MINIMAL GRAPHS OVER RIEMANNIAN SURFACES AND HARMONIC DIFFEOMORPHISMS

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ABSTRACT. We construct a parabolic entire minimal graph S over a finite topology complete Riemannian surface Σ of curvature -1 and infinite area (thus of non-parabolic conformal type). The vertical projection of this graph yields a harmonic diffeomorphism from S onto Σ. The proof uses the theory of divergence lines to construct minimal graphs.

We also generalize a theorem of R. Schoen. Let g_1 and g_2 be two complete metrics on a orientable surface S with compact boundary and suppose
$$\int_{S_r^2} K_{g_2}^- d\sigma_{g_2} \leq C \ln(2 + r)$$
for some $C > 0$ and all $r > 0$. If there is a harmonic diffeomorphism from (S, g_1) to (S, g_2), then (S, g_1) is parabolic.

1. INTRODUCTION

Perhaps Bernstein proved the first global theorem concerning minimal graphs: An entire minimal graph over the euclidean plane $\mathbb{R}^2$ is a plane. This has had a great influence on minimal surface theory and partial differential equations. Among the many different proofs of Bernstein's theorem that followed, that of Heinz [11] used harmonic diffeomorphisms. He proved there is no harmonic diffeomorphism from the disk $\{x^2 + y^2 < 1\}$ onto $\mathbb{R}^2$ with the Euclidean metric. The vertical projection of a minimal graph over a Riemannian manifold is a harmonic diffeomorphism onto its image. Thus Heinz concluded that an entire minimal graph over $\mathbb{R}^2$ is necessarily conformally the complex plane $\mathbb{C}$. The Gauss map of the graph then defines a holomorphic bounded function on $\mathbb{C}$, hence is constant, and the graph is a plane.

Thus the existence of minimal graphs is intimately related to the existence (or non existence) of harmonic diffeomorphisms. Until the last decade, the theory of minimal graphs over surfaces and harmonic diffeomorphisms between surfaces developed considerably, yet independently. Before discussing some of these developments, we state our main results.

In this paper we will construct an entire minimal parabolic graph Σ over any complete Riemannian surface M of sectional curvature -1, finite topology, and infinite area. Thus we obtain a harmonic diffeomorphism of Σ

onto M (Theorem 19). Parabolic here means the annular ends of Σ are conformally $\{z \in \mathbb{C} \mid 1 \le |z|\}$, and infinite area implies M has at least one hyperbolic end: conformally $\{1 \le |z| < c\}$ where $c < +\infty$.

In [22], Schoen proved that there is no harmonic diffeomorphism from the unit disk onto a complete surface of non negative curvature; this is a generalization of Heinz result. We improve Schoen's result. Let S be an orientable surface with a compact boundary. Let g_1 and g_2 be two complete Riemannian metrics on S. Assume that there is a constant $C \ge 0$ such that, for any r,

$$\int_{S_r^2} K_{g_2}^- d\sigma_{g_2} \le C \ln(2 + r)$$

where $K_{g_2}^- = \max\{0, -K_{g_2}\}$ and $S_r^2 = \{p \in S | d_{g_2}(p, \partial S)) < r\}$. If there is a harmonic diffeomorphism $u : (S, g_1) \to (S, g_2)$, then (S, g_1) is parabolic.

Let us now come back to the historical background of our work.

Minimal Graphs. Almost a century before Bernstein proved his theorem, Scherk constructed many interesting minimal graphs over domains in $\mathbb{R}^2$. The best known example is the graph of $\ln \frac{\cos(x)}{\cos(y)}$, over the square $(-\pi/2, \pi/2) \times (-\pi/2, \pi/2)$ in $\mathbb{R}^2$, taking $+\infty$ and $-\infty$ values over opposite sides of the square. The graph is bounded by the four vertical lines over the vertices of the square and extends to a complete doubly periodic minimal surface in $\mathbb{R}^3$ by successive rotations by π about the vertical lines.

Jenkins and Serrin [12] found necessary and sufficient geometric conditions on compact domains in $\mathbb{R}^2$ bounded by piecewise smooth arcs, to find minimal graphs over the domain taking prescribed boundary values (perhaps infinite) on the boundary arcs.

There have been many generalizations of their theorem to domains in Riemannian surfaces. Of interest to us here is the theorem of Pascal Collin and the last author [4], extending the Jenkins-Serrin theory to ideal domains of the hyperbolic plane $\mathbb{H}^2$. They then used this to construct an entire minimal graph over $\mathbb{H}^2$ in $\mathbb{H}^2 \times \mathbb{R}$, conformally the complex plane $\mathbb{C}$. Hence a harmonic diffeomorphism from $\mathbb{C}$ onto $\mathbb{H}^2$. This solved in the negative a conjecture of Schoen and Yau: there is no harmonic diffeomorphism from $\mathbb{C}$ onto $\mathbb{H}^2$. We mention further generalizations of this theorem [16, 13, 6].

To extend the Jenkins-Serrin theorem to higher topology Riemann surfaces, a new idea was needed. We solve this problem in this paper with an idea introduced in the thesis of the first author [15]: divergence lines of sequences of minimal graphs.

Harmonic maps. Harmonic maps between surfaces have long been used to study the Teichmüller space of a Riemann surface. Sampson, Eells, Hartman were among the early pioneers. They showed in [5] the existence of a harmonic map in each homotopy class of maps from M to N, when N has non positive sectional curvatures, and Hartman [10] proved it is unique when the curvature is strictly negative.

For closed hyperbolic surfaces of the same genus, Schoen and Yau [21] proved there is a unique harmonic diffeomorphism between them homotopic to the identity. Wolf, in [24], was able to parametrize Teichmüller space by harmonic diffeomorphisms and described the geometry of its closure (and other analytic properties) in terms of the measured foliations of the Hopf differential of the harmonic diffeomorphism.

Markovic [14] extended this theory to non compact Riemann surfaces of finite analytic type (conformally parabolic and finite topology). He studied the complex structures of such surfaces using quasi conformal harmonic diffeomorphisms.

Harmonic diffeomorphisms from $\mathbb{C}$ into domains of $\mathbb{H}^2$ have been analytically constructed by Au, Tam and Wan [1], Han, Tam, Treibergs and Wan [9] and Tam and Wan [23]. They showed the image of the harmonic map is an ideal polygon of $\mathbb{H}^2$ with $m+2$ vertices, precisely when the Hopf differential is a polynomial of degree m.

M. Wolf [25] realized that harmonic maps into $\mathbb{H}^2$ lead to minimal graphs and multigraphs over domains in $\mathbb{H}^2$. He made a construction of such surfaces using harmonic maps to real trees and measured foliations. This gave many interesting examples. He showed how the measured foliations give information on the growth of the minimal surfaces.

The main question of our study is to understand if the conformal types of two surfaces are related if there is a harmonic diffeomorphism from one to the other. We finish this introduction by stating a conjecture in that direction. There is no harmonic diffeomorphism from the disk onto $\mathbb{R}^2$ with a complete parabolic metric.

The paper is structured as follows. In the second section, we recall some basic definitions about conformal type, topology and geometry of surfaces. Section 3 is devoted to the proof of a non existence result for harmonic diffeomorphism. In Section 4, we gather some results about the minimal surface equation that are used in the next section to prove a Jenkins-Serrin type result. This result is then used in Section 6 to construct a harmonic diffeomorphism from a parabolic surface to a hyperbolic surface with infinite area.

2. Preliminaries

In this section we recall some basic facts about conformal type of surfaces, harmonic maps, the geometry of hyperbolic surfaces.

2.1. **Conformal type.** We refer to [8] for the notions introduced here. First we recall the following definition.

Definition 1. *Let (M, g) be a complete Riemannian manifold with empty boundary. (M, g) is called parabolic if any bounded subharmonic function on M is constant.*

If ∂M is compact, (M, g) is called parabolic if any non negative bounded subharmonic function which vanishes on ∂M vanishes everywhere.

If (M, g) has no boundary and $K \subset M$ is a compact with smooth boundary, it is well known that M is parabolic if and only if $M \setminus \overset{\circ}{K}$ is parabolic.

In dimension 2, the parabolicity is a conformal property. For example, an annulus is parabolic if and only if its conformal modulus is $+\infty$: we recall that any annular domain with one connected compact boundary is conformal to $\{1 \leq z < c\}$, $c \in (1, \infty]$, the conformal modulus of this annular domain in $\frac{1}{2\pi} \ln(c)$.

2.2. Harmonic maps.

A harmonic map $\varphi : (M_1, g_1) \to (M_2, g_2)$ between two Riemannian manifolds is a critical point of the Dirichlet energy functional $E(\varphi) = \frac{1}{2} \int_{M_1} |\varphi^* g_2|^2_{g_1} d\sigma_{g_1}$ where $d\sigma_{g_1}$ is the volume measure. If M_1 has dimension 2, this energy is conformally invariant, so being harmonic only depends on the conformal structure of M_1.

If M_1 and M_2 are surfaces, let us consider conformal parameters z and w on M_1 and M_2 and write their metrics as $g_1 = \lambda^2(z)|dz|^2$ and $g_2 = \sigma^2(w)|dw|^2$. Then a map $\varphi : M_1 \to M_2$ can be written as a function $w = u(z)$. With these notations, the map φ is harmonic if and only if u satisfies the following partial differential equation

$$0 = u_{z\bar{z}} + 2\frac{\sigma_w(u)}{\sigma(u)} u_z u_{\bar{z}}$$

Let S be a surface and M a Riemannian 3-manifold. An isometric immersion $\varphi : S \to M$ is harmonic if and only if φ is minimal. In the case M is a Riemannian product $M = \Sigma \times \mathbb{R}$ and $\pi : M \to \Sigma$ denotes the vertical projection, φ minimal implies that the map $\pi \circ \varphi : S \to \Sigma$ is harmonic.

2.3. Hyperbolic surfaces.

In this paper we will look at orientable surfaces Σ with finite topology and endowed with a complete hyperbolic metric. Let us describe the geometry of the annular ends E of Σ.

If E has finite area then outside some compact E is isometric to the quotient of a horodisk H by a parabolic translation leaving H invariant. The quotient of a horodisk H' contained in H is called a horoannulus of the end. Such an end can be compactified by adding one point p at infinity. These annular ends are parabolic. We call these ends hyperbolic cusp ends.

If E has infinite area, the geometry is the following. First, we have two particular cases:

- Σ could be $\mathbb{H}^2$ and E is just the outside of a compact subset of $\mathbb{H}^2$ or
- Σ could be the quotient of $\mathbb{H}^2$ by a parabolic translation and E is just the quotient of the outside of a horodisk left invariant by the parabolic translation.

If we are not in these two particular case, the picture is the following. Let γ be a geodesic in $\mathbb{H}^2$. If c is an equidistant curve c to γ, we denote C_c the non-convex component of $\mathbb{H}^2 \setminus c$. Then there is an equidistant curve c and a hyperbolic translation T leaving γ invariant such that, outside some

compact, E is isometric to the quotient of C_c by T. Thus, when we will consider such an end E, we will see E as the particular subdomain isometric to the quotient of C_c by T, for example ∂E has constant curvature. The ideal compactification of $\mathbb{H}^2$ passes to the quotient and gives a compactification of E by adding a circle. These annular ends are non-parabolic. In the following, we will focus on this general case. So any non parabolic end will be seen as the quotient of some C_c by T (the other cases can be treated similarly but are exactly the cases studied in [4, 13]).

Since each end can be compactified, the whole surface Σ can be compactified by $\overline{\Sigma}^\infty$. We will denote $\partial_\infty \Sigma = \overline{\Sigma}^\infty \setminus \Sigma$. $\partial_\infty \Sigma$ is made of one point for each parabolic end and one circle for each non-parabolic end. If A is a subset of Σ, $\overline{A}^\infty$ denotes the closure of A in $\overline{\Sigma}^\infty$ and $\partial_\infty A$ is $\overline{A}^\infty \cap \partial_\infty \Sigma$.

3. A non-existence result

Our first result is a characterization "à la Huber" of parabolicity. For a Riemannian metric g on a surface we denote by K_g the sectional curvature and $d\sigma_g$ the area measure.

Proposition 2. *Let (S, g) be a complete Riemannian surface with a compact boundary. We denote $S_r = \{p \in S | d_g(p, \partial S) < r\}$. We assume that there is $C > 0$ such that for any $r > 0$*

$$\int_{S_r} K_g d\sigma_g \geq -C \ln(2 + r).$$

Then (S, g) is parabolic.

Proof. We are going to give an upper-bound for the growth of the area $|S_r|$ of S_r. Let $\ell(r)$ denote the length of $\partial S_r \setminus \partial S$. It is known that ℓ is differentiable almost everywhere and $\ell(b) - \ell(a) \leq \int_a^b \ell'(u)du$ and $\ell'(u) \leq 2\pi\chi(S_u) - \mathcal{K}(u) + \int_{\partial S} \kappa_g ds$ where $\mathcal{K}(u) = \int_{S_u} K_g d\sigma_g$ and $\int_{\partial S} \kappa_g ds$ is the integral of the curvature of ∂S computed with respect to the outward unit normal (see [2] and the references therein). We denote by c this last integral.

From the coarea formula, we have:

$$
\begin{aligned}
|S_r| &= \int_0^r \ell(u)du \\
&\le r\ell(0) + \int_0^r \int_0^u \ell'(v)dvdu \\
&\le r\ell(0) + \int_0^r \ell'(v) \int_v^r dudv \\
&\le r\ell(0) + \int_0^r (r-v)\ell'(v)dv \\
&\le r\ell(0) + \int_0^r (r-v)(2\pi\chi(S_v) + c - \mathcal{K}(v))dv \\
&\le r\ell(0) + \int_0^r (r-v)(2\pi + c + C\ln(2+v))dv \\
&\le r\ell(0) + (2\pi + c)\frac{r^2}{2} + Cr\int_0^r \ln(2+v)dv \\
&\le r\ell(0) + (2\pi + c)\frac{r^2}{2} + Cr^2\ln(2+r)
\end{aligned}
$$

So $\frac{r}{|S_r|}$ is not integrable at $+\infty$ which implies (S,g) is parabolic (see [8]).   □

In [22], Schoen proved that there is no harmonic diffeomorphism from the unit disk onto a complete surface of non negative curvature. The following proposition is an improvement of this result.

Proposition 3. *Let S be an orientable surface with a compact boundary. Let g_1 and g_2 be two complete Riemannian metrics on S. Assume that there is a constant $C \ge 0$ such that for any r*

$$
\int_{S_r^2} K_{g_2}^- d\sigma_{g_2} \le C\ln(2+r)
$$

where $K_{g_2}^- = \max(0, -K_{g_2})$ and $S_r^2 = \{p \in S | d_{g_2}(p, \partial S) < r\}$.
If there is a harmonic diffeomorphism $\varphi : (S, g_1) \to (S, g_2)$ then (S, g_1) is parabolic.

Proof. By changing the orientation on (S, g_1), we can assume that φ preserves the orientation. Let z be a local conformal complex coordinate on (S, g_1) and w be a local conformal complex coordinate on (S, g_2). We denote $g_1 = \lambda^2(z)|dz|^2$, $g_2 = \sigma^2(w)|dw|^2$ and φ by $w = u(z)$. The Jacobian of the map u is then $J(u) = \frac{\sigma^2(f)}{\lambda^2(z)}(|u_z|^2 - |u_{\bar{z}}|^2)$ (see [21, 22]) since u preserves the orientation $J(u) > 0$ and $|u_z| > |u_{\bar{z}}|$.

We then define on S the metric $\hat{g} = \sigma^2(u)|u_z|^2|dz|^2$; this metric is conformal to g_1 and does not depend on the choice of the complex coordinates z

and w. If we compare $\hat{g}$ with the pull-back metric $\varphi^* g_2$ we have

$$
\begin{aligned}
\varphi^* g_2 = \sigma^2(u)|u_z dz + u_{\bar{z}} d\bar{z}|^2 &\leq \sigma^2(u)(|u_z||dz| + |u_{\bar{z}}||d\bar{z}|)^2 \\
&\leq \sigma^2(u)2(|u_z|^2 + |u_{\bar{z}}|^2)|dz|^2 \\
&\leq \sigma^2(u)4|u_z|^2|dz|^2 \\
&\leq 4\hat{g}
\end{aligned}
$$

Since $\varphi^* g_2$ is complete, $\hat{g}$ is complete. Let

$$
S_r^\wedge = \{p \in S \,|\, d_{\hat{g}}(p, \partial S) \leq r\}
$$
$$
S_r^* = \{p \in S \,|\, d_{\varphi^* g_2}(p, \partial S) \leq r\}
$$

Since $\varphi^* g_2 \leq 4\hat{g}$, we have $S_r^\wedge \subset S_{2r}^*$.

The computation of the curvature of $\hat{g}$ (see [21, 22]) gives

$$
K_{\hat{g}} = \frac{-1}{2\sigma^2(u)|u_z|^2} \Delta \ln(\sigma(u)^2|u_z|^2) = K_{g_2}(u) \frac{\sigma^2(u)(|u_z|^2 - |u_{\bar{z}}|^2)}{\sigma^2(u)|u_z|^2} = K_{g_2} \hat{J}(\varphi)
$$

where $\hat{J}(\varphi)$ is the jacobian of $\varphi : (S, \hat{g}) \to (S, g_2)$. Thus

$$
\begin{aligned}
\int_{S_r^\wedge} K_{\hat{g}}^- \, da_{\hat{g}} &= \int_{S_r^\wedge} K_{g_2}^-(\varphi) \hat{J}(\varphi) d\sigma_{\hat{g}} \\
&= \int_{\varphi(S_r^\wedge)} K_{g_2}^- d\sigma_{g_2} \\
&= \int_{S_r^\wedge} K_{\varphi^* g_2}^- d\sigma_{\varphi^* g_2} \\
&\leq \int_{S_{2r}^*} K_{\varphi^* g_2}^- d\sigma_{\varphi^* g_2} \leq C \ln(2r + 2)
\end{aligned}
$$

So, by Proposition 2, $(S, \hat{g})$ is parabolic. Since $\hat{g}$ is conformal to g_1, (S, g_1) is parabolic. $\qquad\square$

As a consequence we have the following corollary

Corollary 4. *Let g_1 and g_2 be two complete metrics on an orientable finite topology surface S with g_2 hyperbolic and of finite area. If there is a harmonic diffeomorphism $\varphi : (S, g_1) \to (S, g_2)$ then (S, g_1) is parabolic.*

There is no harmonic diffeomorphism from $A(r) = \{z \in \mathbb{C} \,|\, 1 \leq |z| < r\}$ to a hyperbolic cusp end (we recall that a hyperbolic cusp end is parabolic).

4. PRELIMINARIES ON THE MINIMAL SURFACE EQUATION

Let Ω be an open subset inside a Riemanniann surface Σ and u be a function on Ω. In the following, we will use the following notations

- G_u is the graph of u in $\Sigma \times \mathbb{R}$,
- W_u is $\sqrt{1 + \|\nabla u\|^2}$,
- X_u is the vectorfield $\frac{\nabla u}{W_u}$,
- N_u is $(X_u, -\frac{1}{W_u})$ the downward unit normal to G_u and

- if γ is a curve in Ω, $F_u(\gamma) = \int_\gamma X_u \cdot \nu$ where ν is a unit normal to γ.

$F_u(\gamma)$ is called the flux across γ. Of course, $F_u(\gamma)$ depends on the choice of ν but, in the following, γ will be often a boundary component of some open subset so ν will be always chosen as the outward pointing unit normal.

4.1. **The minimal surface equation.** The function u solves the minimal surface equation if

$$\text{(MSE)} \qquad 0 = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + \|\nabla u\|^2}}\right) = \operatorname{div} X_u$$

This is equivalent to say that G_u is a minimal surface in $\Omega \times \mathbb{R}$.

We are going to study the Dirichlet problem for the (MSE). This problem has been studied by many different authors. We refer to the works of Jenkins and Serrin [12] for $\mathbb{R}^2$, Nelli and the last author [17], the authors [16] for $\mathbb{H}^2$ and Pinheiro [19] for the the general case. We will gather here some results whose proof can be found in these papers.

The first result is a classical compactness result.

Theorem 5. *Let $(u_n)_n$ be a uniformly bounded sequence of solutions of the* (MSE) *on an open subset Ω of Σ. There is a subsequence of $(u_n)_n$ that converges to a solution u of the* (MSE); *the convergence is smooth on each compact subset of Ω.*

We have introduced the notation $F_u(\gamma)$ for curves in Ω, actually this notion can be extended to subarcs of $\partial\Omega$ if Ω is smooth and u solves (MSE). Indeed, X_u is bounded and, in that case, has vanishing divergence. Actually, we can often extend continuously the value of X_u on $\partial\Omega$.

Lemma 6. *Let Ω be an open subset in Σ and γ a geodesic arc contained in $\partial\Omega$ and u a solution of* (MSE) *in Ω.*

- *If u diverges to $+\infty$ (resp. $-\infty$) as one approaches γ, then X_u extends continuously on γ with $X_u = \nu$ (resp. $X_u = -\nu$).*
- *If $F_u(\gamma) = \ell(\gamma)$ (resp. $-\ell(\gamma)$), then u diverges to $+\infty$ (resp. $-\infty$) as one approaches γ.*

Proof. The first statement is almost contained in Lemma 2.5 in [16] where $u \to +\infty$ on γ implies $F_u(\gamma) = \ell(\gamma)$ is proved. Actually, if $u \to +\infty$ on γ, Proposition 28 implies that X_u is equicontinuous near γ so it extends continuously to γ. The value of X_u on γ is then a consequence of $F_u(\gamma) = \ell(\gamma)$.

The second statement is Lemma 3.6 in [16]. $\qquad\square$

An other result is Lemma 2.7 in [16].

Lemma 7. *Let Ω be an open subset in Σ and γ a geodesic arc contained in $\partial\Omega$ and $(u_n)_n$ a sequence of solutions of* (MSE) *in Ω which extend continuously to γ. If $(u_n)_n$ diverges to $+\infty$ (resp. $-\infty$) on γ while remaining bounded on compact subsets of Ω then $F_{u_n}(\gamma) \to \ell(\gamma)$ (resp. $-\ell(\gamma)$).*

4.2. **Divergence lines.** One important tool for our study is to understand the limit of a sequence of solutions of (MSE). In this section, we present the notion of divergence line that was introduced by the first author in [15] for $\mathbb{R}^2$ and the authors in [16] for $\mathbb{H}^2$.

In the sequel, we consider a complete Riemannian surface Σ, a sequence of open subsets $(\Omega_n)_n \subset \Sigma$ and a sequence $(u_n)_n$ of solutions of (MSE), u_n being defined on Ω_n. First we define the limit open subset

$$\Omega = \bigcup_n \left(\text{interior}\left(\bigcap_{k \geq n} \Omega_k\right)\right)$$

Because of the equicontinuity result given by Proposition 28, we can assume that the sequence $(X_{u_n})_n$ converges to some continuous vectorfield X on Ω (the convergence is locally uniform). So we can define the convergence domain of the sequence as the open subset

$$\mathcal{B} = \mathcal{B}(X) = \{p \in \Omega \,|\, \|X\|(p) < 1\}$$

and the divergence set as $\mathcal{D} = \mathcal{D}(X) = \Omega \setminus \mathcal{B} = \{p \in \Omega \,|\, \|X\|(p) = 1\}$.

Proposition 8. *Let p be a point in Ω.*

- *If $p \in \mathcal{B}$, let D be the connected component of $\mathcal{B}$ containing p. Then $u_n - u_n(p)$ converges on D to a solution of (MSE) (the convergence is locally C^k for any k).*
- *If $p \in \mathcal{D}$, let γ be the geodesic in Ω passing through p and orthogonal to $X(p)$. Then $\gamma \subset \mathcal{D}$ and, for any $q \in \gamma$, $X(q)$ is the unit normal to γ.*

Proof. Let us first assume that $p \in \mathcal{B}$. On D, since $\|X\| < 1$, we have $\nabla u_n \to \frac{X}{\sqrt{1-\|X\|^2}}$. Thus $u_n - u_n(p)$ converges locally in C^1 to a function v. Besides $u_n - u_n(p)$ being a solution of (MSE), Theorem 5 implies that the convergence is locally in C^k for any k and v is a solution of (MSE).

Let us now assume $p \in \mathcal{D}$. Since $\|X\|(p) = 1$, we have $N_{u_n}(p) \to X(p)$. There is $\delta > 0$ such that G_{u_n} contains the geodesic disk of radius δ around (p, u_n) (take δ such that $2\delta \leq d(p, \partial\Omega)$). Moreover, by curvature estimates in [20], the second fundamental form of these graphs is uniformly bounded. As a consequence, after a vertical translation by $-u_n(p)\partial_t$, this sequence of geodesic disks converges to a minimal disk S of radius δ which is orthogonal to $X(p)$ at $(p, 0)$. Let $\theta = \langle N, \partial_t \rangle$ along S, where N is the unit-normal to S. Since S is a limit of graphs $\theta \leq 0$ and $\theta(p, 0) = 0$. Moreover θ is in the kernel of the Jacobi operator: $0 = \Delta_S \theta + (\text{Ric}(N, N) + \|A\|^2)\theta$. So by the maximum principle, $\theta = 0$ along S. This implies that S is contained in some $\gamma \times \mathbb{R}$ where γ is a geodesic of S. Since X is normal to S at $(p, 0)$, γ is normal to X at p as well. So S is a geodesic disk of radius δ in $\gamma \times \mathbb{R}$. This implies that, along the geodesic segment in γ of length 2δ and midpoint p, X is the unit normal to γ. Let $\tilde{\gamma}$ denote the connected component of $\gamma \cap \Omega$ containing p. It is now clear that the subset of points q in $\tilde{\gamma}$ where $X(q)$ is

the unit normal to $\tilde{\gamma}$ is open and closed in $\tilde{\gamma}$ so it is the whole $\tilde{\gamma}$ and the second statement of the proposition is proved. $\square$

The above proposition tells that on each connected component of $\mathcal{B}$ the sequence (u_n) converges up to a vertical translation. We also see that $\mathcal{D}$ is made of geodesics of Ω that we will call divergence lines of X. We notice that since the unit normal to these geodesics is given by X, they are embedded geodesics (perhaps periodic). The next lemma is important in order to describe the possible divergence lines.

Lemma 9. *Let γ be a divergence line, then it is a proper geodesic in Ω.*

Proof. Assume that γ is not a proper geodesic. So we can consider a arc-length parametrization of $\gamma : \mathbb{R}_+ \to \Omega$ and a sequence $(s_i)_i$ in $\mathbb{R}_+$ with $s_{i+1} > s_i + 1$ and $\gamma(s_i) \to p \in \Omega$. Let $r > 0$ be such that the geodesic disk $D(p,r) \subset \Sigma$ is convex, is included in Ω, has area at most r and the length of $\partial D(p,r)$ is at most $7r$. By changing the sequence $(s_i)_i$, we assume that $\gamma(s_i) \in D(p, r/2)$. This implies that the geodesic segment in γ of length r and midpoint $\gamma(s_i)$ is included in $D(p,r)$. We notice that $D(p,r) \subset$ interior$\left(\bigcap_{k \geq n_0} \Omega_k \right)$ for some n_0.

Changing u_n into $u_n - u_n(\gamma(0))$, we assume $u_n(\gamma(0)) = 0$. We are going to estimate the area of G_{u_n} inside $D(p,r) \times (-1,1)$ in two ways.

First let us compute an upper-bound. Let us define $B_n = \{(q,t) \in D(p,r) \times (-1,1) \,|\, t < u_n(p)\}$. We have $G_{u_n} \cap (D(p,r) \times (-1,1)) \subset \partial B_n$ and $\partial B_n \setminus G_{u_n}$ has area at most the one of $\partial(D(p,r) \times (-1,1))$: $2r + 2 \times 7r = 16r$. Since G_{u_n} is area minimizing in $D(p,r) \times \mathbb{R}$ we obtain the area of $G_{u_n} \cap (D(p,r) \times (-1,1))$ is at most $16r$.

Let us now compute a lower-bound. Let $U \Subset \Omega$ be an open subset containing $\gamma[0, s_9]$ (we also have $U \subset$ interior$\left(\bigcap_{k \geq n_0} \Omega_k \right)$ for some n_0). By curvature estimates [20], the curvature of the graphs G_{u_n} over U is uniformly bounded. Let S_n be the connected component of $G_{u_n} \cap (U \times (-1,1))$ containing $(\gamma(0), 0)$. As in the proof of Proposition 8, the sequence $(S_n)_n$ converges to $\tilde{\gamma} \times (-1,1)$ where $\tilde{\gamma}$ is the connected component of $\gamma \cap U$ containing $\gamma(0)$. $\tilde{\gamma}$ contains the geodesic segments of length r and midpoint $\gamma(s_i)$ $(0 \leq i \leq 8)$. So the area of the limit surface inside $D(p,r) \times (-1,1)$ is at least $18r$. This implies that the area of $G_{u_n} \cap (D(p,r) \times (-1,1))$ is at least $17r$ for n large. We thus have a contradiction. $\square$

Now we are interested in arguments that prevent some geodesics from being divergence lines. One of these tools is the following result.

Lemma 10. *Let γ be a divergence line and $p \in \gamma$ be a point. Let $D^+ \subset \Omega$ be a halfdisk centered at p and contained strictly on one side of γ and ν be the outward pointing unit normal along ∂D^+. We assume that $D^+ \subset \mathcal{B}$ and consider $q \in D^+$. Then if $X = \nu$ (resp. $X = -\nu$) along γ then $\lim u_n(p) - u_n(q) = +\infty$ (resp. $-\infty$).*

Proof. Since $D_+ \subset \mathcal{B}$, we can assume that $u_n - u_n(q) \to v$, v being a solution to (MSE). We have $X_v = X$ so $X_v = \nu$ along γ. By Lemma 6, this implies that v takes the value $+\infty$ on γ. Let M be positive. By the continuity of X, there is a point $q' \in D^+$ close to p such that $v(q') - v(q) \geq M$ and a curve c in $\overline{D}^+$ from q' to p such that $c' \cdot X > 0$. So $c' \cdot X_{u_n} > 0$ for large n and $u_n(p) - u_n(q) \geq u_n(q') - u_n(q)$. Then $\liminf u_n(p) - u_n(q) \geq \liminf u_n(q') - u_n(q) \geq M$ which gives the result. $\square$

A consequence of the above lemma is the following. Let c_1 and c_2 be two connected components of $\mathcal{B}$ with a common divergence line in their boundary and X pointing into c_2 along it. Let $p_i \in c_i$ be two points, then $u_n(p_2) - u_n(p_1) \to +\infty$. We state the next result only in the case where the surface is hyperbolic.

Lemma 11. *Let $(\gamma_n)_n$ be a sequence of geodesic arcs of length $2\delta > 0$ with midpoints p_n. Let D_n^+ be a geodesic half-disk with diameter γ_n, of radius δ and strictly on one side of γ_n. We assume that $(D_n^+)_n$ converges to D^+ a geodesic half-disk of center p, radius δ and on one side of a geodesic arc γ. We assume that $(u_n)_n$ is a sequence of solutions of (MSE) on D_n^+ and $X_{u_n} \to X$ on D^+. Moreover we assume one of the following possibilities*

- *either u_n takes the value $+\infty$ (resp. $-\infty$) along γ_n,*
- *or the metric is hyperbolic on D_n^+ and u_n is constant along γ_n.*

In both cases, p is not the end point of some divergence line of X. Moreover, in the first case X takes the value ν (resp. $-\nu$) along γ.

Proof. By Proposition 28, in the first case, the sequence X_{u_n} is uniformly equicontinuous on the halfdisk of radius $\delta/2$. Since u_n takes the value $+\infty$, $X_{u_n} = \nu$ along γ_n. As a consequence, X extends continuously to γ by the value ν. This implies that p is not the end point of some divergence line of X.

In the second case, let us see the halfdisk D_n^+ as a halfdisk in $\mathbb{H}^2$. Since u_n is constant along γ_n we can extend the definition of u_n to the whole geodesic disk by symmetry. As above this implies that X_{u_n} is uniformly equicontinuous on the the halfdisk of radius $\delta/2$ and X_{u_n} is orthogonal to γ_n along it. So X extends continuously to γ and is orthogonal to it. This prevents p from being an endpoint of some divergence line of X. $\square$

5. A JENKINS-SERRIN TYPE RESULT

In this section, we are interested in solving the Dirichlet problem for (MSE) on some particular domains Ω of a complete hyperbolic surface Σ. So let us fix a a complete hyperbolic surface Σ with at least one non-parabolic end.

5.1. **Ideal domains and Jenkins-Serrin conditions.** We first define the notion of polygonal domains.

Definition 12. *A polygonal domain in Σ is a connected open subset Ω such that $\partial_\infty \Omega$ is made of a finite number of points and $\partial \Omega$ is made of a finite number of geodesic arcs.*

If Ω is a polygonal domain, the geodesic arcs in the boundary are called the edges of Ω, the end points of these edges and points in $\partial_\infty \Omega$ are called the vertices of Ω. The natural orientation of $\partial \Omega$ allows us to say that the edge γ_2 is the successor of the edge γ_1 if γ_2 comes just after γ_1 when traveling along $\partial \Omega$.

Let us remark that if $\gamma \subset \partial \Omega$ is a geodesic arc, it could be possible that Ω is on both sides of γ. This implies that γ is part of two edges of Ω and, in the following, this arc has to be counted twice.

Among polygonal domains, we consider particular ones. Let $E_1, \ldots, E_q$ be the non-parabolic ends of Σ ($q \geq 1$) and $p_{q+1}, \ldots, p_{q+n}$ the end-points of the cusp ends.

Definition 13. *An ideal domain Ω in Σ is a polygonal domain such that*

- $\partial_\infty \Omega = \{p_{q+1}, \ldots, p_{p+n}\} \bigcup \cup_{i=1}^{q} \{p_1^i, \ldots, p_{2n_i}^i\}$ *where $\{p_j^i\}_{1 \leq j \leq 2n_i}$ are an even number of points in $\partial_\infty E_i$ cyclically ordered,*
- *the edges of Ω are geodesic lines γ_j^i, $1 \leq i \leq q$ and $1 \leq j \leq 2n_i$, where the end points of γ_j^i are p_j^i and p_{j+1}^i (with $p_{2n_i+1}^i = p_1^i$) and*
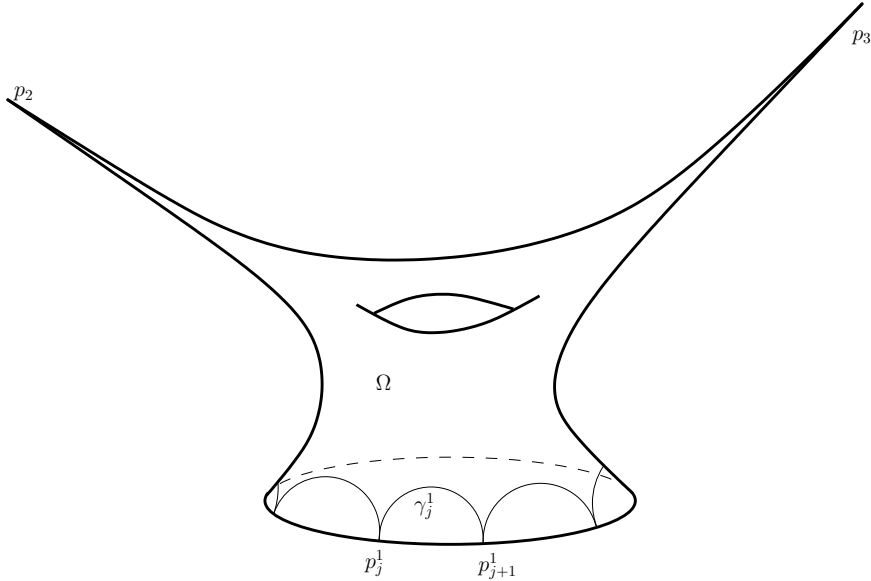- *the edge γ_j^i is included in E_i.*

(see Figure 1)

FIGURE 1. An ideal domain Ω in a hyperbolic surface Σ

As a consequence, the boundary of an ideal domain Ω is made of an even number of edges. In the following, each edge will be labeled "a" or "b"

with the convention : two successive edges have different labels. Since the number of end points on $\partial_\infty E_i$ is even such a labeling is possible.

Definition 14. *Let Ω be an ideal domain in Σ. An inscribed polygonal domain in Ω is a polygonal domain contained in Ω whose vertices are among the ones of Ω.*

Let us notice that the edges of such an inscribed polygonal domain are either closed geodesics or complete geodesics. Besides, Ω is itself a polygonal domain inscribed in Ω. Jenkins-Serrin conditions take into account the "lengths" of boundary components of inscribed polygonal domains. So let us explain how these conditions are defined.

Let Ω be an ideal domain in Σ and consider the vertices p_j^i as in the definition. For each i, j, let $(H_j^i(t))_{t \geq 0}$ be a decreasing family of horodisks centered at p_j^i such that $d(\partial H_j^i(0), H_j^i(t)) = t$. In the cusp end with end point p_i, we also consider a decreasing family of horo-annuli $(H_i(t))_{t \geq 0}$ such that $d(\partial H_i(0), H_i(t)) = t$. Moreover we assume that the horodisks $H_j^i(0)$ and the horo-annuli $H_i(0)$ are disjoint. If $\mathbf{t} = (t_1^1, \ldots, t_{2n_1}^1, t_1^2, \ldots, t_{2n_q}^q, t_{q+1}, \ldots, t_{q+n}) \in \mathbb{R}_+^{N_\Omega}$ where $N_\Omega = n + \sum_{i=1}^q 2n_i$, we define

$$H(\mathbf{t}) = (\cup_{i=1}^q \cup_{j=1}^{2n_i} H_j^i(t_j^i)) \bigcup (\cup_{k=1}^n H_{q+k}(t_{q+k}))$$

This is the union of disjoint horodisks and horo-annuli.

Let us fix a a/b labeling on $\partial\Omega$ and choose $\mathcal{P}$ an inscribed polygonal domain in Ω. The edges of $\mathcal{P}$ can be gathered in three classes: the ones which are edges of Ω labeled a (we denote by $A_\mathcal{P}$ the union of these geodesic lines), the ones which are edges of Ω labeled b (let $B_\mathcal{P}$ be their union), the other ones (let $C_\mathcal{P}$ be their union).

For $\mathbf{t} \in \mathbb{R}_+^{N_\Omega}$, we define $A_\mathcal{P}(\mathbf{t}) = A_\mathcal{P} \setminus H(\mathbf{t})$, $B_\mathcal{P}(\mathbf{t}) = B_\mathcal{P} \setminus H(\mathbf{t})$ and $C_\mathcal{P}(\mathbf{t}) = C_\mathcal{P} \setminus H(\mathbf{t})$. We also denote $\alpha(\mathbf{t}) = \ell(A_\mathcal{P}(\mathbf{t}))$, $\beta(\mathbf{t}) = \ell(B_\mathcal{P}(\mathbf{t}))$ and $\gamma(\mathbf{t}) = \ell(A_\mathcal{P}(\mathbf{t}) \cup B_\mathcal{P}(\mathbf{t}) \cup C_\mathcal{P}(\mathbf{t}))$ where ℓ denotes the length of a curve.

On $\mathbb{R}_+^{N_\Omega}$, we define a partial order by $\mathbf{t}' \geq \mathbf{t}$ if $\mathbf{t}' - \mathbf{t}$ has only non negative components.

Let γ be an edge of $\mathcal{P}$. We notice that if all the components of $\mathbf{t}$ are sufficiently large then γ only intersects the horodisks or horo-annuli in $H(\mathbf{t})$ that are centered at the end points of γ. We assume this is true in the following. Let us understand how the three above quantities evolve when coordinates in $\mathbf{t}$ increase. The edges with the vertex p_k as end-point are included in $C_\mathcal{P}$. So increasing t_k by t, leave $\alpha(\mathbf{t})$ and $\beta(\mathbf{t})$ unchanged and increase $\gamma(\mathbf{t})$ by at least $2t$ (there are at least two edges ending at p_k: it could be one geodesic line counted twice). If $p_j^i \in \partial_\infty \mathcal{P}$, when t_j^i increases to $t_j^i + t$, either $\alpha(\mathbf{t})$ (resp. $\beta(\mathbf{t})$) increases by t or stays unchanged, depending on whether an edge in $A_\mathcal{P}$ (resp. $B_\mathcal{P}$) ends at p_i^j; in any case γ increases by at least $2t$.

As a consequence, $\gamma(\mathbf{t}) - 2\alpha(\mathbf{t})$ and $\gamma(\mathbf{t}) - 2\beta(\mathbf{t})$ is non decreasing with $\mathbf{t}$. So the Jenkins-Serrin conditions $\gamma - 2\alpha > 0$ and $\gamma - 2\beta > 0$ are well

defined for any inscribed polygon $\mathcal{P}$ and means that $\gamma(\mathbf{t}) - 2\alpha(\mathbf{t}) > 0$ and $\gamma(\mathbf{t}) - 2\beta(\mathbf{t}) > 0$ for sufficiently large $\mathbf{t}$.

If $\mathcal{P} = \Omega$, the same argument proves that the condition $\alpha - \beta = 0$ is well defined since the value $\alpha(\mathbf{t}) - \beta(\mathbf{t})$ does not depend on $\mathbf{t}$ for large $\mathbf{t}$.

Remark 1. The above analysis has the following consequence. If γ_1 and γ_2 are two successive edges of $\mathcal{P}$ and none of them is included in $A_{\mathcal{P}}$, we see that $\gamma(\mathbf{t}) - 2\alpha(\mathbf{t}) \to +\infty$ as the components of $\mathbf{t}$ go to $+\infty$. So the condition $\gamma - 2\alpha > 0$ is always satisfied for such an inscribed polygonal domain. So, this condition can be studied only for inscribed polygonal domains $\mathcal{P}$ such that a edges alternate along $\partial\mathcal{P}$. For the $\gamma - 2\beta$ condition, we can focus on inscribed polygonal domains $\mathcal{P}$ such that b edges alternate along $\partial\mathcal{P}$

Lemma 15. *let Ω be an ideal domain in Σ. There is a number $L(\Omega)$ depending only on Ω such the following is true. Let $\{\gamma_i\}_{1 \leq i \leq L}$ be a set of disjoint proper geodesics in Ω which are either closed or with end-points among the vertices of Ω. Then $L \leq L(\Omega)$.*

Proof. First, we look at closed geodesics. If such a geodesic γ bounds a topological disk D then the Gauss-Bonnet formula gives $-|D| = 2\pi$ so none of these geodesics are homotopically trivial. If two of them bound a topological annulus A, the Gauss-Bonnet formula gives $-|A| = 0$ so any two closed geodesics are not homotopic. So there is a constant κ_Σ depending only on the topology of Σ such that the number of closed geodesics in $\{\gamma_i\}_{1 \leq i \leq L}$ is less than κ_Σ.

So now we remove the closed geodesics from $\{\gamma_i\}_{1 \leq i \leq L}$ and we consider the connected components $\mathcal{P}$ of the complement of these geodesics. Applying Gauss-Bonnet formula to Ω, we obtain $-|\Omega| + (N_\Omega - n)\pi = 2\pi\chi(\Omega)$ (let us recall that $N_\Omega - n$ is the number of vertices of Ω on non parabolic ends). So $|\Omega| = (N_\Omega - n)\pi - 2\pi\chi(\Omega)$. From the Gauss-Bonnet formula we also obtain $-|\mathcal{P}| + (n_{\mathcal{P}} + k_{\mathcal{P}})\pi = 2\pi\chi(\mathcal{P})$ where $\mathcal{P}$ is a connected component of $\Omega \setminus (\bigcup_{1 \leq i \leq L} \gamma_i)$, $n_{\mathcal{P}}$ is the number of edges of $\mathcal{P}$ among the edges of Ω and $k_{\mathcal{P}}$ the number of edges among $\{\gamma_i\}_{1 \leq i \leq L}$. As a consequence, the area $|\mathcal{P}|$ is an integer multiple of π and the number of such $\mathcal{P}$ is less than $|\Omega|/\pi = N_\Omega - n - 2\chi(\Omega)$. Besides we have

$$k_{\mathcal{P}}\pi - 2\pi \leq (n_{\mathcal{P}} + k_{\mathcal{P}})\pi - 2\pi\chi(\mathcal{P}) = |\mathcal{P}| \leq |\Omega|;$$

so $k_{\mathcal{P}} \leq 2 + |\Omega|/\pi = 2 + N_\Omega - n - 2\chi(\Omega)$. So summing over all $\mathcal{P}$ and using the estimate of the number of closed geodesics we have

$$L \leq \kappa_\Sigma + (1 + \frac{N_\Omega - n}{2} - \chi(\Omega))(N_\Omega - n - 2\chi(\Omega)).$$

$\square$

5.2. A Jenkins-Serrin theorem.
Let Ω be an ideal domain with a a/b labeling of $\partial\Omega$. We are interested in solving the following Dirichlet problem

on Ω that we call the Jenkins-Serrin-Dirichlet problem:

$$(1) \qquad \begin{cases} \operatorname{div}(X_u) = 0 \text{ on } \Omega \\ u = +\infty \text{ on } A_\Omega \\ u = -\infty \text{ on } B_\Omega. \end{cases}$$

Theorem 16. *Let Ω be an ideal domain with a a/b labeling of $\partial\Omega$. The Jenkins-Serrin-Dirichlet problem has a solution if and only if $\alpha - \beta = 0$ for $\mathcal{P} = \Omega$, and*

$$\gamma - 2\alpha > 0 \quad and \quad \gamma - 2\beta > 0$$

for all other inscribed polygonal domains $\mathcal{P}$. Moreover if the solution exists, it is unique up to an additive constant.

We separate the proof of the above theorem in three parts.

5.2.1. *The conditions are necessary.* Let u be a solution and consider an inscribed polygonal domain $\mathcal{P}$ and $\mathbf{t} \in \mathbb{R}_+^{N_\Omega}$ with large coordinates. The boundary of $\mathcal{P} \setminus H(\mathbf{t})$ is made of $A_\mathcal{P}(\mathbf{t})$, $B_\mathcal{P}(\mathbf{t})$, $C_\mathcal{P}(\mathbf{t})$ and arcs with curvature 1 contained in $\partial H(\mathbf{t})$, we denote by $\Gamma_\mathbf{t}$ the union of these arcs. We notice that $\ell(\Gamma_\mathbf{t})$ goes to 0 as $\mathbf{t} \to \infty$. Since u solves (MSE), Lemma 6 gives

$$(2) \quad \begin{aligned} 0 = F_u(\partial(\mathcal{P} \setminus H(\mathbf{t}))) &= F_u(A_\mathcal{P}(\mathbf{t})) + F_u(B_\mathcal{P}(\mathbf{t})) + F_u(C_\mathcal{P}(\mathbf{t})) + F_u(\Gamma_\mathbf{t}) \\ &= \alpha(\mathbf{t}) - \beta(\mathbf{t}) + F_u(C_\mathcal{P}(\mathbf{t})) + F_u(\Gamma_\mathbf{t}) \end{aligned}$$

Since $\|X_u\| < 1$ along $C_\mathcal{P}(\mathbf{t})$ and $\Gamma_\mathbf{t}$, we have $|F_u(\Gamma_\mathbf{t})| \leq \ell(\Gamma_\mathbf{t}) \xrightarrow[\mathbf{t} \to \infty]{} 0$ and, if $\mathcal{P} \neq \Omega$, $C_\mathcal{P}(\mathbf{t})$ is nonempty then $|F_u(C_\mathcal{P}(\mathbf{t}))| < \ell(C_\mathcal{P}(\mathbf{t})) = \gamma(\mathbf{t}) - \alpha(\mathbf{t}) - \beta(\mathbf{t})$. Moreover the difference $\gamma(\mathbf{t}) - \alpha(\mathbf{t}) - \beta(\mathbf{t}) - |F_u(C_\mathcal{P}(\mathbf{t}))|$ is non decreasing with $\mathbf{t}$. So, if $\mathcal{P} \neq \Omega$, there is $c > 0$ such that for $\mathbf{t}$ large $\gamma(\mathbf{t}) - \alpha(\mathbf{t}) - \beta(\mathbf{t}) - |F_u(C_\mathcal{P}(\mathbf{t}))| \geq c$. Using this in (2), we obtain

$$0 \leq \alpha(\mathbf{t}) - \beta(\mathbf{t}) + (\gamma(\mathbf{t}) - \alpha(\mathbf{t}) - \beta(\mathbf{t}) - c) + \ell(\Gamma_\mathbf{t})$$

which implies $\gamma(\mathbf{t}) - 2\beta(\mathbf{t}) \geq c/2 > 0$ for $\mathbf{t}$ large enough. So $\gamma - 2\beta > 0$ on $\mathcal{P}$. Similar computations give $\gamma - 2\alpha > 0$ on $\mathcal{P}$. If $\mathcal{P} = \Omega$, taking the limit in (2) gives $\lim_{\mathbf{t} \to \infty} \alpha(\mathbf{t}) - \beta(\mathbf{t}) = 0$, so $\alpha - \beta = 0$ for $\mathcal{P} = \Omega$.

Remark 2. If γ is a subarc of $C_\mathcal{P}$ and $\|X_u\| \leq 1 - \delta$ ($\delta > 0$), the constant c appearing in the above proof can be taken equal to $\delta\ell(\gamma)$.

A second remark is that if $\mathcal{P}$ is a polygonal domain as in Definition 12 which is contained in Ω. We can also define $A_\mathcal{P}$, $B_\mathcal{P}$ and $C_\mathcal{P}$ and look at the $\gamma - 2\alpha > 0$ and $\gamma - 2\beta > 0$ conditions. The arguments above also tell us that these conditions are satisfied for such polygonal domains.

5.2.2. *The existence part.* The first step of the existence part of Theorem 16 proof is given by the following result.

Lemma 17. *Let Ω be an ideal domain with a a/b labeling of $\partial\Omega$. For any n, there is a solution to the following Dirichlet problem in Ω*

(3)
$$\begin{cases} \operatorname{div}(X_u) = 0 \ \text{on } \Omega \\ u = n \ \text{on } A_\Omega \\ u = -n \ \text{on } B_\Omega. \end{cases}$$

Proof. We apply the Perron method to solve this Dirichlet problem (see Theorem 2.12 in [7] for harmonic functions). Let us recall its framework. A continuous function w on Ω is called a subsolution of (MSE) if, for any bounded open subset $U \subset\subset \Omega$ with smooth boundary and any solution v to (MSE) on U, $w \leq v$ on ∂U implies $w \leq v$ on U.

A continuous function w on $\overline{\Omega}$ is called a subsolution to (3) if it is a subsolution to (MSE) and $w \leq n$ on A_Ω and $w \leq -n$ on B_Ω. Let $\mathcal{S}$ be the set of all subsolutions to (3). We notice that $w \equiv -n \in \mathcal{S}$ and any $w \in \mathcal{S}$ satisfies $w \leq n$. The Perron method asserts that the function u defined on Ω by $u(p) = \sup_{w\in\mathcal{S}} w(p)$ solves (MSE).

The fact that u satisfies the boundary data of (3) comes from the existence of barriers along the boundary. They can be constructed as follows. Take a point p in $\partial\Omega$ and consider D^+, a geodesic half-disk contained in Ω and centered at p. There exists a solution v of (MSE) on D^+ with boundary data 0 on $\partial D^+ \cap \partial\Omega$ and $2n$ on $\partial D^+ \cap \Omega$. Then, if $p \in A_\Omega$, $n - v \leq u \leq n$ on D^+ since $n - v$ is a subsolution and then $u(p) = n$. If $p \in B_\Omega$, we have $-n \leq u \leq v - n$ and $u(p) = -n$. $\qquad\square$

Let $(u_n)_n$ be the sequence of solutions given by Lemma 17. Let us prove that, up to a subsequence, the sequence converges to a solution of Problem (1). We assume that $X_{u_n} \to X$ and the question is to understand the possible divergence lines of X. Each of them are proper geodesics in Ω and, since u_n is locally constant along $\partial\Omega$, their end points must be among the vertices of Ω (Lemma 11). So divergence lines are either closed geodesics or geodesic lines joining two vertices of Ω.

First let us assume that we have at least one divergence line (the convergence domain $\mathcal{B}(X)$ is not the whole Ω). There are at most a finite number of divergence lines (Lemma 15). Thus $\mathcal{B}(X)$ has a finite number of connected components. Let us define an oriented graph G in the following way. The vertices of G are the connected components of $\mathcal{B}(X)$. A divergence line γ lies in the boundary of two connected components c_1 and c_2 of $\mathcal{B}(X)$ (may be $c_1 = c_2$) and, along γ, X points into one of these connected components, say c_2. We then define an arrow (or oriented edge) e_γ from c_1 to c_2. G is then a finite oriented graph.

Lemma 18. *G has no oriented cycle.*

Proof. Assume $e_{\gamma_1} \cdots e_{\gamma_k}$ is an oriented cycle in G. Let c_i be the initial point of e_{γ_i}; as a cycle, the edge e_{γ_k} has endpoint c_1. Let q_i be a point in c_i. By

Lemma 10, we have

$$0 = (u_n(q_1) - u_n(q_k)) + \sum_{i=1}^{k-1}(u_n(q_{i+1}) - u_n(q_i)) \xrightarrow[n \to \infty]{} +\infty$$

which gives a contradiction. □

So G has a vertex c where all adjacent arrows arrive. The component c is an inscribed polygonal domain $\mathcal{P}$ in Ω. Let $q \in \mathcal{P}$ and define w on c as the limit of $u_n - u_n(q)$. Since $\mathcal{B}(X) \neq \Omega$ and g has no oriented cycle, there is an other vertex c' in G which is joined to c by some edge e_γ. As a consequence if $q' \in c'$, we have $u_n(q) - u_n(q') \to +\infty$ (Lemma 10). Since $u_n \geq -n$, this implies $u_n(q) + n \to +\infty$. We have then proved that $w = -\infty$ on $B_\mathcal{P}$ (the edges of $\mathcal{P}$ among the b-edges of Ω) and $X = -\nu$ along $B_\mathcal{P}$ (Lemmas 6 and 7).

Let $\mathbf{t} \in \mathbb{R}^{N_\Omega}$ be large. As above, the boundary of $\mathcal{P} \setminus H(\mathbf{t})$ splits into $A_\mathcal{P}(\mathbf{t})$, $B_\mathcal{P}(\mathbf{t})$, $C_\mathcal{P}(\mathbf{t})$ and $\Gamma_\mathbf{t}$. So we can compute

$$0 = F_{u_n}(\partial(\mathcal{P} \setminus H(\mathbf{t}))) = F_{u_n}(A_\mathcal{P}(\mathbf{t})) + F_{u_n}(B_\mathcal{P}(\mathbf{t})) + F_{u_n}(C_\mathcal{P}(\mathbf{t})) + F_{u_n}(\Gamma_\mathbf{t})$$

Taking the limit $n \to \infty$ and using $X = -\nu$ along $B_\mathcal{P}(\mathbf{t})$ and $C_\mathcal{P}(\mathbf{t})$ and $\|X\| \leq 1$ along $A_\mathcal{P}(\mathbf{t})$ and $\Gamma_\mathbf{t}$, we obtain

$$\beta(\mathbf{t}) + (\gamma(\mathbf{t}) - \alpha(\mathbf{t}) - \beta(\mathbf{t})) \leq \alpha(\mathbf{t}) + \ell(\Gamma_\mathbf{t})$$

So making $\mathbf{t} \to \infty$, we get $\liminf_{\mathbf{t} \to \infty} \gamma(\mathbf{t}) - 2\alpha(\mathbf{t}) \leq 0$ which is impossible since we assume $\gamma - 2\alpha > 0$ for $\mathcal{P}$. As a consequence, we have proved that there is no divergence line and $\mathcal{B}(X) = \Omega$. Let us notice that we can do the same argument with a vertex c where all adjacent arrows leave and obtain a contradiction with the $\gamma - 2\beta > 0$ property.

Taking $p \in \Omega$, we define w the limit of $u_n - u_n(p)$ on Ω. The function w has the right boundary values. Indeed, because of the values of u_n on $\partial\Omega$, we can be sure that either $w = +\infty$ on A_Ω or $w = -\infty$ on B_Ω (Lemmas 6 and 7). We assume $w = +\infty$ on A_Ω (the other case is similar).

For $\mathbf{t}$ large, we have

$$0 = F_w(\partial(\Omega \setminus H(\mathbf{t}))) = F_w(A_\Omega(\mathbf{t})) + F_w(B_\Omega(\mathbf{t})) + F_w(\Gamma_\mathbf{t})$$

Let us fix some $\mathbf{t}_0$ and assume that $F_w(B_\Omega(\mathbf{t}_0)) \geq -\beta(\mathbf{t}_0) + c$ for some positive c. Then for any $\mathbf{t} \geq \mathbf{t}_0$, using $X_w = \nu$ on A_Ω and $\|X_w\| \leq 1$ on B_Ω, the above equality gives $\alpha(\mathbf{t}) - \beta(\mathbf{t}) \leq -c + \ell(\Gamma_\mathbf{t})$. So $\alpha(\mathbf{t}) - \beta(\mathbf{t}) \leq -c/2 < 0$ for $\mathbf{t}$ large. This gives a contradiction with $\alpha = \beta$ for Ω. So $F_w(B_\Omega(\mathbf{t})) = -\beta(\mathbf{t})$ for large $\mathbf{t}$ and $w = -\infty$ on B_Ω (Lemma 6).

5.2.3. *The uniqueness part.* Let u and v be two solutions of (1) and assume that $u - v$ is not a constant. Let t be a regular value of $u - v$ in the range of $u - v$ and define $D = \{u - v > t\}$, we notice that along $\partial D \cap \Omega$ which is non-empty, $X_u - X_v$ points inside D. Let $\mathbf{t} \in \mathbb{R}_+^{N_\Omega}$ be large and $\delta > 0$ be small. Let $D_{\mathbf{t},\delta}$ be the set of points inside $D \setminus H(\mathbf{t})$ and at distance δ from $\partial\Omega$. The boundary of $D_{\mathbf{t},\delta}$ is made of three parts $\Gamma_{1,\mathbf{t},\delta}$ in $\partial D \cap \Omega$, $\Gamma_{2,\mathbf{t},\delta}$ in

$\partial H(\mathbf{t})$ and $\Gamma_{3,\mathbf{t},\delta}$ in equidistant curves to $\partial\Omega$. Notice that on $\Gamma_{3,\mathbf{t},\delta}$, $X_u - X_v$ goes to 0 as δ goes to 0 since $X_u = X_v$ on $\partial\Omega$. So integrating $\mathrm{div}(X_u - X_v)$ on $D_{\mathbf{t},\delta}$, we obtain

$$0 = \int_{\Gamma_{1,\mathbf{t},\delta}} (X_u - X_v) \cdot \nu + \int_{\Gamma_{2,\mathbf{t},\delta}} (X_u - X_v) \cdot \nu + \int_{\Gamma_{3,\mathbf{t},\delta}} (X_u - X_v) \cdot \nu$$

As $\delta \to 0$, the last term goes to 0. So, with $\Gamma_{1,\mathbf{t}} = \partial D \cap (\Omega \setminus H(\mathbf{t}))$, we have

$$\int_{\Gamma_{1,\mathbf{t}}} (X_u - X_v) \cdot \nu \geq -2\ell(\partial H(\mathbf{t}) \cap \Omega)$$

Letting $\mathbf{t} \to \infty$, we obtain $\int_{\partial D \cap \Omega} (X_u - X_v) \cdot \nu \geq 0$ which contradicts $X_u - X_v$ points inside along $\partial D \cap \Omega$.

Remark 3. In the following, a ideal domain Ω with a a/b labeling that satisfies the conditions of Theorem 16 will be called a Jenkins-Serrin domain. A solution u to the Jenkins-Serrin-Dirichlet problem on Ω will be called a Jenkins-Serrin solution.

5.3. **An example.** In this section, we give an example of a Jenkins-Serrin domain Ω in Σ.

Let $E_1, \ldots, E_p$ be the non parabolic ends of Σ. We recall that E_i is seen as the quotient of some C_c by a hyperbolic translation T.

Let l be an even integer and T_l be the hyperbolic translation such that $T_l^l = T$. Let $p \in \partial_\infty E_i$. Since C_c is invariant by T_l, T_l acts on E_i by isometry. Let us define $p_j^i = T_l^{j-1}(p)$ for $1 \leq j \leq l$. Let $t > 0$ be large and $H(t)$ be a horodisk at p contained in E_i. We define $H_j^i(t) = T_l^{j-1}(H(t))$. There is a value t_l of t such that $H_j^i(t_l)$ is tangent to $H_{j+1}^i(t_l)$. Now we choose l_i even such that $H_j^i(t_{l_i}) \subset E_i$.

Let us consider the ideal domain Ω whose vertices are the cusp end-points of Σ and the p_j^i for $1 \leq j \leq l_i$ and $1 \leq i \leq q$ and the edges are the geodesics joining p_j^i to p_{j+1}^i and passing by the tangency point between $H_j^i(t_l)$ and $H_{j+1}^i(t_l)$.

Let us fix a a/b labeling on $\partial\Omega$, then Ω is a Jenkins-Serrin domain. In order to verify the conditions, we choose the horodisks $H_j^i(t_{l_i})$. For this choice of $\mathbf{t}$, all the edges of Ω are contained in the horodisks so for any inscribed polygonal domain $\mathcal{P}$ we have $\alpha(\mathbf{t}) = 0 = \beta(\mathbf{t})$ and the condition $\alpha = \beta$ is satisfied for $\mathcal{P} = \Omega$. If $\mathcal{P} \neq \Omega$, $C_{\mathcal{P}}(\mathbf{t}) \neq \emptyset$ and then $\gamma(\mathbf{t}) > 0$ which gives $\gamma - 2\alpha > 0$ and $\gamma - 2\beta > 0$.

6. Construction of harmonic diffeomorphisms

The aim of this section is to prove the following theorem

Theorem 19. *Let Σ be an orientable complete hyperbolic surface with finite topology. Then there is a function u defined on Σ solution of the* (MSE) *whose graph in $\Sigma \times \mathbb{R}$ has parabolic conformal type.*

As a consequence, there is a parabolic surface Σ' and a harmonic diffeo-morphism $X : \Sigma' \to \Sigma$.

6.1. **The conformal type.** The first point is that we can control the conformal type of the graph of a Jenkins-Serrin solution .

Proposition 20. *The graph of a Jenkins-Serrin solution is parabolic.*

Proof. Actually we are going to prove that each annular end is parabolic.

Let Ω and u be a Jenkins-Serrin domain and u a Jenkins-Serrin solution. The annular ends of the graph G_u are given by the parts of G_u above the cusp ends of Σ and the ones above non parabolic annular ends.

The annular ends of G_u above cusp ends are parabolic by Corollary 4.

So let us consider E a non parabolic end of Σ. The curve ∂E is contained in Ω and in the homotopy class of $\partial_\infty E$ in $\overline{\Sigma}^\infty$. Then ∂E bounds an annular connected component $D \in \Omega$ whose other boundary components are the edges of Ω with end points in $\partial_\infty E$. Let G_E denote the graph of u above $D \cup \partial E$. We are going to use that G_E is area minimizing in $\overline{D} \times \mathbb{R}$ to prove that G_E has quadratic area growth. Thus the annular ends will be parabolic (see [8]).

Let $\mathbf{t} \in \mathbb{R}_+^N$ be large such that $\partial E \cap H(\mathbf{t}) = \emptyset$. For $r > 0$, let $\mathbf{t} + r$ be the N_Ω-tuple with all coordinates increased by r. Let us notice that $D \setminus H(\mathbf{t} + r)$ contains all points in D at distance less than r from ∂E. Besides the boundary of $D \setminus H(\mathbf{t} + r)$ is made of ∂E, a finite number of geodesic arcs whose lengths are bounded by $r + a_0$ for some constant $a_0 > 0$ and subarcs of horocycles whose lengths go to 0 as $r \to +\infty$.

We define $M = \sup_{\partial E} |u|$. Let $G_E(r)$ denote the part of G_E contained in $\overline{D \setminus H(\mathbf{t} + r)} \times [-M - r, M + r]$. $G_E(r)$ contains all points in G_E at intrinsic distance less than r from its boundary. Let $B(r)$ be the component of $(\overline{D \setminus H(\mathbf{t} + r)} \times [-M - r, M + r]) \setminus G_E(r)$ contained below $G_E(r)$. Let $S(r) = \partial B(r) \setminus G_E(r)$, $S(r)$ is a surface in $\overline{D} \times \mathbb{R}$ with the same boundary as $G_E(r)$ so since G_E is area minimizing $Area(G_E(r)) \leq Area(S(r))$. To estimate the area of $S(r)$, we just say that $S(r) \subset \partial(\overline{D \setminus H(\mathbf{t} + r)} \times [-M - r, M + r])$. Since D has finite area and $\partial(D \setminus H(\mathbf{t} + r))$ has linear growth, we conclude that $Area(S(r))$ has quadratic growth. So G_E has quadratic area growth and is parabolic. $\square$

Remark 4. The arguments used in [4] are different from the above ones. They give a precise description of the asymptotic behaviour of the graph of a Jenkins-Serrin solution.

6.2. **Extension.** Here we explain how a Jenkins-Serrin domain can be "extended" to an other Jenkins-Serrin domain such that the solutions given by Theorem 16 are close on the original domain.

Let us fix a Jenkins-Serrin domain Ω_0 in Σ and let γ_1, γ_2 be two consecutive edges of Ω_0 with γ_1 labeled b and γ_2 labeled a. The connected component D_i of $\Sigma \setminus \gamma_i$ that does not contain Ω_0 is isometric to a hyperbolic half-space.

Let E be the annular end of Σ that contains γ_1 and γ_2. Let β_i be the geodesic ray contained in E which is orthogonal to γ_i and ∂E. Using the disk model for $\mathbb{H}^2$, let P be the hyperbolic halfspace bounded by the geodesic γ joining -1 to $-i$ and containing the origin and let β be the geodesic joining $e^{i\frac{\pi}{4}}$ to $e^{-i3\frac{\pi}{4}}$. Let $\varphi_i : P \to D_i$ be an isometry preserving the orientation such that $\varphi_i(\gamma) = \gamma_i$ and $\varphi_i(\beta \cap P) = \beta_i \cap D_i$.

For $t \in [0, \pi/4]$, let R_t be the ideal rhombus in P with vertices 1, ie^{it}, -1 and $-i$ and R'_t be the ideal rhombus with vertices e^{-it}, i, -1 and $-i$. In Σ, we define $R_{1,t} = \varphi_1(R_t)$ and $R_{2,t} = \varphi_2(R'_t)$. We then consider the new ideal domain $\mathcal{D}_t = \Omega_0 \cup (R_{1,t} \cup R_{2,t}) \cup (\gamma_1 \cup \gamma_2)$. The a/b labeling of $\partial\Omega_0$ induces a natural a/b labeling of $\partial\mathcal{D}_t$ that we consider in the following. In order to lighten the notation, we define $\Omega = \mathcal{D}_0$, $R_1 = R_{1,0}$ and $R_2 = R_{2,0}$ (see Figure 2).
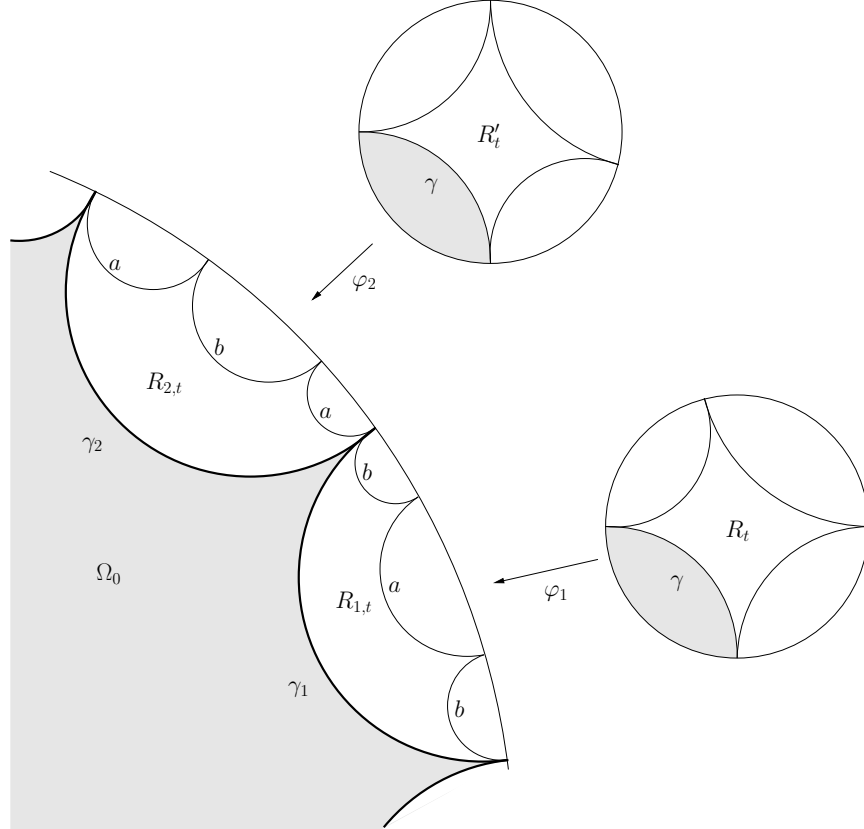

FIGURE 2. The extension of the domain Ω_0

Our aim in this section is to prove the following result

Proposition 21. *Let Ω_0 be a Jenkins-Serrin domain in Σ, $p \in \Omega_0$ be a point and γ_1, γ_2 be two consecutive edges of Ω_0 with γ_1 labeled b and γ_2*

labeled a. Let $\mathcal{D}_t$ be the ideal domain defined above. Then for $t > 0$ small enough, $\mathcal{D}_t$ is a Jenkins-Serrin domain.

Let u be the Jenkins-Serrin solution on Ω_0 and u_t be the one on $\mathcal{D}_t$ with $u(p) = 0 = u_t(p)$. Let K be a compact subset in Ω_0 and ε be a positive number. Then for t small enough, $\|u - u_t\|_{C^2(K)} \leq \varepsilon$.

The first step consists in analyzing the Jenkins-Serrin conditions on $\Omega = \mathcal{D}_0$.

Lemma 22. *Let $\mathcal{P}$ be a polygonal domain inscribed in Ω. If $\mathcal{P}$ is not R_1, R_2, $\Omega \setminus \overline{R}_1$ or $\Omega \setminus \overline{R}_2$, the Jenkins-Serrin conditions are satisfied. For $\mathcal{P} = R_1$ or $\mathcal{P} = \Omega \setminus \overline{R}_2$, we have $\gamma - 2\beta = 0$ and, for $\mathcal{P} = R_2$ or $\mathcal{P} = \Omega \setminus \overline{R}_1$, we have $\gamma - 2\alpha = 0$.*

Proof. Let $\mathcal{P}$ be a polygonal domains inscribed in Ω. Since Ω_0 satisfies the Jenkins-Serrin conditions and R_1 and R_2 are isometric to ideal squares, $\mathcal{P} = R_1, R_2, \Omega \setminus \overline{R}_1$ or $\Omega \setminus \overline{R}_2$ satisfy the stated conditions. Moreover, if $\mathcal{P} = \Omega$, $\alpha - \beta = 0$.

Assume now that $\mathcal{P}$ is not one of these five polygonal domains. By Remark 1, we assume that the a-components alternate along $\partial\mathcal{P}$ (the other case is similar). Let us first notice that, if $\gamma_2 \subset \partial\mathcal{P}$, then $\mathcal{P} = R_2$ which is excluded and, if $\gamma_1 \subset \partial\mathcal{P}$, then $\mathcal{P} \cap R_1 = \emptyset$. Let us introduce some notations

- $A_\mathcal{P}^0 = A_\mathcal{P} \cap \partial\Omega_0$, $B_\mathcal{P}^0 = B_\mathcal{P} \cap \partial\Omega_0$, $C_\mathcal{P}^0 = C_\mathcal{P} \cap (\Omega_0 \cup \gamma_1)$,
- $A_\mathcal{P}^i = A_\mathcal{P} \cap \partial R_i$, $B_\mathcal{P}^i = B_\mathcal{P} \cap \partial R_i$, $C_\mathcal{P}^i = C_\mathcal{P} \cap R_i$,
- $d_i = \gamma_i \cap \mathcal{P}$.

For $\mathbf{t}$ large, we then define

- $\alpha^i(\mathbf{t}) = \ell(A_\mathcal{P}^i \setminus H(\mathbf{t}))$ and $\gamma^i(\mathbf{t}) = \ell((A_\mathcal{P}^i \cup B_\mathcal{P}^i \cup C_\mathcal{P}^i) \setminus H(\mathbf{t}))$, for $i \in \{1, 2, 3\}$ and
- $\delta^i(\mathbf{t}) = \ell(d_i \setminus H(\mathbf{t}))$.

We have $\alpha(\mathbf{t}) = \alpha^0(\mathbf{t}) + \alpha^1(\mathbf{t}) + \alpha^2(\mathbf{t})$ and $\gamma(\mathbf{t}) = \gamma^0(\mathbf{t}) + \gamma^1(\mathbf{t}) + \gamma^2(\mathbf{t})$. So we can compute $\gamma(\mathbf{t}) - 2\alpha(\mathbf{t}) = K^0(\mathbf{t}) + K^1(\mathbf{t}) + K^2(\mathbf{t})$ where

$$K^0(\mathbf{t}) = \gamma^0(\mathbf{t}) + \delta^1(\mathbf{t}) + \delta^2(\mathbf{t}) - 2(\alpha^0(\mathbf{t}) + \delta^2(\mathbf{t})),$$

$$K^1(\mathbf{t}) = \gamma^1(\mathbf{t}) + \delta^1(\mathbf{t}) - 2(\alpha^1(\mathbf{t}) + \delta^1(\mathbf{t})),$$

$$K^2(\mathbf{t}) = \gamma^2(\mathbf{t}) + \delta^2(\mathbf{t}) - 2\alpha^2(\mathbf{t}).$$

Actually $K^0(\mathbf{t})$ (resp. $K^i(\mathbf{t})$) computes $\gamma - 2\alpha$ for $\mathcal{P} \cap \Omega_0$ (resp. $\mathcal{P} \cap R_i$) in Ω_0 (resp. R_i). Since Ω_0, R_1 and R_2 are Jenkins-Serrin domains, these three terms are non-negative (see Section 5.2.1 and Remark 2). Moreover, since the a-components alternate along $\partial\mathcal{P}$ and $\mathcal{P}$ is not Ω, R_1, R_2, $\Omega \setminus \overline{R}_1$ or $\Omega \setminus \overline{R}_2$, $C_\mathcal{P}^0$ is not equal to γ_1. This implies that $\mathcal{P} \cap \Omega_0 \neq \Omega_0$ and $K^0(\mathbf{t}) > 0$ for large $\mathbf{t}$. Thus the condition $\gamma - 2\alpha > 0$ is proved for $\mathcal{P}$. $\square$

The second step of the extension argument consists in proving the first statement of Proposition 21. We notice that the family $\{\mathcal{D}_t\}_t$ is a continuous family of ideal domains in $\Sigma : \Omega = \mathcal{D}_0$ is $\bigcup_{t>0}(\text{interior} \bigcap_{0<s<t} \mathcal{D}_s)$.

Lemma 23. *For $t > 0$ small enough, $\mathcal{D}_t$ is a Jenkins-Serrin domain.*

Proof. If $\mathcal{P}$ is an inscribed polygonal domain in $\mathcal{D}_{t_0}$, we can actually define a unique continuous family $\{\mathcal{P}_t\}_t$ such that $\mathcal{P}_t$ is an inscribed polygonal domain in $\mathcal{D}_t$ such that $\mathcal{P} = \mathcal{P}_{t_0}$. Assume that the a-edges alternate along $\partial \mathcal{P}_t$ and $\mathcal{P}_t$ is not $\mathcal{D}_t$, $R_{2,t}$ or $\mathcal{D}_t \setminus \overline{R}_{1,t}$. We have several cases to study. If $\mathcal{P}_t \subset \Omega_0$ then $\mathcal{P}_t = \mathcal{P}_0$ and the condition $\gamma - 2\alpha > 0$ is satisfied.

We notice that $R_{1,t}$ and $R_{2,t}$ are isometric through an isometry S that send γ_1 to γ_2 and exchanges the labels of the edges. If $R_{1,t} \cup R_{2,t} \subset \mathcal{P}_t$, we have $R_{1,s} \cup R_{2,s} \subset \mathcal{P}_s$ and $\mathcal{P}_t \cap \Omega_0 = \mathcal{P}_s \cap \Omega_0$ for any $s \leq t$. The symmetry S implies that the value of $\gamma - 2\alpha$ for $\mathcal{P}_s$ does not depend on s so $\gamma - 2\alpha > 0$ on $\mathcal{P}_t$.

Let us assume $R_{2,t} \subset \mathcal{P}_t$ and $R_{1,t} \cap \mathcal{P}_t = \emptyset$. A decomposition similar to the one in Lemma 22 proof gives $\gamma(\mathbf{t}) - 2\alpha(\mathbf{t}) = K^0(\mathbf{t}) + K^2(\mathbf{t})$ where $K^0(\mathbf{t}) > 0$ since $\mathcal{P}_t \cap \Omega_0$ does not depend on t and $K^2(\mathbf{t}) > 0$ for $t > 0$ because $R_{2,t} \cap \mathcal{P}_t = R_{2,t}$ which is isometric to R'_t (see Figure 3). So $\gamma - 2\alpha > 0$ is satisfied for $\mathcal{P}_t$.

If we are not in the cases $\mathcal{P}_t \subset \Omega_0$, $R_{1,t} \cup R_{2,t} \subset \mathcal{P}_t$ or $R_{2,t} \subset \mathcal{P}_t$ and $R_{1,t} \cap \mathcal{P}_t = \emptyset$, we can be sure that $C_{\mathcal{P}_t}$ intersects γ_2 or γ_1 since the a-edges alternate. More precisely, there is a compact subset in Ω_0 (close to γ_2 and γ_1) that does not depend on the particular $\mathcal{P}_t$ such that $C_{\mathcal{P}_t} \cap K$ contains a subarc of length at least ε (ε independent of $\mathcal{P}_t$). Let u_0 be the Jenkins-Serrin solution on Ω_0. We have $\|X_{u_0}\| \leq 1 - \delta$ ($\delta > 0$) on K. So by Remark 2, we have $\gamma - 2\alpha \geq \delta\varepsilon > 0$ on $\mathcal{P}_0$. Since the value of $\gamma - 2\alpha$ on $\mathcal{P}_t$ depends continuously on t, we see that $\gamma - 2\alpha > \delta\varepsilon/2$ for $\mathcal{P}_t$ for any $t \leq t_0$ where t_0 does not depend on the particular $\mathcal{P}_t$.

For $\mathcal{P} = \mathcal{D}_t$, $\alpha - \beta = 0$ comes from the fact that Ω_0 is a Jenkins-Serrin domain and S is an isometry from $R_{1,t}$ to $R_{2,t}$ exchanging the label of the edges.

For $\mathcal{P}_t = R_{2,t}$, the condition $\gamma - 2\alpha$ for $t > 0$ can be easily verified on R'_t so the same is true on $\mathcal{P}_t = R_{2,t}$ (see Figure 3).

For $\mathcal{P} = \mathcal{D}_t \setminus \overline{R}_{1,t}$, the condition $\gamma - 2\alpha > 0$ then follows from the fact that Ω_0 is a Jenkins-Serrin domain and the condition $\gamma - 2\alpha > 0$ on R'_t.

The same argument can be done for polygonal domains with alternating b-edges. $\square$

From the above result, there is a Jenkins-Serrin solution u_t on $\mathcal{D}_t$, we consider the one satisfying $u_t(p) = 0$. As t goes to 0, $\mathcal{D}_t$ goes to Ω and, considering a subsequence, X_{u_t} converges to some X on Ω. The description of X is given by the following result.

Lemma 24. *X has exactly two divergence lines : the geodesic lines γ_1 and γ_2. Along γ_1, X points into Ω_0 and along γ_2, X points into R_2. Moreover, $X = \nu$ (resp. $X = -\nu$) along the a-boundary components (resp. b-boundary components) of $\partial\Omega$.*
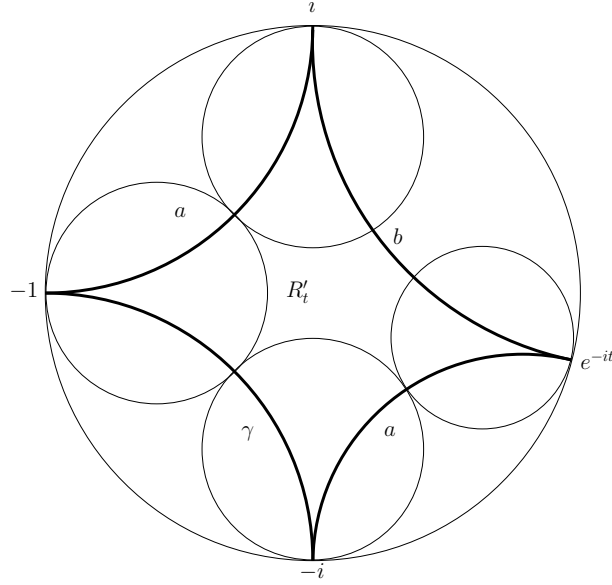
FIGURE 3. The ideal rhombus R'_t with a choice of horodisks proving the $\gamma - 2\alpha > 0$ condition

Proof. First, because of the value of X_{u_t} along $\partial \mathcal{D}_t$, $X_{u_t} = \nu$ (resp. $X_{u_t} = -\nu$) along the a-boundary components (resp. b-boundary components) of $\partial \mathcal{D}_t$. As a consequence $X = \nu$ (resp. $X = -\nu$) along the a-boundary components (resp. b-boundary components) of $\partial \Omega$ (Lemma 11).

If X has no divergence line, then, considering a subsequence, u_t converges to u a solution of (MSE) on Ω. Because of the value of X along $\partial \Omega$, u is then a Jenkins-Serrin solution on Ω which is impossible since Ω is not a Jenkins-Serrin domain. So X must have at least one divergence line.

Moreover the value of X along $\partial \Omega$ implies that the divergences lines are either closed geodesics or proper geodesics ending at vertices of Ω.

As in the proof of Theorem 16, we introduce an oriented graph structure G on the set of connected component of $\mathcal{B}(X)$. Using the same arguments, there is an inscribed polygonal domain $\mathcal{P}$ in Ω which is a connected component of $\mathcal{B}(X)$ where the condition $\gamma - 2\alpha > 0$ is not satisfied. So $\mathcal{P} = R_2$ or $\mathcal{P} = \Omega \setminus \overline{R}_1$ by Lemma 22. There is also an inscribed polygonal domain $\mathcal{P}'$ in Ω which is a connected component of $\mathcal{B}(X)$ where the condition $\gamma - 2\beta > 0$ is not satisfied. So $\mathcal{P}' = R_1$ or $\mathcal{P}' = \Omega \setminus \overline{R}_2$.

This implies that at least γ_1 or γ_2 is a divergence line with the stated value of X along it. Assume only γ_1 is a divergence line (the same can be done for γ_2). This would imply that $\mathcal{B}(X)$ has only two connected components $\mathcal{P}' = R_1$ and $\mathcal{P} = \Omega \setminus \overline{R}_1$. So a subsequence of u_t converges to a solution u of (MSE) on $\Omega \setminus \overline{R}_1$. Because of the value of X along $\partial(\Omega \setminus \overline{R}_1)$, u would be

a Jenkins-Serrin solution on $\Omega \setminus \overline{R}_1$ which is impossible since $\Omega \setminus \overline{R}_1$ is not a Jenkins-Serrin domain by Lemma 22. γ_1 and γ_2 are divergence lines.

The last point consists in proving there are no other divergence lines. We notice that γ_1 and γ_2 split the graph G in three connected components. If one of these three components contains one edge, then a similar argument will prove that Ω_0, R_1 or R_2 is not a Jenkins-Serrin domain. □

We can now finish the proof of Proposition 21.

Proof of Proposition 21. The structure of $\mathcal{B}(X)$ implies that on Ω_0, a subsequence of u_t converges to v a solution of (MSE) on Ω_0. Besides the value of X on $\partial \Omega_0$ implies that v is a Jenkins-Serrin solution on Ω_0. By uniqueness of this solution and since $u(p) = 0 = v(p)$ we have $u = v$ on Ω. Since this limit does not depend on the sequence, this implies that $u_t \to u$ uniformly on each compact subset of Ω_0. So $\|u - u_t\|_{C^2(K)} \leq \varepsilon$ for t small enough.   □

6.3. **The construction.** Using the preceding results, we are ready to prove Theorem 19.

Proposition 25. *Let Ω be a Jenkins-Serrin domain. There is an increasing sequence $(\Omega_n)_n$ of Jenkins-Serrin domains $(\Omega_0 = \Omega)$ and an increasing sequence of compact subsets $(K_n)_n$ of Ω_n such the following is true. Let u_n be the Jenkins-Serrin solution on Ω_n and E_i be the annular ends of Σ; we then have*

- *$K_n \subset \Omega_n$ and $\cup_n K_n = \Sigma$,*
- *$\|u_{n+1} - u_n\|_{C^2(K_n)} \leq \frac{1}{2^n}$,*
- *$(K_n \setminus \overset{\circ}{K}_{n-1}) \cap E_i$ is an annulus and*
- *the graph of u_n over $(K_j \setminus K_{j-1}) \cap E_i$ is an annulus whose conformal modulus is at least 1 for any $n \geq j$.*

Proof. Fix a point $\bar{p} \in \Sigma \setminus \cup_i E_i$, there is a constant $\bar{d} > 0$ such the following is true. There are sequences Ω_n, K_n and u_n such that

- $K_n \subset \Omega_n$, K_n contains $\{p \in \Omega_n | d(p, \partial\Omega_n) \geq 1\}$, $d(\bar{p}, \partial\Omega_n) \geq n\bar{d}$,
- $\|u_{n+1} - u_n\|_{C^2(K_n)} \leq \frac{1}{2^n}$,
- $(K_n \setminus \overset{\circ}{K}_{n-1}) \cap E_i$ is an annulus and
- the graph of u_n over $(K_j \setminus K_{j-1}) \cap E_i$ is an annulus whose conformal modulus is at least 1 for any $n \geq j$.

Clearly this will prove the proposition. The proof of the existence is by induction. So assume that Ω_j, K_j and u_j are constructed for $j \leq n$

Since Ω_n is a Jenkins-Serrin domain, we can gather its edges in a finite number of pairs $\{\gamma_1^i, \gamma_2^i\}$ such that γ_1^i, γ_2^i are consecutive edges and γ_1^i is labeled b and γ_2^i is labeled a. Let ε be positive and apply Proposition 21 successively to the pair γ_1^i, γ_2^i to add perturbed squares along these edges. We obtain a Jenkins-Serrin domain Ω_{n+1} and a solution u_{n+1} such that $\|u_n - u_{n+1}\|_{C^2(K_n)} \leq \varepsilon$. Choosing ε sufficiently small, we can ensure that

$\|u_n - u_{n+1}\|_{C^2(K_n)} \leq \frac{1}{2^n}$ and the graph of u_{n+1} over $(K_j \setminus \overset{\circ}{K}_{j-1}) \cap E_i$ is an annulus whose conformal modulus is at least 1 for $j \leq n$.

Next we can choose a compact subset K_{n+1} containing $\{p \in \Omega_{n+1} | d(p, \partial \Omega_{n+1}) \leq 1\}$ such that $K_{n+1} \setminus \overset{\circ}{K}_n$ is made of annuli in each annular end of Σ. Moreover, by Proposition 20 the graph of u_{n+1} is parabolic. So K_{n+1} can be chosen such that the graph of u_{n+1} over the annuli of $K_{n+1} \setminus \overset{\circ}{K}_n$ has conformal modulus at least 1.

The last point we have to check is that $d(\bar{p}, \partial \Omega_{n+1}) \geq (n+1)\bar{d}$. For this we just have to analyze how the distance between ∂E_i and $\partial \Omega_n$ in E_i evolves. Actually Lemma 29 (see also Figure 4) implies that, in E_i, $d(\partial E_i, \partial \Omega_n) \geq n d_{\kappa_i}$ where κ_i is the curvature of ∂E_i; thus $d(\bar{p}, \partial \Omega_{n+1}) \geq (n+1)\bar{d}$ if $\bar{d} = \min d_{\kappa_i}$. $\qquad \square$

We can now prove our main theorem.

Proof of Theorem 19. Starting with the Jenkins-Serrin domain given in Section 5.3, we apply Proposition 25 to construct Ω_n, K_n and u_n. Since $\cup_n K_n = \Sigma$ and $\|u_n - u_{n+1}\|_{C^2(K_n)} \leq \frac{1}{2^n}$, u_n converges to a solution u of (MSE) on Σ, the convergence is smooth on any compact subsets of Σ.

Since (u_n) converges smoothly to u, the modulus of the graph of u on each annular component of $K_i \setminus \overset{\circ}{K}_{i-1}$ is at least 1. This implies that each annular end of the graph of u has infinite conformal modulus and is parabolic. Thus the graph of u is parabolic. $\qquad \square$

APPENDIX A. AN EQUICONTINUITY RESULT

Let us fix some notations. If $p \in \mathbb{R}^n$, $N \in \mathbb{S}^{n-1}$ and $\delta > 0$, we denote by $D(p, N, \delta)$ the ball in the hyperplane passing through p and normal to N with center p and radius δ. Then we denote by $C(p, N, \delta)$ the cylinder $\{q + sN, q \in D(p, N, \delta) \text{ and } s \in \mathbb{R}\}$. Finally if S is a hypersurface in $\mathbb{R}^n$ and $p \in S$ and $N(p)$ denote the unit normal to S, we denote by $S(p, \delta)$ the connected component of $S \cap C(p, N(p), \delta)$ containing p.

We first begin by recalling a classical result (see for example, Lemma 2.4 in [3] or Lemma 4.1.1 in [18]).

Proposition 26. *Let c and δ be positive, there is $\delta' > 0$ such the following is true. Let S be a hypersurface in $\mathbb{R}^n$ and $p \in S$ such that the second fundamental form of S is bounded by c and $d_S(p, \partial S) \geq \delta$. Then $S(p, \delta')$ is a graph over $D(p, N(p), \delta')$. Moreover, the function v which defines this graph satisfies $v(q) \leq 8c|p-q|^2$, $|\nabla v(q)| \leq 8c|p-q|$ and $|\nabla^2 v| \leq 16c$ for any $q \in D(p, \delta')$.*

A consequence of this local description is the following result.

Proposition 27. *Let $U \subset \Omega$ be two open subsets of $\mathbb{R}^{n-1}$ and c, δ be positive. Let $\mathcal{S}$ be a set of smooth functions on Ω such that, for any $p \in U$ and $u \in \mathcal{S}$, the following is true*

- $d_{G_u}((p, u(p)), \partial G_u) \geq \delta$ and
- the second fundamental form of G_u is bounded by c on the geodesic disk of radius δ and center $(p, u(p))$.

Then the family $\{X_u : U \to \mathbb{R}^{n-1}, u \in \mathcal{S}\}$ is uniformly equicontinuous where U is endowed with the geodesic metric.

Let us recall that the geodesic distance d_γ between two points in U is given by the infimum of the length of curves in U joining the two points. In an open set, d_γ induces the usual topology.

Proof. Proving X_u is uniformly equicontinuous is the same as proving N_u is uniformly equicontinuous. So if $\{N_u : U \to \mathbb{S}^{n-1}, u \in \mathcal{S}\}$ is not uniformly equicontinous, it means that we have two sequences $(p_{1,n})_n$ and $(p_{2,n})_n$ in U and a sequence u_n in $\mathcal{S}$ such that $d_\gamma(p_{1,n}, p_{2,n}) \to 0$, $N_{u_n}(p_{1,n}) \to N_1$ and $N_{u_n}(p_{2,n}) \to N_2$ with $N_1 \neq N_2$. Moreover, by changing the point $p_{2,n}$ by a point along a curve of length at most $d_\gamma(p_{1,n}, p_{2,n}) + \frac{1}{n}$ between $p_{1,n}$ and $p_{2,n}$, we can assume $d_{\mathbb{S}^{n-1}}(N_{u_n}(p_{1,n}), N_{u_n}(p_{2,n})) \leq \pi/2$ and so $\alpha = d_{\mathbb{S}^{n-1}}(N_1, N_2) \leq \pi/2$. By Proposition 26, there is δ' such that on $G_{u_n}(p_{1,n}, \delta')$ the unit normal is at distance less that $\alpha/3$ from $N_{u_n}(p_n)$. Moreover, the translate $G_{u_n}(p_{1,n}, \delta'/2) - p_{1,n} - u_n(p_{1,n})\partial_{x_n}$ converges (after taking a subsequence) in C^1 topology to a graph over $D(0, N_1, \delta'/2)$ along which the unit normal is at distance less than $\alpha/3$ from N_1. By the same argument, $G_{u_n}(p_{2,n}, \delta'/2) - p_{2,n} - u_n(p_{2,n})\partial_{x_n}$ converges in C^1 topology to a graph over $D(0, N_2, \delta'/2)$ along which the unit normal is at distance less than $\alpha/3$ from N_2. Since $0 < d_{\mathbb{S}^{n-1}}(N_1, N_2) < \pi/2$, these two limit graphs intersect and are transverse. Thus $G_{u_n}(p_{1,n}, \delta'/2) - u_n(p_{1,n})\partial_{x_n}$ and $G_{u_n}(p_{2,n}, \delta'/2) - u_n(p_{2,n})\partial_{x_n}$ must intersect and be transverse for n large. This is impossible since at an intersection point the normals have to be the same; indeed, these two surfaces are vertical translates of the same graph. $\square$

In this paper, this has the following consequence.

Proposition 28. *Let $U \subset \Omega \subset \Sigma$ be two open subsets of a Riemannian surface (U with compact closure). Let δ be positive. Let $\mathcal{S}$ be a set of solutions of* (MSE) *on Ω such that for any $p \in U$ and $u \in \mathcal{S}$, $d_{G_u}((p, u(p)), \partial G_u) \geq \delta$. Then the family $\{X_u : U \to T\Sigma, u \in \mathcal{S}\}$ is equicontinuous.*

Proof. Let $p \in U$ and consider a local chart $\varphi : V \in \mathbb{R}^2 \to U$ around p. Because of the hypothesis $d_{G_u}((p, u(p)), \partial G_u) \geq \delta$, curvature estimates for stable minimal surfaces [20] apply to prove that G_u has uniformly bounded second fundamental form near $(p, u(p))$ in $\Sigma \times \mathbb{R}$. This implies that, in $\mathbb{R}^2$, the family $\{u \circ \varphi, u \in \mathcal{S}\}$ satisfies the hypotheses of Proposition 27 for some open set $W \subset V$ ($\varphi^{-1}(p) \in W$). This implies the equicontinuity of $X_{u \circ \varphi}$ at $\varphi^{-1}(p)$ and then the one of X_u at p. $\square$

In the above proposition, if $U \subset\subset \Omega$, the property $d_{G_u}((p, u(p)), \partial G_u) \geq \delta$ is satisfied, so $\{X_u : \Omega \to T\Sigma\}$ is equicontinuous.

Appendix B. A technical lemma

In this section, we prove the following result (see Figure 4).

Lemma 29. *Let κ be in $(0,1)$ then there is a positive constant d_κ such that the following is true. Let c be a complete curve of constant curvature κ in $\mathbb{H}^2$ and γ be a complete geodesic contained in the non-meanconvex side of c. Let γ' be the unique geodesic orthogonal to γ and c. In the halfplane bounded by γ that does not contain c, there are two uniquely determined geodesic rays γ_1 and γ_2 starting from a common point in $p \in \gamma'$ and ending at the endpoints of γ that are orthogonal at p. Then we have*

$$d(c, \gamma_1 \cup \gamma_2) \geq d(c, \gamma) + d_\kappa.$$

Proof. Let us in fact consider the foliation of $\mathbb{H}^2$ by curves $\{c_t\}_{t=\in\mathbb{R}}$ of constant curvature κ and orthogonal to γ' such that $d(c_t, c_{t'}) = |t - t'|$, $c = c_{-d(c,\gamma)}$ and c_o is tangent to γ. There is some $d_\kappa > 0$ such that c_{d_κ} is tangent to γ_1 and γ_2 and contained in the quarter space bouded by γ_1 and γ_2 (notice that d_k only depends on κ). We then have

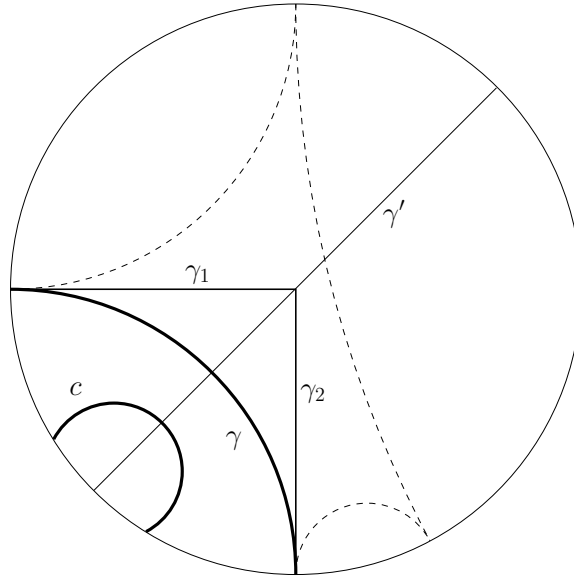$$d(c, \gamma_1 \cup \gamma_2) \geq d(c, c_{d_\kappa}) = d(c, \gamma) + d_\kappa.$$

$\square$

Figure 4. The geodesic rays γ_1 and γ_2 with R'_t drawn

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