Minimal hypersurfaces asymptotic to Simons cones

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Abstract

In this paper, we prove that, up to similarity, there are only two minimal hypersurfaces in $\mathbb{R}^{n+2}$ that are asymptotic to a Simons cone, *i.e.* the minimal cone over the minimal hypersurface $\sqrt{\frac{p}{n}}\mathbb{S}^p\times\sqrt{\frac{n-p}{n}}\mathbb{S}^{n-p}$ of $\mathbb{S}^{n+1}$

1 Introduction

One important property of minimal hypersurfaces is the monotonicity formula. If Σ is a proper minimal hypersurface in $\mathbb{R}^{n+2}$, it says that the quantity

$$\theta(p,r)=\frac{1}{\omega_{n+1}r^{n+1}}\mathrm{Vol}(\Sigma\cap B(p,r))$$

is a non decreasing function of r (here ω_{n+1} is the volume of the unit ball of dimension $n+1$ and $B(p,r)$ denote the ball of $\mathbb{R}^{n+2}$ centered at p and radius r).

Hence we can define the density at infinity of Σ as $\theta_\infty(\Sigma)=\lim_\infty\theta(p,r)$. The monotonicity implies that $\theta_\infty(\Sigma)\ge 1$ and $\theta_\infty(\Sigma)=1$ iff Σ is a hyperplane. One interesting question is to understand the gap between this value 1 and the density at infinity of Σ for Σ not a hyperplane.

When $\theta_\infty(\Sigma)$ is finite, the asymptotic behaviour of Σ is given by a minimal cone which is the limit of a blow-down sequence $(t_i\Sigma)_{i\in\mathbb{N}}$ with $t_i\searrow 0$ (here the limit is in the varifold sense). This cone has density $\theta_\infty(\Sigma)$ so the study of minimal cones is important to understand what are the possible densities at infinity.

In dimension 3 ($n=1$), it is known that $\theta_\infty(\Sigma)\ge 2$ and it is conjectured that this value 2 is only realized by catenoids and singly periodic Scherk surfaces (see [7] for a partial answer by Meeks and Wolf). In dimension 4

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($n = 2$), the proof of the Willmore conjecture by Marques and Neves [6] implies that $\theta_\infty(\Sigma) \geq \pi/2$ if Σ is non planar and this value corresponds to the cone over a Clifford torus. In higher dimension, good candidates for the lowest value of the density at infinity are the one of the cone over product of spheres. More precisely the submanifold

$$S_{n,p} = \sqrt{\frac{p}{n}}\mathbb{S}^p \times \sqrt{\frac{n-p}{n}}\mathbb{S}^{n-p}$$

is a minimal hypersurface of $\mathbb{S}^{n+1}$ (notice that $S_{2,1}$ is a Clifford torus). The cone $C_{n,p}$ over $S_{n,p}$ is a good candidate for the lowest density at infinity; such a cone (and its image by linear isometry) is called a Simons cone. More precisely, if n is even it is conjectured that the density of $C_{n,n/2}$ is a lower bound for the density at infinity of a non planar minimal hypersurface of $\mathbb{R}^{n+2}$ and, if n is odd, the lower bound is given by $C_{n,(n-1)/2}$. The best known result about that question is given by Ilmanen and White in [5]; they obtain lower bounds for the density of some area-minimizing cones under topological assumptions.

The aim of this paper is to understand the minimal hypersurfaces whose asymptotic behaviour is given by $C_{n,p}$. This cone is invariant by the subgroup $O_{p+1}(\mathbb{R}) \times O_{n-p+1}(\mathbb{R})$ of $O_{n+2}(\mathbb{R})$. Actually we are going to prove that a minimal hypersurface asymptotic to $C_{n,p}$ is also invariant by this subgroup. This implies that, up to homotheties and translations, there are only two such hypersurfaces. So our main result can be stated as follows.

Theorem 1. *For any $n \geq 2$ and $1 \leq p \leq n-1$ there are two minimal hypersurfaces $\Sigma_{n,p,\pm} \subset \mathbb{R}^{n+2}$ such the following is true. If Σ is a minimal hypersurface of $\mathbb{R}^{n+2}$ asymptotic to a Simons cone, then $\Sigma = f(\Sigma_{n,p,\pm})$ for some $p \in \{1, \dots, n-1\}$, sign $\pm$ and a similarity f. Moreover $\Sigma_{2p,p,+} = \Sigma_{2p,p,-}$.*

After writing the paper, the author has discovered that the same question was studied by Simon and Solomon in [10]. They got the same result but with the restriction that the cone $C_{n,p}$ is area minimizing that is $n \geq 6$ and, if $n = 6$, $p \notin \{1, 5\}$. So Theorem 1 generalizes their result to any value of n and p.

In dimension 4 ($n = 2$), the proof of the Willmore conjecture by Marques and Neves [6] gives the following corollary which identifies the non planar minimal hypersurfaces with the lowest density at infinity.

Corollary 2. *Let Σ be a minimal hypersurface of $\mathbb{R}^4$ with $\theta_\infty(\Sigma) = \frac{\pi}{2}$ then $\Sigma = f(\Sigma_{2,1,\pm})$ for a similarity f.*

The proof of the main theorem starts with a result of Allard and Almgren [3], which implies that a minimal hypersurface Σ asymptotic to $C_{n,p}$ can be described as a normal graph over $C_{n,p}$ and the function defining the graph has a certain asymptotic. The first part of the proof consists in improving this asymptotic to get a very good description of the behaviour of Σ outside a compact subset. A similar work appears in the paper of Simon and Solomon [10] but we add some extra arguments to deal with low values of n.

Using this description, we are then able to apply the Alexandrov reflection technique [2] to Σ to prove that it possesses a lot of symmetries and then is invariant by $O_{p+1}(\mathbb{R}) \times O_{n-p+1}(\mathbb{R})$. Here we apply this technique to non compact hypersurfaces; this is why we need to know the asymptotic behaviour of Σ (see [8] for a similar situation). This argument is different from the one of Simon and Solomon. The last step of the proof consists in classifying the minimal hypersurfaces invariant by such a group of isometries.

The paper is divided as follow. In Section 2, we recall some definitions and study the Simons cones $C_{n,p}$. We mainly study the minimal surface equation satisfied by normal graphs over $C_{n,p}$. We are interested on the asymptotic behaviour of solutions of this equation. In Section 3, we prove that a minimal hypersurface asymptotic to $C_{n,p}$ is $O_{p+1}(\mathbb{R}) \times O_{n-p+1}(\mathbb{R})$ invariant. This is the main step of the proof of Theorem 1; we also give the proof of Corollary 2. In Section 4, we classify all minimal hypersurfaces that are invariant by $O_{p+1}(\mathbb{R}) \times O_{n-p+1}(\mathbb{R})$. The paper ends with two appendices where we give two results used in Sections 2 and 3.

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2 Preliminar results

2.1 Density at infinity and asymptotic behaviour of minimal hypersurfaces

Let Σ be a proper minimal hypersurface in $\mathbb{R}^{n+2}$. The monotonicity formula tells that the quantity.

$$\theta(p,r) = \frac{1}{\omega_{n+1} r^{n+1}} \mathrm{Vol}(\Sigma \cap B(p,r))$$

is increasing in r; here ω_{n+1} is the volume of the unit ball of dimension $n+1$ and $B(p,r)$ denote the ball of $\mathbb{R}^{n+2}$ centered at p and radius r.

Hence we can define the density at infinity of Σ as $\theta_\infty(\Sigma)=\lim_\infty \theta(p,r)$. This definition does not depend on the point p. Choosing $p\in\Sigma$, we get $\theta_\infty(\Sigma)\ge \lim_0\theta(p,r)=1$ and the equality case in the monotonicity formula says that $\theta_\infty(\Sigma)=1$ if and only if Σ is planar.

Assume now that $\theta_\infty(\Sigma)$ is finite. Then if $(t_i)_{i\in\mathbb{N}}$ is a decreasing sequence converging to 0, there is a subsequence such that the blow-down sequence $(t_i\Sigma)$ converges to C in the varifold sense where C is a cone over a stationary varifold of $\mathbb{S}^{n+1}$. This cone C is called a limit cone of Σ, we also say that Σ is asymptotic to C. *A priori*, the cone C depends on the chosen sequence (t_i) and is not smooth outside the origin. As an example, if Σ is a catenoid, $\theta_\infty(\Sigma)=2$ and C is a plane with multiplicity 2. Notice that in dimension 3 $(n=1)$, except the plane, no minimal cone is smooth outside the origin.

In $\mathbb{R}^4$ $(n=2)$, if $\theta_\infty(\Sigma)<2$; the proof of Theorem A.1 in [6] given by Marques and Neves implies that a limit cone C is smooth outside the origin. So C is a cone over a smooth minimal surface S of $\mathbb{S}^3$. If $\theta_\infty(\Sigma)>1$, S is not an equator of $\mathbb{S}^3$ and has non zero genus. So Theorem B in [6] implies that the area of S is at least $2\pi^2$ and is $2\pi^2$ if and only if S is a Clifford torus. Thus $\theta_\infty(\Sigma)\ge\frac{\pi}{2}$ and $\theta_\infty(\Sigma)=\frac{\pi}{2}$ iff a limit cone C is a cone over a Clifford torus (in that case C does not depend on the blow-down sequence by a result of Allard and Almgren [3]).

2.2 The Simons cones

The aim of this paper is to identify a minimal hypersurface in terms of its limit cone. Actually we are interested to particular minimal cones.

For $n\ge 2$ and $1\le p\le n-1$, let us write $\mathbb{R}^{n+2}=\mathbb{R}^{p+1}\times\mathbb{R}^{n-p+1}$ and consider the submanifold $S_{n,p}=\sqrt{\frac{p}{n}}\mathbb{S}^p\times\sqrt{\frac{n-p}{n}}\mathbb{S}^{n-p}$ which is a minimal hypersurface of $\mathbb{S}^{n+1}$. Let $C_{n,p}$ be the minimal cone over the minimal hypersurfaces $S_{n,p}$. The surface $S_{2,1}$ is the Clifford torus of $\mathbb{S}^3$ and $C_{2p,p}$ is the classical Simons cone [11]. So in the following, we call $C_{n,p}$ a *Simons cone*. Actually, any image of $C_{n,p}$ by a linear isometry is also called a Simons cone.

$C_{n,p}$ can be parametrized by

$$X:\mathbb{R}\times\mathbb{S}^p\times\mathbb{S}^{n-p}\to\mathbb{R}^{n+2};(t,x,y)\mapsto e^t(\sqrt{\frac{p}{n}}x,\sqrt{\frac{n-p}{n}}y).$$

Using this coordinate system, the metric on $C_{n,p}$ is $e^{2t}(dt^2+\frac{p}{n}ds_1^2+\frac{n-p}{n}ds_2^2)$ where ds_1^2 and ds_2^2 are respectively the round metrics on $\mathbb{S}^p$ and $\mathbb{S}^{n-p}$. The unit normal vector to $C_{n,p}$ is given by

$$N(t,x,y) = (\sqrt{\frac{n-p}{n}}x, -\sqrt{\frac{p}{n}}y)$$

Let (e_i) and (f_α) be respectively orthonormal bases of $T_x\mathbb{S}^p$ and $T_y\mathbb{S}^{n-p}$. Then an orthonormal basis of $T_{X(t,x,y)}C_{n,p}$ is given by

$$((\sqrt{\frac{p}{n}}x, \sqrt{\frac{n-p}{n}}y), (e_1,0), \cdots, (e_p,0), (0,f_1), \cdots, (0,f_{n-p}))$$

In this basis, the shape operator S of $C_{n,p}$ is diagonal with

$$\begin{aligned} S((\sqrt{\frac{p}{n}}x, \sqrt{\frac{n-p}{n}}y)) &= 0 \\ S((e_i,0)) &= e^{-t}\sqrt{\frac{n-p}{p}}(e_i,0) \\ S((0,f_\alpha)) &= e^{-t}\sqrt{\frac{p}{n-p}}(0,f_\alpha) \end{aligned}$$

2.3 The minimal surface equation

In the following of the paper, we study minimal hypersurfaces of $\mathbb{R}^{n+2}$ that can be described as normal graphs over a cone $C_{n,p}$. More precisely, such a surface is the image of the following parametrization:

$$Y : (t,x,y) \mapsto e^t\Big((\sqrt{\frac{p}{n}}x, \sqrt{\frac{n-p}{n}}y) + g(t,x,y)(\sqrt{\frac{n-p}{n}}x, -\sqrt{\frac{p}{n}}y)\Big)$$

where g is a smooth function defined on a domain of $\mathbb{R}\times\mathbb{S}^p\times\mathbb{S}^{n-p}$.

Using computations of the preceding section, this hypersurface is minimal if g satisfies to the following partial differential equation:

$$\begin{aligned} 0 = \partial_t\left(\frac{g+g_t}{W}\right) &+ \frac{n}{p(1+\sqrt{\frac{n-p}{p}}g)}\operatorname{div}_1\Big(\frac{\nabla^1 g}{(1+\sqrt{\frac{n-p}{p}}g)W}\Big) \\ &+ \frac{n}{(n-p)(1-\sqrt{\frac{p}{n-p}}g)}\operatorname{div}_2\Big(\frac{\nabla^2 g}{(1-\sqrt{\frac{p}{n-p}}g)W}\Big) \\ &+ \frac{ng + (g+g_t)(n+\frac{n}{p}\frac{n-2p}{\sqrt{\frac{p}{n-p}}}g)}{W(1+\frac{n-2p}{p(n-p)}g - g^2)} \end{aligned} \tag{1}$$

where ∇^1, ∇^2, div_1, div_2 are respectively the gradient and the divergence operator for the round metric with respect to the $x \in \mathbb{S}^p$ and $y \in \mathbb{S}^{n-p}$ variables and W is given by the following expression:

$$W = \Big(1 + (g+g_t)^2 + \frac{n}{p}\frac{|\nabla^1 g|^2}{(1+\sqrt{\frac{n-p}{p}}g)^2} + \frac{n}{n-p}\frac{|\nabla^2 g|^2}{(1-\sqrt{\frac{p}{n-p}}g)^2}\Big)^{\frac{1}{2}}$$

The expression of Equation (1) is long but we notice that it is an elliptic second order equation and moreover it is uniformly elliptic if ∇g is uniformly bounded.

Besides, for most of our arguments, we only need a simplified version of Equation (1). Indeed the function g will be close to 0, so we will use the following form:

$$0 = g_{tt} + \frac{n}{p}\Delta_1 g + \frac{n}{n-p}\Delta_2 g + (n+1)g_t + 2ng + Q(g) \tag{2}$$

where Δ_1 and Δ_2 are respectively the Laplace operator with respect to x and y variables and $Q(g)$ gathers all the nonlinear terms of Equation (1).

2.4 The kernel of the linearized operator

The linearized operator of the minimal surface equation (2) is

$$Lu = u_{tt} + \frac{n}{p}\Delta_1 u + \frac{n}{n-p}\Delta_2 u + (n+1)u_t + 2nu$$

Our analysis of solutions of (1) is based on the asymptotic behaviour of elements in the kernel of L. Such an element in the kernel can be decomposed as the sum of terms of the form $v(t)\Phi(x)\Psi(y)$ where Φ and Ψ are respectively eigenfunctions of the Laplace operator on $\mathbb{S}^p$ and $\mathbb{S}^{n-p}$. The eigenvalues of Δ on $\mathbb{S}^m$ are $-k(k+m-1)$ $(k \geq 0)$. So $(t,x,y) \mapsto v(t)\Phi(x)\Psi(y)$ is in the kernel if v satisfies the following ode for some k and l:

$$0 = v_{tt} + (n+1)v_t + (2n - \frac{n}{p}k(k+p-1) - \frac{n}{n-p}l(l+(n-p)-1))v$$

The asymptotic behaviour of v is given by the roots of

$$0 = \lambda^2 + (n+1)\lambda + 2n - \frac{n}{p}k(k+p-1) - \frac{n}{n-p}l(l+(n-p)-1)$$

In the following, these roots are denoted by $\lambda_{k,l,\pm}$. Actually, we are only interested in roots whose real part is between -2 and 0.

If $k+l=0$, the equation is $0=\lambda^2+(n+1)\lambda+2n$ whose discriminant $(n+1)^2-8n$ is negative if $n<6$ and positive if $n\geq 6$. So the roots are

$$\begin{cases} \lambda_{0,0,\pm} &= \frac{-(n+1)\pm i\sqrt{8n-(n+1)^2}}{2} \text{ if } n<6 \\ \lambda_{0,0,\pm} &= \frac{-(n+1)\pm\sqrt{(n+1)^2-8n}}{2} \text{ if } n\geq 6 \end{cases}$$

If $n\geq 6$, a computation gives $\lambda_{0,0,-}<\lambda_{0,0,+}<-2$. So the real part of $\lambda_{0,0,\pm}$ is between -2 and 0 only for $n=2,3$.

If $k+l=1$, the equation is $0=\lambda^2+(n+1)\lambda+n=(\lambda+1)(\lambda+n)$. So $\lambda_{k,l,+}=-1$ and $\lambda_{k,l,-}=-n$ which lies in $[-2,0)$ if $n=2$.

If $k+l\geq 2$, $2n-\frac{n}{p}k(k+p-1)-\frac{n}{n-p}l(l+(n-p)-1)\leq 0$, so $\lambda_{k,l,+}\geq 0$ and $\lambda_{k,l,-}\leq -(n+1)\leq -3$.

2.5 Asymptotic behaviour of a minimal graph

In this section, we study the asymptotic behaviour of a minimal normal graph over a cone $C_{n,p}$. Actually, we prove an improvement result for the asymptotic behaviour of solutions of (1).

First we recall a classical definition of weighted norm for functions on $\mathbb{R}_+\times\mathbb{S}^p\times\mathbb{S}^{n-p}$. If u is a continuous function on $\mathbb{R}_+\times\mathbb{S}^p\times\mathbb{S}^{n-p}$ and δ is a real number, we define its weighted norm

$$\|u\|_\delta=\sup\{e^{\delta t}|u(t,x,y)|,(t,x,y)\in\mathbb{R}_+\times\mathbb{S}^p\times\mathbb{S}^{n-p}\}$$

when this quantity is finite. When $\|u\|_\delta<+\infty$, we will also write $u=O(e^{-\delta t})$.

We then have the following result that describes the asymptotic behaviour of a solution of (1) with $\|u\|_\delta$ finite for $\delta>0$.

Proposition 3. *Let u be a solution of* (1) *on $\mathbb{R}_+\times\mathbb{S}^p\times\mathbb{S}^{n-p}$ such that ∇u is uniformly bounded and $\|u\|_\delta<+\infty$ with $\delta>0$ and $-2\delta\neq\lambda_{k,l,\pm}$ for all $k,l\geq 0$. Then u can be written $u=v+r$ where $\|v\|_\delta<+\infty$ satisfies to $L(v)=0$ and $\|r\|_{2\delta}<+\infty$.*

Proof. The proof is based on the spectral decomposition of functions on $\mathbb{S}^p\times\mathbb{S}^{n-p}$.

First, since ∇u is uniformly bounded, Equation (1) is uniformly elliptic, so classical elliptic estimates give upper bounds on the derivatives of u: more precisely, for any $m>0$, there is a constant C_m such that for any $s>1$

$$\|\nabla^m u\|_{C^0([s,s+1]\times\mathbb{S}^p\times\mathbb{S}^{n-p})}\leq C_m\|u\|_{C^0([s-1,s+2]\times\mathbb{S}^p\times\mathbb{S}^{n-p})}$$

This implies that for any m, $\|\nabla^m u\|_\delta < +\infty$. Since the term $Q(u)$ in (2) gathers all the nonlinear terms in u we have $\|Q(u)\|_{2\delta} < \infty$ and $\|\nabla^m Q(u)\|_{2\delta} < \infty$.

In the preceding section, we have describe the spectrum of the Laplace operator on the sphere. So let us denote $\lambda_k = k(k+p-1)$ and $\Phi_{k,\alpha}$ an orthonormal basis of the eigenspace of Δ_1 associated to $-\lambda_k$ on $\mathbb{S}^p$. We also denote $\mu_l = l(l+n-p-1)$ and $\Psi_{l,\beta}$ an orthonormal basis of the eigenspace of Δ_2 associated to $-\mu_l$ on $\mathbb{S}^{n-p}$. The multplicity of the $-\lambda_k$ and $-\mu_l$ are respectively bounded by $c(k^p+1)$ and $c(k^{n-p}+1)$. Moreover, we have the following estimates for the L^∞ norm of the eigenfunctions (see [12]):

$$\|\Phi_{k,\alpha}\|_\infty \le c\lambda_k^{\frac{p-1}{4}} \text{ and } \|\Psi_{l,\beta}\|_\infty \le c\mu_l^{\frac{(n-p)-1}{4}}.$$

Now let us define

$$\begin{aligned} g_{k,l,\alpha,\beta}(t) &= \int_{\mathbb{S}^p\times\mathbb{S}^{n-p}} u(t,x,y)\Phi_{k,\alpha}(x)\Psi_{l,\beta}(y)dxdy, \\ f_{k,l,\alpha,\beta}(t) &= -\int_{\mathbb{S}^p\times\mathbb{S}^{n-p}} Q(u)(t,x,y)\Phi_{k,\alpha}(x)\Psi_{l,\beta}(y)dxdy. \end{aligned}$$

$g_{k,l,\alpha,\beta}$ and $f_{k,l,\alpha,\beta}$ are smooth functions on $\mathbb{R}_+$ and, from (2), they satisfy

$$g''_{k,l,\alpha,\beta} - (\lambda_{k,l,+} + \lambda_{k,l,-})g'_{k,l,\alpha,\beta} + (\lambda_{k,l,+} \times \lambda_{k,l,-})g_{k,l,\alpha,\beta} = f_{k,l,\alpha,\beta}$$

Using $\Delta_1\Phi_{k,\alpha} = -\lambda_k\Phi_{k,\alpha}$, $\Delta_2\Psi_{l,\beta} = -\mu_l\Psi_{l,\beta}$ and integration by parts, we get the following estimates for $a, b \in \mathbb{N}$:

$$\begin{aligned} |g_{k,l,\alpha,\beta}(s)| &\le c\frac{\sup_{t=s}|\nabla^{2a+2b}u(t,x,y)|}{(1+\lambda_k)^a(1+\mu_l)^b} \\ |f_{k,l,\alpha,\beta}(s)| &\le c\frac{\sup_{t=s}|\nabla^{2a+2b}Q(u)(t,x,y)|}{(1+\lambda_k)^a(1+\mu_l)^b} \end{aligned}$$

Thus we get

$$\begin{aligned} \|g_{k,l,\alpha,\beta}\|_\delta &\le c\frac{\|\nabla^{2a+2b}u\|_\delta}{(1+\lambda_k)^a(1+\mu_l)^b} \\ \|f_{k,l,\alpha,\beta}\|_{2\delta} &\le c\frac{\|\nabla^{2a+2b}Q(u)\|_{2\delta}}{(1+\lambda_k)^a(1+\mu_l)^b} \end{aligned}$$

From Lemma 10 in Appendix A, we can write

$$g_{k,l,\alpha,\beta}(t) = a_{k,l,\alpha,\beta}e^{t\lambda_{k,l,+}} + b_{k,l,\alpha,\beta}e^{t\lambda_{k,l,+}} + r_{k,l,\alpha,\beta}(t)$$

with some estimates on the different terms. First we notice that $\frac{(2+|\lambda_{k,l,+}|^2+|\lambda_{k,l,-}|^2)^{1/2}}{|\lambda_{k,l,+}-\lambda_{k,l,-}|}$ is uniformly bounded and $|2\delta - \Re(\lambda_{k,l,\pm})|$ is uniformly bounded below far from 0. Thus there is a uniform constant c such that

$$\begin{aligned}\max(|a_{k,l,\alpha,\beta}|, |a_{k,l,\alpha,\beta}|) &\le c(\|g_{k,l,\alpha,\beta}\|_\delta + \|g'_{k,l,\alpha,\beta}\|_\delta + \|f_{k,l,\alpha,\beta}\|_{2\delta}) \\ &\le c\frac{\|\nabla^{2a+2b}u\|_\delta + \|\nabla^{2a+2b+1}u\|_\delta + \|\nabla^{2a+2b}Q(u)\|_{2\delta}}{(1+\lambda_k)^a(1+\mu_l)^b}\end{aligned}$$

and

$$\begin{aligned}\|r_{k,l,\alpha,\beta}\|_{2\delta} &\le c\|f_{k,l,\alpha,\beta}\|_{2\delta} \\ &\le c\frac{\|\nabla^{2a+2b}Q(u)\|_{2\delta}}{(1+\lambda_k)^a(1+\mu_l)^b}\end{aligned}$$

Besides if $\Re(\lambda_{k,l,\pm}) \ge -\delta$, $t \mapsto e^{t\lambda_{k,l,\pm}}$ does not have a finite δ-norm so $a_{k,l,\beta,\alpha}$ or $b_{k,l,\alpha,\beta}$ vanishes. When $\Re(\lambda_{k,l,-}) \le -2\delta$, $t \mapsto e^{t\lambda_{k,l,\pm}}$ has a finite 2δ-norm equal to 1.

Finally we have the following writing

$$\begin{aligned}u = &\sum_{-2\delta \le \Re(\lambda_{k,l,\pm}) \le -\delta} (a_{k,l,\alpha,\beta}e^{t\lambda_{k,l,+}} + b_{k,l,\alpha,\beta}e^{t\lambda_{k,l,-}})\Phi_{k,\alpha}(x)\Psi_{l,\beta}(y) \\ &+ \sum_{\Re(\lambda_{k,l,\pm}) < -2\delta} (a_{k,l,\alpha,\beta}e^{t\lambda_{k,l,+}} + b_{k,l,\alpha,\beta}e^{t\lambda_{k,l,-}})\Phi_{k,\alpha}(x)\Psi_{l,\beta}(y) \\ &+ \sum r_{k,l,\alpha,\beta}(t)\Phi_{k,\alpha}(x)\Psi_{l,\beta}(y)\end{aligned}$$

First we notice that the first sum is finite and is an element of the kernel of L, this is the expected function v. Let us see that the two other sums converge and have finite 2δ-norms. Let $A(t,x,y)$ be the second sum. In the following computation, we use the expressions of λ_k and μ_l, their multiplicities and the L^∞ estimates on $\Phi_{k,\alpha}$ and $\Psi_{l,\beta}$.

$$\begin{aligned}\|A\|_{2\delta} \le C &\sum_{\Re(\lambda_{k,l,\pm}) < -2\delta} (|a_{k,l,\alpha,\beta}| + |b_{k,l,\alpha,\beta}|)\lambda_k^{\frac{p-1}{4}}\mu_l^{\frac{(n-p)-1}{4}} \\ &\le C\sum_{k,l,\alpha,\beta} \frac{\|\nabla^{2a+2b}u\|_\delta + \|\nabla^{2a+2b+1}u\|_\delta + \|\nabla^{2a+2b}Q(u)\|_{2\delta}}{(1+\lambda_k)^a(1+\mu_l)^b}\lambda_k^{\frac{p-1}{4}}\mu_l^{\frac{(n-p)-1}{4}} \\ &\le C(\|\nabla^{2a+2b}u\|_\delta + \|\nabla^{2a+2b+1}u\|_\delta + \|\nabla^{2a+2b}Q(u)\|_{2\delta})\sum_{k,l}\frac{(1+k^{\frac{3p}{2}})(1+l^{\frac{3(n-p)}{2}})}{(1+k^2)^a(1+l^2)^b} \\ &< +\infty\end{aligned}$$

if a and b are chosen such that $2a - \frac{3p}{2} \geq 2$ and $2b - \frac{3(n-p)}{2} \geq 2$. The study of the last sum works the same. $\square$

Remark. From the proof, the function v in the kernel of L can be actually written as a finite sum of terms of the form $e^{\lambda t}\Phi(x)\Psi(y)$ with $-2\delta < \Re(\lambda) \leq -\delta$.

A second remark is that the function r is a solution of $L(r)+Q(v+r) = 0$ with $\|Q(v,r)\|_{2\delta} < +\infty$. So elliptic estimates give that $\|\nabla^m r\|_{2\delta} < +\infty$.

3 Symmetries of minimal hypersurfaces asymptotic to Simons cones

Let Σ be a minimal hypersurface of $\mathbb{R}^{n+2}$ which has a Simons cone as limit cone. If f is a similarity (composition of an isometry and a homothety) of $\mathbb{R}^{n+2}$, $f(\Sigma)$ is also a minimal hypersurface asymptotic to a Simons cone. The following result says that up to similarities, there are two such hypersurfaces (at n and p fixed) and even one when $n = 2p$.

Theorem 1. *For any $n \geq 2$ and $1 \leq p \leq n-1$, there are two minimal hypersurfaces $\Sigma_{n,p,\pm}$ in $\mathbb{R}^{n+2}$ such that the following is true. If Σ is a minimal hypersurface of $\mathbb{R}^{n+2}$ with a Simons cone as limit cone, then $\Sigma = f(\Sigma_{n,p,\pm})$ for some $p \in \{1,\dots,n-1\}$, sign $\pm$ and a similarity f. Moreover $\Sigma_{2p,p,-} = \Sigma_{2p,p,+}$.*

Actually, $C_{n,p} = f(C_{n,n-p})$ for a certain isometry f of $\mathbb{R}^{n+2}$ so $\Sigma_{n,p,\pm} = \Sigma_{n,n-p,\mp}$.

In the case $n = 2$, we have a corollary of this which comes from the proof of the Willmore conjecture by Marques and Neves [6].

Corollary 2. *Let Σ be a minimal hypersurface of $\mathbb{R}^4$ whose density at infinity is $\theta_\infty(\Sigma) = \frac{\pi}{2}$. Then $\Sigma = f(\Sigma_{2,1,\pm})$ for a similarity f.*

Proof. As explained in Section 2.1, $\theta_\infty(\Sigma) = \frac{\pi}{2}$ implies that Σ is asymptotic to the cone over a Clifford torus so Theorem 1 applies. $\square$

In order to prove Theorem 1 we first notice that, using an isometry, we can assume that the limit cone in $C_{n,p}$. The cone $C_{n,p}$ is invariant by the subgroup $O_{n,p} = O_{p+1}(\mathbb{R}) \times O_{n-p+1}(\mathbb{R})$ of $O_{n+2}(\mathbb{R})$. The following result is the main step of the proof of Theorem 1. It says that a minimal hypersurface with $C_{n,p}$ as limit cone is also invariant by the subgroup $O_{n,p}$.

Theorem 4. *Let Σ be a minimal hypersurface of $\mathbb{R}^{n+2}$ which has $C_{n,p}$ as limit cone. Then there is $x_0 \in \mathbb{R}^{n+2}$ such that the translated hypersurface $\Sigma - x_0$ is invariant by $O_{p+1}(\mathbb{R}) \times O_{n-p+1}(\mathbb{R})$.*

The rest of this section is devoted to the proof of this result.

3.1 Asymptotic behaviour of Σ

In this section we study the asymptotic behaviour of a minimal surface with $C_{n,p}$ as limit cone.

Proposition 5. *Let Σ be a minimal hypersurface of $\mathbb{R}^{n+2}$ with $C_{n,p}$ as limit cone. Then there is $x_0 \in \mathbb{R}^{n+2}$ such that, outside a compact set, the translate $\Sigma - x_0$ can be described as the normal graph of a function g over a subdomain of $C_{n,p}$. Moreover the function g can be written $g(t,x,y) = u(t) + f(t,x,y)$ where u is in the kernel of L and $\|f\|_\delta < +\infty$ for some $\delta > 2$.*

Proof. First, we use a result of Allard and Almgren [3] and Simon [9] which implies that outside a compact set, the hypersurface Σ can be described as the normal graph of a function g over $C_{n,p}$ and the function g is defined on $[\underline{t}, +\infty) \times \mathbb{S}^p \times \mathbb{S}^{n-p}$ and satisfies $\|g\|_\varepsilon < +\infty$ for some $\varepsilon > 0$. The result of Allard and Almgren applies since all Jacobi functions on $S_{n,p}$ comes from Killing vectorfields of $\mathbb{S}^{n+1}$ (see Section 6 in [3]). Decreasing slightly ε if necessary, we can assume that $-2\varepsilon \neq \lambda_{k,l,\pm}$ and apply Proposition 3. So $g = v + r$ with v in the kernel of L with decay between $-\varepsilon$ and -2ε and $\|r\|_{2\varepsilon} < +\infty$. If there is no element in the kernel of L with decay between $-\varepsilon$ and -2ε, we get $\|g\|_{2\varepsilon} < \infty$; in that case we have then improved the decay of g. So we can iterate this argument until we get a first non vanishing element in the kernel.

The first decay of elements in the kernel is given by $\lambda_{1,0,+} = \lambda_{0,1,+} = -1$. Besides, the eigenfunctions $\Phi_{1,\alpha}$ and $\Psi_{1,\beta}$ are the coordinates functions so g can be written

$$g(t,x,y) = e^{-t}(a_1x_1 + \cdots + a_{p+1}x_{p+1} + b_1y_1 + \cdots + b_{n-p+1}y_{n-p+1}) + r(t,x,y)$$

with $\|r\|_{1+\varepsilon} < +\infty$ for some $\varepsilon > 0$. This can also be written

$$g(t,x,y) = e^{-t}(X_0, N(t,x,y)) + r(t,x,y)$$

The first term can be interpreted as a translation. More precisely, in the parametrization Y, a term $(X_0, N)N$ appears. So the translated hypersurface $\Sigma - X_0$ can be expressed as the normal graph of a function w over $C_{n,p}$ with the following estimates $\|w\|_{1+\varepsilon} < +\infty$ for some $\varepsilon > 0$.

From now on, we study the asymptotic behaviour of $\Sigma - X_0$ as a normal graph over $C_{n,p}$. We still call Σ this translated hypersurface.

If we apply Proposition 3, we get the following writing $w = v + r$ with v in the kernel of L with a decay between $-1-\varepsilon$ and $-2-2\varepsilon$ and $\|r\|_{2+2\varepsilon} < +\infty$. If $n > 3$, all $\lambda_{k,l,\pm}$ are outside the segment $[-2-2\varepsilon, -1-\varepsilon]$ for ε close enough to 0, so v is vanishing and the proposition is proved. If $n = 3$, $\lambda_{0,0,\pm} = -2 \pm i\sqrt{2}$ is the only possibilities. This value comes from the constant functions on $\mathbb{S}^p$ and $\mathbb{S}^{n-p}$ so v only depends on t, so the proposition is proved.

When $n = 2$, we have two possibilities, $\lambda_{0,0,\pm} = -\frac{3}{2} \pm i\frac{\sqrt{7}}{2}$ coming from constant functions on $\mathbb{S}^p$ and $\mathbb{S}^{n-p}$ and $\lambda_{1,0,-} = \lambda_{0,1,-} = -2$ from the coordinate functions. So w can be written

$$w(t,x,y) = ae^{-\frac{3t}{2}}\cos(\frac{\sqrt{7}t}{2} + \varphi_0) + e^{-2t}(X_1, N(t,x,y)) + r(t,x,y).$$

Let us prove that actually X_1 is vanishing. To prove this, we use a flux argument. Let us recall that if Ω is a subset of Σ with smooth boundary and ν denote the normal to $\partial\Omega$ tangent to Σ, then the flux of ν across $\partial\Omega$ vanishes; more precisely:

$$\int_{\partial\Omega} \nu = 0$$

We apply this result to the bounded subset Ω_{t_0} of Σ whose boundary is the hypersurface $\{t = t_0\}$. Using the above expression of w in Appendix B, we estimate this flux (see Equation (7)) and we get

$$0 = \int_{\partial\Omega_{t_0}} \nu = -\int_{\mathbb{S}^1\times\mathbb{S}^1} \frac{1}{2}(X_1, N)N + O(e^{-2\varepsilon t}).$$

Taking the limit $t \to +\infty$ and taking the scalar product with X_1, we get that $(X_1, N) = 0$ for all $(x,y) \in \mathbb{S}^1 \times \mathbb{S}^1$: so $X_1 = 0$. This finishes the proof of the proposition. □

We recall that the derivatives of f also have finite δ-norms.

3.2 Alexandrov reflection

Let Σ be a minimal hypersurface in $\mathbb{R}^{n+2}$ with $C_{n,p}$ as limit cone. We translate Σ such that the asymptotic behaviour of Proposition 5 is true (the translated hypersurface is still named Σ). In this section, we use this asymptotic behaviour to prove that Σ is invariant by $O_{p+1}(\mathbb{R}) \times O_{n-p+1}(\mathbb{R})$ and then prove Theorem 4.

Let us denote the coordinates of $\mathbb{R}^{n+2}$ by $(x_1,\cdots,x_{p+1},y_1,\cdots,y_{n-p+1})$. Actually, we are going to prove that Σ is symmetric with respect to $\{x_1=0\}$. If $f\in O_{p+1}(\mathbb{R})$, $f(\Sigma)$ satisfies the same hypotheses as Σ so $f(\Sigma)$ will be symmetric with respect to $\{x_1=0\}$ and then Σ will be symmetric with respect to $f^{-1}(\{x_1=0\})$. All these symmetries imply that Σ is $O_{p+1}(\mathbb{R})$-invariant. For the $O_{n-p+1}(\mathbb{R})$-invariance, the proof is similar by exchanging p by $n-p$.

Outside a compact, the hypersurface Σ is the normal graph a function g that can be written as in Proposition 5 $g(t,x,y)=f(t)+O(e^{-\delta t})$ with $\delta>2$. The first coordinate of the point $Y(t,x,y)$ is given by $e^t(\sqrt{\frac{p}{n}}+g(t,x,y)\sqrt{\frac{n-p}{n}})x_1$. In the following we are interested to the following subset of Σ:

$$\Sigma_{t_0,a}=Y(\{(t,x,y)\in\mathbb{R}\times\mathbb{S}^p\times\mathbb{S}^{n-p}\,|\,t\geq t_0,e^t(\sqrt{\frac{p}{n}}+g(t,x,y)\sqrt{\frac{n-p}{n}})x_1>a\}).$$

So a point of Σ is in $\Sigma_{t_0,a}$ if it is sufficiently far from the origin and its first coordinate is larger than a.

We denote by π the projection map of $\mathbb{R}^{p+1}$ on $\{x_1=0\}$. We have a first lemma that describes $\Sigma_{t_0,a}$.

Lemma 6. *There are t_0 and $c>0$ such that for any $a>0$ the map $(\pi,id):\mathbb{R}^{n+2}\to\{x_1=0\}\times\mathbb{R}^{n-p+1}$ is injective on $\Sigma_{t_a,a}$ where*

$$t_a=\max(t_0,\ln\frac{c}{a})$$

Proof. t_0 is assumed to be sufficiently large so that e^{-t_0} is small enough and all the computations below are right. We denote $x=(x_1,\pi(x))$. If the map is not injective, we have (t,x,y) and (t',x',y') $(t'\geq t)$ such that

$$e^t(\sqrt{\frac{p}{n}}+g\sqrt{\frac{n-p}{n}})\pi(x)=e^{t'}(\sqrt{\frac{p}{n}}+g'\sqrt{\frac{n-p}{n}})\pi(x') \tag{3}$$

$$e^t(\sqrt{\frac{n-p}{n}}-g\sqrt{\frac{p}{n}})y=e^{t'}(\sqrt{\frac{n-p}{n}}-g'\sqrt{\frac{p}{n}})y' \tag{4}$$

with $g=g(t,x,y)$ and $g'=g(t',x',y')$.

From (4), $y=y'$ and $e^t(\sqrt{\frac{n-p}{n}}-g\sqrt{\frac{p}{n}})=e^{t'}(\sqrt{\frac{n-p}{n}}-g'\sqrt{\frac{p}{n}})$. So if $h=e^t(\sqrt{\frac{n-p}{n}}-f(t)\sqrt{\frac{p}{n}})$ and $h'=e^{t'}(\sqrt{\frac{n-p}{n}}-f(t')\sqrt{\frac{p}{n}})$, we get $h'-h=O(e^{(1-\delta)t})(|t-t'|+|x-x'|)$

So $e^t(t'-t) \leq O(e^{(1-\delta)t})(|t-t'|+|x-x'|)$. Then

$$|t'-t| \leq ce^{-\delta t}|x-x'|$$

Thus

$$e^{t'}(\sqrt{\frac{p}{n}}+g'\sqrt{\frac{n-p}{n}}) = e^t(\sqrt{\frac{p}{n}}+g\sqrt{\frac{n-p}{n}}) + O(e^{(1-\delta)t})|x-x'|$$

Using this in (3), we get

$$e^t\sqrt{\frac{p}{n}}|\pi(x-x')| = O(e^{(1-\delta)t})|x-x'|$$

On the hemisphere $\mathbb{S}^p \cap \{x_1 > 0\}$, we have $|\pi(x-x')| \geq c\min(x_1,x_1')|x-x'|$ for some constant $c > 0$. This implies $e^t\sqrt{\frac{p}{n}}\min(x_1,x_1')|x-x'| = O(e^{(1-\delta)t})|x-x'|$. Thus

$$a|x-x'| \leq ce^{(1-\delta)t}|x-x'|$$

Since $\delta > 2$ it implies $x = x'$ and then $t = t'$ if

$$t \geq \ln\frac{c}{a} > \frac{1}{\delta-1}\ln\frac{c}{a}$$

$\square$

This lemma implies that large part of Σ can be described as graph in the x_1 direction. For $a > 0$, we denote by S_a the symmetry with respect to $x_1 = a$.

Lemma 7. *There are constants t_0, b and c such that for any $a > 0$ the image of $\Sigma_{t_a,a}$ by S_a does not intersect Σ where*

$$t_a = \max(t_0, b\ln\frac{c}{a})$$

Proof. As above, t_0 is chosen such that e^{-t_0} is sufficiently small. We have

$$S_a(Y(t,x,y)) = \begin{pmatrix} 2a - e^t(\sqrt{\frac{p}{n}}+g\sqrt{\frac{n-p}{n}})x_1 \\ e^t(\sqrt{\frac{p}{n}}+g\sqrt{\frac{n-p}{n}})\pi(x) \\ e^t(\sqrt{\frac{n-p}{n}}-g\sqrt{\frac{p}{n}})y \end{pmatrix}$$

So $|S_a(Y((t,x,y))| \geq e^t(\sqrt{\frac{n-p}{n}} - g\sqrt{\frac{p}{n}}) \geq ce^t$. Thus $S_a((Y(t,x,y))$ is outside a large ball if $t > t_0$ is large. So we can care only about the part of Σ which is parametrized by the normal graph and with large t: if $S_a(Y(t,x,y))$ is inside Σ, this point can be written $Y(t',x',y')$ with t' large.

If $a \leq e^t(\sqrt{\frac{p}{n}} + g\sqrt{\frac{n-p}{n}})x_1 \leq 3a/2$, it is clear that $S_a(Y(t,x,y))$ is not in Σ because of Lemma 6 applied with $a' = a/2$.

Now we assume that $e^t(\sqrt{\frac{p}{n}} + g\sqrt{\frac{n-p}{n}})x_1 \geq 3a/2$ and we have

$$S_a(Y(t,x,y)) = Y(t',x',y') \tag{5}$$

For $(x,y) \in \mathbb{R}^{p+1} \times \mathbb{R}^{n-p+1}$, let $Q(x,y) = (n-p)|x|^2 - p|y|^2$. We have $Q(S_a(Y(t,x,y))) = Q(Y(t',x',y'))$ thus

$$\begin{aligned} e^{2t'}(2g'\sqrt{p(n-p)} + g'^2(n-2p)) =& (n-p)4a(a - e^t(\sqrt{\frac{p}{n}} + g\sqrt{\frac{n-p}{n}})x_1) \\ &+ e^{2t}(2g\sqrt{p(n-p)} + g^2(n-2p)) \end{aligned}$$

Using $e^t(\sqrt{\frac{p}{n}} + g\sqrt{\frac{n-p}{n}})x_1 \geq 3a/2$, this gives

$$e^{2t'}(2g'\sqrt{p(n-p)} + g'^2(n-2p)) - e^{2t}(2g\sqrt{p(n-p)} + g^2(n-2p)) \leq -(n-p)2a^2 \tag{6}$$

From (5), we also have

$$e^t(\sqrt{\frac{n-p}{n}} - g\sqrt{\frac{p}{n}})y = e^{t'}(\sqrt{\frac{n-p}{n}} - g'\sqrt{\frac{p}{n}})y'$$

As in the Lemma 6, this gives $|t'-t| \leq ce^{-\delta t}$. Using this in (6), we finally get

$$ce^{(2-\delta)t} \geq (n-p)2a^2$$

Lemma 7 is then proved since $\delta > 2$. $\square$

Now we can apply the Alexandrov reflection procedure to prove the following result.

Lemma 8. *The surface Σ is symmetric with respect to $\{x_1 = 0\}$.*

Proof. First we denote by $\Sigma_a = \Sigma \cap \{x_1 > a\}$. Let also t_a be given by Lemma 7. If $a > 0$ is large, Σ_a is a subset of the part of Σ which is a normal graph. Besides $|Y(t,x,y)| \geq a$ so t is large on Σ_a if a is large. This implies that for a sufficiently large $\Sigma_a = \Sigma_{t_a,a}$. So from Lemma 7, $S_a(\Sigma_a) \cap \Sigma = \emptyset$ for a large.

For any $a > 0$, $\Sigma_a \setminus \Sigma_{t_a,a}$ is a bounded subset, so if there is some $a' > 0$ such that $S_{a'}(\Sigma_{a'}) \cap \Sigma \neq \emptyset$, there is a first contact point between $S_a(\Sigma_a)$ and Σ: there is $a_0 > 0$ and $p_0 \in \Sigma \cap (S_{a_0}(\Sigma_{a_0}) \cup \partial\Sigma_{a_0})$ such that $S_{a_0}(\Sigma_{a_0})$ lies on one side of Σ near p_0. So applying the maximum principle at p_0, we get $S_{a_0}(\Sigma_{a_0}) \subset \Sigma$ which is not possible by Lemma 7.

This implies that Σ_0 is a graph in the x_1 direction and $\Sigma \cap \{x_1 < 0\}$ lies on one side of $S_0(\Sigma_0)$ in $\{x_1 < 0\}$. We notice that $S_0(\Sigma)$ has the same asymptotic behaviour as Σ. Thus, applying the same argument to $S_0(\Sigma)$, we get that $\Sigma \cap \{x_1 < 0\}$ is also a graph in the x_1 direction. Now because of the asymptotic behaviour of Σ the first coordinate of the normal to Σ changes its sign. So Σ is normal to $\{x_1 = 0\}$ and the maximum principle implies that $S_0(\Sigma) = \Sigma$. $\square$

4 Minimal hypersurfaces invariant by $O_{p+1}(\mathbb{R}) \times O_{n-p+1}(\mathbb{R})$

In order to finish the proof of Theorem 1, we need to understand all the minimal hypersurfaces that are invariant by $O_{p+1}(\mathbb{R}) \times O_{n-p+1}(\mathbb{R})$. This study has been partially done by Bombieri, de Giorgi and Giusti in [4] in the case $n = 2p \geq 6$. It has been completed by Alencar, Barros, Palmas, Reyes and Santos [1]; here for sake of completeness, we write the part of the study which is necessary for our result. We want to prove that up to homotheties there is two minimal hypersurfaces invariant by $O_{p+1}(\mathbb{R}) \times O_{n-p+1}(\mathbb{R})$ with $C_{n,p}$ as limit cone.

So we are looking for a pair of functions a, b defined on an interval I such that the hypersurface parametrized by

$$X : I \times \mathbb{S}^p \times \mathbb{S}^{n-p} \longrightarrow \mathbb{R}^{n+2}; (t, x, y) \longmapsto (a(t)x, b(t)y)$$

is minimal.

The hypersurface is minimal if a and b satisfy to a certain ordinary differential equation:

$$0 = a''b' - b''a' + (a'^2 + b'^2)\Big((n-p)\frac{a'}{b} - p\frac{b'}{a}\Big)$$

Since being minimal is invariant by homotheties, $(\lambda a, \lambda b)$ is a solution if (a, b) is a solution. In other to use this property we introduce new parameters by these expressions:

$$(a, b) = e^{\rho}(\cos\theta, \sin\theta)$$
$$(a', b') = e^{r}(\cos\varphi, \sin\varphi)$$

The above ode is then equivalent to

$$\begin{cases} \rho' &= e^{r-\rho}\cos(\theta-\varphi) \\ \theta' &= -e^{r-\rho}\sin(\theta-\varphi) \\ \varphi' &= e^{r-\rho}\frac{(n-2p)\cos(\theta-\varphi)+n\cos(\theta+\varphi)}{\sin 2\theta} \end{cases}$$

So, changing the time parameter, we get the following system

$$\begin{cases} \rho' &= \sin 2\theta\cos(\theta-\varphi) \\ \theta' &= -\sin 2\theta\sin(\theta-\varphi) \\ \varphi' &= (n-2p)\cos(\theta-\varphi)+n\cos(\theta+\varphi) \end{cases}$$

So we are let to understand the flow lines of

$$(\theta, \varphi)' = Y(\theta, \varphi) = (-\sin 2\theta\sin(\theta-\varphi), (n-2p)\cos(\theta-\varphi)+n\cos(\theta+\varphi))$$

We denote by Y_1 and Y_2 the two components of the vectorfield. First we remark that $Y_1(k\pi/2, \varphi) = 0$ so the subsets $\{k\pi/2 \le \theta \le (k+1)\pi/2\}$ are stable. Moreover, we have $Y(\theta+\pi, \varphi) = -Y(\theta, \varphi)$, $Y(\theta, \varphi+\pi) = -Y(\theta, \varphi)$ and $Y(-\theta, -\varphi) = Y(\theta, \varphi)$. So we need to understand the vectorfield on $[0, \pi/2] \times (-\pi/2, \pi/2]$.

In this subset, the singular points are the following

- a saddle point $(\pi/2, 0)$ with stable direction $(0, 1)$ and unstable one $(p+1, p-n)$,
- a saddle point $(0, \pi/2)$ with stable direction $(0, 1)$ and unstable one $(n+1-p, -p)$ and
- a stable nodal or focal point (θ_0, θ_0) where $\theta_0 \in (0, \pi/4]$ satisfies $\cos\theta_0 = \sqrt{\frac{p}{n}}$ (if $n \le 2$, the roots of $dY(\theta_0, \theta_0)$ are conjugate complex numbers with negative real parts and, if $n \ge 3$ the roots are negative real numbers).

The properties of the vectorfield Y are summarized in the following proposition (see also Figure 1).

Proposition 9. *The vectorfield Y satisfies to the following properties :*

- *if $\varphi \in (-\pi/2, 0)$, $Y_2(\theta, \varphi) > 0$ for all $\theta \in [0, \pi/2]$ and*
- *there is a continuous decreasing surjective function $t : [0, \theta_0] \to [0, \pi/2]$ such that Y points inside $[\varphi, \varphi + t(\varphi)]^2$ along its boundary for $\varphi \in [0, \theta_0]$.*

Proof. We have $Y_2(\theta, \varphi) = 2(n-p)\cos\varphi\cos\theta - 2p\sin\varphi\sin\theta$, so the first property is clear.

For the second property, we first notice that, for $\theta, \varphi \in [0, \pi/2]$, $Y_1(\theta, \varphi)$ has the same sign as $\varphi - \theta$. We remark also that $Y_2(\theta, \varphi) = Y_2(\varphi, \theta)$. Moreover

$$\begin{aligned} Y_2(\varphi + t, \varphi) &= (n - 2p + n\cos 2\varphi)\cos t - n\sin 2\varphi \sin t \\ &= n(\cos 2\varphi - \cos 2\theta_0)\cos t - n\sin 2\varphi \sin t \end{aligned}$$

So, for $\varphi \in [0, \theta_0]$, $t \mapsto Y_2(\varphi + t, \varphi)$ is non increasing for $t \in [0, \pi/2]$. It vanishes for $t = t(\varphi) = \arctan(\frac{\cos 2\varphi - \cos 2\theta_0}{\sin 2\varphi})$. Thus is non negative for $t \in [0, t(\varphi)]$. When $\psi \geq \theta_0$, $t \mapsto Y_2(\psi + t, \psi)$ is non increasing for $t \in [-\pi/2, 0]$. Since $Y_2(\varphi, \varphi + t(\varphi)) = Y_2(\varphi + t(\varphi), \varphi) = 0$, it implies that $Y_2(\varphi + t(\varphi) - t, \varphi + t(\varphi))$ is non positive for $t \in [-t(\varphi), 0]$. This finishes the proof of the second item. □

The above properties are sufficient to describe all the integral curves of Y passing trough a point in $(\theta, \varphi) \in (0, \pi/2) \times (-\pi/2, \pi/2)$. We have four possibilities :

- an integral curve starting from $(\theta_0, \theta_0 - \pi)$ and ending at (θ_0, θ_0),
- an integral curve starting from $(\theta_0, \theta_0 + \pi)$ and ending at (θ_0, θ_0),
- the unstable manifold starting from $(\pi/2, 0)$ and ending at (θ_0, θ_0) or
- the unstable manifold starting from $(0, \pi/2)$ and ending at (θ_0, θ_0).

The behaviour of θ and φ along the unstable manifolds close to $(\pi/2, 0)$ and $(0, \pi/2)$ implies that ρ has a limit when the time parameter goes to $-\infty$. This implies that these two integral curves generate minimal hypersurfaces that extend smoothly near this endpoint.

The behaviour of θ and φ near (θ_0, θ_0) (and also $(\theta_0, \theta_0 \pm \pi)$) implies that ρ grows linearly when the time parameter is close to $\pm\infty$. This implies that all these integral curves generate proper minimal hypersurfaces whose asymptotic behaviour is given by twice the cone $C_{n,p}$ in the first two cases and once the cone $C_{n,p}$ in the last two cases.

Since we study minimal hypersurfaces asymptotic to once the cone $C_{n,p}$, there is only two possibilities that correspond to the two unstable manifolds (when $n = 2p$, extra symmetries of Y implies that the two integral curves are symmetric to each other). Moreover, we know that the density at infinity of $C_{n,p}$ is less than 2 so these two hypersurfaces are embedded. These two hypersurfaces are precisely the hypersurfaces $\Sigma_{n,p,\pm}$ that appear in the statement of Theorem 1.

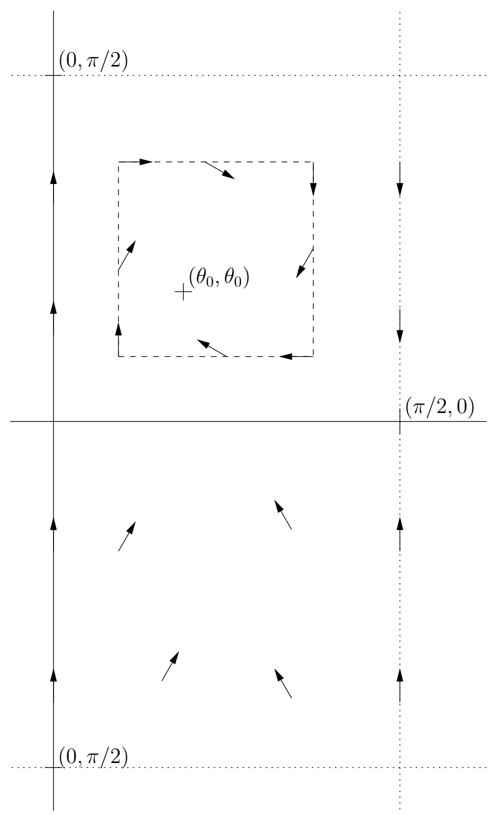

Figure 1: The vector field Y in $[0, \pi/2] \times [-\pi/2, \pi/2]$

A An ODE lemma

In this appendix we prove the following lemma about solutions of linear ode.

Lemma 10. *Let g and f be two smooth functions on $\mathbb{R}_+$ such that*

$$g'' - (\lambda + \mu)g' + \lambda\mu g = f$$

We assume that $\|g\|_{1,\beta}$ and $\|f\|_\delta$ are finite where $\delta \neq \Re(\lambda), \Re(\mu)$. Then g can be written $g(t) = ae^{\lambda t} + be^{\mu t} + v(t)$ with the following estimates:

$$\max(|a|,|b|) \leq c\frac{(2+|\lambda|^2+|\mu|^2)^{1/2}}{|\lambda-\mu|}(|g(0)|+|g'(0)|+|f(0)|)$$
$$+ 2\max(0,(\delta+\Re(\lambda))^{-1},(\delta+\Re(\mu))^{-1})\|f\|_\delta$$
$$\|v\|_\delta \leq 2(|\delta+\Re(\lambda)|^{-1}+|\delta+\Re(\mu)|^{-1})\|f\|_\delta$$

for some universal constant $c > 0$.

Proof. As a solution of such an ode, g can be written

$$g(t) = ae^{\lambda t} + be^{\mu t} + e^{\lambda t}\int_0^t f(u)e^{-\mu t}du + e^{\lambda t}\int_0^t f(u)e^{-\mu u}du$$

With a and b solution of

$$\begin{cases} g(0) &= a+b \\ g'(0) &= \lambda a + \mu b + 2f(0) \end{cases}$$

So $\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\mu-\lambda}\begin{pmatrix} \mu & -1 \\ \lambda & 1 \end{pmatrix}\begin{pmatrix} g(0) \\ g'(0)-2f(0) \end{pmatrix}$ so

$$\max(|a|,|b|) \leq c\frac{(2+|\lambda|^2+|\mu|^2)^{1/2}}{|\lambda-\mu|}(|g(0)|+|g'(0)|+|f(0)|).$$

If $\delta + \Re(\lambda) > 0$

$$\begin{aligned}\int_0^t f(u)e^{-\lambda t}du &= \int_0^{+\infty} f(u)e^{-\lambda t}du + \int_{+\infty}^t f(u)e^{-\lambda t}du \\ &= A + \int_{+\infty}^t f(u)e^{-\lambda t}du\end{aligned}$$

with $|A| \leq \int_0^{+\infty}|f|_\delta e^{-(\delta+\Re(\lambda))u}du \leq \frac{|f|_\delta}{\delta+\Re(\lambda)}$ and

$$\Big|\int_{+\infty}^t f(u)e^{-\lambda u}du\Big| \leq \frac{|f|_\delta}{\delta+\Re(\lambda)}e^{-(\delta+\Re(\lambda))t}.$$

If $\delta + \Re(\lambda) < 0$, we have:

$$\Big|\int_0^t f(u)e^{-\lambda u}du\Big| \leq \frac{2|f|_\delta}{-(\delta+\Re(\lambda))}e^{-(\delta+\Re(\lambda))t}.$$

This finally gives $g = ae^{\lambda t} + be^{\mu t} + v$ with the expected estimates. $\square$

B A flux computation

In this appendix, we make the computation of the flux used in the proof of Proposition 5. So we use some notation introduced in this proof.

We are in the case $n = 2$, so the hypersurface Σ is parametrized by

$$Y(t,\theta,\varphi) = e^t\big(R(\theta,\varphi) + w(t,\theta,\varphi)N(\theta,\varphi)\big)$$

where

$$R(\theta,\varphi) = \frac{1}{\sqrt{2}}\begin{pmatrix}\cos\theta\\ \sin\theta\\ \cos\varphi\\ \sin\varphi\end{pmatrix} \quad \text{and} \quad N(\theta,\varphi) = \frac{1}{\sqrt{2}}\begin{pmatrix}\cos\theta\\ \sin\theta\\ -\cos\varphi\\ -\sin\varphi\end{pmatrix}$$

Moreover, we notice that w and w_t are $O(e^{-\frac{3t}{2}})$ and w_θ and w_φ are $O(e^{-2t})$. We also define

$$E_\theta(\theta,\varphi) = \begin{pmatrix}-\sin\theta\\ \cos\theta\\ 0\\ 0\end{pmatrix} \quad \text{and} \quad E_\varphi(\theta,\varphi) = \begin{pmatrix}0\\ 0\\ -\sin\varphi\\ \cos\varphi\end{pmatrix}$$

We notice that $R, E_\theta, E_\varphi, N$ is an oriented orthonormal basis. We have

$$\begin{aligned} Y_t &= e^t\big(R + (w+w_t)N\big)\\ Y_\theta &= e^t\big(\frac{1+w}{\sqrt{2}}E_\theta + w_\theta N\big)\\ Y_\varphi &= e^t\big(\frac{1-w}{\sqrt{2}}E_\varphi + w_\varphi N\big) \end{aligned}$$

So the cross product of X_t, X_θ and X_φ is

$$\begin{aligned} \bigwedge(Y_t, Y_\theta, Y_\varphi) &= e^{3t}\big(\frac{1-w^2}{2}N - \frac{1-w}{\sqrt{2}}w_\theta E_\theta - \frac{1+w}{\sqrt{2}}w_\varphi E_\varphi - (w+w_t)R\big)\\ &= e^{3t}\big(\frac{1}{2}N - \frac{w+w_t}{2}R - \frac{w_\theta}{\sqrt{2}}E_\theta - \frac{w_\varphi}{\sqrt{2}}E_\varphi + O(e^{-\frac{5t}{2}})\big) \end{aligned}$$

So the unit normal $n(t,\theta,\varphi)$ to the graph has the following expression

$$n = N - (w+w_t)R - \sqrt{2}w_\theta E_\theta - \sqrt{2}w_\varphi E_\varphi + O(e^{-\frac{5t}{2}})$$

Then to get an expression of the normal ν to the boundary of Ω_t, we compute

$$\bigwedge(n, Y_\theta, Y_\varphi) = e^{2t}(-\frac{1}{2}R + \frac{w+w_t}{2}N + O(e^{-\frac{5t}{2}}))$$

So $\nu = -R + (w+w_t)N + O(e^{-\frac{5t}{2}})$. Besides the surface element along $\partial\Omega_t$ can be estimated by $e^{2t}(\frac{1}{2} + O(e^{-3t}))d\theta d\varphi$. So the flux F of ν is given by

$$\begin{aligned} F &= e^{2t}\int_0^{2\pi}\int_0^{2\pi}(-R + (w+w_t)N + O(e^{-\frac{5t}{2}}))(\frac{1}{2} + O(e^{-3t}))d\theta d\varphi \\ &= \frac{e^{2t}}{2}\int_0^{2\pi}\int_0^{2\pi} -R + (w+w_t)N + O(e^{-\frac{5t}{2}})d\theta d\varphi \end{aligned}$$

We notice that the integral of R and N vanishes, so because of the expression of w we get the following estimates

$$\begin{aligned} F &= \frac{e^{2t}}{2}\int_0^{2\pi}\int_0^{2\pi} -e^{2t}(X_1, N)N + O(e^{-(2+2\varepsilon)t})d\theta d\varphi \\ &= -\int_0^{2\pi}\int_0^{2\pi}\frac{1}{2}(X_1, N)N d\theta d\varphi + O(e^{-2\varepsilon t}) \end{aligned} \tag{7}$$

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