Adding one handle to half-plane layers

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Abstract

In this paper, we build properly embedded singly periodic minimal surfaces which have infinite total curvature in the quotient by their period. These surfaces are constructed by adding a handle to the toroidal half-plane layers defined by H. Karcher. The technics that we use are to solve a Jenkins-Serrin problem over a strip domain and to consider the conjugate minimal surface to the graph. To construct the Jenkins Serrin graph, we solve in fact the maximal surface equation and use an other conjugation technic.

Introduction

In a preceding paper [MRT], the author with M. Rodriguez and M. Traizet has constructed a family of properly embedded singly periodic minimal surfaces with an infinite number of Scherk-ends. These surfaces are built as conjugate surfaces to Jenkins-Serrin graphs over unbounded convex polygonal domains which are neither a strip nor a half-plane.

In this paper, we study the case where this unbounded convex polygonal domain is a strip. If we do exactly the same construction as in [MRT] for a strip, we will prove that the minimal surface that we obtain is actually doubly periodic. In fact, this give a new construction of well-known examples of properly embedded doubly periodic minimal tori with parallel ends. These surfaces are called *toroïdal half-plane layers* by H. Karcher who has built them in [Ka]. In the classification of properly embedded doubly periodic minimal tori with parallel ends made by J. Pérez, M. Rodriguez and M. Traizet in [PRT], these surfaces are denoted by $M_{\theta,\alpha,\frac{\pi}{2}}$ (see also [Ro1]).

The aim of this paper is to modify the above example in such a way that it loses its second period. The idea is to add one handle to this minimal surface. F. S. Wei [We] has been the first one to add handles to half-plane layers but he can only do it in a periodic way thus the surface is still doubly periodic. In [RTW], W. Rossman, E. C. Thayer and M. Wohlgemuth have also added handles to the toroïdal half-plane layers in a periodic way. In fact we will prove:

Theorem 1. *There exists a properly embedded singly periodic minimal surface in $\mathbb{R}^3$ whose quotient in $\mathbb{R}^2 \times \mathbb{S}^1$ has genus 1, an infinite number of parallel Scherk ends and two limit ends.*

We notice that, recently, L. Hauswirth and F. Pacard [HP] have constructed examples of properly embedded minimal surface in $\mathbb{R}^3$ with two limit ends asymptotic to half Riemann surfaces. Their construction consists in gluing a Costa-Hoffmann-Meeks surface of small genus between two half Riemann surfaces. These are the first examples of non periodic embedded minimal surfaces with limit ends. These examples illustrate the study of properly embedded minimal surface with finite genus and infinite number of ends developed by several authors. We refer to T. Colding and W. Minicozzi [CM] and W. H. Meeks, J. Pérez, and A. Ros [MPR1] and [MPR2].

Let us give some explanations on our construction. Recall how Scherk's singly periodic surface can be built. Let the function u be defined on the unit square $(-1/2, 1/2)^2$ by : $u(x, y) = \ln(\cos(\pi x)) - \ln(\cos(\pi y))$. The function u is a solution to the minimal surface equation which takes the value $+\infty$ on two opposite sides of the square and $-\infty$ on the other sides. The construction of such functions was generalized by H. Jenkins and J. Serrin in [JS]. The graph of u is a minimal surface bounded by four vertical straight-lines over the vertices of the square. This graph is a fundamental piece for the Scherk's doubly periodic surface. The conjugate surface to this graph is a minimal surface bounded by four horizontal symmetry curves lying in two horizontal planes at distance 1 from each other. By reflecting about one of the two symmetry planes, we get a fundamental domain for Scherk's singly periodic surface which has period $(0, 0, 2)$.

This construction was generalized by H. Karcher [Ka] and others authors. In this paper we replace the unit square by a strip. We see the boundary of the strip as the union of infinitely many unitary edges. So we want to find a solution to the minimal surface equation which take on the boundary the value $\pm\infty$ in alternating the sign on every edge. In the second section, we build such a solution and we describe the surface that we obtain when we consider the conjugate surface to the graph: as we said above this surface was already built by H. Karcher and is called half-plane layer.

In the last section, we prove that we can add a handle to a half-plane layer. This time, we work on a punctured strip. The solution to the minimal surface equation we need to construct is multi-valuated in such a way that its graph is bounded by a vertical straight-line over the removed point. The conjugate surface Σ to this graph is *a priori* periodic but we prove that the period vanishes. Besides it is bounded by horizontal symmetry curves which lie in the planes $\{z = 0\}$ and $\{z = 1\}$. By reflecting about one of this two symmetry planes we obtain a fundamental piece of the surface announced in Theorem 1. The vertical straight-line over the removed point corresponds in Σ to a closed horizontal curve in the handle. To build the multivalued minimal graph, we add one step to Karcher's technics. In fact, we solve a Dirichlet problem for the maximal surface equation on the punctured strip. By a conjugation technic, the solution to this Dirichlet problem gives us the multivalued graph. Besides, the use of maximal surface allows us to manage simply the vertical periods that could appear in the construction.

In the next section, we recall some tools that we use for the study of the Dirichlet problem associated to the minimal surface equation. We also introduce the conjugation between minimal and maximal surfaces.

The author would like to thank Martin Traizet for many helpful discussions and for having talked him about this problem that was asked by F. S. Wei.

1 Preliminaries

In this paper, we build minimal surfaces as graphs of functions. So in this section, we recall some facts about the minimal surface equation.

Let u be a function on a domain $\Omega \subset \mathbb{R}^2$. The graph of u is a minimal surface if u satisfies the minimal surface equation:

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \tag{$*$}$$

In their article [JS], H. Jenkins and J. Serrin study the Dirichlet problem with infinite boundary data associated to this partial differential equation in bounded domain. They give necessary and sufficient conditions to ensure existence of a solution.

One tools which is introduced is the conjugate 1-form:

$$\mathrm{d}\Psi_u = \frac{u_x}{\sqrt{1+|\nabla u|^2}}\mathrm{d}y - \frac{u_y}{\sqrt{1+|\nabla u|^2}}\mathrm{d}x$$

$\mathrm{d}\Psi_u$ is closed since u satisfies $(*)$. Then the function Ψ_u is locally defined and is called the conjugate function. Ψ_u has a geometric meaning since it is the third coordinate of the conjugate minimal surface to the graph of u

expressed in the x, y coordinates. If $v = \Psi_u$, we have $|\nabla v| < 1$, then $v = \Psi_u$ is 1-Lipschitz continuous. Besides the conjugate function v is a solution of the maximal surface equation:

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = 0 \tag{$**$}$$

Any solution v to $(**)$ satisfies $|\nabla v| < 1$. Then such a solution extends continuously to the boundary. When the function v can be written $v = \Psi_u$, some tools to compute v along the boundary are given by lemmas in [JS]. Let v be a solution to $(**)$ and let us define the following 1-form:

$$\mathrm{d}\Phi_v = \frac{v_y}{\sqrt{1-|\nabla v|^2}}\mathrm{d}x - \frac{v_x}{\sqrt{1-|\nabla v|^2}}\mathrm{d}y$$

$\mathrm{d}\Phi_v$ is closed because of $(**)$. Besides the function Φ_v is a solution of $(*)$. If u is a solution of $(*)$ and $v = \Psi_u$, we have $u = \Phi_v$.

In this paper, we need to study the convergence of sequences of solutions of $(*)$ or $(**)$. Because of the correspondence $u \longleftrightarrow \Psi_u$ and $v \longleftrightarrow \Phi_v$ the study is the same for both equations. Let us recall some points of the study of the convergence of a sequence (u_n) of solutions of $(*)$. Here the convergence we consider is the C^∞ convergence on every compact set. One can find proofs of all the facts below in [Ma2, Ma3].

Let p be a point in Ω such that $|\nabla u_n|(p) \to +\infty$. Passing to a subsequence, we can assume that $\nabla u_n/|\nabla u_n|(p) \to \nu$ where ν is an unitary vector in $\mathbb{R}^2$. Let L' be the straight-line which passes by p and is normal to ν, we denote by L the connected component of $L' \cap \Omega$ which contains p. Then for every point q in L we have $|\nabla u_n|(q) \to +\infty$ and $\nabla u_n/|\nabla u_n|(q) \to \nu$. L is called a divergence line of (u_n). Let us fix the orientation of L such that ν points to the right-hand side of L. Then if $q_1, q_2 \in L$, the convergence of the gradient implies:

$$\lim_{n\to+\infty} \int_{[q_1,q_2]} \mathrm{d}\Psi_{u_n} = |q_1q_2| \tag{1}$$

where $|q_1q_2|$ is the distance between q_1 and q_2.

To prove the convergence of a subsequence of (u_n), it suffices to prove that there is no divergence line. Indeed, if it is the case, the sequence $(|\nabla u_n|)$ is bounded at every point; then some estimates on the derivatives imply that we can ensure the convergence of a subsequence of $(u_n - u_n(p))$ for a point p in Ω. Since we shall study only Dirichlet problem with infinite boundary data, we notice that the vertical translation by $u_n(p)$ does not matter. We shall use some results that were developed in [Ma2] to prove that there is no divergence line.

Let us study a sequence (v_n) of solutions of $(**)$. Since $|\nabla v_n| < 1$, we consider points p where $\nabla v_n \to \nu$ with ν a unitary vector of $\mathbb{R}^2$. The divergence line L becomes the straight-line which is spanned by ν and, if $q_1, q_2 \in L$, Equation (1) becomes

$$\lim_{n\to+\infty} \int_{[q_1,q_2]} \mathrm{d}v_n = |q_1 q_2|$$

2 The first family: the toroïdal half-plane layers

In this section, we build a family of doubly periodic properly embedded minimal tori with parallel ends. These surfaces are called half-plane layers and were defined by H. Karcher [Ka] (see also [Ro1, Ro2]). Here these surfaces are constructed as the conjugate surface to a Jenkins-Serrin graph over an unbounded domain.

2.1 The Dirichlet problem

We begin in specifying the domains we shall consider.

Let P be a polygon in $\mathbb{R}^2$, we say that P is *a convex unitary* $2k$*-gon* if P is convex, has $2k$ edges and each one has unitary length.

Let Ω be a bounded polygonal domain, we say that Ω is a *unitary* $2k$*-polygonal domain* if the polygon $P = \partial\Omega$ is a convex unitary $2k$-gon. Let Ω be such a polygonal domain. We denote by $e_1, \cdots, e_{2k}$ the $2k$ edges of $\partial\Omega$ (the edges are numbered with respect to the orientation of $\partial\Omega$).

One can consider the following Dirichlet problem :

Problem 1. *To find a solution u of $(*)$ in Ω such that u takes the value $+\infty$ on e_{2p} and $-\infty$ on e_{2p-1} for $p \in \{1, \cdots, k\}$*

It is known that this Dirichlet problem has a solution if and only if $\partial\Omega$ is not special (see the definition below, [PT] and [MRT]). Besides the solution is unique up to a constant.

Let P be a convex unitary $2k$-gon, P is said to be special if $k \geq 3$ and there exist $v, w \in \mathbb{S}^1$ such that two edges of P are equal to $\pm v$ and all other edges are equal to $\pm w$.

We want to study this Dirichlet problem when the domain Ω becomes a strip. We notice that the case when Ω becomes an unbounded convex polygonal domain which is not a strip is studied in [MRT]. Let Ω be a strip of width $y_0 > 0$. We can normalize Ω to be $\mathbb{R} \times (0, y_0)$. For every $n \in \mathbb{Z}$, let $a_0(n)$ (resp. $a_1(n)$) denote the point $(2n, 0)$ (resp. $(2n+1, 0)$). Let $x_0 \in [-1, 1]$, then for every $n \in \mathbb{Z}$ we denote by $b_0(n)$ (resp. $b_1(n)$) the point $(x_0 + 2n - 1, y_0)$ (resp. $(x_0 + 2n, y_0)$) (see Figure 1). Hence the Dirichlet problem we will study is:

Problem 2. *To find a solution u of $(*)$ in Ω such that u takes the value $+\infty$ on $(a_0(n), a_1(n))$ and $(b_1(n), b_0(n+1))$ and the value $-\infty$ on $(a_1(n), a_0(n+1))$ and $(b_0(n), b_1(n))$ for every $n \in \mathbb{Z}$.*

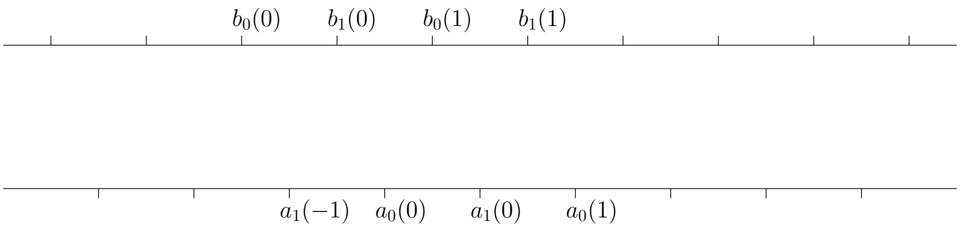

Figure 1:

This Dirichlet problem is parameterized by the parameter (x_0, y_0) which fixes the domain Ω and the boundary value. When (x_0, y_0) describes $[-1, 1] \times \mathbb{R}_+^*$, it is clear that we get all the Dirichlet problem in a strip that can be seen as the limit of Problem 1. Besides, the Dirichlet problem parameterized by $(-1, y_0)$ and $(1, y_0)$ are the same. Thanks to Collin's example [Co], it is known that we do not have uniqueness of solutions to this Dirichlet problem. Then to get uniqueness, we add an other boundary condition. This condition has an important geometrical meaning that we will explain soon. Thus for $(x_0, y_0) \in [-1, 1] \times \mathbb{R}_+$, we denote by $\mathcal{D}(x_0, y_0)$ the following Dirichlet problem:

Problem 3 ($\boldsymbol{\mathcal{D}(x_0, y_0)}$). *To find a solution u of $(*)$ in Ω such that u takes the value $+\infty$ on $(a_0(n), a_1(n))$ and $(b_1(n), b_0(n+1))$ and the value $-\infty$ on $(a_1(n), a_0(n+1))$ and $(b_0(n), b_1(n))$ for every $n \in \mathbb{Z}$ and*

$$\int_{a_0(0)}^{b_1(0)} \mathrm{d}\Psi_u = 1 \tag{2}$$

Let u be a solution of this Dirichlet problem. Since Ω is simply connected, the conjugate function Ψ_u is well defined; we fix Ψ_u by $\Psi_u(a_0(0)) = 0$. So because of the value of u on the boundary, for every n, if $p \in [a_1(n-1), a_1(n)]$, $\Psi_u(p) = |a_0(n)p|$ (see [JS]) ($|pq|$ denotes the Euclidean distance between p and q). The graph of Ψ_u on $\mathbb{R} \times \{0\}$ is then serrate. Because of (2), $\Psi_u(b_1(0)) = 1$ and, with the value of u on the boundary, for every $n \in \mathbb{Z}$ and $p \in [b_1(n-1), b_1(n)]$, $\Psi_u(p) = |b_0(n)p|$.

To build a solution to $\mathcal{D}(x_0, y_0)$, the idea is to truncate the domain Ω, solve a Jenkins-Serrin problem and take the limit. Thus we need to choose our truncation such that we can manage the hypothesis (2) and solve the Jenkins-Serrin problem. Since we know that the hypotheses of Jenkins-Serrin theorem [JS] are satisfied for Problem 1, we will use unitary polygonal domain. We need a lemma.

Lemma 2. *Let (x, y) be in $[-1, 1] \times \mathbb{R}_+$ such that $x^2 + y^2 > 1$. There exists a convex polygonal path γ in $[-1, +\infty) \times [0, y]$ that joins $(0, 0)$ to (x, y) and is composed of an odd number of unitary length edges.*

Proof. First there are two simple cases : $x = \pm 1$. If $x = 1$, we begin γ by one edge that joins $(0, 0)$ to $(1, 0)$ then we joins $(1, 0)$ to $(1, y)$ by passing by the point $p = (t, y/2)$; t is chosen such that the distance between p and $(1, 0)$ is an integer. When $x = -1$, this is the same idea : we end γ by an edge that joins $(0, y)$ to $(-1, y)$ and complete γ as an isosceles triangle (see Figure 2).

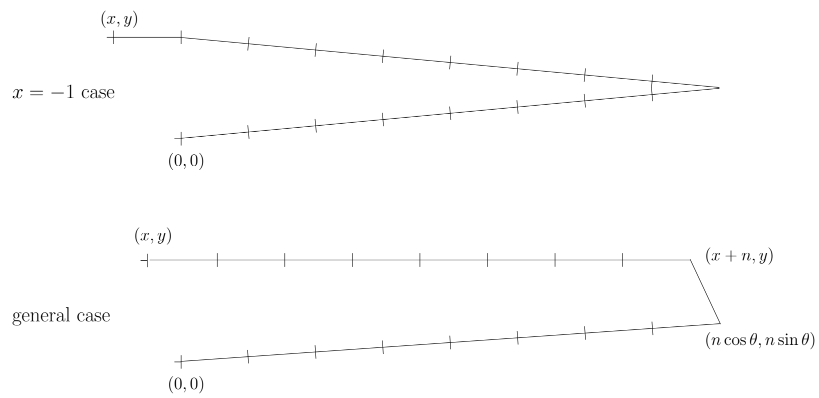

Figure 2:

Now, we assume that $x^2 < 1$. Let $n \in \mathbb{N}$ be a large integer and $\theta > 0$ be a small angle. We try to define γ as follow : the n first unitary edges of γ joins $(0, 0)$ to $(n\cos\theta, n\sin\theta)$, the n last unitary edges joins $(x + n, y)$ to (x, y), then, if the distance between $(n\cos\theta, n\sin\theta)$ and $(x+n, y)$ is one, we can complete γ by one unitary edge and γ will have $2n+1$ edges. So the idea is that we can choose n and θ such that this construction works and γ has all the expected properties (see Figure 2).

We want to chose n and θ such that :

$$(x+n(1-\cos\theta))^2+(y-n\sin\theta)^2=1$$

Then, with $t=1/n$, this equation becomes

$$(x^2+y^2-1)t^2+2\left(x(1-\cos\theta)-y\sin\theta\right)t+2(1-\cos\theta)=0$$

For small $\theta>0$, the discriminant is $4\theta^2(1-x^2)+o(\theta^2)>0$. Then one solution is:

$$t=\theta\left(\frac{y+\sqrt{1-x^2}}{x^2+y^2-1}\right)+o(\theta)$$

Hence there is small θ such that $1/t$ is an integer and we can complete γ. Let us see that γ satisfies the desired properties. First:

$$n\sin\theta=\frac{\sin\theta}{t}=\frac{x^2+y^2-1}{y+\sqrt{1-x^2}}+o(1)$$

Since $x^2-1<0$, $n\sin\theta<y$ then γ is in $[-1,+\infty)\times[0,y]$. Besides

$$\frac{y-n\sin\theta}{(x+n)-n\cos\theta}=\frac{1}{x}\left(y-\frac{x^2+y^2-1}{y+\sqrt{1-x^2}}\right)+o(1)$$

So when $x>0$, $\dfrac{y-n\sin\theta}{(x+n)-n\cos\theta}>\tan\theta$ for small θ; hence γ is convex. When $x\le 0$, γ is clearly convex. $\square$

Now we can solve the Dirichlet problem.

Theorem 3. *Let (x_0,y_0) be in $[-1,1]\times\mathbb{R}_+$. There exists a solution to $\mathcal{D}(x_0,y_0)$ if and only if $x_0^2+y_0^2>1$. Besides, when $x_0^2+y_0^2>1$, the solution is unique up to an additive constant.*

Proof. First, since $|\mathrm{d}\Psi_u|<1$, (2) proves that the condition $x_0^2+y_0^2>1$ needs to be satisfied for having a solution.

We now assume that $x_0^2+y_0^2>1$, we will prove the existence of a solution. We shall build a sequence (Ω_n) of bounded convex unitary $2k$-polygonal domain in Ω such that the parallelogram $a_0(n)b_1(n)b_0(-n+1)a_1(-n)$ is included in Ω_n. We build Ω_n as follows. From Lemma 2, there exists a convex polygonal arc γ that joins $a_0(n)$ to $b_1(n)$ with an odd number of unitary edges in $[2n-1,+\infty)\times[0,y_0]$. Let s be the symmetry with respect to the middle point of $[a_1(0),b_1(0)]$ (*i.e.* the point $((x_0+1)/2,y_0/2)$). The polygonal arc $s(\gamma)$ joins $b_0(-n+1)$ to $a_1(-n)$ in $(-\infty,-2n+1]\times[0,y_0]$. We then define Ω_n as the polygonal domain bounded by the following polygonal arc : the segment $[a_1(-n),a_0(n)]$, the arc γ, the segment $[b_1(n),b_0(-n+1)]$ and the arc $s(\gamma)$. By construction, Ω_n is a convex polygonal domain with an even number of unitary edges and the parallelogram $a_0(n)b_1(n)b_0(-n+1)a_1(-n)$ is included in Ω_n. Besides we notice that Ω_n is symmetric with respect to the middle point of $[a_1(0),b_1(0)]$.

Since $x_0^2+y_0^2>1$, the polygonal domain Ω_n is not special (see [MRT]) so there exists a solution u_n of $(*)$ in Ω_n that takes the value $+\infty$ on $[a_0(0),a_1(0)]$ and alternately the values $+\infty$ and $-\infty$ on each edge of $\partial\Omega_n$. Let Ψ_n be the conjugate function to u_n, we fix Ψ_n by $\Psi_n(a_0(0))=0$. By construction, the number of edges between $a_0(0)$ and $b_1(0)$ in $\partial\Omega_n$ is odd then $\Psi_n(b_1(0))=1$. Besides, by maximum principle, $0\le\Psi_n\le 1$ in Ω_n. We restrict the function u_n to the parallelogram $a_0(n)b_1(n)b_0(-n+1)a_1(-n)$ and we study the convergence of the sequence (u_n) on this increasing sequence of domain. Let us notice that on each edge of the parallelogram which is included in $\partial\Omega$, the function u_n takes the value that we prescribe in our Dirichlet problem $\mathcal{D}(x_0,y_0)$.

Let us study the lines of divergence of this sequence on the limit domain Ω. Because of the value on the boundary the end points of a line of divergence must be vertices of Ω (see Lemma A.3 in [Ma2]). Let L be a line of divergence and let p and q in L. Thus, we have for a subsequence:

$$\lim_{n\to+\infty}|\Psi_n(p)-\Psi_n(q)|=|pq| \tag{3}$$

Since for every n, $0\le\Psi_n\le 1$, this implies that $|pq|\le 1$. So L must have two end-points: one in $\{y=0\}$ and one in $\{y=y_0\}$. On each vertex of Ω, Ψ_n takes the value 0 or 1. If we apply (3) with p and q the end-points of L, we can conclude that one end-point is $a_i(k)$ and the other one is $b_{1-i}(l)$ with $i\in\{0,1\}$ and $1=|a_i(k)b_{1-i}(l)|$. But since $x_0^2+y_0^2>1$, the distance between one $a_i(k)$ and one $b_{1-i}(l)$ is greater than 1. Hence the sequence has no divergence line.

So a sub-sequence of (u_n) converges in Ω to a solution of $(*)$. Since the sequence u_n takes the good boundary value on each edges for every n, the function u takes the expected boundary value. Moreover:

$$\int_{[a_0(0),b_1(0)]}\mathrm{d}\Psi_u=\lim_{n\to+\infty}\int_{[a_0(0),b_1(0)]}\mathrm{d}\Psi_n=\Psi_n(b_1(0))-\Psi_n(a_0(0))=1$$

This ends the proof of the existence.

We finish by the uniqueness proof. As we explain above, if u and u' are two solutions to $\mathcal{D}(x_0, y_0)$, the boundary value of Ψ_u and $\Psi_{u'}$ are the same. Then Ψ_u and $\Psi_{u'}$ are two bounded solutions of the maximal surface equation with the same boundary value (Ψ_u and Ψ'_u are bounded since every point in Ω is at a bounded distance from the boundary). The uniqueness result in [Ma1] (Corollary 2.4) implies that $u - u'$ is constant. $\square$

Let us make one remark. In the construction, we have $0 \le \Psi_n \le 1$, then the solution u satisfies $0 \le \Psi_u \le 1$.

2.2 The half-plane layers

Using the solution to $\mathcal{D}(x_0, y_0)$, we are then able to build a doubly periodic properly embedded minimal tori with parallel ends.

Let (x_0, y_0) be in $[-1, 1] \times \mathbb{R}_+$ such that $x_0^2 + y_0^2 > 1$. Let u be the solution to $\mathcal{D}(x_0, y_0)$ given by Theorem 3. First, because of the boundary value of u, the boundary of the graph of u is composed of vertical straight lines over each vertex of Ω.

Let t by the translation by the vector $(-2, 0)$. The function $u \circ t$ which is defined in Ω is also a solution to $\mathcal{D}(x_0, y_0)$. Hence there exists a constant $k \in \mathbb{R}$ such that $u \circ t = u + k$. The graph of u is then periodic with $(2, 0, k)$ as period. Besides $u \circ t = u + k$ implies $\Psi_u \circ t = \Psi_u + k'$; the boundary values show that $k' = 0$.

Let Σ denote the conjugate minimal surface to the graph of u. Σ is the conjugate surface to a graph over a convex domain, by R. Krust Theorem, Σ is also a graph; it is then embedded. Since $0 \le \Psi_u \le 1$, Σ is included in $\mathbb{R}^2 \times [0, 1]$.

Let us study the boundary of Σ. In the neighborhood of each vertex p of Ω, the graph of u is bounded by a vertical straight-line. In Σ, the conjugate of this vertical straight-line is an horizontal symmetry curve. This curve is in the $\{z = 0\}$ plane when $p = a_0(k)$ or $p = b_0(k)$ (because $\Psi_u(p) = 0$) and it is in the $\{z = 1\}$ plane when $p = a_1(k)$ or $p = b_1(k)$. The boundary of Σ is then composed of all these horizontal curves, they are drawn in Figure 3.

Since the graph of u is periodic, the minimal surface Σ is also periodic. Let X_1^*, X_2^* and X_3^* be the three functions on Ω that give the three coordinates of Σ. We have $X_3^* = \Psi_u$. The period of Σ is given by:

$$(X_1^*, X_2^*, X_3^*)\,(2, y_0/2) - (X_1^*, X_2^*, X_3^*)\,(0, y_0/2)$$

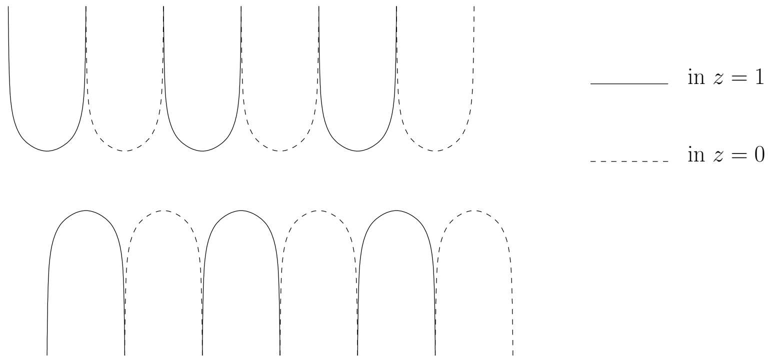

Figure 3:

Since $\Psi_u \circ t = \Psi_u$, the third coordinate of the period is zero. As Σ is a graph, the horizontal part of the period does not vanish: Σ has a non-zero horizontal period.

Since Σ is embedded in $\mathbb{R}^2 \times [0,1]$ with boundary in $\{z=0\}$ and $\{z=1\}$, we can extend Σ into $\widetilde{\Sigma}$ by symmetry with respect to all the horizontal planes $\{z=n\}$ $(n \in \mathbb{Z})$. The surface $\widetilde{\Sigma}$ is then a doubly periodic embedded minimal surface. The two periods are the horizontal one that comes from Σ and the vertical period $(0,0,2)$ that comes from the horizontal symmetries.

2.3 Scherk-type ends

In this subsection, we prove that the annular ends of the above surface are Scherk-type. In fact this result is local; so it applies to the surfaces we build in the next section.

Let us consider, for example, the end given by a neighborhood of $(a_0(n), a_1(n))$. Let γ be a circle-arc in Ω that joins $a_0(n)$ to $a_1(n)$. Since u converges to $+\infty$ on $(a_0(n), a_1(n))$, taking γ close to $(a_0(n), a_1(n))$, we can assume that, over the subset D of Ω bounded by $\gamma \cup (a_0(n), a_1(n))$, the normal to the graph of u is close to $(0,-1,0)$. Let us consider the conjugate surface to the graph of u over D; we complete it by symmetry and we get in the quotient of $\mathbb{R}^3$ by $(0,0,2)$ a complete annulus A bounded by a compact curve. On A, the Gauss map is close to $(0,-1,0)$. Thus by a result of J. L. Barbosa and M. do Carmo [BC], A is stable. By D. Fischer-Colbrie [Fi], this implies that A has finite total-curvature. Finally the classification of annular ends by W. Meeks and H. Rosenberg in [MR] says that A is Scherk-type since the limit normal to the end is horizontal.

3 How to add one handle ?

In this section, we shall construct a family of singly-periodic properly embedded minimal tori with an infinite number of parallel ends and two limit ends. First, let us consider one of the preceding examples and consider it as an embedded minimal surface with infinite total curvature in $\mathbb{R}^3/T$ with T the vertical period (it has genus 0). Then the idea to build our new family is to had one handle to this periodic minimal surface (see Figure 4). With this handle the surface loses its horizontal period so it has an infinite number of ends. As we said in the introduction, F. S. Wei [We], W. Rossman, E. C. Thayer and M. Wohlgemuth [RTW] have added in a periodic way an infinite number of handles to the most symmetric half-plane layers; we refer to their papers for pictures of such surfaces.

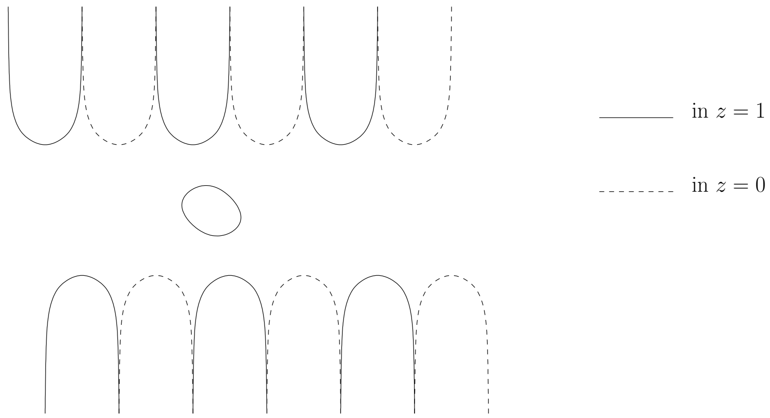

Figure 4:

As in the preceding section, to make this construction, we begin in solving a Dirichlet problem and then we consider the conjugate minimal surface to the graph.

3.1 The Dirichlet problem

Let (x_0, y_0) be in $[-1, 1] \times \mathbb{R}_+$, we denote by Ω the polygonal domain associated to (x_0, y_0) as in Section 2. Let c be the middle point of $[a_1(0), b_1(0)]$, the coordinates of c are $(\frac{x_0+1}{2}, \frac{y_0}{2})$.

We shall solve a Dirichlet problem for the maximal surface equation in $\Omega\backslash\{c\}$.

Theorem 4. *Let (x_0, y_0) be in $[-1,1]\times\mathbb{R}_+$. We assume that $(x_0+1)^2+y_0^2 > 4$. Then there exists a solution v to $(**)$ with the following boundary values:*

- *if $p \in [a_1(n-1), a_1(n)]$, $v(p) = |a_0(n)p|$*
- *if $p \in [b_1(n-1), b_1(n)]$, $v(p) = |b_0(n)p|$*
- *$v(c) = 1$*

Besides the solution v is unique.

Proof. First let us remark that the condition $(x_0+1)^2+y_0^2 > 4$ is necessary since it says that the distance between $a_0(0)$ and c is greater than 1, and $v(c) - v(a_0(0)) = 1$.

We notice that since a solution is bounded in the boundary and Ω is a strip a solution v is bounded in Ω; hence the uniqueness of the solution is a consequence of Theorem 2.2 in [Ma1].

Let us now prove the existence part of the theorem. Let n be an integer and consider the domain Ω_n that we defined in Theorem 3 proof. We also have the solution u_n to $(*)$ and its conjugate Ψ_n. Let us denote φ the boundary value of Ψ_n on $\partial\Omega_n$.

Since Ψ_n satisfies $|\nabla\Psi_n| < 1$ in Ω_n, we get that for every $p, q \in \partial\Omega_n$:

$$|\varphi(p) - \varphi(q)| \leq d_{\Omega_n}(p, q) \text{ and} \tag{4}$$
$$|\varphi(p) - \varphi(q)| < d_{\Omega_n}(p, q) \text{ if } [p, q]\backslash\partial\Omega_n \neq \emptyset \tag{5}$$

Here, d_{Ω_n} denotes the intrinsic distance in Ω_n, but since Ω_n is convex this distance is the classical Euclidean distance in $\mathbb{R}^2$ (see also Theorem 1 in [KM]).

Since c is at a distance greater than 1 from $a_0(0)$ and $c \in [0,1]\times\mathbb{R}_+$, c is at a distance greater than 1 from every point in $\partial\Omega_n$ where φ vanishes. This implies that:

$$\forall p \in \partial\Omega_n, \quad |\varphi(p) - 1| < d_{\Omega_n}(p, c) \tag{6}$$

Then equations (4), (5) and (6) implies that there exists a solution v_n of $(**)$ in $\Omega_n\backslash\{c\}$ which have φ as boundary value in $\partial\Omega_n$ and $v_n(c) = 1$; this result is a consequence of Theorem 5 (see Theorem 1 in [KM] and Theorem 4.1 in [BS]). We remark that, by maximum principle, we have $0 \leq v_n \leq 1$.

Theorem 5. *Let $D \subset \mathbb{R}^2$ be a bounded domain. Let $\mathcal{S} \subset D$ be a finite set. Let $\varphi : \partial D \cup \mathcal{S} \to \mathbb{R}$ be a function such that*

$$\forall p, p' \in \partial D \cup \mathcal{S}, \quad p \neq p', \quad |\varphi(p) - \varphi(p')| \leq d_D(p, p') \tag{7}$$

where the inequality is strict whenever the segment $[p,p']$ is not contained in ∂D. Then there exists a function $v : D \to \mathbb{R}$ which satisfies $(**)$ in $D \setminus \mathcal{S}$, with boundary data $v = \varphi$ on $\partial D \cup \mathcal{S}$. (Here d_D is the intrinsic distance in D.)

The solution v will be constructed as the limit of the sequence (v_n). We consider the restriction of the function v_n to the parallelogram $a_0(n)b_1(n)b_0(-n+1)a_1(-n)$ minus the point c. This increasing sequence of domains converges to $\Omega\backslash\{c\}$. We notice that, as in Theorem 3 proof, on each edges of Ω, v_n takes the value which is prescribed in the Dirichlet problem for v.

Let us study divergence lines of the sequence (v_n). Let L be such a line. Because of the boundary value of v_n on $\partial\Omega$, the end-points of L can only be vertices of Ω or the point c. Let p and q be in L; since L is a line of divergence, we have for a subsequence:

$$\lim_{n\to+\infty} |v_n(p) - v_n(q)| = |pq| \tag{8}$$

Since $0 \le v_n \le 1$, this equation implies that $|pq| \le 1$ and the length of L needs to be less than 1: L has two end-points. First let us assume that one end-point of L is c. If the other end-point p is $a_1(k)$ or $b_1(k)$, $v_n(c)-v_n(p) = 0$ then Equation (8) implies $|cp| = 0$ which is impossible. So the other end-point p is $a_0(k)$ or $b_0(k)$. So $v_n(c) - v_n(q) = 1$ and by (8) $|cp| = 1$; but $(x_0 + 1)^2 + y_0^2 > 4$, then $|cp| > 1$. Hence one end-point of L is in $\{y = 0\}$ and the other one is in $\{y = y_0\}$. Since $x_0^2 + y_0^2 > 1$, the same arguments as in Theorem 3 prove that such a divergence line do not exist. Hence the sequence (v_n) has no divergence line and for a subsequence (v_n) converges to a solution v of $(**)$ which takes the expected boundary value on $\partial\Omega$ and c. □

We recall that s denotes the symmetry with respect to the point c. For every $n \in \mathbb{N}$ we have $s(a_0(n)) = b_0(-n+1)$ and $s(a_1(n)) = b_1(-n)$. So, if v is the function given by Theorem 4, $v \circ s$ is also a solution to this Dirichlet problem, thus $v \circ s = v$.

Let t be the translation in $\mathbb{R}^2$ of vector $(-2, 0)$. Let us study the sequence $(v_n) = (v \circ t^n)$. Since $t(a_i(k)) = a_i(k-1)$ and $t(b_i(k)) = b_i(k-1)$, the function v_n is defined on $\Omega\backslash\{t^{-n}(c)\}$ and its boundary value $v_n(p)$ is $|a_0(k)p|$ in $[a_1(k-1), a_1(k)]$ and $|b_0(k)p|$ in $[b_1(k-1), b_1(k)]$. The sequence of domains converges to Ω. The discussion in Theorem 3 proof proves that (v_n) has no divergence line. So, for a subsequence, we can assume that (v_n) converges to v_∞ a solution in Ω with boundary value:

$$v_\infty(p) = \begin{cases} |a_0(k)p| \text{ in } [a_1(k-1), a_1(k)] \\ |b_0(k)p| \text{ in } [b_1(k-1), b_1(k)] \end{cases}$$

Let u_0 be the solution to $\mathcal{D}(x_0, y_0)$ given by Theorem 3, v_∞ has the same boundary value as Ψ_{u_0}, by uniqueness of the solution $v_\infty = \Psi_{u_0}$. Besides this implies that Ψ_{u_0} is the only possible limit for a subsequence of (v_n) then the sequence (v_n), itself, converges to Ψ_{u_0}.

Let us make another remark. When the parameter (x_0, y_0) moves in its allowed domain, the domain $\Omega\backslash\{c\}$ and the boundary value for v change continuously. Hence the discussion about divergence lines in Theorem 4 proof shows that the solution v depends continuously of the parameter (x_0, y_0).

3.2 The minimal graph

Let (x_0, y_0) satisfy the hypotheses of Theorem 4 and v be the associated solution of $(**)$ in Ω. The 1-form $\mathrm{d}\Phi_v$ locally defines a function u which is a solution of $(*)$. In $\Omega\backslash\{c\}$, the function u can be seen as a multi-valuated function *i.e.* when we make a turn around c, we add a constant k to u. The graph of u has a vertical period.

Because of the boundary value of v along $\partial\Omega$, the function u takes the value $+\infty$ on the edges $[a_0(n), a_1(n)]$ and $[b_1(n), b_0(n+1)]$ and the value $-\infty$ on $[a_1(n), a_0(n+1)]$ and $[b_0(n), b_1(n)]$ $(n \in \mathbb{Z})$ (see Lemma 4 in [MRT]). A part of the boundary of the graph of v is then composed of vertical straight-line over the vertices of Ω. The last boundary part of the graph is given by the behaviour near c. In fact the graph of v is bounded by a a vertical straight-line over c: to see this, we consider u as a well defined function on the universal cover of a neighborhood of c and we apply Theorem 4.2 in [Ma2], u satisfies the two hypotheses since the graph of u has a vertical period and, in Ω, $v \le 1 = v(c)$. Besides the equation $v \circ s = v$ implies that $u \circ s = u + k/2$; this equation is written on the universal cover of $\Omega\backslash\{c\}$.

The convergence $v \circ t^n \to \Psi_{u_0}$ implies that $(u \circ t^n)$ converges to u_0. In a certain sense, it says that the asymptotic behaviour of the graph of u is given by the graph of u_0.

Besides, as v, the function u depends continuously in the parameter $(x_0, y_0) \in [-1, 1] \times \mathbb{R}_+^* \cap \{(x_0+1)^2 + y_0^2 > 4\}$.

3.3 The conjugate surface : half-plane layer with one handle

In this subsection, we study the conjugate surface to the graph of u. The surface that we obtain is the one that was announced in Theorem 1.

Let Σ be this conjugate. Since the graph of u has a vertical period, Σ is also periodic; but in fact we have:

Proposition 6. *The periods of Σ vanish.*

Proof. Let γ be the circle of center c and radius ε in Ω; γ is a generator of the homotopy group of $\Omega\backslash\{c\}$. Let $\mathrm{d}X_1^*$, $\mathrm{d}X_2^*$ and $\mathrm{d}X_3^*$ the three coordinate 1-forms in $\Omega\backslash\{c\}$ of the surface Σ. These 1-forms depend only in the first derivatives of u which is why they are well defined on $\Omega\backslash\{c\}$. The period of Σ is then given by:

$$\int_\gamma (\mathrm{d}X_1^*, \mathrm{d}X_2^*, \mathrm{d}X_3^*)$$

Since $\mathrm{d}X_3^* = \mathrm{d}\Psi_u = \mathrm{d}v$ the third coordinate of the period vanishes. The equation $u \circ s = u + k/2$ implies that:

$$s^*(\mathrm{d}X_1^*, \mathrm{d}X_2^*) = -(\mathrm{d}X_1^*, \mathrm{d}X_2^*)$$

Since $s(\gamma)$ is in the same homotopy class of $\pi_1(\Omega\backslash\{c\})$ as γ, we obtain.

$$\begin{aligned}\int_\gamma (\mathrm{d}X_1^*, \mathrm{d}X_2^*) = \int_{s(\gamma)} (\mathrm{d}X_1^*, \mathrm{d}X_2^*) &= \int_\gamma s^*(\mathrm{d}X_1^*, \mathrm{d}X_2^*) \\ &= \int_\gamma -(\mathrm{d}X_1^*, \mathrm{d}X_2^*) \\ &= -\left(\int_\gamma (\mathrm{d}X_1^*, \mathrm{d}X_2^*)\right)\end{aligned}$$

Then the two horizontal periods vanish. $\square$

Σ has then the topology of an annulus. Besides since $u \circ s = u + k/2$, the surface Σ is symmetric with respect to a vertical axis. We denote by S this symmetry.

Since $0 \le v \le 1$, the surface Σ is included in $\mathbb{R}^2 \times [0,1]$. Let us study the boundary of Σ. First, in a neighborhood of each vertex p of Ω, the graph of u is bounded by a vertical straight-line over p. Then the conjugate of this line in Σ is a horizontal symmetry curve. Because of the value of v on the vertices, if $p = a_0(n)$ or $p = b_0(n)$, the curve is in the $z = 0$ plane; if $p = a_1(n)$ or $p = b_1(n)$, the curve is in the $z = 1$ plane. The last boundary component of Σ comes from the conjugate of the vertical line over

c. This conjugate is a horizontal closed curve Γ in the $z = 1$ plane: the curve is closed since the period vanishes and then it is the conjugate of a segment included in the vertical line, Γ is in $\{z = 1\}$ since $v(c) = 1$. Besides Γ is embedded since it is locally convex and has total curvature 2π. The convexity comes from the fact that it is the conjugate of a vertical segment in the boundary of a graph and the total curvature is 2π since this vertical segment correspond to one turn around the point c then the gauss map which is horizontal describes only one time $\mathbb{S}^1$. Because of the symmetry of Σ, the Jordan curve Γ is symmetric with respect to S. All these boundary curves are drawn in Figure 4.

The last property of Σ is the following.

Proposition 7. *The surface Σ is embedded.*

Proof. Let Σ_1 be the conjugate of the part of the graph of u which is over $\Omega \cap \{y < y_0/2\}$. Σ_1 is a graph over a domain D_1 since $\Omega \cap \{y < y_0/2\}$ is convex; so Σ_1 is embedded. Let Σ_2 be the conjugate of the part of the graph of u which is over $\Omega \cap \{y > y_0/2\}$. Σ_2 is also a graph over a domain that we denote by D_2.

Let us study what happens over $y = y_0/2$. Let $\mathcal{C}_-$ be the projection in the horizontal plane $\{z = 0\} = \mathbb{R}^2$ of the conjugate to the curve in the graph of u which is over $(-\infty, (x_0 + 1)/2) \times \{y_0/2\}$, we parameterize $\mathcal{C}_-$ by $(-\infty, (x_0 + 1)/2)$. In the same way, let $\mathcal{C}_+$ be the horizontal projection of conjugate to the curve in the graph of u which is over $((x_0 + 1)/2, +\infty) \times \{y_0/2\}$, $\mathcal{C}_+$ is parameterized by $((x_0 + 1)/2, +\infty)$.

Let us describe the boundary component of D_1 that corresponds to the boundary component $y = y_0/2$ of $\Omega \cap \{y < y_0/2\}$. This boundary component is composed of the union of $\mathcal{C}_-$, one half of the horizontal projection of Γ and $\mathcal{C}_+$. The boundary component of D_2 associated to $\{y = y_0\}$ is the union of $\mathcal{C}_-$, the other half of the projection of Γ and $\mathcal{C}_+$. Since along $\mathcal{C}_-$ and $\mathcal{C}_+$ the domains D_1 and D_2 are not on the same side of the boundary, the union of $\mathcal{C}_-$, $\mathcal{C}_+$ and the horizontal projection of Γ is embedded in $\mathbb{R}^2$.

Since the periods of Σ vanish, there exist two functions X_1^* and X_2^* on $\Omega \backslash \{c\}$ that give the first two coordinates of Σ. We obtain:

$$\mathcal{C}_-'(x) = (\frac{\partial}{\partial x} X_1^*, \frac{\partial}{\partial x} X_2^*)(x, y_0/2)$$

Let $n \in \mathbb{N}$, we have $\mathcal{C}_-(x - 2n) = (X_1^*, X_2^*) \circ t^n(x, y_0/2)$. The convergence of $u \circ t^n$ to u_0 implies that $(\frac{\partial}{\partial x} X_1^*, \frac{\partial}{\partial x} X_2^*) \circ t^n$ converges to $(\frac{\partial}{\partial x} X_{01}^*, \frac{\partial}{\partial x} X_{02}^*)$ on Ω where X_{01}^* and X_{02}^* are the two first coordinates of the conjugate surface to u_0. This implies that $||\mathcal{C}'_-||$ is bounded in $\mathbb{R}_-$ then $\mathcal{C}_-$ is Lipschitz continuous. Besides:

$$\begin{aligned}\lim_{n\to+\infty}\mathcal{C}_-(-2n+2)-\mathcal{C}_-(-2n) &= \lim_{n\to+\infty}\int_{-2n}^{-2n+2}\mathcal{C}'_-(x)\mathrm{d}x\\ &= \lim_{n\to+\infty}\int_0^2\left(\frac{\partial X_1^*}{\partial x},\frac{\partial X_2^*}{\partial x}\right)\circ t^n(x,y_0/2)\mathrm{d}x\\ &= \int_0^2\left(\frac{\partial}{\partial x}X_{0}{}_1^*,\frac{\partial}{\partial x}X_{0}{}_2^*\right)(x,y_0/2)\mathrm{d}x\end{aligned}$$

The last integral is the horizontal period of the conjugate surface to the graph of u_0 so this vector is non zero. Since this limit does not vanish and $\mathcal{C}_-$ is Lipschitz continuous, we obtain

$$\lim_{x\to-\infty}\mathcal{C}_-(x)=\infty$$

Since the surface Σ is symmetric, we have $\mathcal{C}_+(x)=S(\mathcal{C}_-(x_0+1-x))$ then $\lim_{x\to+\infty}\mathcal{C}_+(x)=\infty$. Let us compactify $\mathbb{R}^2$ into a sphere $\mathbb{S}^2$, the curves $\mathcal{C}_-$ and $\mathcal{C}_+$ joins at infinity. Hence the union of $\mathcal{C}_-$, $\mathcal{C}_+$ and Γ cuts the sphere into three connected components. Since D_1 and D_2 are connected and $\mathcal{C}_-$, $\mathcal{C}_+$ and Γ are in their boundary, each D_i is included in one of these three connected components. Over a neighborhood of the point $\mathcal{C}_-(-1)$, Σ is graphical thus D_1 and D_2 do not lie in the same connected component of $\mathbb{S}^2\backslash(\mathcal{C}_-\cup\mathcal{C}^+\cup\Gamma)$. Hence D_1 and D_2 do not intersect themselves: Σ_1 and Σ_2 are disjointed and so $\Sigma=\overline{\Sigma_1\cup\Sigma_2}$ is embedded. □

Finally Σ is an embedded minimal surface in $\mathbb{R}^2\times[0,1]$ with boundary in $\{z=0\}$ and $\{z=1\}$. Besides we can extend Σ into $\widetilde{\Sigma}$ by symmetry with respect to the horizontal plane $\{z=n\}$. $\widetilde{\Sigma}$ is then an embedded minimal surface in $\mathbb{R}^3$ with one vertical period $(0,0,2)$. The quotient surface is a torus with an infinite number of parallel Scherk ends (see Sub-section 2.3). Theorem 1 is then proved.

Let us also remark that because of the convergence of $(u\circ t^n)$ to u_0, we can say that Σ has in a certain sense an asymptotic behaviour given by the half-plane layer parametrized by (x_0,y_0). It implies, for example, that Σ has infinite total curvature.

We notice that, since v depends continuously on (x_0,y_0), we can say that, in a certain sense, Σ depends continuously on (x_0,y_0). In particular, compact parts of Σ depend continuously.

3.4 Two particular cases

In this subsection, we study two particular range for the parameter (x_0, y_0).

3.4.1 $x_0 = 1$ and $y_0 > 0$

In this case, the point c is $(1, y_0/2)$. Besides the point $a_i(n)$ has the same abscissa as $b_i(n)$. Then, in the Dirichlet problem for v, we have two symmetries for the boundary value φ:

1. $\varphi(x, 0) = \varphi(x, y_0)$ for every $x \in \mathbb{R}$.
2. $\varphi(x, y) = \varphi(2 - x, y)$ for every $x \in \mathbb{R}$ and $y \in \{y_0\}$.

Because of the uniqueness in Theorem 4, the solution v has the same symmetries *i.e.* $v(x, y) = v(x, y_0 - y)$ and $v(x, y) = v(2 - x, y)$ for every $(x, y) \in \Omega \backslash \{c\}$.

For the surface Σ, this implies that we have two vertical orthogonal planes of symmetry. Let us observe the intersection of Σ with the plane that corresponds to the symmetry $v(x, y) = v(2 - x, y)$. This curve is the conjugate of the curve which is over $\{x = 1\}$ in the graph of u. This curve has two components which are symmetric with respect to the other plane and each one joins a boundary component in the plane $\{z = 1\}$ to Γ which is also in $\{z = 1\}$. Thus, following the terminology of W. Rossman, E. C. Thayer and M. Wohlgemuth in [RTW], the handle of the surface $\widetilde{\Sigma}$ is a $'-'$ type handle.

We notice that combining the two planar symmetries we recover the symmetry with respect to a vertical axis. Besides, to build the surface Σ we only need one fourth of this surface. This fourth of a surface can be seen as the conjugate of the graph of u over $[1, +\infty) \times [0, y_0/2]$ which is convex. So this fourth of Σ is a graph over a domain, the symmetries implies that this domain is included in an angular sector of angle $\pi/2$. Then, by symmetry, the whole surface Σ is a graph and is embedded. In this case, it is easier to see the embeddedness.

We remark that it seems possible (but difficult) to prove the embeddedness of Σ for all (x_0, y_0) in using the embeddedness in the particular case $x_0 = 1$ and the continuity in the parameter.

3.4.2 $x_0 = -1$ and $y_0 > 2$

In this case, the point c is $(0, y_0/2)$ and the point $a_i(n)$ has the same abscissa as $b_i(n+1)$. Thus, we also have two symmetries for the boundary value of v.

Because of the uniqueness of the solution, v satisfies: $v(x,y) = v(x, y_0 - y)$ and $v(x,y) = v(-x,y)$ for every $(x,y) \in \Omega \backslash \{c\}$.

So we have also two vertical orthogonal planes of symmetry for the minimal surface Σ. The intersection of Σ with the plane that corresponds to $v(x,y) = v(-x,y)$ is then the conjugate of the curve which is over $\{x = 0\}$ in the graph of u. This conjugate curve has two connected components which are symmetric with respect to the other symmetry. Both components join a boundary curve of Σ which is in $\{z = 0\}$ to $\Gamma \subset \{z = 1\}$. Then, this times, the handle of $\widetilde{\Sigma}$ is a $'+'$ type handle (see [RTW]).

Thus, when x_0 goes from 1 to -1, the surface $\widetilde{\Sigma}$ is continuously deformed from a surface with a $'-'$ type handle into a surface with a $'+'$ type handle.

As in the preceding particular case, it is easy to prove the embeddedness by using the symmetries.

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