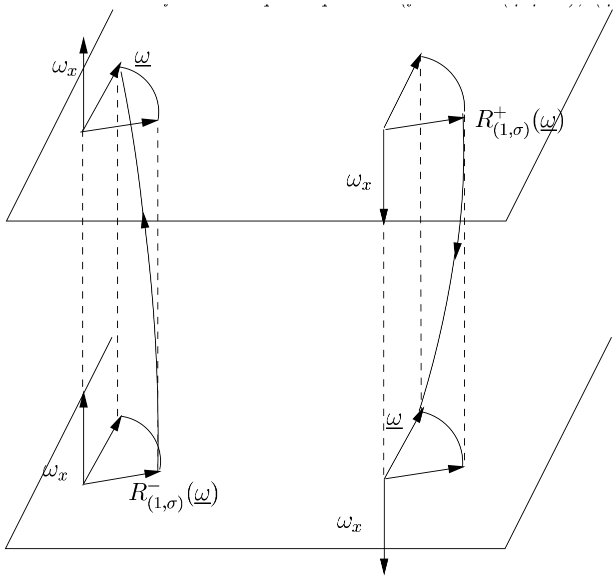

FIGURE 4.1. Action of the transport operator (formulae (4.4.1), (4.4.1))

4.4.2 Auxiliary equation

We now study the following auxiliary problem, which will be useful to evaluate the limit of the current $\underline{J}^\alpha$ given by (4.4.4). Find $f(x,\omega)$ such that:

$$\mathcal{A}^{0*} f(x,\omega) = g(x,\omega)\,,\ (x,\omega) \in [0,1] \times \mathbb{S}^2 \tag{4.4.3}$$

$$f_+(x,\omega) = \mathcal{B}^* f_-(x,\omega)\,,\ (x,\omega) \in \mathcal{S}_+, \tag{4.4.4}$$

where $\mathcal{A}^{0*}$ is the formal adjoint of $\mathcal{A}^0$:

$$\mathcal{A}^{0*} f = -v_x \frac{\partial f}{\partial x} + (\underline{v} \times B)\nabla_{\underline{v}} f = -\left(|v|\omega_x \frac{\partial f}{\partial x} + B(\underline{\xi}) \frac{\partial f}{\partial \omega}(e_x \times \omega) \right) .$$

We recall that both $\mathcal{A}^{0*}$ and $\mathcal{A}^0$ operate only on $(x,\omega) \in [0,1] \times \mathbb{S}^2$, and not on $\underline{\xi} \in \mathbb{R}^2$, $|v| > 0$ (see (4.4.1)), which are therefore omitted in the following discussion. Similarly, $\mathcal{B}^*$ only operate on $\omega \in \mathbb{S}^2$. We have:

Lemma 4.4.4 *Let $g \in L^2([0,1] \times \mathbb{S}^2)$ and let $G = G(x,\sigma,\underline{\omega})$ be defined by:*

$$G(x,\sigma,\underline{\omega}) = \begin{cases} \dfrac{1}{|v|\,|\omega_x|} \displaystyle\int_x^1 g(x',\sigma,R^+_{(x'-x,\sigma)}(\underline{\omega}))\,dx'\ , & \sigma = +1, \\[2ex] \dfrac{1}{|v|\,|\omega_x|} \displaystyle\int_0^x g(x',\sigma,R^-_{(x-x',\sigma)}(\underline{\omega}))\,dx'\ , & \sigma = -1. \end{cases} \tag{4.4.4}$$

Note that G also depends on $|v|$ and $\underline{\xi}$. Suppose that $\sqrt{|\omega_x|}G$ belongs to $L^2([0,1]\times\mathbb{S}^2)$ and that its trace G_- on Γ_- belongs to $L^2(\mathcal{S}_-)$ for almost every $(|v|,\underline{\xi}) \in \mathbb{R}_+ \times \mathbb{R}^2_{\underline{\xi}}$. Then, problem (4.4.3), (4.4.4) has a solution f such that $\sqrt{|\omega_x|}f \in L^2([0,1]\times\mathbb{S}^2)$ for almost every $(|v|,\underline{\xi}) \in \mathbb{R}_+ \times \mathbb{R}^2_{\underline{\xi}}$ if and only if the condition:

$$\int_0^1 \int_{\mathbb{S}^2} g(x,\omega)\, dx\, d\omega = 0 \tag{4.4.4}$$

holds. Furthermore, all solutions in this space are equal to f, up to an additive function of $\underline{\xi}$ and $|v|$.

Proof:

Using the change of variables (4.4.1), we are led to:

$$|v|\omega_x \frac{\partial f^\dagger}{\partial x} = -g^\dagger. \tag{4.4.4}$$

Integrating (4.4.2) with respect to x, we obtain:

$$f^\dagger(x,\sigma,\underline{\omega}^\dagger) = \begin{cases} f^\dagger(1,\sigma,\underline{\omega}^\dagger) + \dfrac{1}{|v|\omega_x}\displaystyle\int_x^1 g^\dagger(x',\sigma,\underline{\omega}^\dagger)\, dx'\ , & \sigma = +1, \\ \\ f^\dagger(0,\sigma,\underline{\omega}^\dagger) + \dfrac{-1}{|v|\omega_x}\displaystyle\int_0^x g^\dagger(x',\sigma,\underline{\omega}^\dagger)\, dx'\ , & \sigma = -1. \end{cases}\ , \tag{4.4.4}$$

With the original variables, this gives:

$$f(x,\sigma,\underline{\omega}) = \begin{cases} f_+(1,\sigma,R^+_{(1-x,\sigma)}(\underline{\omega})) + G(x,\sigma,\underline{\omega})\ , & \sigma = +1, \\ f_+(0,\sigma,R^-_{(x,\sigma)}(\underline{\omega})) + G(x,\sigma,\underline{\omega})\ , & \sigma = -1. \end{cases} \tag{4.4.4}$$

If we find f_+ in $L^2(\mathcal{S}_+)$ such that the f given by (4.4.2) satisfies the boundary conditions (4.4.4), then f is a solution of problem (4.4.3), (4.4.4) with $\sqrt{|\omega_x|}f \in L^2([0,1]\times\mathbb{S}^2)$. This follows from the assumption on G, and from the fact that $R^+(\underline{\omega})$ is a rotation of $\underline{\omega}$, and gives:

$$\int_{|\underline{\omega}|\le 1} |f_+(1,\sigma,R^+_{(1-x,\sigma)}(\underline{\omega}))|^2\, d\underline{\omega} = \int_{|\underline{\omega}|\le 1} |f_+(1,\sigma,\underline{\omega})|^2\, d\underline{\omega}$$

(Note that, by the parameterization $\omega = (\sigma,\underline{\omega})$, we have $|\omega_x|\, d\omega = d\underline{\omega}$).

Next, we show that there actually exists such an $f_+ \in L^2(\mathcal{S}_+)$. We also show that f_+ is unique up to an additive constant, which proves the last statement of the lemma, since $\underline{\xi}$ and $|v|$ are mere parameters throughout this proof. Evaluating (4.4.2) at $x = 0, 1$ we can write $f_- = M_+ f_+ + G_-$ where G_- is the trace of G on Γ_-. Thus, by means of the boundary condition (4.4.4) we have:

$$f_+ = \mathcal{B}^*(M_+ f_+ + G_-). \tag{4.4.4}$$

Using the same notations as in the proof of Lemma 4.2.2, (4.4.2) can be written:

$$\begin{array}{c} f_+(1) - \mathcal{B}^*(1)M_+(f_+(0)) = \mathcal{B}^*(1)G_-(1), \\ f_+(0) - \mathcal{B}^*(0)M_+(f_+(1)) = \mathcal{B}^*(0)G_-(0), \end{array}, \tag{4.4.4}$$

and so we have

$$\begin{array}{c} f_+(1) - (\mathcal{B}^*(1)M_+\mathcal{B}^*(0)M_+)(f_+(1)) = h(1), \\ f_+(0) - (\mathcal{B}^*(0)M_+\mathcal{B}^*(1)M_+)(f_+(0)) = h(0), \end{array}, \tag{4.4.4}$$

with

$$\begin{array}{c} h(1) = \mathcal{B}^*(1)M_+\mathcal{B}^*(0)G_-(0) + \mathcal{B}^*(1)G_-(1), \\ h(0) = \mathcal{B}^*(0)M_+\mathcal{B}^*(1)G_-(1) + \mathcal{B}^*(0)G_-(0). \end{array}.$$

We consider now on the first of (4.4.2), the treatment of the second one being similar. Using the expression (4.2) of $\mathcal{B}^*$, we can write:

$$\mathcal{B}^*(1)M_+\mathcal{B}^*(0)M_+ = \beta(0)\beta(1)(J^*M_+)^2 + \mathcal{L}(1), \tag{4.4.4}$$

where $\mathcal{L}(1)$ is a compact operator on $L^2(\mathcal{S}_+(1))$. Moreover, equation (4.4.2) can be written

$$(I - \mathcal{G}(1))(f_+(1)) = k(1), \tag{4.4.4}$$

where

$$\mathcal{G}(1) \quad = \quad (I - \beta(0)\beta(1)(J^*M_+)^2)^{-1}\mathcal{L}(1), \tag{4.4.5}$$

$$k(1) \quad = \quad (I - \beta(0)\beta(1)(J^*M_+)^2)^{-1}h(1). \tag{4.4.6}$$

It is not difficult to check that $\mathcal{G}(1)$ is a compact operator on $L^2(\mathcal{S}_+(1))$, which is positive (i.e. if $\phi \in L^2(\mathcal{S}_+(1))$, $\phi \geq 0$, then $\mathcal{G}(1)\phi > 0$) and such that the constant functions on $\mathcal{S}_+(1)$ are eigenfunction associated with the eigenvalue 1. Therefore, by the Krein-Rutman Theorem, the Null-Space $N(I - \mathcal{G}(1))$ is of dimension 1.

By the Fredholm Alternative, equation (4.4.2) has a solution f_+ in $L^2(\mathcal{S}_+(1))$ if and only if $k(1) \in N(I - \mathcal{G}^*(1))^\perp$. It is not difficult to check that $N(I - \mathcal{G}^*(1))$

also consists of constant functions. Therefore, the solvability condition for equation (4.4.2) reads $(k(1), 1)_{L^2(\mathcal{S}_+(1))} = 0$, or:

$$(h(1), (I - \beta(0)\beta(1)(M_-J)^2)^{-1}1)_{L^2(\mathcal{S}_+(1))} = 0. \tag{4.4.6}$$

But, since $(I - \beta(0)\beta(1)(M_-J)^2)1 = 1 - \beta(0)\beta(1)$, condition (4.4.2) is also written $(h(1), 1)_{L^2(\mathcal{S}_+(1))} = 0$, or, by duality,

$$(G_-(0), \mathcal{B}(0)M_-\mathcal{B}(1)1)_{L^2(\mathcal{S}_-(0))} + (G_-(1), \mathcal{B}(1)1)_{L^2(\mathcal{S}_-(1))} = 0. \tag{4.4.6}$$

But, $\mathcal{B}(0)M_-\mathcal{B}(1)1 = 1$ and $\mathcal{B}(1)1 = 1$. So (4.4.2) gives:

$$(G_-(0), 1)_{L^2(\mathcal{S}_-(0))} + (G_-(1), 1)_{L^2(\mathcal{S}_-(1))} = 0, \tag{4.4.6}$$

which is written explicitly:

$$\int_0^1 \int_{|\underline{\omega}|\le 1} \Big(g(x, 1, R^+_{(x,1)}(\underline{\omega})) + g(x, -1, R^-_{(1-x,-1)}(\underline{\omega}))\Big) \frac{1}{|\omega_x|}\, d\underline{\omega}\, dx = 0. \tag{4.4.6}$$

Using the change of variables $\underline{\omega}' = R^+_{(x,1)}(\underline{\omega})$ for the first term and $\underline{\omega}' = R^-_{(1-x,-1)}(\underline{\omega})$ for the second one, condition (4.4.2) reads:

$$\int_0^1 \int_{|\underline{\omega}|\le 1} (g(x, 1, \underline{\omega}) + g(x, -1, \underline{\omega})) \frac{1}{|\omega_x|}\, d\underline{\omega}\, dx = 0 \tag{4.4.6}$$

which is exactly the expression of the solvability condition (4.4.4) written by means of the parameterization $\omega(\sigma, \underline{\omega})$.

The existence of $f_+(1)$ and its uniqueness up to an additive constant (i.e. belonging to the Null-Space of $(I - \mathcal{G}(1))$) are thus proved, under the solvabilty condition (4.4.4). Similarly, the existence of $f_+(0)$ and its uniqueness, up to an additive constant, are proved under the same condition. That the two arbitrary constants for $f_+(1)$ and $f_+(0)$ are equal follows easily by considering equations (4.4.2). ■

Remark 4.4.2 The present technique, based on the reduction to an integral equation at the boundary, is mainly due to [2]. However, we note that the regularity $L^2(\mathcal{S}_-)$ of the trace does not imply the regularity $L^2([0, 1] \times \mathbb{S}^2)$ for the function inside the domain because of the weight $|\omega_x|$. This point seems to have been overlooked in [2] and is connected to the question of characterizing the range of the trace operator for solutions of first order problems, which is still mainly open up to now.

Lemma 4.4.5 *The functions $g = \omega_i$, $(i = y, z)$ satisfy the assumptions of Lemma 4.4.4.*

Proof:
Let G_y be the function associated to $g = \omega_y$ by (4.4.4). We need to show that $\sqrt{|\omega_x|}G_y \in L^2([0,1] \times \mathbb{S}^2)$ and $(G_y)_- \in L^2(\mathcal{S}_-)$. The proof is obviously similar for $g = \omega_z$. After straightforward computations, we obtain

$$G_y(x, \sigma, \underline{\omega}) = \begin{cases} \dfrac{1}{B(\underline{\xi})}[\omega_y \sin b(1-x) + \omega_z(\cos b(1-x) - 1)]\, dx' \ , \ \sigma = +1, \\ \\ -\dfrac{1}{B(\underline{\xi})}[\omega_y \sin bx + \omega_z(1 - \cos bx)]\, dx' \ , \ \sigma = -1. \end{cases} ,$$

With the Hypothesis 4.3.1 (iii) on B, it is not difficult to check that G_y satisfies the required hypothesis. ■

We note that the magnetic field operator has removed the singularity $|\omega_x|^{-1}$ that would otherwise be expected. The magnetic field thus contributes to maintain a finite diffusivity (see the introduction).

By Lemma 4.4.4, there exist functions $D_i(x, \omega; \underline{\xi}, \varepsilon)$, $(i = y, z)$, solutions of problem (4.4.3), (4.4.4) with right-hand-side $g = \omega_i$, unique up to additive functions of $\underline{\xi}$ and ε. We suppose that D_i satisfy the following regularity requirements:

Hypothesis 4.4.2 *(i) D_i, $(i = y, z)$, belong to $L^2([0,1] \times \mathbb{S}^2)$ for almost every $(\underline{\xi}, \varepsilon) \in \mathbb{R}^2_{\underline{\xi}} \times \mathbb{R}^+_\varepsilon$ and are C^1 bounded functions on Θ away from the set $\{v_x = 0\}$.*
(ii) The functions $\omega_i D_j(x, \omega; \underline{\xi}, \varepsilon)$ belongs to $L^1([0,1] \times \mathbb{S}^2)$ and $\int_0^1 \int_{\mathbb{S}^2} \omega_i D_j \, dx \, d\omega$ is a C^1 function of $(\underline{\xi}, \varepsilon) \in \mathbb{R}^2_{\underline{\xi}} \times (0, \infty)_\varepsilon$, uniformly bounded on $\mathbb{R}^2_{\underline{\xi}} \times [0, \infty)_\varepsilon$, and tending to 0 as $\varepsilon \to 0$.

Hypothesis 4.4.2 can be viewed as a regularity assumption on the data i.e. on the magnetic field B, the boundary scattering kernel K and the accommodation coefficient β. We do not look for explicit conditions on these data because the developments would be technical and of rather limited interest. We only remark that hypothesis 4.4.2 is not empty, because it is satisfied at least in the case of isotropic scattering. In this case, (see [26]), the scattering kernel K is equal to the constant π^{-1}, and we have for $\omega_x > 0$:

$$\begin{aligned} D_z &= \frac{1}{4\pi B}\,\omega_y^\dagger\left(\frac{1-\beta_0}{1-\beta_0\beta_1}(1-\cos b)+\cos b-\cos bx\right) \\ &\quad+\frac{1}{4\pi B}\,\omega_z^\dagger\left(\frac{1+\beta_0}{1-\beta_0\beta_1}\sin b-\sin b+\sin bx\right), \end{aligned}$$

with $\underline{\omega}^\dagger = R^-_{(x,\sigma)}(\underline{\omega})$ and similarly for $\omega_x < 0$, and for D_y. From hypothesis 4.3.1 (ii), (iii) and hypothesis 4.2.3 (ii), $\underline{D}$ is bounded and thus , $\underline{\omega}\,\underline{D}$ is clearly integrable in (x,ω). Furthermore, formula (4.34) and (4.35) of [26] shows that $\int_0^1\int_{\mathbb{S}^2}\underline{\omega}\,\underline{D}\,dx\,d\omega$ is a C^1 function of $\underline{\xi}$, ε for $\underline{\xi}\in\mathbb{R}^2$ and $\varepsilon > 0$, as soon as B and the accommodation coefficient β are C^1 and that it is bounded and tends to 0 as $\varepsilon\to 0$. Finally, away from the plane $\{v_x = 0\}$, all derivative of D_z are smooth, showing that Hypothesis 4.4.2 is satisfied.

From Hypothesis 4.4.2 (ii), we deduce that the diffusivity tensor (4.1.1) is defined and is a C^1 function of $(\underline{\xi},\varepsilon)\in\mathbb{R}^2_{\underline{\xi}}\times\mathbb{R}^+_\varepsilon$. We also note that the definition of D_{ij} does not depend on the arbitrary additive function of $\underline{\xi}$ and ε which enters in the definition of D_j.

4.4.3 The current equation

Let us define the current $\underline{J}^\alpha(\underline{\xi},\varepsilon,t) = (J^\alpha_y, J^\alpha_z)$ as follows:

$$\begin{aligned} \underline{J}^\alpha(\underline{\xi},\varepsilon,t) &= \frac{|v|}{\alpha}\int_0^1\int_{\mathbb{S}^2}\underline{v}f^\alpha(x,\underline{\xi},|v|,\omega,t)\,dx\,d\omega \\ &= \frac{2\varepsilon}{\alpha}\int_0^1\int_{\mathbb{S}^2}\underline{\omega}f^\alpha(x,\underline{\xi},\varepsilon,\omega,t)\,dx\,d\omega. \end{aligned} \tag{4.4.4}$$

We first prove the following technical lemma:

Lemma 4.4.6 *Let $\varphi(x,v)$ be a C^1 function. We have:*

$$\begin{aligned} &\sqrt{2\varepsilon}\int_0^1\int_{\mathbb{S}^2}(\nabla_{\underline{v}}\varphi)(x,\sqrt{2\varepsilon}\omega)\,dx\,d\omega = \frac{\partial}{\partial\varepsilon}J_\varphi, \\ &J_\varphi(\varepsilon) = 2\varepsilon\int_0^1\int_{\mathbb{S}^2}\underline{\omega}\,\varphi(x,\sqrt{2\varepsilon}\omega)\,dx\,d\omega \end{aligned} \tag{4.4.4}$$

Proof:

We only compute the y component. We note that

$$\frac{\partial\varphi}{\partial v_y} = \frac{\partial\varphi}{\partial|v|}\omega_y + \frac{1}{|v|}\frac{\partial\varphi}{\partial\omega}(e_y - \omega_y\omega), \tag{4.4.4}$$

where $(\partial\varphi/\partial\omega)(e_y - \omega_y\omega)$ denotes the derivative of φ with respect to ω acting on the tangent vector $e_y - \omega_y\omega$ Then, integrating (4.4.3) with respect to x and ω and using that

$$\int_{\mathbb{S}^2} \frac{\partial\varphi}{\partial\omega}(e_y - \omega_y\omega)\, d\omega = -\int_{\mathbb{S}^2} \varphi(\omega)\, \mathrm{div}_{\mathbb{S}^2}\,(e_y - \omega_y\omega)\, d\omega = -\int_{\mathbb{S}^2} \varphi(\omega)(-2\omega_y)\, d\omega,$$

which is deduced from Stoke's Theorem on the sphere, we have:

$$\sqrt{2\varepsilon}\int_0^1\int_{S^2} \frac{\partial\varphi}{\partial v_y}\, dx\, d\omega =$$

$$= \sqrt{2\varepsilon}\left(\int_0^1\int_{S^2} \frac{\partial\varphi}{\partial|v|}\omega_y dx\, d\omega + \int_0^1\int_{S^2} \frac{1}{|v|}\frac{\partial\varphi}{\partial\omega}(e_y - \omega_y\omega) dx\, d\omega\right)$$

$$= \sqrt{2\varepsilon}\left(\frac{\partial}{\partial|v|}\int_0^1\int_{S^2} \omega_y\varphi\, dx\, d\omega + \frac{2}{|v|}\int_0^1\int_{S^2} \varphi\omega_y\, dx\, d\omega\right)$$

$$= \sqrt{2\varepsilon}\left(\sqrt{2\varepsilon}\frac{\partial}{\partial\varepsilon}\int_0^1\int_{S^2} \omega_y\varphi\, dx\, d\omega + \frac{2}{\sqrt{2\varepsilon}}\int_0^1\int_{S^2} \varphi\omega_y\, dx\, d\omega\right)$$

$$= \left(2\varepsilon\frac{\partial}{\partial\varepsilon}\int_0^1\int_{S^2} \omega_y\varepsilon\, dx\, d\omega + 2\int_0^1\int_{S^2} \varepsilon\omega_y\, dx\, d\omega\right) = \frac{\partial J_\varphi}{\partial\varepsilon}.$$

■

We denote by Θ' the position-energy space $\Theta' = \mathbb{R}^2_{\underline{\xi}} \times \mathbb{R}^+_\varepsilon$ and by $d\theta'$ its volume element $d\theta' = d\underline{\xi}\, d\varepsilon$. We note that $dv = |v|^2 d|v|\, d\omega = \sqrt{2\varepsilon}\, d\varepsilon\, d\omega$.

Lemma 4.4.7 *As α approaches 0, the current $\underline{J}^\alpha(\underline{\xi}, \varepsilon, t)$ converges in the sense of distribution to $\underline{J}(\underline{\xi}, \varepsilon, t)$ given by (4.1.2). More precisely, for every $\underline{\psi} = (\psi_y, \psi_z) \in C^1(\Theta' \times [0, T], \mathbb{R}^2)$ with compact support in $\mathbb{R}^2_{\underline{\xi}} \times (0, \infty)_\varepsilon \times [0, T[$, we have:*

$$\lim_{\alpha\to 0}\int_0^T\int_{\Theta'} \underline{J}^\alpha \cdot \underline{\psi}\, d\theta'\, dt = \int_0^T\int_{\Theta'} F\left(\nabla_{\underline{\xi}} - \underline{E}\frac{\partial}{\partial\varepsilon}\right) \cdot (\mathbb{D}^T\underline{\psi})\, dt\, d\theta', \tag{4.4.4}$$

where $\mathbb{D}^T$ denotes the transpose of $\mathbb{D}$.

We note that the right-hand side of equation (4.4.7) is the weak form of that of equation (4.1.2).

Proof:

We use the weak formulation (4.4.2) with $\phi = \sqrt{2\varepsilon}\underline{\psi}(\underline{\xi}, \varepsilon, t) \cdot \underline{D}(x, \omega; \underline{\xi}, \varepsilon)\ \chi_\rho(v_x)$ for test function. Since $\underline{D}$ is not smooth at $v_x = 0$, we use the truncation function χ_ρ

which is smooth (see Lemma 4.3.2), vanishes identically for $|v_x| \le \rho$ and is equal to 1 for $|v_x| \ge 2\rho$. Hypothesis 4.4.2 provides all the necessary assumptions to allow the passage $\rho \to 0$ in (4.4.2). Therefore, this passage will be omitted in the following proof and we just use $\underline{\psi} \cdot \underline{D}$ as a test function as if it were smooth. Because of (4.4.3), (4.4.4), we have for $i \in \{y, z\}$:

$$\left(v_x \frac{\partial}{\partial x} - (\underline{v} \times B) \cdot \nabla_{\underline{v}} \right) (\sqrt{2\varepsilon}\psi_i \, D_i) = \sqrt{2\varepsilon}\psi_i \left(v_x \frac{\partial}{\partial x} - (\underline{v} \times B) \cdot \nabla_{\underline{v}} \right) (D_i)$$

$$= -\sqrt{2\varepsilon}\psi_i \, \omega_i \, , \quad \text{in } \Theta,$$

and

$$(\sqrt{\varepsilon}\psi_i \, D_i)_+ - \mathcal{B}^*(\sqrt{\varepsilon}\psi_i \, D_i)_- = \sqrt{\varepsilon}\psi_i[(D_i)_+ - \mathcal{B}^*(D_i)_-] = 0 \, , \quad \text{on } \Gamma \, .$$

Therefore:

$$\frac{1}{\alpha} \int_0^T \int_\Theta f^\alpha \left(v_x \frac{\partial}{\partial x} - (\underline{v} \times B) \cdot \nabla_{\underline{v}} \right) (\sqrt{2\varepsilon}\underline{\psi} \cdot \underline{D}) \, dt \, d\theta$$

$$-\frac{1}{\alpha} \int_0^T \int_{\Gamma_+} |v_x| \, f^\alpha (\, (\sqrt{2\varepsilon}\underline{\psi} \cdot \underline{D})_+ - \mathcal{B}^*(\sqrt{2\varepsilon}\underline{\psi} \cdot \underline{D})_- \, dt \, d\Gamma$$

$$= -\frac{1}{\alpha} \int_0^T \int_\Theta f^\alpha \, \underline{\omega} \cdot \underline{\psi}\sqrt{2\varepsilon} \, dt \, d\theta = -\int_0^T \int_{\Theta'} \underline{J}^\alpha \cdot \underline{\psi} \, dt \, d\theta' \, ,$$

which a posteriori justifies the introduction of the auxiliary function $\underline{D}$. Thus, the weak formulation (4.4.2) yields:

$$\int_0^T \int_{\Theta'} \underline{J}^\alpha \cdot \underline{\psi} \, dt \, d\theta' = \alpha \int_0^T \int_\Theta \sqrt{2\varepsilon} f^\alpha \underline{D} \cdot \frac{\partial}{\partial t} \underline{\psi} \, dt \, d\theta$$
$$+\alpha \int_\Theta \sqrt{2\varepsilon} f_I \underline{D} \cdot \underline{\psi}|_{t=0} \, d\theta + \int_0^T \int_\Theta f^\alpha (\underline{v} \cdot \nabla_{\underline{\xi}} - \underline{E} \cdot \nabla_{\underline{v}})(\sqrt{2\varepsilon}\underline{D} \cdot \underline{\psi}) \, dt \, d\theta. \tag{4.4.4}$$

Now, we let α tend to 0. Because $\underline{D} \in L^2([0,1] \times \mathbb{S}^2)$, for almost every $(\underline{\xi}, \varepsilon) \in \mathbb{R}^2_{\underline{\xi}} \times \mathbb{R}^+_\varepsilon$, the first and second terms on the right hand side of (4.4.3) converge to 0. Now, the limit of the last term exists and we have, because of Lemma 4.4.3:

$$\lim_{\alpha \to 0} \int_0^T \int_\Theta f^\alpha \underline{v} \cdot \nabla_{\underline{\xi}} (\sqrt{2\varepsilon}\underline{D} \cdot \underline{\psi}) \, dt \, d\theta = \int_0^T \int_\Theta f^0 \nabla_{\underline{\xi}} \cdot [2\varepsilon \underline{\omega}(\underline{D} \cdot \underline{\psi})] \, dt \, d\theta$$

$$= \int_0^T \int_{\Theta'} F \nabla_{\underline{\xi}} \cdot \left((2\varepsilon)^{3/2} \int_0^1 \int_{\mathbb{S}^2} \underline{\omega}(\underline{D} \cdot \underline{\psi})\, dx\, d\omega \right) dt\, d\theta'$$

$$= \int_0^T \int_{\Theta'} F \nabla_{\underline{\xi}} \cdot [\mathbb{D}^T \underline{\psi}]\, dt\, d\theta', \qquad (4.4.4)$$

Similarly, using (4.4.6), we have:

$$\lim_{\alpha \to 0} \int_0^T \int_{\Theta} f^\alpha (\underline{E} \cdot \nabla_{\underline{v}})(\sqrt{2\varepsilon}\underline{D} \cdot \underline{\psi})\, dt\, d\theta = \int_0^T \int_{\Theta} f^0 (\underline{E} \cdot \nabla_{\underline{v}})(\sqrt{2\varepsilon}\underline{D} \cdot \underline{\psi})\, dt\, d\theta$$

$$= \int_0^T \int_{\Theta'} F \underline{E} \cdot \left(\sqrt{2\varepsilon} \int_0^1 \int_{\mathbb{S}^2} \nabla_{\underline{v}} (\sqrt{2\varepsilon}\underline{D} \cdot \underline{\psi})\, dx\, d\omega \right) dt\, d\theta'$$

$$= \int_0^T \int_{\Theta'} F \underline{E} \cdot \frac{\partial}{\partial \varepsilon} \left((2\varepsilon)^{3/2} \int_0^1 \int_{\mathbb{S}^2} \underline{\omega}(\underline{D} \cdot \underline{\psi})\, dx\, d\omega \right) dt\, d\theta'$$

$$= \int_0^T \int_{\Theta'} F \underline{E} \cdot \frac{\partial}{\partial \varepsilon} (\mathbb{D}^T \underline{\psi})\, dt\, d\theta' . \qquad (4.4.4)$$

Lemma 4.4.7 follows by collecting (4.4.3) and (4.4.3). ■

4.4.4 The continuity equation

To complete the proof of Theorem 4.1.1, it remains to prove that equations (4.1.1) and (4.1.3) hold true in the weak sense. This is the object of the following:

Lemma 4.4.8 *For any test function $\psi(\underline{\xi}, \varepsilon, t)$ belonging to $C^2(\Theta' \times [0, T])$, with compact support in $\mathbb{R}^2_{\underline{\xi}} \times (0, \infty)_\varepsilon \times [0, T[$, we have:*

$$\int_0^T \int_{\Theta'} \left(4\pi\sqrt{2\varepsilon} F \frac{\partial \psi}{\partial t} + \underline{J} \cdot \left(\nabla_{\underline{\xi}} - \underline{E} \frac{\partial}{\partial \varepsilon} \right) \psi \right) dt\, d\theta'$$
$$+ \int_{\Theta'} 4\pi\sqrt{2\varepsilon} F_I \psi|_{t=0}\, d\theta' = 0. \qquad (4.4.4)$$

Note that equation (4.4.8) is the weak form of equations (4.1.1) and (4.1.3).

Proof:
We first define the function:

$$F^\alpha(\underline{\xi}, \varepsilon, t) = \frac{1}{4\pi} \int_0^1 \int_{\mathbb{S}^2} f^\alpha(x, \underline{\xi}, |v|, \omega, t) dx\, d\omega, \qquad (4.4.4)$$

which weakly converges to F as α tends to 0. Dividing equation (4.4.2) by α, and using $\phi = \psi(\underline{\xi}, \varepsilon, t)$ as a test function, we obtain the weak form of the continuity equation, which looks exactly similar to (4.4.8) except that F and $\underline{J}$ are replaced by F^α and $\underline{J}^\alpha$. Using the weak convergence of F^α and $\underline{J}^\alpha$ to F and $\underline{J}$ respectively (see Lemma 4.4.7), allows to pass to the limit in the continuity equation for F^α and to obtain equation (4.4.8). ■

4.5 Properties of the diffusivity

In the next two propositions, we prove that the diffusion tensor $\mathbb{D}$ defined by (4.1.1) satisfies the Onsager relation $\mathbb{D}(B)^T = \mathbb{D}(-B)$ and that it is positive definite, i.e. $(\mathbb{D}Y, Y) > 0$.

Proposition 4.5.1 *The diffusion tensor* $\mathbb{D}$ *satisfies* $\mathbb{D}(B)^T = \mathbb{D}(-B)$.

Proof:
For $f(x,\omega)$, we define the transformation $\mathcal{J}f$: $\mathcal{J}f(x,\omega) = f(x,-\omega)$. We make the dependence of $\mathcal{A}^0$ and $\mathbb{D}$ upon B explicit by writing $\mathcal{A}^0(B)$ and $\mathbb{D}(B)$. We begin by noting that $\mathcal{J}f$ is a solution of ${\mathcal{A}^0}^*(-B)\mathcal{J}f = \mathcal{J}g$, $\mathcal{J}f_+ = \mathcal{B}^*\mathcal{J}f_-$ if and only if the function f is solution of $\mathcal{A}^0(B)f = g$, $f_- = \mathcal{B}f_+$. This follows from the reciprocity relation (4.2.1), and from:

$$ {\mathcal{A}^0}^*(-B)\mathcal{J}f = \mathcal{J}\mathcal{A}^0(B)f \quad , \quad \mathcal{A}^0(B)\mathcal{J}f = \mathcal{J}{\mathcal{A}^0}^*(-B)f. \tag{4.5.0} $$

Now, we have (see (4.2) for the definition of $(\cdot,\cdot)_\mathcal{S}$):

$$ ({\mathcal{A}^0}^*(B)D_i(B), \mathcal{J}D_j(-B))_\mathcal{S} = (D_i(B), \mathcal{A}^0(B)\mathcal{J}D_j(-B))_\mathcal{S}. $$

But on the one hand:

$$ \left({\mathcal{A}^0}^*(B)D_i(B), \mathcal{J}D_j(-B)\right)_\mathcal{S} = (\omega_i, \mathcal{J}D_j(-B))_\mathcal{S} = -(\omega_i, D_j(-B))_\mathcal{S} $$

and on the other hand, with (4.5):

$$ \begin{aligned} \big(D_i(B), \mathcal{A}^0(B)\mathcal{J}D_j(-B)\big)_\mathcal{S} &= \left(D_i(B), \mathcal{J}{\mathcal{A}^0}^*(-B)D_j(-B)\right)_\mathcal{S} \\ &= (D_i(B), \mathcal{J}\omega_j)_\mathcal{S} = -(D_i(B), \omega_j)_\mathcal{S} \end{aligned} $$

Therefore, $(D_i(B), \omega_j)_\mathcal{S} = (\omega_i, D_j(-B))_\mathcal{S}$, which is, up to the multiplication by $(2\varepsilon)^{3/2}$, the result to be proved. ■

Proposition 4.5.2 *The diffusion tensor $\mathbb{D}$ is positive definite: more precisely, for all $\underline{\xi} \in \mathbb{R}^2$ and all $\varepsilon_0 > 0$, there exists $C = C(\varepsilon_0) > 0$ such that:*

$$(\mathbb{D}Y, Y) = \sum_{i,j=1}^{2} \mathbb{D}_{ij} Y_i Y_j \geq C|Y|^2 = C \sum_{i=1}^{2} Y_i^2 , \quad \forall\, Y, \underline{\xi} \in \mathbb{R}^2 , \quad \forall \varepsilon \geq \varepsilon_0 \,. \tag{4.5.0}$$

Proof:

Let $Y = (y_1, y_2)$ such that $|Y| > 0$. From the definition of $\mathbb{D}$ we obtain that:

$$(\mathbb{D}Y, Y) = (2\varepsilon)^{3/2} \int_0^1 \int_{\mathbb{S}^2} \left(\sum_{i=1}^{2} \omega_i y_i \right) \left(\sum_{i=1}^{2} D_i y_i \right) dx \, d\omega. \tag{4.5.0}$$

Define $\Phi(x, \omega)$ as follows:

$$\Phi(x, \omega) = \sum_{i=1}^{2} y_i D_i(x, \omega). \tag{4.5.0}$$

Then:

$$\sum_{i=1}^{2} \omega_i y_i = \mathcal{A}^{0^*} \Phi(x, \omega),$$

and equation (4.5) reads:

$$(\mathbb{D}Y, Y) = (2\varepsilon)^{3/2} (\mathcal{A}^{0^*} \Phi, \Phi)_{\mathcal{S}}$$

$$= (2\varepsilon)^{3/2} \left(\int_{\mathcal{S}_-} |\omega_x| \, |\Phi_-|^2 \, d\omega - \int_{\mathcal{S}_+} |\omega_x| \, |\Phi_+|^2 \, d\omega \right) \tag{4.5.0}$$

$$\geq (2\varepsilon_0)^{3/2} \left(|\Phi_-|^2_{L^2(\mathcal{S}_-)} - |\mathcal{B}^* \Phi_-|^2_{L^2(\mathcal{S}_+)} \right) \geq 0,$$

Now, if there exists Y such that $(\mathbb{D}Y, Y) = 0$, the corresponding Φ satisfies:

$$|\mathcal{B}^* \Phi_-|^2_{L^2(\mathcal{S}_+)} = |\Phi_-|^2_{L^2(\mathcal{S}_-)}. \tag{4.5.0}$$

From the transposition of equation (4.2) to the adjoint operator $\mathcal{B}^*$, equation (4.5) is possible only if Φ_- is a constant function of ω, on each connected component of $\mathcal{S}_-$ (i.e. an element of $\mathcal{C}^-$, see section 4.2). Then, $\Phi_- = J\Phi_+$ is the same constant. Denoting by $\Phi_0 = \Phi|_{x=0}$, $\Phi_1 = \Phi|_{x=1}$ and using (4.4.2), we have, for $\omega_x > 0$:

$$\Phi_0 = \Phi_1 + \frac{1}{|v||\omega_x|} \int_0^1 \underline{y} \cdot R^+_{(x',\sigma)}(\underline{\omega}) \, dx'$$

$$= \Phi_1 + \frac{1}{B}[y_1(\omega_1 \sin b + \omega_2(\cos b - 1)) + y_2(-\omega_1(\cos b - 1) + \omega_2 \sin b)].$$

Obviously, for Φ_0 and Φ_1 to be independent of $\underline{\omega}$, the only possibility is that $y_1 = y_2 = 0$, which contradicts the fact that $|Y| > 0$. This ends the proof of (4.5.2). ■

5

Diffusion Driven by Collisions II

In this chapter, we wave the assumption of no collisions of electrons against the neutral molecules of the gas in the ionic thruster model. The electrons mass is much smaller compared with the one of neutral molecules, hence the collision operator which describes the scattering is of Boltzmann type. Moreover, we shall assume that collision are isotrope, i.e. the scattering cross section is constant in x and ω. Our goal is to prove that the drift-diffusion type model derived in the previous chapter can be extended to the collisional case. We shall see how the introduction of collisions does not change the philosophy of the proof bringing only some difficulties in the generalization of the proof of the existence of the solution of the auxiliary problem.

We shall divide the chapter in the following steps. In section 5.1 we introduce the kinetic equation governing our model and write the new scaled version of our model. It turns out that the collision operator appear at the leading order in the scaled equation. The various assumptions and properties of the boundary operator are recalled in section 5.2. In section 5.3, we prove some properties of the collision operator, which will be used in order to generalize the results of the previous chapter. Then, in section 5.4, existence and uniqueness of the solution of the evolution problem are proved. In section 5.5 we show that, to the limit $\alpha \to 0$, the solution of the evolution problem tends towards a function independent of the position variable x and of the velocity direction ω. Moreover,we prove the existence of the solution of the auxiliary problem and we derive the drift-diffusion equation. Finally, in section 5.6 we show that the Onsager relation and the positivity of the diffusion matrix still hold true.

5.1 Model, Scaling and Main Theorem

As previously, we study the evolution of electrons in between two parallel plates and subject to crossed electric and magnetic fields. Electrons are also subject to elastic collision with the neutral molecules of the host medium and are reflected and diffused by the boundaries. The electron distribution function $\hat{f}(\hat{X}, \hat{v}, \hat{t})$ must then satisfy the following Boltzmann equation:

$$\frac{\partial \hat{f}}{\partial \hat{t}} + \hat{v}_x \frac{\partial \hat{f}}{\partial \hat{x}} + \hat{v}_y \frac{\partial \hat{f}}{\partial \hat{y}} + \hat{v}_z \frac{\partial \hat{f}}{\partial \hat{z}} - \frac{q}{m}\left(\hat{E}_y \frac{\partial \hat{f}}{\partial \hat{v}_y} + \hat{E}_z \frac{\partial \hat{f}}{\partial \hat{v}_z}\right)$$
$$-\frac{q}{m}\hat{B}_x\left(\hat{v}_z \frac{\partial \hat{f}}{\partial \hat{v}_y} - \hat{v}_y \frac{\partial \hat{f}}{\partial \hat{v}_z}\right) = \hat{\mathcal{L}}\hat{f}, \tag{5.1.0}$$

where $\hat{X} \in \hat{\Omega} = \{[0, l] \times \mathbb{R}^2\}$, $\hat{v} \in \mathbb{R}^3$ and $-q < 0$ and m are the electron charge and mass. Furthermore, the operator $\hat{\mathcal{L}}$ is the isotropic linear collision operator acting from the space $L^2(\Theta)$ in itself and given by:

$$\hat{\mathcal{L}}(\hat{f})(\hat{x}, \hat{\underline{\xi}}, \hat{v}) = \int_{\mathbb{R}^3} \hat{\phi}_0(\hat{\underline{\xi}}, \hat{v}', \hat{v})\,(\hat{f}' - \hat{f})\,\delta\left(\frac{|\hat{v}'|^2}{2} - \frac{|\hat{v}|^2}{2}\right) d\hat{v}'. \tag{5.1.0}$$

where we recall that the position-velocity set Θ is given by $\{[0, 1] \times \mathbb{R}^2\} \times \mathbb{R}^+ \times S^2$ and where $\hat{f}' = \hat{f}(\hat{x}, \hat{\underline{\xi}}, \hat{v}')$ is the density of those particles which in the collision change the velocity from v' to v. The function δ is a Dirac mass and yields to elastic collisions, i.e. only those particles which in the collision change their velocity direction but not their velocity modulus are taken into account in the integral. The function $\hat{\phi}_0(\hat{\underline{\xi}}, \hat{v}', \hat{v})$ is the probability of an electron to change its velocity from $\hat{v}'$ to $\hat{v}$ and it is the so called scattering cross section. Here we shall consider only isotropic collisions, in other words $\hat{\phi}_0$ will be assumed to be constant with respect to $\hat{x}$, $\hat{\omega}'$ and $\hat{\omega}$.

Recalling that our model is based on the assumption that the distance between the plates is small compared with their dimension, we introduce the scaling of section 4.1 and let $\mathcal{L}_0 = (\alpha^2 t_0)^{-1}$ be a typical scale for the collision operator $\mathcal{L}$, where $\alpha = l/L$ and t_0 is the typical time scale. Hence the adimensional collision operator is given by:

$$\mathcal{L} = \mathcal{L}_0 \hat{\mathcal{L}}.$$

This scaling gives for the transport part of the Boltzmann equation the same adimensional equation of the collision-less case. Moreover, the chosen typical scale for the scattering cross section (or equivalently for the mean free path, or again for the relaxation time) is such that the collision operator will be of the leading order in the adimensional Boltzmann equation:

$$\alpha^2 \frac{\partial f^\alpha}{\partial t} + \alpha(\underline{v} \cdot \nabla_{\underline{\xi}} f^\alpha - \underline{E} \cdot \nabla_{\underline{v}} f^\alpha) \\ + v_x \frac{\partial f^\alpha}{\partial x} - (\underline{v} \times B) \cdot \nabla_{\underline{v}} f^\alpha = \mathcal{L} f^\alpha \tag{5.1.0}$$

where we use the same notations as in section 4.1: $\underline{\xi} = (y, z)$, $\underline{v} = (v_y, v_z)$, $\underline{E} = (E_y, E_z)$ and $(\underline{v} \times B) = (0, v_z B, -v_y B)$. Moreover, $f^\alpha = f^\alpha(X, v, t) = f^\alpha(x, \underline{\xi}, |v|, \omega, t)$ is the distribution function at a time $t > 0$ of the electrons which are in a position $x \in [0, 1], \underline{\xi} \in \mathbb{R}^2$ with velocity modulus $|v| \in \mathbb{R}^+$ and velocity direction $\omega \in S^2$.

The operator $\mathcal{L}$ reads now:

$$\mathcal{L}(f)(x, \underline{\xi}, v) = \int_{\mathbb{R}^3} \phi_0(\underline{\xi}, v', v)\,(f' - f)\,\delta\left(\frac{|v'|^2}{2} - \frac{|v|^2}{2}\right) dv'. \tag{5.1.0}$$

Equation (5.1) is equipped with conservative boundary conditions describing the incoming flux of particles with respect to the outgoing flux. Let us briefly recall the notations for the boundary conditions and boundary operators, already introduced in section 4.1. The boundary condition a $x = \{0, 1\}$ are given by:

$$f_-(X, v) = \mathcal{B} f_+(X, v) = \beta J f_+(X, v) + (1 - \beta)\mathcal{K}(f_+)(X, v), \quad (X, v) \in \Gamma^-, \tag{5.1.0}$$

where the accommodation coefficient $\beta = \beta(x, \underline{\xi}, |v|)$ may depend upon $x(= 0, 1)$, $\underline{\xi}$, $|v|$, and is such that $0 \leq \beta < 1$.

The mirror reflection operator J acting on $L^2(\Gamma^+)$ with values on $L^2(\Gamma^-)$ is given according to the formula:

$$J f_+(x, \underline{\xi}, |v|, \omega) = f_+(x, \underline{\xi}, |v|, \omega_*), \tag{5.1.0}$$

with $\omega_* = (-\omega_x, \omega_y, \omega_z)$ the specular reflection velocity. We remark that, the adjoint operator of J, J^*, acts from $L^2(\Gamma^-)$ to $L^2(\Gamma^+)$ and it is also a mirror reflection operator. Hence, J and J^* satisfy $J^* J = I_{\Gamma^+}$, $J J^* = I_{\Gamma^-}$.

The operator $\mathcal{K}(f_+)(X,v)$ is the diffusive reflection operator acting on the set of functions defined on Γ^+ with values on functions defined on Γ^- and it is given by:

$$\mathcal{K}(f_+)(X,v) = \int_{\{\omega' \in S^2, (X,v')\in\Gamma^+\}} K(X,|v|;\omega' \to \omega)\, f_+(X,|v|\omega')\, |\omega'_x|\, d\omega'. \qquad (5.1.0)$$

The dependence of the kernel K with respect to $X=(x,y,z)$, $x=\{0,1\}$, $(y,z)\in\mathbb{R}^2$ and $|v|$ will be omitted, otherwise specified. The quantity $K(\omega'\to\omega)\,|\omega_x| d\omega$ is the probability of an electron impinging on a plane at a position (y,z) with velocity modulus $|v|$ and velocity direction ω' to be reflected with new velocity angle ω belonging to the solid angle $d\omega$ (and the same velocity modulus). We define the adjoint operator of $\mathcal{K}$, $\mathcal{K}^*$ as follows, for $f \in L^2(\Gamma^-)$:

$$\mathcal{K}^*(f)(X,v) = \int_{\Gamma^-} K(X,|v|;\omega\to\omega')\, |\omega'_x|\, f(X,|v|\omega')\, d\omega',\ (X,v)\in\Gamma^+. \qquad (5.1.0)$$

We recall that, the adjoint operator of $\mathcal{B}$, $\mathcal{B}^*$, from $L^2(\Gamma^-)$ to $L^2(\Gamma^+)$ is given by:

$$\mathcal{B}^* f_-(X,v) = \beta J^* f_-(X,v) + (1-\beta)\mathcal{K}^*(f_-)(X,v), \qquad (X,v)\in\Gamma^- \qquad (5.1.0)$$

We state now the main theorem which is going to be proved in the following sections. Our goal is to show that the limit of f^α when α goes to 0 is a function of the longitudinal coordinate $\underline{\xi}$, of the energy $\varepsilon = |v|^2/2$ and of the time t, $F(\underline{\xi},\varepsilon,t)$, which obeys a diffusion equation in the position-energy space. It is easily seen the analogy with theorem 4.1.1.

Theorem 5.1.1 *Under some hypotheses listed later on (namely hypotheses 5.2.1, 5.2.2, 4.3.1, 5.4.1, 5.5.1), f^α converges to f^0 as $\alpha\to 0$ in the weak star topology of $L^\infty([0,T],L^2(\Theta))$ for any $T>0$, where $f^0(X,v,t)=F(\underline{\xi},|v|^2/2,t)$ and $F(\underline{\xi},\varepsilon,t)$ is a distributional solution of the problem:*

$$4\pi\sqrt{2\varepsilon}\frac{\partial F}{\partial t} + \left(\nabla_{\underline{\xi}} - \underline{E}\frac{\partial}{\partial\varepsilon}\right)\cdot \underline{J} = 0, \qquad (5.1.1)$$

$$\underline{J}(\underline{\xi},\varepsilon,t) = -\mathbb{D}\left(\nabla_{\underline{\xi}} - \underline{E}\frac{\partial}{\partial\varepsilon}\right) F(\underline{\xi},\varepsilon,t), \qquad (5.1.2)$$

$$F|_{t=0} = F_I, \qquad (5.1.3)$$

in the domain $(\underline{\xi},\varepsilon)\in\mathbb{R}^2\times(0,\infty)$ and where F_I is a suitable given initial data (see hypothesis 5.4.1). The diffusion tensor $\mathbb{D}=\mathbb{D}(\underline{\xi},\varepsilon)$ is given by

$$\mathbb{D}(\underline{\xi},\varepsilon) = (2\varepsilon)^{3/2}\int_0^1\int_{\mathbb{S}^2} \underline{D}(x,\omega;\underline{\xi},\varepsilon)\underline{\omega}\, dx\, d\omega, \qquad (5.1.3)$$

where $\underline{\omega} = (\omega_y, \omega_z)$*,* $\underline{D} = (D_y, D_z)$*,* $\underline{D}\,\underline{\omega}$ *is the tensor product* $(D_i\omega_j)_{i,j\in\{y,z\}}$ *and finally* $D_i(x,\omega;\underline{\xi},|v|^2/2)$*,* $(i = y,z)$ *is a solution of the problem*

$$\begin{cases} -v_x \dfrac{\partial D_i}{\partial x} + (\underline{v} \times B) \cdot \nabla_{\underline{v}} D_i = \mathcal{L} D_i + \omega_i \, , \text{ in } \Theta \, , \\ \\ (D_i)_+ = \mathcal{B}^*(D_i)_- \, , \text{ on } \Gamma \, , \end{cases} \tag{5.1.3}$$

unique, up to an additive function of $\underline{\xi}$ *and* ε*.*

5.2 Assumptions on boundary operators

Before proving the various steps, we introduce the assumptions on the diffusion kernel K. We shall assume in addition of all the hypotheses listed in the collision-less case that the kernel of the diffusion operator is rotationally invariant for any rotation of axis ω_x. This additional assumption is necessary when proving the existence of the solution of the auxiliary problem. For the sake of completeness we recall all the assumptions:

Hypothesis 5.2.1 *We assume that the kernel* K *satisfies the following properties:*
(i) positivity:

$$K(\omega' \to \omega) > 0, \tag{5.2.0}$$

(ii) flux conservation, $x = 0, 1$*:*

$$\int_{\mathcal{S}_-(x)} K(\omega' \to \omega)\, |\omega_x|\, d\omega = 1, \tag{5.2.0}$$

(iii) reciprocity relation:

$$K(\omega' \to \omega) = K(-\omega \to -\omega'), \quad \forall\, (\omega, \omega') \in S^2. \tag{5.2.0}$$

As proved, from Hypothesis 5.2.1 it follows the *normalization identity*, $x = 0, 1$:

$$\int_{\mathcal{S}_+(x)} K(\omega' \to \omega)\, |\omega'_x|\, d\omega' = 1. \tag{5.2.0}$$

and the *Darrozes-Guiraud inequality*, $x = 0, 1$:

$$\int_{\mathcal{S}_-(x)} |f_-(x,\omega)|^2\, |\omega_x|\, d\omega \le \int_{\mathcal{S}_+(x)} |f_+(x,\omega)|^2\, |\omega_x|\, d\omega. \tag{5.2.0}$$

We remark that the operator $\mathcal{K}^*$ satisfies hypothesis 5.2.1.

Let us briefly recall the definitions of the boundary operators $Q_\pm$:

$$Q_\pm f(x,\omega) = \frac{1}{\pi}\int_{\mathcal{S}_\pm(x)} |\omega'_x|\, f(x,\omega')\, d\omega' \,,\ \omega\in\mathcal{S}_\pm(x)\,, x=0,1\,, \tag{5.2.0}$$

and its orthogonal complement $P_\pm = \mathrm{I}_{\mathcal{S}_\pm} - Q_\pm$, where $\mathrm{I}_{\mathcal{S}_\pm}$ is the identity.

We also assume that:

Hypothesis 5.2.2 *(i) The operator* $\mathcal{K}$ *is a compact operator from* $L^2(\mathcal{S}^+)$ *to* $L^2(\mathcal{S}^-)$.
(ii) There exists a constant $k<1$ *such that:*

$$\|\mathcal{K}P_+\|_{\mathcal{L}(L^2(\mathcal{S}^+),L^2(\mathcal{S}^-))} \le k < 1\,, |v|\in\mathbb{R}^+\,,\ \underline{\xi}\in\mathbb{R}^2. \tag{5.2.0}$$

(iii) There exists $\beta_0<1$ *such that* $0\le\beta\le\beta_0<1\,,\ \underline{\xi}\in\mathbb{R}^2\,,\ |v|>0\,,\ x=0,1.$

It follows that the operator $\mathcal{K}^*$ is compact from $L^2(\mathcal{S}^-)$ to $L^2(\mathcal{S}^+)$, and

$$\|\mathcal{B}P_+\|_{\mathcal{L}(L^2(\mathcal{S}^+),L^2(\mathcal{S}^-))} \le \sqrt{\beta_0+(1-\beta_0)k^2} = k_0. \tag{5.2.0}$$

We remark that Lemmas 4.2.2 and 4.2.3 still hold true.

In addition to the above assumptions we shall need also the following

Hypothesis 5.2.3 *The diffusion kernel* $\mathcal{K}$ *is rotationally invariant, i.e. for every rotation* R *of axis* ω_x*:*

$$K(\omega'\to\omega) = K(R\omega'\to R\omega). \tag{5.2.0}$$

We now notice that, if $f\in D(\mathcal{A}^\alpha)$, then $Q_-f_- = JQ_+f_+$ (thanks to Lemma 4.2.3). Thus, there exists a single function $q(f)=q(x,\underline{\xi},|v|)$, $x=0,1$, $\underline{\xi}\in\mathbb{R}^2$, $|v|>0$ such that:

$$q = Q_-f_- \quad \text{on } \Gamma^-\,, \qquad q = Q_+f_+ \quad \text{on } \Gamma^+.$$

Other results proved in the previous sections will be recalled when needed.

5.3 The collision operator $\mathcal{L}$

In this section we list the assumptions on the collision operator $\mathcal{L}$ and study its main properties. We consider isotrope collisions, then $\phi_0(x,\xi,v',v)$ is constant with respect to x and to the directions ω' and ω. Again, this assumption turns out to be crucial when proving the existence of the solution of the auxiliary problem. We first remark that $\mathcal{L}$ may be written as follows (using the definition of Dirac mass):

$$\mathcal{L}(f)(x,\underline{\xi},|v|,\omega) = \int_{S^2} \phi_0(\underline{\xi},|v|)\,(f(|v|\omega') - f(|v|\omega))\,|v|\,d\omega' \tag{5.3.0}$$

where we have replaced $v \in \mathbb{R}^3$ by $|v|\omega \in \mathbb{R}^+ \times S^2$. We shall denote $\Phi = \Phi(\underline{\xi},|v|) = \phi_0(\underline{\xi},|v|)\,|v|$, omitting the dependence of ϕ on $\underline{\xi},|v|$. We remark that Φ depends on ξ and $|v|$. We assume that the scattering cross section Φ satisfies the following:

Hypothesis 5.3.1 *There exist two constants, c_1 and c_2, such that $0 < c_1 < c_2$ and*

$$c_1 \le \Phi \le c_2.$$

We define:

$$\lambda = \int_{S^2} \Phi\,d\omega = \Phi 4\pi.$$

Remark that λ may depend on $\underline{\xi}$ and $|v|$. Moreover, the operator $\mathcal{L}f$ can be splitted in the 'gain' part L^+f and of the 'lost' part $-\lambda f$:

$$\mathcal{L}f = L^+f - \lambda f \quad , \qquad L^+f = \Phi \int_{S^2} f\,d\omega = \lambda[f]. \tag{5.3.0}$$

where $[f]$ denotes the mean of the function f with respect to ω. We have the following lemma.

Lemma 5.3.1 *If Hypothesis 5.3.1 holds, then:*
(i) For every $f,g \in L^2(\Theta)$:

$$(\mathcal{L}(f),g)_\Theta = -\frac{1}{2}\int_\Theta\int_{S^2} \Phi(f'-f)(g'-g)\,d\omega'\,d\theta. \tag{5.3.0}$$

(ii) $\mathcal{L}$ is bounded, i.e. $\exists M$ such that $\|\mathcal{L}\| \le M$.
(iii) $\mathcal{L}$ is self-adjoint, i.e. $(\mathcal{L}f,g)_\Theta = (f,\mathcal{L}g)_\Theta \quad \forall f,g \in L^2(\Theta)$.
(iv) $\mathcal{L}$ is a dissipative operator, i.e. $(\mathcal{L}f,f)_\Theta \le 0$ for every $f \in L^2(\Theta)$.
(v) The Null-Space of $\mathcal{L}$, $N(\mathcal{L})$, is composed of constant functions on ω:

$$N(\mathcal{L}) = \{f \in L^2(\Theta), f \text{ constant in } \omega\}.$$

Proof:
(i) Relation (5.3.1) follows easily applying a change of variables in (5.3).
(ii) We prove that there exists a constant M such that $|\mathcal{L}f|_{L^2(\Theta)} \le M\,|f|_{L^2(\Theta)}$.

$$|\mathcal{L}f|^2_{L^2(\Theta)} = \int_\Theta \left(\int_{S^2} \Phi(f'-f)\,d\omega'\right)^2 d\theta\,d\omega$$

$$\leq \int_\Theta \left(\int_{S^2} \Phi^2 d\omega' \right) \left(\int_{S^2} (f'-f)^2 \, d\omega' \right) d\theta$$

$$\leq 4\pi c_2^2 \int_\Theta \int_{S^2} (f'-f)^2 \, d\theta \, d\omega'$$

$$\leq 8\pi c_2^2 \int_0^1 \int_{\mathbb{R}^2} \int_{\mathbb{R}^+} \left(\int_{S^2} f'^2 \, d\omega' + \int_{S^2} f^2 d\omega \right) dx \, d\underline{\xi} \, d|v| = M^2 |f|^2_{L^2(\Theta)}$$

with $M = 4\sqrt{\pi} c_2$.

(iii) It follows easily from relation (5.3.1).

(iv) Relation (5.3.1) with $f = g$ and Hypothesis 5.3.1 give $(\mathcal{L}(f), f)_\Theta \leq 0$.

(v) The Null-Space of $\mathcal{L}$ is defined as:

$$N(\mathcal{L}) = \{ f \in L^2(\Theta), (\mathcal{L}f, \varphi)_\Theta = 0 \quad \forall\, \varphi \in L^2(\Theta) \}.$$

If, in particular, we take $\varphi = f$, then we have:

$$(\mathcal{L}f, f)_\Theta = -\frac{1}{2} \int_\Theta \int_{S^2} \Phi (f'-f)^2 \, d\omega' \, d\theta = 0.$$

As $\Phi \geq c_1 > 0$, it must be:

$$\int_\Theta \int_{S^2} (f'-f)^2 \, d\omega' \, d\theta = 0,$$

which implies that f is constant on S^2.

Conversely, if f is constant in ω, then $\mathcal{L}f = 0$. ■

Moreover, the operator $\mathcal{L}$ satisfies also the following *coercivity* relation:

Lemma 5.3.2 *For every $f \in L^2(\Theta)$,*

$$-(\mathcal{L}f, f)_\Theta \geq c_1 |f - [f]|^2_{L^2(\Theta)} \tag{5.3.0}$$

Proof:

Let $f \in L^2(\Theta)$, then by means of (5.3.1), we have:

$$-(\mathcal{L}f, f)_\Theta = \frac{1}{2} \int_\Theta \int_{S^2} \Phi (f'-f)^2 \, d\omega' \, d\theta \geq c_1 \frac{1}{2} \int_\Theta \int_{S^2} (f'-f)^2 \, d\omega' \, d\theta$$

$$= c_1 \frac{1}{2} \int_\Theta \int_{S^2} (f')^2 d\omega' \, d\theta + c_1 \frac{1}{2} \int_\Theta \int_{S^2} f^2 \, d\omega' \, d\theta - c_1 \int_\Theta \int_{S^2} f' f \, d\omega' \, d\theta$$

$$= c_1 4\pi |f|^2_{L^2(\Theta)} - c_1 \int_0^1 \int_{\mathbb{R}^2} \int_{\mathbb{R}^+} \left(\int_{S^2} f \, d\omega \right)^2 dx \, d\underline{\xi} \, d|v|.$$

Replacing f by $f-[f]$ in the above inequality, we obtain:

$$-(\mathcal{L}(f-[f]), f-[f])_\Theta \geq c_1 |f-[f]|^2_{L^2(\Theta)}$$
$$-c_1 \int_0^1 \int_{\mathbb{R}^2} \int_{\mathbb{R}^+} \left(\int_{S^2} f - [f]\, d\omega \right)^2 dx\, d\underline{\xi}\, d|v|. \tag{5.3.0}$$

The left hand side of (5.3) is equal to $-(\mathcal{L}f, f)_\Theta$; whereas, the second member of the right hand side of (5.3) is equal to zero. Hence, relation (5.3.2) is proved. ■

5.4 The transport operator

For $(X, v) \in \Theta$, we recall the definition of the transport operator $\mathcal{A}^\alpha$:

$$\mathcal{A}^\alpha f(X,v) = \underline{v} \cdot \nabla_{\underline{\xi}} f(X,v) - \underline{E} \cdot \nabla_{\underline{v}} f(X,v)$$
$$+\frac{1}{\alpha}\left(v_x \frac{\partial f}{\partial x}(X,v) - (\underline{v} \times B) \cdot \nabla_{\underline{v}} f(X,v) \right), \tag{5.4.0}$$

and of its domain $D(\mathcal{A}^\alpha)$:

$$D(\mathcal{A}^\alpha) = \{ f \in L^2(\Theta),\ \mathcal{A}^\alpha f \in L^2(\Theta), P_+ f_+ \in L^2(\Gamma^+),$$
$$Q_+ f_+ \in L^2_{loc}(\Gamma^+), f_- = \mathcal{B} f_+ \}.$$

In this definition, the spaces $L^2_{loc}(\Gamma^\pm)$ are given by (4.3), and are equipped with the family of semi-norms $|\cdot|_{\Gamma^\pm, R}$, see (4.3).

We recall the hypothesis done on the initial data f_I. This hypothesis is necessary in order to avoid initial layers.

Hypothesis 5.4.1 *We suppose that there exists a function F_I such that*

$$f_I(x, \underline{\xi}, v) = F_I(\underline{\xi}, |v|^2/2)$$

and that f_I satisfies

$$f_I \in L^2(\Theta)\ , \quad (\underline{v} \cdot \nabla_{\underline{\xi}} - \underline{E} \cdot \nabla_{\underline{v}}) f_I \in L^2(\Theta).$$

We note that hypothesis 5.4.1 implies that $f_I \in D(\mathcal{A}^\alpha)$ for all $\alpha > 0$. We now consider the following evolution problem:

$$\begin{cases} \alpha \dfrac{\partial}{\partial t} f^\alpha + \mathcal{A}^\alpha f^\alpha = \dfrac{1}{\alpha} \mathcal{L} f^\alpha \,, \quad \text{in } \Theta, \\ f_-^\alpha = \mathcal{B} f_+^\alpha \,, \quad \text{on } \Gamma, \\ f^\alpha|_{t=0} = f_I \,, \end{cases} \tag{5.4.0}$$

which is also written:

$$\begin{cases} \alpha \dfrac{\partial}{\partial t} f^\alpha + \mathcal{A}^\alpha f^\alpha = \dfrac{1}{\alpha} \mathcal{L} f^\alpha \,, \quad \text{in } \Theta, \\ \\ f^\alpha(0) = f_I \in D(\mathcal{A}^\alpha) \,. \end{cases} \tag{5.4.0}$$

where we assume that also hypothesis 4.3.1 is satisfied.

The first result to be proved is the existence and uniqueness of the solution f^α of the adimensional evolution problem (5.4).

Theorem 5.4.1 *Fixed α, for any $T > 0$, there exists a unique solution f^α of the evolution problem (5.4). Moreover, $f^\alpha \in C^0([0,T], D(\mathcal{A}^\alpha)) \cap C^1([0,T], L^2(\Theta))$.*

We start by the derivation of a Green formula for the evolution problem (5.4). We define the space $H_0(\mathcal{A}^\alpha)$ as follows:

$$\begin{aligned} H_0(\mathcal{A}^\alpha) &= \{f \in L^2(\Theta), \mathcal{A}^\alpha f \in L^2(\Theta), \mathcal{L} f \in L^2(\Theta), f_+ \in L^2(\Gamma^+)\} \\ &= \{f \in L^2(\Theta), \mathcal{A}^\alpha f \in L^2(\Theta), \mathcal{L} f \in L^2(\Theta), f_- \in L^2(\Gamma^-)\}. \end{aligned}$$

Moreover, since $\mathcal{L}$ is a bounded and self-adjoint operator, and $D(\mathcal{A}^\alpha) \cap D(\mathcal{L}) = D(\mathcal{A}^\alpha)$, we deduce, by means of the same procedures used in section 4.3, that:

Lemma 5.4.1 *Under Hypothesis 4.3.1, for $f, g \in H_0(\mathcal{A}^\alpha)$, f and g with compact support with respect to v, we have:*

$$\begin{aligned} (\mathcal{A}^\alpha f, g)_\Theta - \frac{1}{\alpha} (\mathcal{L} f, g)_\Theta &= -(f, \mathcal{A}^\alpha g)_\Theta - \frac{1}{\alpha} (f, \mathcal{L} g)_\Theta \\ &+ \frac{1}{\alpha} \left(\int_{\Gamma^+} |v_x|\, f_+ g_+ \, d\Gamma - \int_{\Gamma^-} |v_x|\, f_- g_- \, d\Gamma \right) . \end{aligned} \tag{5.4.0}$$

We remark that if $g = f$ in (5.4.1), then:

$$(\mathcal{A}^\alpha f, f)_\Theta + (f, \mathcal{A}^\alpha f)_\Theta = \frac{1}{\alpha}\left(\int_{\Gamma^+} |v_x|\, f_+^2 \, d\Gamma - \int_{\Gamma^-} |v_x|\, f_-^2 \, d\Gamma\right)$$

Therefore, it follows directly from lemmas 4.3.2 and 4.3.3 that:

Lemma 5.4.2 *If $f \in D(\mathcal{A}^\alpha)$, then there exists a constant $C > 0$ such that:*

$$|P_- f_-|^2_{L^2(\Gamma^-)} \leq |P_+ f_+|^2_{L^2(\Gamma^+)} \leq \frac{2\alpha}{1-k_0}(\mathcal{A}^\alpha f, f)_\Theta \leq C\alpha |f|^2_{\mathcal{A}^\alpha}. \tag{5.4.0}$$

where k_0 is given by (5.2) and

$$|q|^2_{\Gamma,R} \leq C\left(\alpha |f|^2_{\mathcal{A}^\alpha} + R|f|^2\right) \tag{5.4.0}$$

We can now prove theorem 5.4.1.

Proof (of Theorem 5.4.1):

From Lemma 5.4.2 follows that the transport operator $\mathcal{A}^\alpha$ is closed (see lemma 4.3.4). Moreover, as a consequence of the Darrozes-Guiraud inequality, $\mathcal{A}^{\alpha *}$ is an accretive operator. Thus, thanks to the Lumer-Phillips theorem, $-\mathcal{A}^\alpha$ generates a strongly continuous semigroup of contractions. Hence, being $\mathcal{L}$ a bounded operator, by means of the perturbation theory of semigroups ([53]), we obtain that $-\mathcal{A}^\alpha + \alpha^{-1}\mathcal{L}$ is the generator of a strongly continuous semigroup for every α. ■

5.5 Convergence towards the macroscopic model

This section is devoted to the convergence proof and is divided in subsections. We first prove, in section 5.5.1, that as α approaches 0 the solution f^α converges to a function f^0 independent on x and ω. Then, in section 5.5.2, is given a new proof of the existence of the solution of the auxiliary problem. And we conclude by the proof of the validity of the SHE model in section 5.5.3.

5.5.1 A priori estimates and convergence

The next step is to prove that as α approaches 0, the solution f^α of the evolution problem (5.4) tends to a function F which is independent of x and ω. We first need to prove some a priori estimates.

Lemma 5.5.1 *(i) The solution f^α of problem (5.4) is bounded in $L^\infty((0,T), L^2(\Theta))$ by a constant C depending only on the data; i.e.*

$$|f^\alpha|_{L^\infty((0,T);L^2(\Theta))} \leq C. \tag{5.5.0}$$

(ii) There exists a constant C only depending on the data and such that:

$$\int_0^T |P_+ f_+^\alpha|^2_{L^2(\Gamma^+)}\, ds \leq C\alpha^2. \tag{5.5.0}$$

(iii) There exists a constant C, only depending on R, M and on the data, such that:

$$\int_0^T |q(f^\alpha)|^2_{\Gamma^+,R} ds \leq C_{R,M}. \tag{5.5.0}$$

(iv) There exists a constant C only depending on the data , on R and M and such that:

$$\int_0^T c_1 |f^\alpha - [f^\alpha]|^2_{L^2(\Theta)}\, ds \leq -\int_0^T (\mathcal{L}f^\alpha, f^\alpha)_\Theta ds \leq C\alpha^2. \tag{5.5.0}$$

Proof:

(i)-(ii) Multiplying problem (5.4) by f^α, integrating and considering relation (5.4.2) and the dissipativity of $\mathcal{L}$, we obtain:

$$\alpha \left(|f^\alpha(t)|^2_{L^2(\Theta)} - |f_I|^2_{L^2(\Theta)} \right) \leq -\frac{1-k_0^2}{2\alpha} \int_0^t |P_+ f_+^\alpha|^2\, ds$$

which immediately gives (i) and (ii).

(iii) From relation (5.4.2), using the fact that $\mathcal{A}^\alpha f^\alpha = -\alpha \frac{\partial}{\partial t} f^\alpha + \frac{1}{\alpha}\mathcal{L}f^\alpha$ and with the same procedures applied in the proof of lemma 4.4.1, we obtain:

$$\int_0^T |q(f^\alpha)|^2_{\Gamma^+,R} ds \leq -\alpha^2 \int_0^T \left(\chi_R \frac{\partial}{\partial t} f^\alpha, \chi_R f^\alpha sgn(v_x)\phi \right)_\Theta ds$$

$$+CR\int_0^T |f^\alpha|^2_{L^2(\Theta)} ds + \alpha\frac{1}{\alpha}\int_0^T (\mathcal{L}f^\alpha \chi_R, f^\alpha sgn(v_x)\phi\chi_R)_\Theta\, ds$$

$$\leq -\alpha^2 \left[\int_\Theta |f^\alpha|^2 sgn(v_x)\phi\chi_R^2\, dx\, d\underline{\xi}\, dv \right]_0^T + C\left((R+M)T|f_I|^2_{L^2(\Theta)} \right) \leq C_{R,M}$$

(iv) Multiplying by $\alpha^{-1} f^\alpha$ and integrating problem (5.4) we obtain:

$$-\frac{1}{\alpha^2}\int_0^T (\mathcal{L}f^\alpha, f^\alpha)_\Theta\, ds = |f_I|^2_{L^2(\Theta)} - |f^\alpha(t)|^2_{L^2(\Theta)} - \frac{1}{\alpha}\int_0^t (\mathcal{A}^\alpha f^\alpha, f^\alpha)_\Theta\, ds$$

$$\leq |f_I|^2_{L^2(\Theta)} - \frac{1-k_0}{2\alpha^2}\int_0^t |P_+ f_+^\alpha|^2_{L^2(\Gamma^+)}\, ds$$

and considering the sup for $t \in (0,T)$, we have:

$$-\int_0^T (\mathcal{L}f^\alpha, f^\alpha)_\Theta\, ds \leq C\alpha^2,$$

Finally, recalling Lemma 5.3.2, relation (5.5.1) is proved. ■

Remark 5.5.1 As a consequence of Lemma 5.5.1, as α tends to 0, there exists a subsequence, still denoted by f^α, of solutions of problem (5.4), which converges in $L^\infty(0,T;L^2(\Theta))$ weak star to a function f^0. Furthermore, using the diagonal extraction process, the subsequence of $q(f^\alpha)$ converges to a function $q(x,\underline{\xi},|v|,t)$ with $x = 0,1$ in $L^2(0,T,L^2(\gamma \times B_R))$ weak star for any R, where B_R is the ball centered at 0 and of radius R in the velocity space. Also, from (5.5.1), the traces $P_+f_+^\alpha$ and $P_-f_-^\alpha$ converge in $L^2(0,T;L^2(\Gamma^+))$ and $L^2(0,T;L^2(\Gamma^-))$ (respectively) strongly towards zero. Finally, from estimates (5.5.1) we obtain also that the limit function f^0 is independent on ω in Θ, i.e. $f^0 = f^0(x,\underline{\xi},|v|,t)$.

With the same arguments of section 4.4.1, we deduce that the traces $f_\pm^0$ of f^0 on $\Gamma^\pm$ satisfy:

$$P_-f_-^0 = P_+f_+^0 = 0,$$
$$Q_-f_-^0 = Q_+f_+^0 = q,$$

and so: $f^0|_\Gamma = q$, where $q = q(x,\underline{\xi},|v|,t)$, with $x = 0,1$, is independent of ω.

We now write the weak formulation of problem (5.4)

Lemma 5.5.2 *Let f^α be the solution of problem (5.4). Then, f^α is a weak solution , i.e. for any test function $\phi \in C_0^1([0,T]\times\Theta)$, with compact support in Θ such that $\phi(\cdot,\cdot,T) = 0$, we have:*

$$\int_0^T\int_\Theta f^\alpha\left(\alpha\frac{\partial}{\partial t}\phi + (\underline{v}\cdot\nabla_{\underline{\xi}}\phi - \underline{E}\cdot\nabla_{\underline{v}}\phi)\right) dt\, d\theta$$

$$+\frac{1}{\alpha}\int_0^T\int_\Theta f^\alpha\left(v_x\frac{\partial}{\partial x}\phi - (\underline{v}\times B)\cdot\nabla_{\underline{v}}\phi + \mathcal{L}\phi\right) dt\, d\theta + \alpha\int_\Theta f_I\phi|_{t=0}\, d\theta \qquad (5.5.0)$$

$$= \frac{1}{\alpha}\left(\int_0^T\int_{\Gamma^+} |v_x|\, f_+^\alpha(\phi_+ - \mathcal{B}^*\phi_-)\, dt\, d\gamma\right).$$

Proof:
Multiplying equation (5.4), using the Green Formula (5.4.1) and the boundary conditions (5.1), equation (5.5.2) is proved. ■

We can now prove the main result of this section.

Lemma 5.5.3 *The limit function f^0 is a function only of $\underline{\xi}, |v|, t$, i.e.*

$$f^0 = f^0(\underline{\xi}, |v|, t).$$

Proof:
Multiplying (5.5.2) by α, and using a test function ϕ with compact support in Θ, we get:

$$\begin{aligned} \alpha^2 \int_0^T \int_\Theta f^\alpha \frac{\partial}{\partial t}\phi \, dt \, d\theta + \alpha \int_0^T \int_\Theta f^\alpha (\underline{v} \cdot \nabla_{\underline{\xi}} \phi - \underline{E} \cdot \nabla_{\underline{v}} \phi) \, dt \, d\theta \\ + \int_0^T \int_\Theta f^\alpha \left(v_x \frac{\partial}{\partial x}\phi - (\underline{v} \times B) \cdot \nabla_{\underline{v}} \phi + \mathcal{L}\phi \right) \, dt \, d\theta \\ + \alpha^2 \int_\Theta f_I \phi|_{t=0} \, d\theta = 0 \end{aligned} \qquad (5.5.0)$$

Hence, when α goes to 0 in (5.5.1), thanks to Remark 5.5.1, we get:

$$\int_0^T \int_\Theta f^0 \left(v_x \frac{\partial}{\partial x}\phi - (\underline{v} \times B) \cdot \nabla_{\underline{v}} \phi + \mathcal{L}\phi \right) \, dt \, d\theta = 0. \qquad (5.5.0)$$

This is equivalent to say that f^0 is a distributional solution of the equation $\mathcal{A}f^0 = \mathcal{L}f^0$, where, as f^0 is independent on ω (see remark 5.5.1), $\mathcal{L}f^0 = 0$ and $\mathcal{A}$ is given by:

$$\mathcal{A}f = v_x \frac{\partial f}{\partial x}.0 \qquad (5.5.0)$$

Thus, f^0 is a solution of the problem:

$$\mathcal{A}f^0 = 0 \, , \qquad f^0|_\Gamma = q, \qquad (5.5.0)$$

where $q = q(x, \xi, |v|, t)$ is independent of ω and $x = 0, 1$.
Hence, integrating along the trajectories, it is easily seen that the solution f^0 of problem (5.5.1) is independent on x too, and is given by $f^0(X, v, t) = q(\underline{\xi}, |v|, t)$. ■

From now on we shall denote $F(\underline{\xi}, \varepsilon, t) = f^0(\underline{\xi}, |v|, t)$, where $\varepsilon = |v|^2/2$ is the kinetic energy (per unit mass).

5.5.2 Auxiliary equation

As seen in section 4.4.2, in order to derive the diffusivity tensor $\mathbb{D}$, we need to prove the existence of the solution of an auxiliary problem.

Let us recall the definition of the following operators, for $f \in L^2(\Theta)$:

$$\mathcal{A}^0 f = v_x \frac{\partial f}{\partial x} - (\underline{v} \times B) \cdot \nabla_{\underline{v}} f = |v|\omega_x \frac{\partial f}{\partial x} + B \frac{\partial f}{\partial \omega}(e_x \times \omega) \tag{5.5.0}$$

and

$$\mathcal{A}^{0*} f = -v_x \frac{\partial f}{\partial x} + (\underline{v} \times B) \cdot \nabla_{\underline{v}} f = -|v|\omega_x \frac{\partial f}{\partial x} - B \frac{\partial f}{\partial \omega}(e_x \times \omega) \tag{5.5.0}$$

where $\dfrac{\partial f}{\partial \omega}(e_x \times \omega)$ is the differential of f with respect to $\omega \in S^2$ acting on the tangent vector to S^2, $e_x \times \omega$. We recall that $\mathcal{A}^0$ and $\mathcal{A}^{0*}$ only operates on the variables $(x, \omega) \in [0,1] \times S^2$, leaving $\underline{\xi} \in \mathbb{R}^2$ and $|v| \geq 0$ as mere parameters. We therefore only consider the dependence of f on $(x, \omega) \in [0,1] \times S^2$.

Moreover,the sphere S^2 can be parameterized by $\omega = (\sigma, \underline{\omega})$, where $\underline{\omega} = (\omega_y, \omega_z)$ and $\sigma = \omega_x / |\omega_x| \in \{-1, +1\}$. The fact that $\sigma = \pm 1$ recalls that we need two maps to parameterize the sphere in this way. Next, we denote by $R^+_{(x,\sigma)}(\underline{\omega})$ the rotation of $\underline{\omega}$ about the x-axis of angle bx, where:

$$b = \frac{B}{|v|\omega_x}, \quad \omega_x = \sigma \sqrt{1 - \omega_y^2 - \omega_z^2}.$$

In other words, $\underline{\omega}^\dagger = R^+_{(x,\sigma)}(\underline{\omega}) = (\omega_y^\dagger, \omega_z^\dagger)$ is given by:

$$\begin{cases} \omega_y^\dagger = \omega_y \cos bx - \omega_z \sin bx \\ \\ \omega_z^\dagger = \omega_y \sin bx + \omega_z \cos bx \end{cases} \tag{5.5.0}$$

We note that $\underline{\omega}^\dagger$ also depends on $|v|$ and $\underline{\xi}$, but we shall not stress this dependence otherwise needed. Similarly, $R^-_{(x,\sigma)}(\underline{\omega})$ is the rotation of angle $-bx$. We have:

$$\underline{\omega}^\dagger = R^+_{(x,\sigma)}(\underline{\omega}) \text{if and only if} \underline{\omega} = R^-_{(x,\sigma)}(\underline{\omega}^\dagger),$$

and

$$R^-_{(x',\sigma)} R^+_{(x,\sigma)} = R^+_{(x-x',\sigma)} = R^-_{(x'-x,\sigma)}.$$

Let us define the operator $\mathcal{I}$ which to a function $f(x,\sigma,\underline{\omega})$ associates $f(x,\sigma,-\underline{\omega})$:

$$\mathcal{I}f(x,\sigma,\underline{\omega}) = f(x,\sigma,-\underline{\omega}).$$

We now prove the following consequence of hypothesis 5.2.3.

Lemma 5.5.4 *Let $f \in L^2(\Gamma^{\pm})$ be an odd function in $\underline{\omega}$ for every $x \in [0,1]$ and for every $\sigma \in \{-1,+1\}$, i.e. $\mathcal{I}f(x,\sigma,\underline{\omega}) = f(x,\sigma,-\underline{\omega})$, if the diffusion kernel satisfies (5.2.3), then the boundary operator $\mathcal{B}$ is odd:*

$$\mathcal{B}\mathcal{I}f = \mathcal{I}\mathcal{B}f\ , \quad \forall\ f \in L^2(\Gamma^-).$$

The same result holds for $\mathcal{B}^*$.

Proof:

It is clear that $\mathcal{I}Jf = J\mathcal{I}f$. For what regards the operator $\mathcal{K}$ we have:

$$\mathcal{K}(\mathcal{I}f) = \int_{\Gamma^+} K(\omega' \to \omega) f(\sigma', -\underline{\omega}')|\omega'_x|\, d\omega'$$

$$= \int_{\Gamma^+} K(R^{-1}\omega'' \to \omega) f(\omega'')|\omega''_x|\, d\omega''$$

where R is the rotation of axis ω_x and angle π and it is such that $R\omega' = \omega''$. Moreover, there exists a $\tilde{\omega}$ such that $R^{-1}\tilde{\omega} = \omega$. Thus,

$$\mathcal{K}(\mathcal{I}f) = \int_{\Gamma^+} K(R^{-1}\omega'' \to R^{-1}\tilde{\omega}) f(\omega'')\, |\omega''_x|\, d\omega''$$

and being K invariant for any rotation of axis ω_x, and recalling that R is a rotation of axis ω_x and angle π, we obtain:

$$\int_{\Gamma^+} K(\omega'' \to -\omega) f(\omega'')|\omega''_x|\, d\omega'' = \mathcal{I}(\mathcal{K}f)$$

Thus, for the boundary operator $\mathcal{B}$ we obtain:

$$\mathcal{B}\mathcal{I}f = \beta J\mathcal{I}f + (1-\beta)\mathcal{K}\mathcal{I}f = \beta\mathcal{I}Jf + (1-\beta)\mathcal{I}\mathcal{K}f = \mathcal{I}\mathcal{B}f.$$

■

We now prove the main result of this section, that is the existence and uniqueness, up to an additive function of $\underline{\xi}$ and $|v|$, of the solution of the stationary problem:

$$\begin{cases} \mathcal{A}^{0*}f + \lambda f = \lambda[f] + g. \\ f_+ = \mathcal{B}^* f_- \end{cases} \tag{5.5.0}$$

Proposition 5.5.1 *Let $g \in L^2([0,1] \times S^2)$ be an odd function with respect to $\underline{\omega}$ for every $x \in [0,1]$. Define $G = G(x, \sigma, \underline{\omega})$ by:*

$$G(x,\sigma,\underline{\omega}) = \begin{cases} \dfrac{1}{|v|\,|\omega_x|} \displaystyle\int_x^1 e^{-\gamma(x'-x)} g(x', \sigma, R^+_{(x'-x,\sigma)}(\underline{\omega}))\, dx' \ , \quad \sigma = +1 \\ \\ \dfrac{1}{|v|\,|\omega_x|} \displaystyle\int_0^x e^{\gamma(x-x')} g(x', \sigma, R^-_{(x-x',\sigma)}(\underline{\omega}))\, dx' \ , \quad \sigma = -1 \end{cases} \tag{5.5.0}$$

where $\gamma = \lambda/(|v|\omega_x)$. Note that G also depends on $|v|$ and $\underline{\xi}$. Assume that the trace G_- on Γ^- belongs to $L^2(\mathcal{S}^-)$ for a.e. $(|v|, \underline{\xi}) \in \mathbb{R}^+ \times \mathbb{R}^2$. Then the stationary problem (5.5.2) has a unique solution f such that $f \in L^2([0,1] \times S^2)$ for a.e. $(|v|, \underline{\xi}) \in \mathbb{R}^+ \times \mathbb{R}^2$ and f satisfying:

$$\int_0^1 \int_{S^2} f(x,\omega)\, dx\, d\omega = 4\pi \int_0^1 [f](x)\, dx = 0, \tag{5.5.0}$$

Furthermore, all solutions in this space are equal to f, up to an additive function of $\underline{\xi}$ and $|v|$.

Proof:

We look for a solution f of problem (5.5.2) satisfying (5.5.1). Assume that for all $x \in [0,1]$, $[f](x) = 0$. This yields us to find a solution of:

$$\begin{cases} \mathcal{A}^{0*} f + \lambda f = g \\ f_+ = \mathcal{B}^* f_- \end{cases} \tag{5.5.0}$$

If such a solution satisfies $[f](x) = 0\ \forall x \in [0,1]$, then it is also solution of (5.5.2) and it satisfies (5.5.1).

Applying the change of variables (5.5.2), equation (5.5.2) reads:

$$-|v|\omega_x \frac{\partial f^\dagger}{\partial x} + \lambda f^\dagger = g^\dagger \tag{5.5.0}$$

where $f^\dagger(x,\sigma,\underline{\omega}^\dagger) = f(x,\sigma,R^+_{(x,\sigma)}(\underline{\omega}^\dagger))$ and $g^\dagger(x,\sigma,\underline{\omega}^\dagger) = g(x,\sigma,R^+_{(x,\sigma)}(\underline{\omega}^\dagger))$. Integrating with respect to x we get:

$$f^\dagger(x,\sigma,\underline{\omega}^\dagger) = \begin{cases} e^{-\gamma(1-x)} f^\dagger(1,\sigma,\underline{\omega}^\dagger) + G^\dagger(x,\sigma,\underline{\omega}^\dagger) \ , \quad \sigma = +1 \\ \\ e^{\gamma x} f^\dagger(0,\sigma,\underline{\omega}^\dagger) + G^\dagger(x,\sigma,\underline{\omega}) \ , \qquad \sigma = -1 \end{cases} \tag{5.5.0}$$

where $\gamma = \lambda/|v|\omega_x$, $G^\dagger$ is given by

$$G^\dagger(x,\sigma,\underline{\omega}^\dagger) = \begin{cases} \dfrac{1}{|v|\,|\omega_x|}\displaystyle\int_x^1 e^{-\gamma(x'-x)} g^\dagger(x',\sigma,\underline{\omega}^\dagger)\,dx' \ , & \sigma = +1 \\ \\ \dfrac{-1}{|v|\,|\omega_x|}\displaystyle\int_0^x e^{\gamma(x-x')} g^\dagger(x',\sigma,\underline{\omega}^\dagger)\,dx' \ , & \sigma = -1 \end{cases} \tag{5.5.0}$$

Back to the original variables we have:

$$f(x,\sigma,\underline{\omega}) = \begin{cases} e^{-\gamma(1-x)} f_+(1,\sigma,R^+_{(1-x,\sigma)}(\underline{\omega})) + G(x,\sigma,\underline{\omega}) \ , & \sigma = +1 \\ \\ e^{\gamma x} f_+(0,\sigma,R^-_{(x,\sigma)}(\underline{\omega})) + G(x,\sigma,\underline{\omega}) \ , & \sigma = -1 \end{cases} \tag{5.5.0}$$

where $f_+(0)$ and $f_+(1)$ have to be determined by means of the boundary conditions. Evaluating (5.5.2) for $x = 0, 1$, we get:

$$f_-(1,\sigma,\underline{\omega}) = e^{-|\gamma|} f_+(0,\sigma,R^-_{(1-x,\sigma)}(\underline{\omega})) + G_-(1,\sigma,\underline{\omega}) \ , \quad \sigma = -1 \tag{5.5.0}$$

$$f_-(0,\sigma,\underline{\omega}) = e^{-|\gamma|} f_+(1,\sigma,R^+_{(x,\sigma)}(\underline{\omega})) + G_-(0,\sigma,\underline{\omega}) \ , \quad \sigma = +1 \tag{5.5.0}$$

where,

$$G_-(1,\sigma,\underline{\omega}) = \frac{1}{|v|\ |\omega_x|}\int_0^1 e^{-|\gamma|(1-x')} g(x',\sigma,R^-_{(1-x',\sigma)}(\underline{\omega}))\,dx' \ , \quad \sigma = -1$$

$$G_-(0,\sigma,\underline{\omega}) = \frac{1}{|v|\ |\omega_x|}\int_0^1 e^{-|\gamma|x'} g(x',\sigma,R^+_{(x',\sigma)}(\underline{\omega}))\,dx' \ , \quad \sigma = +1$$

and which can be written compactly:

$$f_- = M_+ f_+ + G_- \tag{5.5.0}$$

where in particular M_+ is a bounded operator of norm strictly smaller than 1, and which maps functions belonging to $L^2(\mathcal{S}^+)$ into functions belonging to $L^2(\mathcal{S}^-)$ (see section 4.4.1). Considering the boundary conditions, omitting the dependence on σ and $\underline{\omega}$, we have:

$$f_+(1) = \mathcal{B}_1^* f_-(1) = \mathcal{B}_1^* M_+ f_+(0) + \mathcal{B}_1^* G_-(1)$$

$$f_+(0) = \mathcal{B}_0^* f_-(0) = \mathcal{B}_0^* M_+ f_+(1) + \mathcal{B}_0^* G_-(0)$$

where $\mathcal{B}_1^*$ is the boundary operator defined on the plane $x = 1$ and $\mathcal{B}_0^*$ is the one defined on $x = 0$. Hence:

$$\begin{aligned} f_+(1) &= \mathcal{B}_1^* M_+ \mathcal{B}_0^* M_+ f_+(1) + \mathcal{B}_1^* M_+ \mathcal{B}_0^* G_-(0) + \mathcal{B}_1^* G_-(1) \\ f_+(0) &= \mathcal{B}_0^* M_+ \mathcal{B}_1^* M_+ f_+(0) + \mathcal{B}_0^* M_+ \mathcal{B}_1^* G_-(1) + \mathcal{B}_0^* G_-(0). \end{aligned} \tag{5.5.0}$$

Therefore, as $\|\mathcal{B}^*\| = \|\mathcal{B}_0^*\| = \|\mathcal{B}_1^*\| = 1$ and as $\|M_+\| < 1$, we have that:

$$f_+(1) = (I - \mathcal{B}_1^* M_+ \mathcal{B}_0^* M_+)^{-1} (\mathcal{B}_1^* M_+ \mathcal{B}_0^* G_-(0) + \mathcal{B}_1^* G_-(1))$$

$$f_+(0) = (I - \mathcal{B}_0^* M_+ \mathcal{B}_1^* M_+)^{-1} (\mathcal{B}_0^* M_+ \mathcal{B}_1^* G_-(1) + \mathcal{B}_0^* G_-(0)).$$

The existence and uniqueness of the solution of problem (5.5.2) is proved. We still have to show that the solution f given by (5.5.2) is such that $[f](x) = 0$ for every $x \in [0,1]$. It will follows that f given by (5.5.2) is also solution of (5.5.2) and it satisfies (5.5.1).

Being $g(x,\sigma,\underline{\omega})$ odd with respect to $\underline{\omega}$, it is clear that also $G(x,\sigma,\underline{\omega})$ is odd with respect to $\underline{\omega}$ and for every $x \in [0,1]$. Thus, $G_- \mathcal{I} f = \mathcal{I} G_- f$. Moreover, because $|\gamma|$ depends only on $|\omega_x|$, the operator M_+ is such that $M_+ \mathcal{I} f = \mathcal{I} M_+ f$, too. Finally, from lemma 5.5.4 and from relations (5.5.2), it follows that:

$$\mathcal{I} f_+(1) = -f_+(1)\ , \quad \mathcal{I} f_+(0) = -f_+(0).$$

Hence, considering the explicit formulation of the solution f given by (5.5.2), we have that f is odd with respect to $\underline{\omega}$ for every $x \in [0,1]$ and $\sigma \in \{-1,+1\}$. Therefore, $[f](x,\sigma) = 0$ and the lemma is proved. We remark that with the same procedures applied to prove that $-\mathcal{A}^\alpha$ is the generator of a semigroup, it is possible to prove that $-\mathcal{A}^{0^*}$ generates a semigroup too. Hence equation (5.5.2) may be seen as the resolvent equation for the operator $-\mathcal{A}^{0^*}$, and the solution f given by (5.5.2) belongs to $L^2([0,1] \times S^2)$ for a.e. $(\underline{\xi}, \varepsilon) \in \mathbb{R}^2_{\underline{\xi}} \times \mathbb{R}^+_\varepsilon$. ■

In the remainder of the paper, we need g to be equal to ω_y and ω_z. We have:

Lemma 5.5.5 *The functions $g = \omega_y$ and $g = \omega_z$ satisfy the assumptions of Proposition 5.5.1.*

Proof:

Let G_y be the function associated to $g = \omega_y$ by (5.5.1). We need to show that

$(G_y)_- \in L^2(\mathcal{S}_-)$. The proof is obviously similar for $g = \omega_z$. We get by straight forward computations:

$$G_y(x,\sigma,\underline{\omega}) = \begin{cases} \dfrac{1}{B^2+\lambda^2}[\lambda\omega_y - B\omega_z + e^{-\gamma(1-x)}(B\omega_y+\lambda\omega_z)\sin b(1-x) \\ \qquad -e^{-\gamma(1-x)}(\lambda\omega_y - B\omega_z)\cos b(1-x)]\,,\ \sigma=+1 \\ \dfrac{-1}{B^2+\lambda^2}[-\lambda\omega_y + B\omega_z + e^{\gamma x}(B\omega_y+\lambda\omega_z)\sin bx \\ \qquad +e^{\gamma x}(\lambda\omega_y - B\omega_z)\cos bx]\,,\ \sigma=-1 \end{cases},$$

with the Hypothesis 4.3.1 (iii) on B, it is an easy matter to check that G_y satisfies the required hypothesis.
Moreover, it is clear that ω_y, ω_z are odd function with respect to $\underline{\omega}$. ■

By Proposition 5.5.1, there exist functions $D_y(x,\omega;\underline{\xi},\varepsilon)$, $D_z(x,\omega;\underline{\xi},\varepsilon)$, solutions of problem (5.5.2) with right-hand-side $g=\omega_y$ and $g=\omega_z$ respectively, unique up to additive functions of $\underline{\xi}$ and ε. In addition, we need the following regularity for D_y, D_z:

Hypothesis 5.5.1 *(i) D_y, D_z are C^1 bounded functions on $\Theta\backslash\{v_x=0\}$.*
(ii) The functions $\omega_i D_j(x,\omega;\underline{\xi},\varepsilon)$ belongs to $L^1([0,1]\times S^2)$ and

$$\int_0^1\int_{S^2}\omega_i D_j\,dx\,d\omega$$

is a C^1 function of $(\underline{\xi},\varepsilon)\in\mathbb{R}^2_{\underline{\xi}}\times\mathbb{R}^+_\varepsilon$.

Remark 5.5.2 Hypothesis 5.5.1 can be viewed as a regularity assumption on the data: the magnetic field B, the boundary scattering kernel K and the accommodation coefficient β. We do not look for explicit condition on these data because the developments would be technical and of rather limited interest. We confine ourselves to noting that hypothesis 5.5.1 is not empty, because it is satisfied at least in the case of isotropic scattering. ■

From Hypothesis 5.5.1 (ii), we deduce that the diffusivity tensor:

$$\mathbb{D}_{ij} = (2\varepsilon)^{3/2} \int_0^1 \int_{S^2} \omega_j \, D_i(x,\omega;\underline{\xi},\varepsilon)\, dx\, d\omega \ , \quad i,j \in \{y,z\},$$

is defined and is a C^1 function of $(\underline{\xi},\varepsilon) \in \mathbb{R}^2_{\underline{\xi}} \times \mathbb{R}^+_\varepsilon$. We also note that the definition of D_{ij} does not depend on the arbitrary additive function of $\underline{\xi}$ and ε which enters in the definition of D_j.

5.5.3 The continuity equation

The last steps of the proof of theorem 5.1.1 are to define the current J^α, to prove that it admit a limit and to derive the weak formulation of the drift diffusion model (5.1.1).

Let us introduce the current $\underline{J}^\alpha(\underline{\xi},\varepsilon,t) = (J^\alpha_y, J^\alpha_z)$ as follows:

$$\begin{aligned} \underline{J}^\alpha(\underline{\xi},\varepsilon,t) &= \frac{|v|}{\alpha} \int_0^1 \int_{S^2} \underline{v} f^\alpha(x,\underline{\xi},|v|,\omega,t)\, dx\, d\omega \\ &= \frac{2\varepsilon}{\alpha} \int_0^1 \int_{S^2} \underline{\omega} f^\alpha(x,\underline{\xi},\varepsilon,\omega,t)\, dx\, d\omega. \end{aligned}$$

We remark that lemma 4.4.6 still holds. Moreover, denoting by Θ' the position-energy space $\Theta' = \mathbb{R}^2 \times \mathbb{R}^+$ and by $d\theta'$ its volume element $d\theta' = d\underline{\xi}\, d\varepsilon$ (note that $dv = |v|^2 d|v|\, d\omega = \sqrt{2\varepsilon}\, d\varepsilon\, d\omega$), we have:

Lemma 5.5.6 *As α approaches 0, the current $\underline{J}^\alpha(\underline{\xi},\varepsilon,t)$ converges in the sense of distribution to $\underline{J}^0(\underline{\xi},\varepsilon,t)$. More precisely, for every $\psi \in C^1(\Theta' \times [0,T],\mathbb{R}^2)$ with compact support in $\mathbb{R}^2 \times \mathbb{R}^+ \times [0,T[$, we have:*

$$\lim_{\alpha\to 0} \int_0^T \int_{\Theta'} \underline{J}^\alpha \cdot \underline{\psi}\, d\theta'\, dt = \int_0^T \int_{\Theta'} \underline{J}^0 \cdot \underline{\psi}\, d\theta'\, dt \tag{5.5.0}$$

where

$$\underline{J}^0 = -\mathbb{D} \cdot \left(\nabla_{\underline{\xi}} - \underline{E} \frac{\partial}{\partial \varepsilon} \right) F^0. \tag{5.5.0}$$

Proof:

Analogous to the the proof of lemma 4.4.7 with $\underline{D}$ solution of (5.5.1). ■

It remains to prove that equations (5.1.1) and (5.1.2) hold in a weak sense. Once defined the function F^α,

$$F^\alpha(\underline{\xi},\varepsilon,t) = \frac{1}{4\pi} \int_0^1 \int_{\mathbb{S}^2} f^\alpha(x,\underline{\xi},|v|,\omega,t) dx\, d\omega, \tag{5.5.0}$$

the proof is the same as the one of lemma 4.4.8, thus we just quote the following

Lemma 5.5.7 *For any test function* ψ *belonging to* $C^2(\Theta' \times [0,T])$*, with compact support in* $\mathbb{R}^2 \times \mathbb{R}^+ \times [0,T[$*, we have:*

$$\int_0^T \int_{\Theta'} \left(4\pi\sqrt{2\varepsilon} F^0 \frac{\partial \psi}{\partial t} + \underline{J}^0 \cdot \left(\nabla_{\underline{\xi}} - \underline{E}\frac{\partial}{\partial \varepsilon}\right) \psi \right) dt\, d\theta' \\ + \int_{\Theta'} 4\pi\sqrt{2\varepsilon} F_I \psi|_{t=0}\, d\theta' = 0. \tag{5.5.0}$$

The main theorem 5.1.1 is proved also in the collision case.

5.6 Properties of the diffusivity

Last but not least, we prove that the diffusion tensor $\mathbb{D}$ still satisfies the Onsager relation and still is positive definite.

Proposition 5.6.1 *The diffusion tensor* $\mathbb{D}$ *satisfies the Onsager relation:*

$$\mathbb{D}(B)^T = \mathbb{D}(-B).$$

Proof:

For $f(x,\omega)$, we define the transformation $\mathcal{J}f$: $\mathcal{J}f(x,\omega) = f(x,-\omega)$. We make the dependence of $\mathcal{A}^0$ and $\mathbb{D}$ upon B explicit by writing $\mathcal{A}^0(B)$ and $\mathbb{D}(B)$. We begin by noting that $\mathcal{J}f$ is a solution of ${\mathcal{A}^0}^*(-B)\mathcal{J}f - \mathcal{L}\mathcal{J}f = \mathcal{J}g$, $\mathcal{J}f_+ = \mathcal{B}^*\mathcal{J}f_-$ if and only if the function f is solution of $\mathcal{A}^0(B)f - \mathcal{L}f = g$, $f_- = \mathcal{B}f_+$. This follows from the reciprocity relation (5.2.1), and from relation:

$$\begin{aligned} ({\mathcal{A}^0}^*(-B) - \mathcal{L})\mathcal{J}f &= \mathcal{J}(\mathcal{A}^0(B) - \mathcal{L})f \\ (\mathcal{A}^0(B) - \mathcal{L})\mathcal{J}f &= \mathcal{J}({\mathcal{A}^0}^*(-B) - \mathcal{L})f. \end{aligned} \tag{5.6.0}$$

Now, the proof follows exactly as the proof of proposition 4.5.1. ■

Proposition 5.6.2 *The diffusion tensor* $\mathbb{D}$ *is positive definite: there exists* $C > 0$ *such that:*

$$(\mathbb{D}Y, Y) = \sum_{i,j=1}^{2} \mathbb{D}_{ij} Y_i Y_j \geq C|Y|^2 = C\sum_{i=1}^{2} Y_i^2. \tag{5.6.0}$$

Proof:

Let $Y = (y_1, y_2) = y_i$ such that $|Y| > 0$, and $\underline{D} = (D_y, D_z) = (D_1, D_2)$. From the definition of $\mathbb{D}$ we obtain that:

$$(\mathbb{D}Y, Y) = C \int_0^1 \int_{S^2} \left(\sum_{i=1}^{2} \omega_i y_i \right) \left(\sum_{i=1}^{2} D_i y_i \right) dx\, d\omega, \tag{5.6.0}$$

where $C = (2\varepsilon)^{3/2}$. Define $V(x, \omega)$ as follows:

$$V(x, \omega) = \sum_{i=1}^{2} y_i D_i(x, \omega), \tag{5.6.0}$$

then:

$$\sum_{i=1}^{2} \omega_i y_i = (\mathcal{A}^{0^*} - \mathcal{L})V(x, \omega),$$

and equation (5.6) reads:

$$(\mathbb{D}Y, Y) = C((\mathcal{A}^{0^*} - \mathcal{L})V, V)_{\mathcal{S}}$$

$$\geq C \left(\int_{\mathcal{S}^-} |\omega_x| \, |V_-|^2 \, d\omega - \int_{\mathcal{S}^+} |\omega_x| \, |V_+|^2 \, d\omega \right) \tag{5.6.0}$$

$$= C \left(|V|^2_{L^2(\mathcal{S}^-)} - |\mathcal{B}V_-|^2_{L^2(\mathcal{S}^+)} \right) \geq 0,$$

Now, if there exists Y such that $(\mathbb{D}Y, Y) = 0$, the corresponding V satisfies:

$$|\mathcal{B}^* V_-|^2_{L^2(\mathcal{S}^+)} = |V_-|^2_{L^2(\mathcal{S}^-)}. \tag{5.6.0}$$

Equation (5.6) is possible only if V_- is a constant function of ω, on each connected component of $\mathcal{S}^-$ (i.e. an element of $\mathcal{C}^-$, see Section 4.2). Then, $V_- = JV_+$ is the same constant. Denoting by $V_0 = V|_{x=0}$, $V_1 = V|_{x=1}$ and using (5.5.2), we have, for $\omega_x > 0$:

$$V_0 = e^{-\gamma} V_1 + \frac{1}{|v||\omega_x|} \int_0^1 e^{-\gamma x'} \underline{y} \cdot R^+_{(x',\sigma)}(\underline{\omega})\, dx'$$

As in the proof of proposition 4.5.2, for V_0 to be independent of $\underline{\omega}$, the only possibility is that $y_1 = y_2 = 0$, which contradicts the fact that $|Y| > 0$, ending the proof of the lemma. ■

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