Complements to quasi-periodic minimal surfaces

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Abstract

This paper is an appendix to [5]. We give precisions on the surfaces constructed in this previous paper.

Introduction

In [5], M. Traizet and the author construct a familly of properly embedded minimal surfaces in $\mathbb{R}^2 \times \mathbb{S}^1$. These surfaces have an infinite number of ends, two limit ends and finite or infinite genus. The construction is based on the choice of a parameter $0 < \ell < 1$ and a sequence $(p_i)_{i\in\mathbb{Z}}$ of intergers.

The authors explain that their surfaces can be seen as the "gluing" of fundamental domains of Karcher and Wei surfaces: the sequence $(p_i)_{i\in\mathbb{Z}}$ prescribes in which order the gluing of fundamental domains is made. In this paper, we give sense to this affirmation.

In fact, we want to give precisions on the behaviour of these surfaces, so this paper can be understood as an appendix to [5]. We have essentially two results. The first one deals with the ends of the surface. We know that each end is of Scherk type. Our result gives some uniformity on the surface for the behaviour of these ends (see Theorem 4).

Our second result (Theorem 9) tells us that for small ℓ, the surfaces can be sliced by parallel vertical planes such that each obtained component is like the fundamental domain of either Karcher surface or Wei surface i.e. same genus, four parallel Scherk type ends and bounded by two closed curves (see Figure 3).

In the first section, we quickly explain how the surfaces are built in [5]; for more explanations, we recommend the reading of [5]. We also give a first result which is a gradient bound for the function used to construct our surfaces.

The second section deals with the behaviour of Scherk type ends and mainly the uniformity of this behaviour.

The third section contains a precision of Proposition 10 in [5]. This result says that for a fixed ℓ there exists a constant which bounds the Gauss curvature of all the surfaces built with this ℓ parameter. Our precision explains how this constant changes with ℓ.

In the last section, we explain in which sense surfaces in [5] can be viewed as gluing of fundamental domains of Karcher and Wei surfaces.

1 Prelimiaries and notations

1.1 The graphs

In this paper, we consider minimal surfaces as graphs of functions over a domain in $\mathbb{R}^2$. The graph of u is a minimal surface if u satisfies the minimal graph equation :

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)=0$$

The solutions of this equation will be constructed as follow. Let v be a function over a domain in $\mathbb{R}^2$. We say that v satisfies the maximal graph equation if :

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right)=0 \tag{1}$$

We notice that $|\nabla v|$ needs to be less than 1. When v is such a solution, we can locally define a solution u of the minimal graph equation by :

$$\mathrm{d}u=\mathrm{d}\Phi_v=\frac{v_y}{\sqrt{1-|\nabla v|^2}}\mathrm{d}x-\frac{v_x}{\sqrt{1-|\nabla v|^2}}\mathrm{d}y$$

The function v and u are said to be canjugated. For many tools on the study of these two equations, we refer to [5].

We notice that when a minimal surface is a graph, we always choose as normal the downward pointing normal.

1.2 The surfaces

In the section, we explain some steps of the construction in [5] and fix some notations.

Let ℓ be a real number in $(0,1)$. Let Ω be the strip $\mathbb{R}\times(-\ell,\ell)$. We remark that Ω depends on ℓ but in the following we shall not make this precision. For k in $\mathbb{Z}$, let $a_k^+ = (k, \ell)$ and $a_k^- = (k, -\ell)$. We then define the function φ on the segment $[a_{2k-1}^\pm, a_{2k+1}^\pm]$ by $\varphi(p) = |p - a_{2k}^\pm|$. The function φ is piecewise affine on $\partial\Omega$, with value 0 at $a_{2k}^\pm$ and 1 at $a_{2k+1}^\pm$.

Let $(p_i)_{i\in\mathbb{Z}}$ be a strictly increasing sequence of integers. If we denote by $\eta(\ell)$ the quantity $1 - \sqrt{1-\ell^2}$, we have the following result (see Proposition 1 in [5]):

Proposition 1. *For every sequence $(q_i)_{i\in\mathbb{Z}}$ such that, for all $i \in \mathbb{Z}$, $q_i \in \big(2p_i - \eta(\ell), 2p_i + \eta(\ell)\big)$, there exists a solution v of the maximal surface equation such that $v|_{\partial\Omega} = \varphi$ and, for every $i \in \mathbb{Z}$, $v(q_i) = 0$. Besides the solution is unique for that boundary data.*

In the following, we shall denote sometimes by $v[q_i, i \in \mathbb{Z}]$ this solution. Because of the uniqueness, we have :

$$v(x, y) = v(x, -y) \tag{2}$$

Let us now define the following three 1-forms :

$$\mathrm{d}X_1^* = \frac{-v_x v_y \mathrm{d}x + (1-(v_y)^2)\mathrm{d}y}{\sqrt{1-|\nabla v|^2}} \tag{3}$$

$$\mathrm{d}X_2^* = \frac{-(1-(v_x)^2)\mathrm{d}x + v_x v_y \mathrm{d}y}{\sqrt{1-|\nabla v|^2}} \tag{4}$$

$$\mathrm{d}X_3^* = \mathrm{d}v \tag{5}$$

These three 1-forms are closed so X_1^*, X_2^* and X_3^* are locally well defined and give a minimal immersion. Since $X_3^* = v$, this function is well defined on $\Omega \setminus \{q_i, i \in \mathbb{Z}\}$. Let γ_i be a small circle around q_i. The quantity $\displaystyle\int_{\gamma_i} \mathrm{d}X_2^*$ vanishes because of the symmetry (2) of the function v. Then the function X_2^* is well defined on $\Omega \setminus \{q_i, i \in \mathbb{Z}\}$. We then define the period by :

$$F_i(q_j, j \in \mathbb{Z}) = \int_{\gamma_i} \mathrm{d}X_1^*$$

Then there exists $\eta_0 < \eta(\ell)$ which depends only on ℓ such that the following proposition is true (see Propositions 5 and 6 in [5]).

Proposition 2. *There exists a sequence $(q_i)_{i\in\mathbb{Z}}$, such that $|q_i - 2p_i| \leq \eta_0$ and, for every $j \in \mathbb{Z}$, $F_j(q_i, i \in \mathbb{Z}) = 0$.*

The proposition says that the sequence $(q_i)_{i\in\mathbb{Z}}$ can be chosen such that X_1^* is also well defined on $\Omega\setminus\{q_i, i\in\mathbb{Z}\}$. In the following, we shall say that the sequence of singularities $(q_i)_{i\in\mathbb{Z}}$ solves the Period Problem for the data $(p_i)_{i\in\mathbb{Z}}$ or simply solves the Period Problem without precising the data.

We notice that the functions X_k^* depends on ℓ and $(q_i)_{i\in\mathbb{Z}}$; in the following, we shall precise these parameters only when it is necessary. Besides, these functions are defined up to a constant, so, in general, we choose $(X_1^*, X_2^*, X_3^*)((-1,0) = (0,0,v(-1,0))$.

When the Period Problem is solved, the map (X_1^*, X_2^*, X_3^*) defines a minimal embedding whose image is a minimal surface in $\mathbb{R}^2\times[0,1]$. The boundary of this surface is included in $\{z=0\}$ and $\{z=1\}$. Hence by completed by symmetry with respect to horizontal planes, we get a complete minimal surface $\mathcal{M}[q_i, i\in\mathbb{Z}]$. We notice that the surface $\mathcal{M}[q_i, i\in\mathbb{Z}]$ is periodic of period $(0,0,2)$; then it can be seen as a surface in $\mathbb{R}^2\times\mathbb{S}^1$. In the following, we shall use both points of view. If u is the conjugate function to u, the image of the immersion is the conjugate surface to the minimal graph of u.

We notice that this construction can also be made when $(p_i)_{\in I}$ is a strictly increasing sequence of integers with I finite or $I=\mathbb{N},\ -\mathbb{N}$.

In [5] (see Proposition 10), it is proved that for every ℓ there exists a constant $C(\ell)$ such that the curvature of the surface $\mathcal{M}[q_i, i\in\mathbb{Z}]$ is bounded by $C(\ell)$ for every sequence $(q_i)_{i\in\mathbb{Z}}$ that solves the period problem.

1.3 Bounded gradient

In this subsection, we give the first result on the solutions v we are interested in. Let us denote by $\Omega[q_i, i\in\mathbb{Z}]$ the set $\Omega\setminus\{q_i, i\in\mathbb{Z}\}$. For $\varepsilon>0$, we denote by $\Omega[q_i, i\in\mathbb{Z}]_\varepsilon$ the set of point in $\Omega[q_i, i\in\mathbb{Z}]$ which are a distance at least ε from $\partial\Omega$ and the points q_i. We then have

Proposition 3. *Let $\eta_1<\eta(\ell)$ and $(q_i)_{i\in\mathbb{Z}}$ be such that, for every $i\in\mathbb{Z}$, $|q_i-2p_i|\le\eta_1$. Let v be the solution $v[q_i, i\in\mathbb{Z}]$ given by Proposition 1. Then for every $\varepsilon>0$ there exists $c>0$ such that $|\nabla v|<1-c$ in $\Omega[q_i, i\in\mathbb{Z}]_\varepsilon$.*

Proof. Let us fix $\varepsilon>0$. If the proposition is not true there exists a sequence P_n in $\Omega[q_i, i\in\mathbb{Z}]_\varepsilon$ such that $|\nabla v|(P_n)\to 1$.

Let us write $P_n=(2k_n+x_n, y_n)$ where $x_n\in[-1,1]$ and $k_n\in\mathbb{Z}$. Let us consider $v_n(x,y)=v(x+2k_n,y)$. We have $v_n=v[q_i^n, i\in\mathbb{Z}]$ where $q_i^n=q_i-2k_n$.

Let $\mathcal{S}_n\subset\mathbb{R}$ be the set $\{q_i^n, i\in\mathbb{Z}\}$. Since for every $n\in\mathbb{N}$ and $i\in\mathbb{Z}$, $q_{i+1}^n-q_i^n>2(1-\eta_1)$, the sequence $(\mathcal{S}_n)_{n\in\mathbb{N}}$ is uniformly locally finite *i.e.*

for every compact subset $K \in \mathbb{R}$ there exists $N_K \in \mathbb{N}$ such that, for every $n \in \mathbb{N}$, $\#(S_n \cap K) \leq N_K$. Then we can assume that the sequence $(S_n)_{n\in\mathbb{N}}$ converges to $S_\infty \subset \mathbb{R}$ for the Hausdroff topology on compact subsets. S_∞ can be written $\{q_i^\infty, i \in I\}$ where I is one of the following possibilities : $\{0, \cdots, p\}$ (possibly empty), $\mathbb{N}$, $-\mathbb{N}$ or $\mathbb{Z}$; and such that, for every $i \in I$, $q_i^\infty < q_{i+1}^\infty$.

Since for every $i \in \mathbb{Z}$ $|q_i - 2p_i| \leq \eta_1$, the study of divergence lines of $(v_n)_{n\in\mathbb{N}}$ proves that $v_n \to v[q_i^\infty, i \in I]$ and the convergence is smooth on compact subsets of $\Omega[q_i^\infty, i \in I]$ (see [5]). Then ∇v_n is uniformly bounded far from 1 on $\Omega[q_i^\infty, i \in I]_{\varepsilon/2} \cap ([-1,1] \times \mathbb{R})$. This contradicts $|\nabla v_n|(x_n, y_n) \to 1$ and the proposition is proved since $(x_n, y_n) \in \Omega[q_i^\infty, i \in I]_{\varepsilon/2} \cap ([-1,1] \times \mathbb{R})$ for great n. □

2 Behaviour far from x=0

In this section, the ℓ parameter is fixed.

Let us consider the Karcher and Wei surfaces (see Figure 1), these surfaces are invariant by some vector $(0, t, 0)$ and the plane $x = 0$ is one of their symmetry planes. Far from this plane, the two surfaces look like infinitely many parallel planes. Because of the periodicity this picture is uniform in y. In this section, we prove that this behaviour is also true for the surfaces $\mathcal{M}[q_i, i \in \mathbb{Z}]$.

Theorem 4. *Let $\varepsilon > 0$. There exists $x_0 = x_0(\ell, \varepsilon) > 0$ such that, for every $(q_i)_{i\in\mathbb{Z}}$ which solves the Period Problem, at every point P of $\mathcal{M}[q_i, i \in \mathbb{Z}] \cap \{|x| \geq x_0\}$, the normal $N(P)$ satisfies $d(N(P), e_y) < \varepsilon$ or $d(N(P), -e_y) < \varepsilon$ where e_y is the vector $(0, 1, 0)$.*

Proof. Since the surfaces $\mathcal{M}[q_i, i \in \mathbb{Z}]$ are symmetric with respect to the plane $x = 0$, we can restrict the study to $\mathcal{M}[q_i, i \in \mathbb{Z}] \cap \{x \geq 0\}$ *i.e.* the image of $\Omega_+ = \Omega \cap \{y \geq 0\}$ by (X_1^*, X_2^*, X_3^*). Let $\alpha > 0$ be small and, for every $k \in \mathbb{Z}$, define $T_\alpha(k)$ the following small triangle : $(x, y) \in \mathbb{R} \times (0, \ell)$ is in $T_\alpha(k)$ if $k \leq x \leq k+1$, $(y-\ell) > -(x-k)\tan\alpha$ and $(y-\ell) > (x-k-1)\tan\alpha$. We then define $D_\alpha = \bigcup_{k\in\mathbb{Z}} T_\alpha(k)$. Because of Lemma 1 in [3], for every ε, there exists $\alpha > 0$ such that, for every $P \in D_\alpha$, $|\nabla v(P) - (1,0)| < \varepsilon$ or $|\nabla v(P) + (1,0)| < \varepsilon$. Then for every $\varepsilon > 0$, there exists $\alpha > 0$ such that, for every $P \in D_\alpha$, the normal N to $\mathcal{M}[q_i, i \in \mathbb{Z}]$ at the point $(X_1^*, X_2^*, X_3^*)(P)$ satisfies $d(N, e_y) < \varepsilon$ or $d(N, -e_y) < \varepsilon$.

Let us fix $\varepsilon > 0$. If the theorem is not true, for every $n \in \mathbb{N}$, there exists a sequence of singularities $(q_i^n)_{i\in\mathbb{Z}}$ such that the Period Problem is solved and a point P_n in $\Omega_+ \setminus \{q_i, i \in \mathbb{Z}\}$ such that :

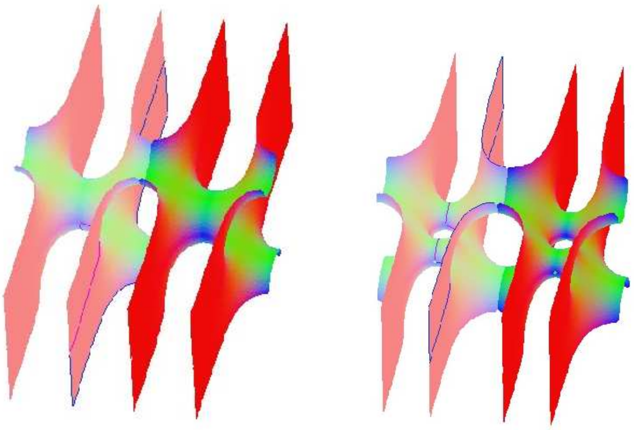

Figure 1: Left : one of the Karcher surfaces. Right : one of Wei surfaces. A fundamental domain is highlighted for each. Both surfaces extend periodically vertically and horizontally. Computer images made by the authors using J. Hoffman's MESH software.

- $X_1^*(P_n) > n$ and
- $d(N_n, e_y) \geq 2\varepsilon$ and $d(N_n, -e_y) \geq 2\varepsilon$, where N_n is the normal to $\mathcal{M}[q_i^n, i \in \mathbb{Z}]$ at the point $Q_n = (X_1^*, X_2^*, X_3^*)(P_n)$.

Besides, by translating the sequence $(q_i^n)_{i\in\mathbb{Z}}$ by an even integer, we can assume that the point P_n is in $-1 \leq x \leq 1$.

Let us consider the horizontal translation T_n such that $T_n(Q_n) \in \{(0,0)\}\times[0,1]$. Since the curvature of all the $\mathcal{M}[q_i^n, i \in \mathbb{Z}]$ is uniformly bounded by a constant $C(\ell)$ (Proposition 10 in [5]), we can assume that $T_n(Q_n) \to Q_\infty$ and $T_n(\mathcal{M}[q_i^n, i \in \mathbb{Z}]) \to \Sigma$ where Σ is an embedded minimal surface. Because of the hypothesis on N_n, the normal N_∞ to Σ at Q_∞ satisfies $d(N_\infty, e_y) \geq 2\varepsilon$ and $d(N_\infty, -e_y) \geq 2\varepsilon$. We notice that, for every $n \in \mathbb{N}$, the normal to $\mathcal{M}[q_i^n, i \in \mathbb{Z}]$ at a point in $\{z = 0\} \cup \{z = 1\}$ is horizontal. Hence if $Q_\infty \in \{z = 0\} \cup \{z = 1\}$, N_∞ is horizontal then Σ is transverse to the corresponding horizontal plane. So we can ensure that there exists $\eta > 0$ and a point $B_\infty \in \Sigma$ such that :

- $d_\Sigma(Q_\infty, B_\infty) \leq 2\eta$,
- $B_\infty \in \mathbb{R}^2 \times [\eta, 1-\eta]$ and

- the normal N'_∞ to Σ at B_∞ satisfies $d(N'_\infty, e_y) \geq 3\varepsilon/2$ and $d(N'_\infty, -e_y) \geq 3\varepsilon/2$.

When $Q_\infty \notin \{z = 0\} \cup \{z = 1\}$, we can choose $B_\infty = Q_\infty$.

By construction of Σ, for every n in $\mathbb{N}$, we can choose a point B_n in $\mathcal{M}[q_i^n, i \in \mathbb{Z}]$ such that the sequence $(T_n(B_n))_{n\in\mathbb{N}}$ converges to B_∞. Besides we can assume that, for every $n \in \mathbb{N}$:

- $d_{\mathcal{M}[q_i^n, i\in\mathbb{Z}]}(Q_n, B_n) \leq 3\eta$,
- $B_n \in \mathbb{R}^2 \times [\eta/2, 1 - \eta/2]$ and
- the normal N'_n to $\mathcal{M}[q_i^n, i \in \mathbb{Z}]$ at B_n satisfies $d(N'_n, e_y) \geq \varepsilon$ and $d(N'_n, -e_y) \geq \varepsilon$.

For every n in $\mathbb{N}$, we can write $B_n = (X_1^*, X_2^*, X_3^*)(A_n)$. Because of the first item above, $d(A_n, P_n) < 3\eta$ then A_n is in $(-(1 + 3\eta), 1 + 3\eta) \times (-3\eta, \ell)$. Because of the second item and since $X_3^* = v[q_i^n, i \in \mathbb{Z}]$ is 1-Lipschitz continuous, A_n is at least at a distance $\eta/2$ from the points $a_k^\pm$ and the singularities q_i^n $(i \in \mathbb{Z})$. The third item says that there exists $\alpha > 0$ such that, for every $n \in \mathbb{N}$, A_n is outside D_α (see Figure 2).

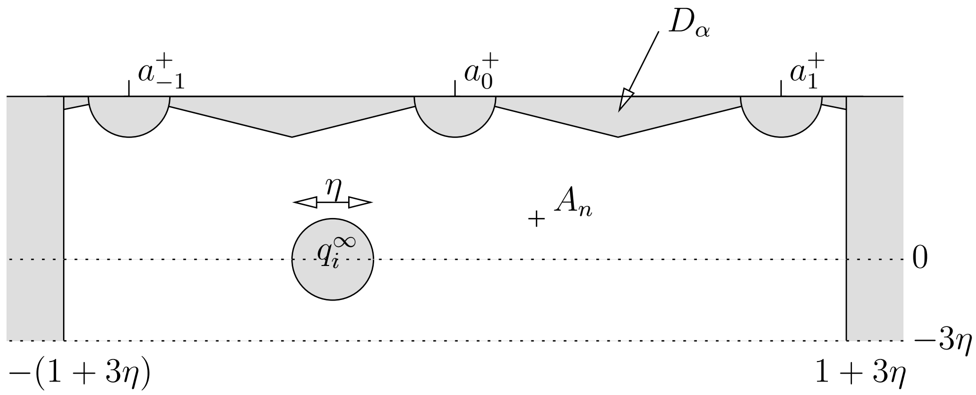

Figure 2:

Let $\mathcal{S}_n \subset \mathbb{R}$ be the set $\{q_i^n, i \in \mathbb{Z}\}$. As in the preceding section, the sequence $(\mathcal{S}_n)_{n\in\mathbb{N}}$ is uniformly locally finite. Then we can assume that the sequence $(S_n)_{n\in\mathbb{N}}$ converges to $S_\infty \subset \mathbb{R}$ in the Hausdroff topology on compact subsets. S_∞ can be written $\{q_i^\infty, i \in I\}$ where I is one of the following possibilities : $\{0, \cdots, p\}$ (possibly empty), $\mathbb{N}$, $-\mathbb{N}$ or $\mathbb{Z}$; and such that, for every $i \in I$, $q_i^\infty < q_{i+1}^\infty$.

The study of divergence lines of $(v[q_i^n, i \in \mathbb{Z}])_{n\in\mathbb{N}}$ proves that $v[q_i^n, i \in \mathbb{Z}] \to v[q_i^\infty, i \in I]$ and the convergence is smooth on compact subsets of $\Omega \setminus \{q_i^\infty, i \in I\}$. Besides $(q_i^\infty)_{i\in I}$ solves the Period Problem. Then $X_1^*[q_i^n, i \in \mathbb{Z}] \to X_1^*[q_i^\infty, i \in I]$ and $X_1^*[q_i^n, i \in \mathbb{Z}]$ is uniformly bounded on compact subset in $\Omega \setminus \{q_i^\infty, i \in I\}$.

Because of the constraint on the position of A_n, A_n moves in a compact subset of $\Omega \setminus \{q_i^\infty, i \in I\}$. So there exists a constant M such that, for every $n \in \mathbb{N}$, $X_1^*(A_n) \le M$. This gives us :

$$\begin{aligned} n \le X_1^*(P_n) &= X_1^*(P_n) - X_1^*(A_n) + X_1^*(A_n) \\ &\le |X_1^*(P_n) - X_1^*(A_n)| + X_1^*(A_n) \\ &\le 3\eta + M \end{aligned}$$

The last inequality comes from the first item above. We then have a contradiction. □

Corollary 5. *There exists a constant $M(\ell)$ such that, for every $(q_i)_{i\in\mathbb{Z}}$ which solves the Period Problem, the curvature K of $\mathcal{M}[q_i, i \in \mathbb{Z}]$ satisfies*

$$|K(P)| \le \frac{M(\ell)}{|x(P)|^2}$$

Proof. We already know that the curvature of $\mathcal{M}[q_i, i \in \mathbb{Z}]$ is uniformly bounded thus we have to prove the inequality for great x.

Let us fix $\varepsilon = \frac{1}{100}$. Theorem 4 gives one x_0 which depends only on ℓ. Let $(q_i)_{i\in\mathbb{Z}}$ which solves the Period Problem. Let Σ be a connected component of $\mathcal{M}[q_i, i \in \mathbb{Z}] \cap \{|x| \ge x_0\}$. By Theorem 4 the image of Σ by the Gauss map is small thus Σ is stable (see [1]) and then for every P on Σ :

$$|K(P)| \le \frac{c}{d(P, \partial\Sigma)^2} \le \frac{c}{(|x| - x_0)^2}$$

where c is a universal constant (see [6]). The corollary is then proved. □

3 Bounded curvature

In [5], it is proved that there exists a constant $C(\ell)$ such that the curvature is uniformly bounded by $C(\ell)$ on every surface $\mathcal{M}[q_i, i \in \mathbb{Z}]$ which is built with the ℓ parameter. In this section, we precise the behaviour of this constant in ℓ.

In fact, we need mainly an improvement of Proposition 9 in [5]. We use the notation $\Omega_L = (-L, L) \times (-\ell, \ell)$.

Proposition 6. *There exists $\kappa > 0$ such that the following is true. Let ℓ be in $(0,1)$ and consider $q \in (-\eta(\ell), \eta(\ell))$ and $\mathcal{S} \subset (-2, -2+\eta(\ell)) \cup (2-\eta(\ell), 2)$. Let v be a solution of the maximal graph equation in $\Omega_2 \setminus (\{q\} \cup \mathcal{S})$ with boundary value φ on $[-2,2] \times \{-\ell, \ell\}$ and 0 at $\{q\} \cup \mathcal{S}$; besides, we require that $0 \le v \le 1$ on the two vertical edges. Let u be the conjugate function to v. Let γ be a small circle around q. Then* $\left| \int_\gamma \mathrm{d}u \right| \ge \kappa \ell$.

Proof. Because of Proposition 9 in [5], if the proposition is not true the exists a sequence $(\ell_n)_{n\in\mathbb{N}}$ which tends to 0 and, for each $n \in \mathbb{N}$, a point q_n, a set $\mathcal{S}_n$ and a function v_n as above such that if u_n is the conjugate to v_n :

$$\left| \int_\gamma \mathrm{d}u_n \right| \le \frac{\ell_n}{n} \tag{6}$$

Let us consider for each n the homothetic of all these data by $1/\ell_n$. We then get, in the domain $(-2/\ell_n, 2/\ell_n) \times (-1,1)$, a point $\tilde{q}_n \in (-\eta(\ell_n)/\ell_n, \eta(\ell_n)/\ell_n)$, a set $\tilde{\mathcal{S}}_n$ which is outside $[-1/\ell_n, 1/\ell_n]$ and a function $\tilde{v}_n$ which is a solution of the maximal graph equation and is defined by $\tilde{v}_n(x,y) = \frac{1}{\ell_n} v_n(\ell_n x, \ell_n y)$. We notice that $0 \le \tilde{v}_n \le 1/\ell_n$ and, on $[-1/\ell_n, 1/\ell_n] \times \{-1, 1\}$, the function $\tilde{v}_n$ takes the value: $\tilde{v}_n(x, \pm 1) = |x|$.

Let us study the divergence lines of $(\tilde{v}_n)_{n\in\mathbb{N}}$. First, since $\ell_n \to 0$, $\frac{\eta(\ell_n)}{\ell_n} = \frac{1}{2}\ell_n + o(\ell_n)$; hence $\tilde{q}_n \to (0,0)$. The limit domain is then $\mathbb{R} \times (-1,1) \setminus \{(0,0)\}$. Since $0 \le \tilde{v}_n$, a divergence line L needs to have at least one end-point. Because of the value of $\tilde{v}_n$ on $\partial\mathbb{R} \times (-1,1)$ and Lemma 3 in [5], the only possible end-points are $(0,-1)$, $(0,0)$ and $(0,1)$. For these three points, $\lim \tilde{v}_n$ is 0 then L can not be a segment joining two of them. This implies that the end-point can not be $(0,-1)$ and $(0,1)$ and L is either $\mathbb{R}_+ \times \{0\}$ or $\mathbb{R}_- \times \{0\}$. Let us prove that these two half-lines are in fact the divergence lines of the sequence.

Let $\tilde{u}_n$ be the conjugate to $\tilde{v}_n$. Let us fix x such that $|x| > 9$. Let $n \in \mathbb{N}$ be enough large such that $1/\ell_n - |x| > 9$. The distance from (x,y) to the boundary of $(-2/\ell_n, 2\ell_n) \times (-1,1) \setminus (\{\tilde{q}_n\} \cup \tilde{\mathcal{S}}_n)$ is then less than 1. Besides $\tilde{u}_n$ takes the value $+\infty$ on $(0, 1/\ell_n) \times \{-1\}$ and $(-1/\ell_n, 0) \times \{1\}$ and the value $-\infty$ on $(0, 1/\ell_n) \times \{1\}$ and $-1/\ell_n, 0) \times \{-\}$. So the distance in the graph of $\tilde{u}_n$ from the point $(x, y, \tilde{u}_n(x,y))$ to the boundary of the graph is larger than 8. Then by Lemma 1 in [3], we have:

$$|\partial_x \tilde{v}_n|(x,y) = \left|\frac{\partial_y \tilde{u}_n}{\sqrt{1+|\nabla \tilde{u}_n|^2}}\right|(x,y) \geq 1 - \frac{4}{(|x|-1)^2} \tag{7}$$

$$|\partial_y \tilde{v}_n|(x,y) = \left|\frac{\partial_x \tilde{u}_n}{\sqrt{1+|\nabla \tilde{u}_n|^2}}\right|(x,y) \leq \frac{2}{|x|-1} \tag{8}$$

Besides because of the value of $\tilde{u}_n$ on the boundary $\partial_y \tilde{u}_n$ is positive if $x < 0$ and negative if $x > 0$. Because of (8), $\tilde{v}_n(x,0) \geq |x| - \frac{2}{|x|-1}$.

Let us assume for example that $\mathbb{R}_+ \times \{0\}$ is not a divergence line, then there exists $c > 0$ such that $\tilde{v}_n(1,0) \leq 1-c$, for every $n \in \mathbb{N}$. Thus for every $x > 1$, $\tilde{v}_n(x,0) \leq 1 - c + |x-1| = x - c$. Let us chose $x > 1$ such that $\frac{2}{|x|-1} < c$, then for large n, we get :

$$x - \frac{2}{|x|-1} \leq \tilde{v}_n(x,0) \leq x - c < x - \frac{2}{|x|-1}$$

We have our contradiction and the divergence lines are known.

Then on $\mathbb{R} \times (-1,0)$ and $\mathbb{R} \times (0,1)$, a subsequence of $(\tilde{v}_n)$ converges to a solution v of the maximal graph equation which takes on $\mathbb{R} \times \{-1,0,1\}$ the value $v(x,y) = |x|$, Let u_- and u_+ the respective conjugate of v on $\mathbb{R} \times (-1,0)$ and $\mathbb{R} \times (0,1)$. u_- takes the value $+\infty$ on $\mathbb{R}_-^* \times \{0\} \cup \mathbb{R}_+^* \times \{-1\}$ and the value $-\infty$ on $\mathbb{R}_-^* \times \{-1\} \cup \mathbb{R}_+^* \times \{0\}$. Such a solution of the minimal graph equation is unique (see Theorem 1 in [2] or Theorem 5.1 in [4]) and the graph of u_- is a part of an helicoid : $u_-(x,y) = -x\tan(\pi(y+1/2))$. In the same way, we obtain that $u_+(x,y) = -x\tan(\pi(y-1/2))$.

Let us now fix x_0 be greater than 9 and consider Γ the rectangle with vertices $A_1 = (x_0, 1/2)$, $A_2 = (-x_0, 1/2)$, $A_3 = (-x_0, -1/2)$ and $A_4 = (x_0, -1/2)$. For large n, because of (6), we have:

$$\frac{1}{n} \geq \left|\int_{\gamma/\ell_n} \mathrm{d}\tilde{u}_n\right| = \left|\int_{\Gamma} \mathrm{d}\tilde{u}_n\right|$$

Using (7), let us estimate the last term :

$$\begin{aligned}\int_\Gamma \mathrm{d}\tilde{u}_n &= \int_{[A_1,A_2]} \mathrm{d}\tilde{u}_n + \int_{[A_2,A_3]} \mathrm{d}\tilde{u}_n + \int_{[A_3,A_4]} \mathrm{d}\tilde{u}_n + \int_{[A_4,A_1]} \mathrm{d}\tilde{u}_n \\ &= \int_{[A_1,A_2]} \mathrm{d}u_+ + \int_{1/2}^{-1/2} \partial_y \tilde{u}_n(-x_0,y)\mathrm{d}y + \int_{[A_3,A_4]} \mathrm{d}u_- \\ &\qquad\qquad + \int_{-1/2}^{1/2} \partial_y \tilde{u}_n(x_0,y)\mathrm{d}y + o(1) \\ &\leq \int_{[A_1,A_2]} \mathrm{d}u_+ + \int_{1/2}^{-1/2} \sqrt{\frac{(x_0-1)^2}{4} - 1}\,\mathrm{d}y + \int_{[A_3,A_4]} \mathrm{d}u_- \\ &\qquad\qquad + \int_{-1/2}^{1/2} -\sqrt{\frac{(x_0-1)^2}{4} - 1}\,\mathrm{d}y + o(1) \\ &\leq -\sqrt{(x_0-1)^2 - 4} + u_+(A_2) - u_+(A_1) + u_-(A_4) - u_-(A_3) + o(1)\end{aligned}$$

By the expression of u_+ and u_-, the two terms $u_+(A_2) - u_+(A_1)$ and $u_-(A_4) - u_-(A_3)$ vanishes. This gives us a contradiction; the proposition is proved. □

We now can give our bound on the curvature of the surface $\mathcal{M}[q_i, i \in \mathbb{Z}]$.

Theorem 7. *There exists $C_0 > 0$ such that, for every ℓ and every sequence of singularities $(q_i)_{i\in\mathbb{Z}}$ that solves the Period Problem, the curvature of the surface $\mathcal{M}[q_i, i \in \mathbb{Z}]$ is uniformly bounded by* $\dfrac{C_0}{\ell^2}$.

Proof. Let ℓ be positive and $(q_i)_{i\in\mathbb{Z}}$ a sequence of singularities that solves the Period problem. By symmetry it suffices to bound the curvature of the graph M of the conjugate u to $v[q_i, i \in \mathbb{Z}]$. We recall u is defined on $\Omega^+ = \mathbb{R} \times (0, \ell)$.

By proposition 6, there exists κ such that $|\int_{\gamma_i} du| \geq \kappa\ell$ for $i \in \mathbb{Z}$. We apply Lemma 5 in [5] with $C = 100$ and obtain a $\delta_1 < 1$ such that $|\nabla u| \geq 100$ in $D(q_i, \delta_1\ell)$, $i \in \mathbb{Z}$. We apply Lemma 6 in [5] with again $C = 100$ and obtain a $\delta_2 < 1$ such that $|\nabla u| \geq 100$ in $D(a_k^+, \delta_2\ell)$, $k \in \mathbb{Z}$. We take $\delta = \min\{\delta_1, \delta_2\}$, we notice that δ does not depend on ℓ or $(q_i)_{i\in\mathbb{Z}}$. Fix some $i \in \mathbb{Z}$. Let U be the graph of u above the half disk $D(q_i, \delta\ell) \cap \Omega^+$. Since $|\nabla u| \geq 100$, the Gauss image of U is included in the spherical domain $\mathbb{S}^2 \cap \{|z| \leq 1/100\}$. The boundary of U consists of a vertical segment, two horizontal segments and a helix-like looking curve which is a graph on $\mathbb{S}^1(q_i, \delta\ell) \cap \Omega^+$. Completing by all symmetries, we obtain a minimal surface Σ which is bounded by two helix-like looking curves, and which is complete in the cylinder $D(q_i, \delta\ell) \times \mathbb{R}$. The surface Σ is of course not a graph anymore. However its Gauss image is still included in $\mathbb{S}^2 \cap \{|z| < 1/100\}$. As the spherical area of this domain is less than 2π, Σ is stable by the theorem of Barbosa Do Carmo [1]. Consider now a point $(x, y) \in D(q_i, (\delta/2)\ell)$ and let $p = (x, y, u(x, y))$ be the corresponding point on Σ. Since $p \in \Sigma$ is at distance more than $(\delta/2)\ell$ from the boundary of Σ, the theorem of Schoen [6] ensures that the Gauss curvature at p is bounded by $c/((\delta/2)\ell)^2$ for some universal constant c. The same argument gives the same estimate for the Gauss curvature when $(x, y) \in D(a_k^+, \delta/2)$, $k \in \mathbb{Z}$.

Assume now that $(x, y) \in \Omega^+$ is at distance more than $(\delta/2)\ell$ from all points q_i and all points a_k^+. Let again $p = (x, y, u(x, y))$. If $y > (\delta/4)\ell$, then the distance of p to the boundary of M is greater than $(\delta/4)\ell$ (because $u = \pm\infty$ on the top edges). Since M is a graph, it is stable, so the Gauss curvature at p is bounded by $c/((\delta/4)\ell)^2$.

It remains to understand the case $0 < y < \delta/4$. There exists i such that $q_i < x < q_{i+1}$. Consider the box $(q_i, q_{i+1}) \times (-(\delta/2)\ell, (\delta/2)\ell)$. As this is a simply connected domain of Ω, u is well defined on it. Let V be the graph of u on this box. The distance of $p = (x, y, u(x, y))$ to the boundary of V is greater than $(\delta/4)\ell$. Since V is stable, we conclude again that the Gauss curvature at p is bounded by $c/((\delta/4)\ell)^2$. Then the constant $C = 16c/\delta^2$ is an answer to the theorem. $\square$

4 Cutting the surface

In this section we prove that the heuristic idea which says that the surface $\mathcal{M}[q_i, i \in \mathbb{Z}]$ is built as a "gluing" of fundamental domains of Karcher and Wei surfaces is true. We study what happens when we cut these surfaces by vertical planes $\{y = c\}$.

Let us fix some notations. Let $(q_i)_{i\in\mathbb{Z}}$ be a sequence of singularities that solves the Period Problem, we denote by v the function $v[q_i, i \in \mathbb{Z}]$. Near a point $a_k^\pm$, the graph of $u = \Phi_v$ is bounded by a vertical straight-line which becomes in $\mathcal{M}[q_i, i \in \mathbb{Z}]$ a strictly convex curve $C_k^\pm$ in $\{z = 0 \text{ or } 1\}$ of total curvature π. On $C_k^\pm$, the normal goes from $(0, 1, 0)$ to $(0, -1, 0)$ thus, on C_k^+, there is one and only one point on it where the normal is either $(1, 0, 0)$ if k is odd or $(-1, 0, 0)$ if k is even and the symmetric on C_k^-. We denote by $x_k^\pm$ and $y_k^\pm$ the first two coordinates of this point. We recall that C_k^+ and C_k^- are symmetric with respect to the plane $\{x = 0\}$; this implies $x_k^+ = -x_k^-$ and $y_k^+ = y_k^-$.

Near a singularity q_i the graph of Φ_v is bounded by a vertical straight-line which becomes a closed convex curve in $\mathcal{M}[q_i, i \in \mathbb{Z}]$. We denote by γ_i this curve. With these notations, we have the following proposition.

Proposition 8. *For every k, the intersection $\mathcal{M}[q_i, i \in \mathbb{Z}] \cap \{y = y_k^\pm\}$ consists in either one or two closed curves.*

Proof. To study $\mathcal{M}[q_i, i \in \mathbb{Z}] \cap \{y = y_k^\pm\}$ consists in studying $X_2^* = y_k^\pm$ in Ω. By symmetry of v, we can restrict our study of $X_2^* = y_k^\pm$ to $\Omega \cap \{y \geq 0\}$. We have $\dfrac{\partial X_2^*}{\partial x} = \dfrac{-(1-v_x^2)}{\sqrt{1-|\nabla v|^2}} < 0$ then there is at most one point on each line $\{y = y_0\}$ where $X_2^* = y_k^\pm$ (where $0 < y_0 < \ell$). Besides because of Proposition 3, for every ε, ∇v is bounded on $\Omega[q_i, i \in \mathbb{Z}]_\varepsilon$; hence there exists exactly one point in each line $\{y = y_0\}$ where $X_2^* = y_k^\pm$. Then $\{X_2^* = y_k^\pm\}$ is a graph of the y-coordinate over $(0, \ell)$. By definition of $y_k^\pm$, one end point of $\{X_2^* = y_k^\pm\}$ is a_k^+; at this point, $X_2^* = y_k^\pm$ is normal to $\{y = \ell\}$. The set has an other end point on $\{y = 0\}$ since ∇v is bounded on $\Omega[q_i, i \in \mathbb{Z}]_\varepsilon$. If this end point is a q_i, the intersection $\mathcal{M}[q_i, i \in \mathbb{Z}] \cap \{y = y_k^\pm\}$ consists then in two closed curves. If the end point is not a q_i, $\{X_2^* = y_k^\pm\}$ extend smoothly across $y = 0$ by symmetry and $\mathcal{M}[q_i, i \in \mathbb{Z}] \cap \{y = y_k^\pm\}$ consists in one closed curve. $\square$

We have the following result, which says that we can cut the surfaces $\mathcal{M}[q_i, i \in \mathbb{Z}]$ by vertical planes to get pieces which are similar to the fundamental domains of Karcher and Wei surfaces.

Theorem 9. *There exists $0 < \ell_0 < 1$ such that, for every $\ell < \ell_0$ and every $(q_i)_{i \in \mathbb{Z}}$ a sequence of singularity that solves the Period Problem for the sequence $(p_i)_{i \in \mathbb{Z}}$, if k is odd, the intersection $\mathcal{M}[q_i, i \in \mathbb{Z}] \cap \{y = y_k^\pm\}$ consists in one closed curve. Besides if $2p_i < k$, we have $\gamma_i \subset (y_k^\pm, +\infty) \times \mathbb{S}^1$.*

Let ℓ be less than ℓ_0. Take a surface $\mathcal{M}[q_i, i \in \mathbb{Z}]$ and slice it by the vertical planes $\{y = y_k^\pm\}$ for all odd k. The theorem says that each connected component of this cutting is like the fundamental domain of either Karcher surface or Wei surface: same genus, four Scherk type ends and the boundary is composed of two closed curves. Besides the model surface is prescribed by the sequence $(p_i)_{i \in \mathbb{Z}}$ (see Figure 3).

Proof. In fact, we shall prove that for small ℓ, the point in $\Omega \cap \{y = 0\}$ where $X_2^* = y_k^\pm$ is in $(k - \ell, k + \ell)$. Since all the q_i are outside $(k - \sqrt{1-\ell^2}, k + \sqrt{1-\ell^2})$, this will prove the theorem.

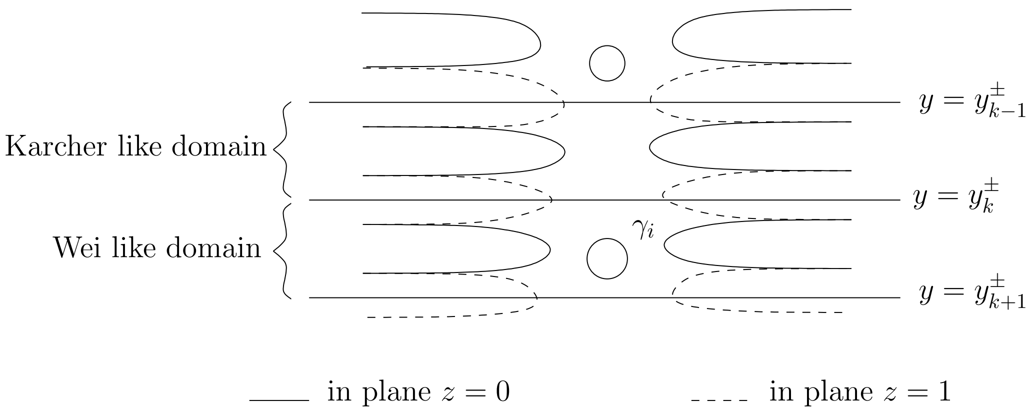

Figure 3:

If it is false, there exists a decreasing sequence $(\ell_n)_{n\in\mathbb{N}}$ which tends to 0 and, for every n, a sequence $(q_i^n)_{i\in\mathbb{Z}}$ of singularities that solves the Period problem and a odd number k_n such that $X_2^* - y^{\pm}_{k_n}$ is either positive or negative on $(k_n - \ell_n, k_n + \ell_n) \times \{0\}$. In the following, we assume that these quantity is positive.

For every $n \in \mathbb{N}$, let us translate $\Omega \setminus \{q_i^n, i \in \mathbb{Z}\}$ by $(-k_n, 0, 0)$ and expand it by $1/\ell_n$. We get $\mathbb{R} \times (-1, 1) \setminus \{(q_i^n - k_n)/\ell_n, i \in \mathbb{Z}\}$. Doing the same transformation on v_n, we get $\tilde{v}_n$ a solution of the maximal graph equation which is defined by :

$$\tilde{v}_n(x, y) = 1 + \frac{v_n(x + k_n, y) - 1}{\ell_n}$$

We notice that we have conserved the equality $\max \tilde{v}_n = 1$. On the boundary, $\tilde{v}_n$ takes the value $1 - |x - k/\ell_n|$ where $k \in \mathbb{N}$ is even and $k - 1 \leq x\ell_n \leq k + 1$.

Let Σ_n be the complete minimal surface in $\mathbb{R}^3$ which is the image of $\mathcal{M}[q_i^n, i \in \mathbb{Z}]$ by the translation of vector $(-x^+_{k_n}, -y^+_{k_n}, 0)$ then by the homothety of center $(0, 0, 1)$ and ratio $1/\ell_n$. On $\mathbb{R} \times (-1, 1) \setminus \{(q_i^n - k_n)/\ell_n, i \in \mathbb{Z}\}$, we can define the three functions $\tilde{X}^*_{1,n}$, $\tilde{X}^*_{2,n}$ and $\tilde{X}^*_{3,n}$ such that the image by these functions after being completed by symmetries is Σ_n. These functions are defined by:

$$\tilde{X}^*_{1,n}(x,y) = \frac{X^*_1(x+k_n,y) - x^+_n}{\ell_n}$$
$$\tilde{X}^*_{2,n}(x,y) = \frac{X^*_2(x+k_n,y) - y^+_n}{\ell_n}$$
$$\tilde{X}^*_{3,n}(x,y) = \tilde{v}_n(x,y)$$

We notice that the point $(0,0,1) \in \Sigma_n$ is the image of the point $(x^+_{k_n}, y^+_{k_n}, 1) \in \mathcal{M}[q^n_i, i \in \mathbb{Z}]$, then the convex horizontal curve in Σ_n that pass by $(0,0,1)$ comes from the behaviour of $\tilde{X}^*_{1,n}$, $\tilde{X}^*_{2,n}$ and $\tilde{X}^*_{3,n}$ near $(0,1) \in \mathbb{R} \times [-1,1]$. Then the hypothesis $X^*_2 - y^{\pm}_{k_n} > 0$ on $(k_n - \ell_n, k_n + \ell_n) \times \{0\}$ becomes $\tilde{X}^*_{2,n} > 0$ on $(-1,1) \times \{0\}$.

Let us study the convergence of the sequence $(\tilde{v}_n)_{n\in\mathbb{N}}$. First of all, since for all i, q^n_i is outside $(k_n - \sqrt{1-\ell_n^2}, k_n + \sqrt{1-\ell_n^2})$, we have $(q^n_i - k_n)/\ell_n$ outside $(-\sqrt{1-\ell_n^2}/\ell_n, \sqrt{1-\ell_n^2}/\ell_n)$. So $\mathbb{R} \times (-1,1) \setminus \{(q^n_i - k_n)/\ell_n, i \in \mathbb{Z}\}$ converges to $\mathbb{R} \times (-1,1)$. Since for every n, $\tilde{v}_n \le 1$, each divergence line L has at least one end-point. Besides, we know that on $(-1/\ell_n, 1/\ell_n) \times \{-1,1\}$ the boundary value of $\tilde{v}_n$ is $1 - |x|$. This implies that the only possible end-points are $(0,-1)$ and $(0,1)$. Hence L must be the segment between these two points and :

$$2 = |L| = \lim_{n\to+\infty} |\tilde{v}_n(0,-1) - \tilde{v}_n(0,1)| = 0$$

This is a contradiction and we have proved that the sequence $(\tilde{v}_n)_{n\in\mathbb{N}}$ has no divergence line. Then a subsequence can be assumed to converge to a solution $\tilde{v}$ of the maximal graph equation. On the boundary $\mathbb{R} \times \{-1,1\}$, $\tilde{v}$ takes the value $1 - |x|$, such a solution is unique, and if $\tilde{u}$ is the conjugate to $\tilde{v}$: $\tilde{u}(x,y) = x\tan(\pi y/2)$ *i.e.* the graph of $\tilde{u}$ is a piece of an helicoid (see Theorem 1 in [2] or Theorem 5.1 in [4]). We notice that because of the uniqueness of the limit $\tilde{u}$, we can ensure that the whole sequence $(\tilde{v}_n)_{n\in\mathbb{N}}$ converges to $\tilde{v}$.

Let us now study the sequence $(\Sigma_n)_{n\in\mathbb{N}}$. Because of Theorem 7, we know that the curvature of the surfaces Σ_n is uniformly bounded by a constant C_0. Then we can assume that a subsequence of (Σ_n) converges to a minimal surface Σ. We shall prove that Σ is in fact the catenoid which corresponds to $\tilde{v}$.

We know that for every n the point $(0,0,1)$ is in Σ_n and at this point the normal to Σ_n is always $-e_x = (-1,0,0)$ then $(0,0,1) \in \Sigma$ and the normal to Σ at this point is $-e_x$. This implies that there exists $\varepsilon > 0$, $\eta > 0$ and a point B such that

- $d_\Sigma(B,(0,0,1)) \leq 2\eta$,
- $B \in \mathbb{R}^2 \times (0, 1-\eta)$ and
- $d(N(B), -e_x) < \varepsilon$ where N is the Gauss map on Σ.

Since Σ is the limit of $(\Sigma_n)_{n\in\mathbb{N}}$. There exist $B_n \in \Sigma_n$ such that

- $d_{\Sigma_n}(B_n,(0,0,1)) \leq 2\eta$,
- $B_n \in \mathbb{R}^2 \times (0, 1-\eta)$ and
- $d(N_n(B_n), -e_x) < \varepsilon$ where N_n is the Gauss map on Σ_n.

Since $B_n \in \mathbb{R}^2 \times (0, 1-\eta)$, there exists $A_n \in \mathbb{R} \times (0,1)$ such that $B_n = (\tilde{X}^*_{1,n}, \tilde{X}^*_{2,n}, \tilde{X}^*_{3,n})(A_n)$. Since $\tilde{v}_n(A_n) = \tilde{X}^*_{3,n}(A_n) \leq 1-\eta$, A_n is at a distance at most η from the point $(0,1)$. Since $d_{\Sigma_n}(B_n,(0,0,1)) \leq 2\eta$, the point A_n is at a distance less than 2η from $(0,1)$. The property $d(N_n(B_n), -e_x) < \varepsilon$ implies that there exist $\alpha > 0$ such that A_n is outside the set $\{(x,y) \in [-1,1] \times (-1,1) \,|\, |y-1| < |x| \tan\alpha\}$. Then the sequence (A_n) moves in a compact subset of $\mathbb{R} \times (-1,1)$. Since the convergence $\tilde{v}_n \to \tilde{v}$ is smooth on compact subset, the surface Σ near the point B is the catenoid which corresponds to $\tilde{v}$. Then Σ is the catenoid which corresponds to $\tilde{v}$. As above the uniqueness of the limit implies that the whole sequence $(\Sigma_n)_{n\in\mathbb{N}}$ converges to the catenoid Σ.

Then $(\tilde{X}^*_{1,n}, \tilde{X}^*_{2,n}, \tilde{X}^*_{3,n})$ which is associated to the $\tilde{v}_n$ converges to the map $(\tilde{X}^*_1, \tilde{X}^*_2, \tilde{X}^*_3)$ associated to $\tilde{v}$ and normalized such that :

$$\lim_{t\to 1} (\tilde{X}^*_1, \tilde{X}^*_2, \tilde{X}^*_3)(0,t) = (0,0,1)$$

Then $\tilde{X}^*_2(0,0) = 0$ and $\tilde{X}^*_2$ is negative on $(0,1)$ which contradicts the hypothesis on $\tilde{X}^*_{2,n}$. The theorem is proved. $\square$

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