A height estimate for constant mean curvature graphs and uniqueness results

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Abstract

In this paper, we give a height estimate for constant mean curvature graphs. Using this result we prove two uniqueness results for the Dirichlet problem associated to the constant mean curvature equation on unbounded domains.

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Introduction

Surfaces with constant mean curvature are mathematical models of soap films. These surfaces appear as interfaces in isoperimetric problems. There are different points of view on constant mean curvature surfaces, one is to consider them as graphs.

Let Ω be a domain in $\mathbb{R}^2$. The graph of a function u over Ω has constant mean curvature $H > 0$ if it satisfies the following partial differential equation:

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 2H \tag{CMC}$$

The graph of such a solution is called an H-graph and has an upward pointing mean curvature vector.

The corresponding Dirichlet problem is to solve (CMC) on Ω with prescribed boundary data. For bounded domains, after the work of J. Serrin [Se1], J. Spruck has given in [Sp] a general answer to the existence and uniqueness questions. His results are of Jenkins-Serrin type [JS] since infinite data are allowed.

On unbounded domains, there are few constructions of solutions. The examples are due to P. Collin [Co] and R. Lopéz [Lo1] for graphs over a strip and R. Lopéz [Lo1, Lo2] for graphs with zero boundary data.

In this paper, we investigate the uniqueness question for the Dirichlet problem. To get uniqueness, we need a control of solutions of the Dirichlet problem, which will enables us to bound the distance between two solutions with the same boundary data.

Our main result (Theorem 2) provides such a control. We call this result a "height estimate" since it bounds the difference of heights between two components of the boundary of an H-graph. The idea of the proof is that if the difference between heights is too big, we can move a sphere of radius $\frac{1}{H}$ through the H-graph, which gives a contradiction with the maximum principle.

With this height estimate, we can bound the difference between two solutions of the same Dirichlet problem under certain hypotheses on the boundary values. For example, if we are on a strip, we prove the uniqueness of possible solutions for Lipschitz boundary data.

The first section of the paper is devoted to the statement and the proof of the height estimate.

In Section 2, we give consequences of the height estimate for solutions of the constant mean curvature equation on unbounded domains.

In the last section, we prove two uniqueness results for the Dirichlet problem on unbounded domains.

1 The height estimate

In this first section, we give a height estimate for solutions of the constant mean curvature equation (CMC). This estimate is designed for solutions on unbounded domains.

First, we need some notations and remarks. If Ω is a domain in $\mathbb{R}^2$ and u is a function which is defined on Ω, we define $F:\Omega\to\mathbb{R}^2$ to be the map:

$$F(x,y)=\big(x,u(x,y)\big)$$

Let us explain what kind of domain we shall consider in the following. For $a>\frac{1}{H}$ and $b>0$, we set $R_{a,b}=[-a,a]\times[-b,b]$. Let $\Omega\subset R_{a,b}$ be a domain with piecewise smooth boundary. We suppose that Ω satisfies the following three hypotheses :

1. Ω is connected,
2. $\partial\Omega\cap\{-a\}\times[-b,b]$ and $\partial\Omega\cap\{a\}\times[-b,b]$ are non empty,
3. $\partial\Omega\cap[-a,a]\times\{-b\}=\emptyset$ and $\partial\Omega\cap[-a,a]\times\{-b\}=\emptyset$.

Let Λ denote the set of the closures of connected components of $\partial\Omega \cap \overset{\circ}{R}_{a,b}$, where $\overset{\circ}{R}_{a,b} = (-a,a) \times (-b,b)$. Let $\gamma \in \Lambda$ be one of these boundary components, γ is homeomorphic either to a circle or to $[0,1]$. If it is homeomorphic to a segment, either it joins $\{-a\} \times [-b,b]$ to $\{a\} \times [-b,b]$ (by connectedness, there are exactly two such components) or the two end points are on the same edge of the rectangle $R_{a,b}$.

Let $c : [0,1] \to R_{a,b}$ be a continuous path with $c(0) \in [-a,a] \times \{-b\}$, $c(1) \in [-a,a] \times \{b\}$ and $c(t) \in \overset{\circ}{R}_{a,b}$ for $0 < t < 1$. We denote by J_c the set of connected components of $[0,1] \cap c^{-1}(\overline{\Omega})$. Let $j \in J_c$, there exist $e_j, o_j \in (0,1)$ such that $j = [e_j, o_j]$. There is a total order on J_c. Let $j, j' \in J_c$, we write $j \lhd j'$ if $o_j < e_{j'}$; the order $\unlhd$ is then a total order. We remark that J_c has a minimum j_{min} and a maximum j_{max}. We then have the following lemma.

Lemma 1. *Let Ω and c be as above. Let $j \in J_c$ with $j \neq j_{min}$. We consider $j' \lhd j$ and denote by γ the element of Λ to which $c(e_j)$ belongs. Then there exists j'' with $j' \unlhd j'' \lhd j$ such that $c(o_{j''})$ belongs to γ.*

Proof. We define $j_0 = \sup\{i \in J_c \,|\, i \lhd j\}$. There are two possibilities. First, $j_0 \lhd j$, in this case $j_0 \unrhd j'$ and $c(o_{j_0}, e_j)$ is a curve outside $\overline{\Omega}$. Because of the different cases for γ, $c(o_{j_0})$ is then in γ. The second possibility is $j_0 = j$. This implies that there exists $i \in J_c$ with $o_i < e_j$ and $e_j - o_i$ as small as we want. The point $c(e_j)$ is at a non zero distance from the complement of γ in $\partial\Omega$. Since c is uniformly continuous, there exists $j' \unlhd i \lhd j$ with $c(o_i) \in \gamma$. $\square$

If c is injective, the $c[e_j, o_j]$ are the connected components of $c\big([0,1]\big) \cap \overline{\Omega}$.

We denote by Δ_1 the connected component of $R_{a,b} \backslash \overline{\Omega}$ that contains $(0,-b)$ and Δ_2 the one that contains $(0,b)$. For $i \in \{1,2\}$, let γ_i denote the element of Λ that are included in the boundary of Δ_i. γ_1 and γ_2 are the two elements of Λ that are homeomorphic to a segment and join $\{-a\} \times [-b,b]$ to $\{a\} \times [-b,b]$.

Let us give a last definition. If A and B are two compact sets in $\mathbb{R}$, we define the distance between A and B by:

$$d(A,B) = \min\{|p-q| \mid p \in A, q \in B\}$$

We are then able to give our height estimate.

Theorem 2. *Let $a > \frac{1}{H}$ and $b > 0$ be real numbers. We consider a domain $\Omega \subset \mathbb{R}_{a,b}$ with piecewise smooth boundary that satisfies the above hypotheses 1., 2. and 3.. We define Λ, γ_1 and γ_2 as above. Let $\Lambda = \Lambda_1 \cup \Lambda_2$ be a partition of Λ such that $\gamma_1 \in \Lambda_1$ and $\gamma_2 \in \Lambda_2$. For $i \in \{1,2\}$, let Γ_i denote* *the part of the boundary* $\bigcup_{\gamma\in\Lambda_i}\gamma$*. Let* u *be a solution of* (CMC) *on* Ω *which is continuous on* $\overline{\Omega}$*. We then have the following upper bound :*

$$d\big(F(\Gamma_1),F(\Gamma_2)\big)\leq\frac{2}{H}$$

where F *is defined at the beginning of this section.*

First we shall prove a weaker version of this result

Theorem 2'. Let $a>\frac{1}{H}$ and $b>0$ be real numbers. We consider $\Omega\subset\mathbb{R}_{a,b}$, Λ and $\Lambda=\Lambda_1\cup\Lambda_2$ a partition as in Theorem 2. For $i\in\{1,2\}$, we define $\Gamma_i=\bigcup_{\gamma\in\Lambda_i}\gamma$. Let u be a solution of (CMC) on Ω which is continuous on $\overline{\Omega}$. We then have the following upper bound :

$$d\big(F(\Gamma_1),F(\Gamma_2)\big)\leq 2a$$

Proof. The idea of the proof is that, if the estimate on the distance does not hold, we would be able to move a sphere of radius $\frac{1}{H}$ through the graph of u and this gives a contradiction with the maximum principle. So let us assume that the distance $d\big(F(\Gamma_1),F(\Gamma_2)\big)$ is larger than $2a$.

The first part of the proof consists in finding the place where the sphere will be located.

Since γ_1 and γ_2 join $\{-a\}\times[-b,b]$ to $\{a\}\times[-b,b]$ in $R_{a,b}$, $F(\gamma_1)$ and $F(\gamma_2)$ join $\{-a\}\times\mathbb{R}$ to $\{a\}\times\mathbb{R}$ in $[-a,a]\times\mathbb{R}$. Let γ be in Λ_1 and (x,z) be a point in $F(\gamma)$. Since $d\big(F(\Gamma_1),F(\Gamma_2)\big)>2a$, no point of $F(\Gamma_2)$ has z as second coordinate, then, if $\gamma'\in\Lambda_2$, $F(\gamma')$ is either above $F(\gamma)$ (*i.e.* $\min_{F(\gamma')}z\geq\max_{F(\gamma)}z$) or below (*i.e.* $\max_{F(\gamma')}z\leq\min_{F(\gamma)}z$). Then γ defines a partition $\Lambda_2=\Lambda_2^-(\gamma)\cup\Lambda_2^+(\gamma)$ with $\Lambda_2^-(\gamma)$ (resp. $\Lambda_2^+(\gamma)$) is the set of $\gamma'\in\Lambda_2$ such that $F(\gamma')$ is below (resp. above) $F(\gamma)$. In the same way, $\gamma\in\Lambda_2$ defines a partition $\Lambda_1=\Lambda_1^-(\gamma)\cup\Lambda_1^+(\gamma)$.

In the following, we assume that $\gamma_1\in\Lambda_1^-(\gamma_2)$ ($F(\gamma_1)$ is below $F(\gamma_2)$). If $\gamma_1\in\Lambda_1^+(\gamma_2)$, the proof is the same by exchanging the labels 1 and 2.

We then define:

$$u_1=\max\left\{u(x,y)\,\middle|\,(x,y)\in\bigcup_{\gamma\in\Lambda_1^-(\gamma_2)}\gamma,-\frac{1}{H}\leq x\leq\frac{1}{H}\right\}$$

We consider a point $(x_1,y_1)\in\bigcup_{\gamma\in\Lambda_1^-(\gamma_2)}\gamma$ such that $u(x_1,y_1)=u_1$. Let $g_1\in\Lambda_1^-(\gamma_2)$ denote the boundary component that contains (x_1,y_1). We then define:

$$u_2=\min\left\{u(x_1,y)\,\middle|\,(x_1,y)\in\bigcup_{\gamma\in\Lambda_2^+(g_1)}\gamma\right\}$$

$\gamma_2 \in \Lambda_2^+(g_1)$ and, since γ_2 join $\{-a\} \times [-b, b]$ to $\{a\} \times [-b, b]$, there is a point in γ_2 with x_1 as first coordinate; this proves that u_2 exists. We have $u_2 > u_1$ and:

Fact 1. For all $z \in (u_1, u_2)$, there exists y such that $(x_1, y) \in \Omega$ and $u(x_1, y) = z$.

Let us prove this fact. We consider $c : [-b, b] \to R_{a,b}$ defined by $c(t) = (x_1, t)$. We consider the set J_c with its order. Let $j_0 \in J_c$ be such that $c(e_{j_0})$ or $c(o_{j_0})$ is $(x_1, y_1) \in g_1$. We then define:

$$j_1 = \min\{j \rhd j_0 \,|\, u(c(o_j)) \geq u_2\}$$

Since $c(o_{j_{max}}) \in \gamma_2$, $u(c(o_{j_{max}})) \geq u_2$; then the segment j_1 exists. Besides $j_0 \lhd j_1$. First let us prove that $u(c(e_{j_1}) \leq u_1$. Let γ denote the element of Λ to which $c(e_{j_1})$ belongs. By Lemma 1, there exists $i \in J_c$ with $j_0 \unlhd i \lhd j_1$ such that $c(o_i) \in \gamma$. We have $u(c(o_i)) < u_2$ by definition of j_1. If $\gamma \in \Lambda_1$, $u(c(o_i)) < u_2$ implies that $\gamma \in \Lambda_1^-(\gamma_2)$ and then $u(c(e_{j_1})) \leq u_1$ (definition of u_1). If $\gamma \in \Lambda_2$, $u(c(o_i)) < u_2$ implies that γ belongs to $\Lambda_2^-(g_1)$ (definition of u_2) and $u(c(e_{j_1})) \leq u_1$. Now, since $c([e_{j_1}, o_{j_1}])$ is connected and included in $\overline{\Omega}$, $(u_1, u_2) \subset u \circ c(e_{j_1}, o_{j_1})$ and this proves Fact 1.

Let t be in $\mathbb{R}$ and D_t be the closed disk in $[-a, a] \times \mathbb{R}$ with center $(0, u_1 + t)$ and radius $\frac{1}{H}$. D_0 contains the point (x_1, u_1). The diameter of D_t is $\frac{2}{H}$ which is less than $2a$, hence we have:

$$D_t \cap F(\Gamma_1) \neq \emptyset \Longrightarrow D_t \cap F(\Gamma_2) = \emptyset$$
$$D_t \cap F(\Gamma_2) \neq \emptyset \Longrightarrow D_t \cap F(\Gamma_1) = \emptyset$$

We define:

$$t_0 = \inf \{t > 0 \,|\, D_t \cap F(\Gamma_1) = \emptyset\}$$

By compactness, $D_{t_0} \cap F(\Gamma_1) \neq \emptyset$ and then $D_t \cap F(\Gamma_2) = \emptyset$ for $0 \leq t \leq t_0$.

Fact 2. We have $u_1 + t_0 < u_2$.

Actually, if $u_1 + t_0 \geq u_2$, $t' = u_2 - u_1$ is less than t_0 and $D_{t'}$ contains the point of $F(\Gamma_2)$ that realizes u_2. This implies that $D_{t'} \cap F(\Gamma_1) = \emptyset$ and contradicts the definition of t_0.

By compactness, there exists $t_1 > t_0$ such that $u_1 + t_1 < u_2$, $D_{t_1} \cap F(\Gamma_1) = \emptyset$ and $D_t \cap F(\Gamma_2) = \emptyset$ for all $0 \leq t \leq t_1$.

Fact 3. Let $\gamma \in \Lambda$ be a boundary component, then there are no $Z_1, Z_2 \in \mathbb{R}$ such that there exist $X_1, X_2 \in [-\frac{1}{H}, \frac{1}{H}]$ with $(X_i, Z_i) \in F(\gamma)$ and:

$$Z_1 < u_1 + t_1 - \sqrt{\frac{1}{H^2} - X_1^2} \leq u_1 + t_1 + \sqrt{\frac{1}{H^2} - X_2^2} < Z_2$$

($F(\gamma)$ can not have points above and below the disk D_{t_1})

First we suppose that $\gamma \in \Lambda_1$. Since $Z_2 > u_1$, the definition of u_1 implies that γ belongs to $\Lambda_1^+(\gamma_2)$. Since γ_2 joins $\{-a\} \times [-b, b]$ to $\{a\} \times [-b, b]$, $F(\gamma_2)$ has a point of coordinates (x_1, z); by definition of u_2, $z \geq u_2$. Then the second coordinate of every point of $F(\gamma)$ needs to be larger than u_2, this contradicts $Z_1 < u_1 + t_1$. Now if $\gamma \in \Lambda_2$, D_t does not intersect $F(\Gamma_2)$ for $0 \leq t \leq t_1$, so letting go down the disk D_t from t_1 to 0, we get $Z_1 \leq u_1$ and then γ belongs to $\Lambda_2^-(g_1)$. This implies that the second coordinate of every point of $F(\gamma)$ is smaller than u_1. This contradicts $Z_2 > u_1 + t_1$ and proves Fact 3.

The idea is now to consider a suitable sphere of radius $\frac{1}{H}$ projecting onto the disk D_{t_1}. Let S_v denote the sphere of radius $\frac{1}{H}$ and center $(0, v, u_1 + t_1)$. When v changes, S_v moves in an horizontal cylinder with vertical section D_{t_1}. For far from zero negative v, S_v is out $\overline{\Omega} \times \mathbb{R}$. Since $D_{t_1} \cap F(\Gamma_1) = \emptyset$ and $D_{t_1} \cap F(\Gamma_2) = \emptyset$, S_v does not intersect the boundary of the graph of u for any v. The graph of u splits $\overline{\Omega} \times \mathbb{R}$ into two connected components: $\mathcal{G}^+$, above the graph, and $\mathcal{G}^-$ which is below. Since $u_1 \leq u_1 + t_1 \leq u_2$, there exists v such that S_v intersects the graph of u by Fact 1.

We start with far from zero negative v and let v increase until v_0 which is the first contact between the graph and the sphere. This first contact does not occur in the boundary of the graph since the sphere never intersects it. Then, since the graph is not a piece of a sphere because of the size of Ω, the maximum principle implies that, in the neighborhood of the contact point, the sphere S_{v_0} is in $\mathcal{G}^-$ (we recall that the mean curvature vector of the graph points to $\mathcal{G}^+$ because of the equation (CMC)). But, in fact, we have:

Fact 4. In the neighborhood of the contact point, the sphere S_{v_0} is in $\mathcal{G}^+$.

This fact is not obvious since, because of the shape of the domain Ω, the sphere keeps getting in and out $\overline{\Omega} \times \mathbb{R}$. We consider the first contact point $p = (x, y, z)$; we know that $(x, z) \in D_{t_1}$. We define $c : s \mapsto (x, s) \in R_{a,b}$ and consider J_c. We have $(x, s, z) \in \mathcal{G}^-$ for $s < y$ near y since the sphere S_{v_0} is in $\mathcal{G}^-$ in a neighborhood of p. Then there exists:

$$s_0 = \min\left\{s \geq -b \,|\, (x, s, z) \in \mathcal{G}^-\right\}$$

Since p is the first contact point, there is $j \in J_c$ such that $s_0 = e_j$. We know that D_{t_1} is above $F(\gamma_1)$ then $(x, e_{j_{min}}, z) \in \mathcal{G}^+$, so $j \rhd j_{min}$. Then there exists $j' \lhd j$ such that $c(e_j)$ and $c(o_{j'})$ belong to the same element of Λ. Then, if $(x, e_j, z) \in \mathcal{G}^-$, $(x, o_{j'}, z) \in \mathcal{G}^-$, this is due to Fact 3. This implies that $(x, s, z) \in \mathcal{G}^-$ for $s < o_{j'}$ near $o_{j'}$ and then $s_0 < e_j$; we have our contradiction.

This ends the proof of $d\big(F(\Gamma_1), F(\Gamma_2)\big) \leq 2a$. $\square$

We remark that Theorem 2' will be sufficient for most of the applications and Theorem 2 is just an improvement. So let us replace $2a$ by $\frac{2}{H}$ to get Theorem 2.

Proof of Theorem 2. Let us consider $a' > 0$ with $\frac{1}{H} < a' < a$. The idea of the proof is to apply Theorem 2' to a well chosen connected component of $\Omega \cap R_{a',b}$. Let D^i denote the connected components of $\Omega \cap R_{a',b}$. First we remark that, among these components, there are ones that satisfy the hypotheses 1., 2. and 3. . For example, since γ_1 joins $\{-a\} \times [-b, b]$ to $\{a\} \times [-b, b]$, one connected component of $\gamma_1 \cap R_{a',b}$ joins $\{-a'\} \times [-b, b]$ to $\{a'\} \times [-b, b]$; then a D^i that has this component in its boundary satisfies the three hypotheses. A component of $\Omega \cap R_{a',b}$ that satisfies the hypotheses is called a good component and the other ones are the bad ones; we rename these good components $D^1, \dots, D^k$. There is only a finite number of such components since the length of the part of $\partial\Omega$ in $R_{a',b}$ is finite.

Let us consider a good component D^i. As defined at the beginning of this section, a set of boundary component Λ^i is associated to D^i. In Λ^i there are two particular elements: these are the two boundary components which are homeomorphic to a segment and joins $\{-a'\} \times [-b, b]$ to $\{a'\} \times [-b, b]$. To avoid any confusion, we denote these components by γ_α^i and γ_β^i (γ_α^i is a part of the boundary of the connected component of $R_{a',b} \backslash D^i$ that contains $(0, -b)$ and γ_β^i is the other one). Each element of Λ^i is a part of an element of Λ, then we get a partition $\Lambda^i = \Lambda_1^i \cup \Lambda_2^i$: an element of Λ^i is in Λ_1^i (resp. Λ_2^i) if it is a part of an element of Λ_1 (resp. Λ_2). Now the proof consists in applying Theorem 2' to a component D^i such that $\gamma_\alpha^i \in \Lambda_1^i$ and $\gamma_\beta^i \in \Lambda_2^i$.

To each good component D^j, we can associate a real number which is the second coordinate of the end point of γ_α^j in $\{-a\} \times [-b, b]$. In the following, we order the good components with respect to this real number and rename the good components $D^1, \dots, D^k$ with respect to this order. The order is the same if we consider the second coordinate of $\gamma_\alpha^j \cup \{a\} \times [-b, b]$, $\gamma_\beta^j \cup \{-a\} \times [-b, b]$ or $\gamma_\beta^j \cup \{a\} \times [-b, b]$.

Let D be a bad component of $\Omega \cap R_{a',b}$. In fact, it is a bad component only because of Hypothesis 2.; then, in D, there is no path from $\{-a'\} \times [-b, b]$ to $\{a'\} \times [-b, b]$. This implies that, as in Figure 1, there exists a path $c : [0, 1] \to R_{a',b}$ that joins $(0, -b)$ to $(0, b)$, is outside all the bad components and such that there exist $0 < e_1 < o_1 < e_2 < \cdots < e_k < o_k < 1$ with:

$$c([0, 1]) \cap \Omega = \bigcup_i c(e_i, o_i)$$

and $c(e_i, o_i) \subset D^i$. First we remark that $c(e_1)$ is in γ_1 so it is in $\gamma_\alpha^1 \in \Lambda_1^1$ and $c(o_k)$ is in γ_2 so it is in $\gamma_\beta^k \in \Lambda_2^k$. Then there exists:

$$i_0 = \min \left\{ i \,\middle|\, \gamma_\beta^i \in \Lambda_2^i \right\}$$

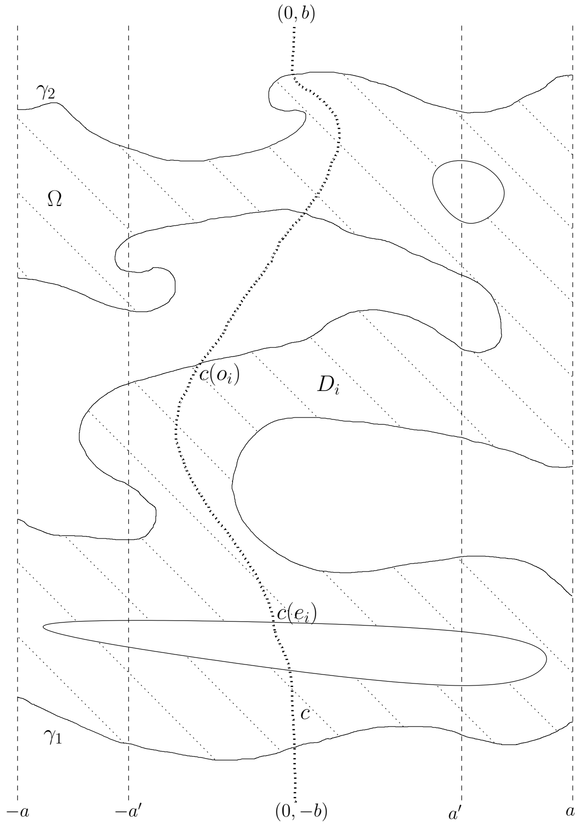

Figure 1:

Let us prove that the good component D^{i_0} is such that $\gamma_\alpha^{i_0} \in \Lambda_1^{i_0}$. First we assume that it is not the case, *i.e.* $\gamma_\alpha^{i_0} \in \Lambda_2^{i_0}$. Since $\gamma_\alpha^1 \in \Lambda_1^1$, $i_0 > 0$. $\gamma_\alpha^{i_0}$ is a part of an element γ of Λ_2 then $c(e_{i_0}) \in \gamma_\alpha^{i_0} \subset \gamma$. Besides we know, by Lemma 1, that $c(o_{i_0-1})$ belongs to the same component as $c(e_{i_0})$. Then $c(o_{i_0-1}) \in \gamma$ and $\gamma_\beta^{i_0-1} \subset \gamma$; this implies that $\gamma_\beta^{i_0-1} \in \Lambda_2^{i_0-1}$. This is a contradiction with the definition of i_0.

Now, we apply Theorem 2' to D^{i_0}. For $j \in \{1, 2\}$, we write $\Gamma'_j = \bigcup_{\gamma \in \Lambda_j^{i_0}} \gamma$. Then we have $\gamma_\alpha^{i_0} \subset \Gamma'_1$ and $\gamma_\beta^{i_0} \subset \Gamma'_2$; so we can apply the theorem and we get:

$$d\big(F(\Gamma_1'), F(\Gamma_2')\big) \leq 2a'$$

Besides, we have $\Gamma_1' \subset \Gamma_1$ and $\Gamma_2' \subset \Gamma_2$ then:

$$d\big(F(\Gamma_1), F(\Gamma_2)\big) \leq d\big(F(\Gamma_1'), F(\Gamma_2')\big) \leq 2a'$$

This inequality is true for every $a' > \frac{1}{H}$, so:

$$d\big(F(\Gamma_1), F(\Gamma_2)\big) \leq \frac{2}{H}$$

□

2 Some consequences of Theorem 2

The aim of this section is to give some consequences of Theorem 2 for solutions of the Dirichlet problem associated to the constant mean curvature equation (CMC) on unbounded domains.

First we explain what kind of domains we shall consider. Let b_- and $b_+ : \mathbb{R}_+ \to \mathbb{R}$ be two continuous functions such that, for every $x \geq 0$, $b_-(x) < b_+(x)$. We are interested in domains of the type $\Omega = \{(x,y) \in \mathbb{R}_+ \times \mathbb{R} \,|\, b_-(x) < y < b_+(x)\}$. When u is a solution of (CMC) on Ω and continuous on $\overline{\Omega}$, we define two continuous functions f_- and f_+ by $f_\pm(x) = u(x, b_\pm(x))$. f_- and f_+ are the boundary values of u.

Finally, if $x \in \mathbb{R}_+$, we define $I_x = \{x\} \times [b_-(x), b_+(x)]$. We then have the following height estimate:

Proposition 3. *Let b_- and b_+ be continuous functions on $\mathbb{R}_+$ with $b_-(x) < b_+(x)$. We define $\Omega = \{(x,y) \in \mathbb{R}_+ \times \mathbb{R} \,|\, b_-(x) < y < b_+(x)\}$. Let u be a solution of* (CMC) *on Ω and continuous on $\overline{\Omega}$. We consider $x_0 > \frac{2}{H}$ and M such that:*

$$\min_{[x_0 - \frac{2}{H}, x_0 + \frac{2}{H}]} (f_-, f_+) \geq M \qquad (1)$$

Then:

$$\min_{I_{x_0}} u \geq M - \frac{3}{H}$$

Proof. Let ε be a positive number. Since f_- and f_+ are continuous, there exists $\eta > 0$ such that:

$$\min_{[x_0 - \frac{2}{H} - \eta, x_0 + \frac{2}{H} + \eta]} (f_-, f_+) \geq M - \varepsilon$$

Let us assume that there is a piecewise smooth injective path $c : [0, 1] \to \Omega \bigcap [x_0 - \frac{2}{H} - \eta, x_0] \times \mathbb{R}$ that joins $I_{x_0 - \frac{2}{H} - \eta}$ to I_{x_0} such that:

$$u \circ c(t) < M - \varepsilon - \frac{2}{H} \tag{2}$$

We consider the domain D bounded by c, the curve $y = b_-(x)$ for $x \in [x_0 - \frac{2}{H} - \eta, x_0]$, a segment included in $I_{x_0 - \frac{2}{H} - \eta}$ and one included in I_{x_0}; D satisfies the three hypotheses of Section 1. In fact, since the function b_- is only continuous, the boundary of D is not piecewise smooth, but Theorem 2 can be applied because of the shape of D. Because of (1) and (2), Theorem 2 is not satisfied; then the above assumption can not be realized. Finally, this implies that there exists a path $c_1 : [0, 1] \to \Omega \bigcap [x_0 - \frac{2}{H} - \eta, x_0] \times \mathbb{R}$ that joins the curve $y = b_-(x)$ to the curve $y = b_+(x)$ such that $u \circ c_1(t) \geq M - \varepsilon - \frac{2}{H}$.

By the same arguments, there exists a path $c_2 : [0, 1] \to \Omega \bigcap [x_0, x_0 + \frac{2}{H} + \eta] \times \mathbb{R}$ that joins the curve $y = b_-(x)$ to the curve $y = b_+(x)$ such that $u \circ c_2(t) \geq M - \varepsilon - \frac{2}{H}$.

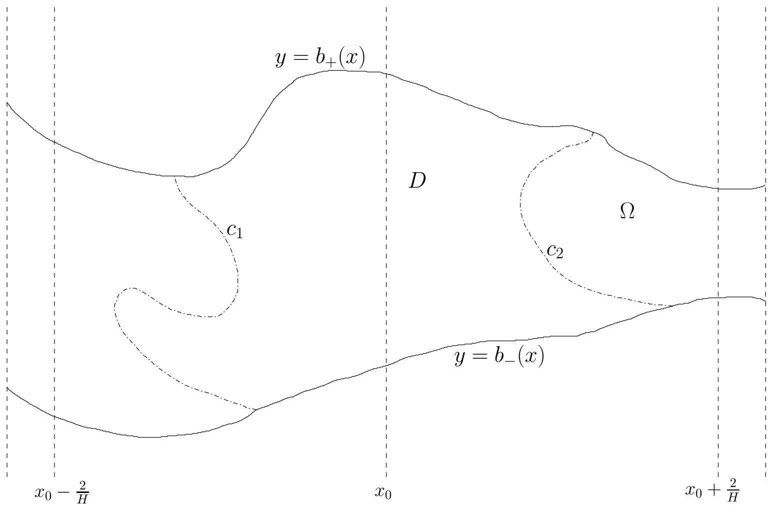

Figure 2:

Now the domain D bounded by c_1, c_2, a piece of $y = b_-(x)$ and a piece of $y = b_+(x)$ contains I_{x_0} (see Figure 2). Besides on the boundary of D, u is everywhere larger than $M - \varepsilon - \frac{2}{H}$ by (1) and above. Then by a classical height estimate [Se2], u is larger than $M - \varepsilon - \frac{3}{H}$ in D. This gives us:

$$\min_{I_{x_0}} u \geq M - \varepsilon - \frac{3}{H} \tag{3}$$

Letting ε tend to zero, we get the expected result. $\square$

We also have a simple upper-bound in this case.

Proposition 4. *Let b_- and b_+ be continuous functions on $\mathbb{R}_+$ with $b_-(x) < b_+(x)$. We define $\Omega = \{(x,y) \in \mathbb{R}_+ \times \mathbb{R} \,|\, b_-(x) < y < b_+(x)\}$. Let u be a solution of* (CMC) *on Ω and continuous on $\overline{\Omega}$. We consider $x_0 > \frac{1}{2H}$ and M such that:*

$$\max_{[x_0 - \frac{1}{2H}, x_0 + \frac{1}{2H}]} (f_-, f_+) \leq M$$

Then:

$$\max_{I_{x_0}} u \leq M$$

Proof. Let ε be a positive number. Since f_- and f_+ are continuous, there exist $\eta > 0$ such that:

$$\max_{[x_0 - \frac{1}{2H} - \eta, x_0 + \frac{1}{2H} + \eta]} (f_-, f_+) \leq M + \varepsilon$$

Let us consider the cylinder of radius $\frac{1}{2H}$ which is centered on the axis $\{x = x_0\} \cap \{z = t\}$. For big t, the cylinder is above the graph and we can let t decrease. Until $t = M + \frac{1}{2H} + \varepsilon$, the cylinders can not touch the boundary of the graph of u. The maximum principle then says us that, for $t = M + \frac{1}{2H} + \varepsilon$, the cylinder is still above the graph; so we get:

$$\max_{I_{x_0}} u \leq M + \varepsilon$$

Letting ε goes to zero, we get the expected result. $\square$

Let $f : I \to \mathbb{R}$ be a function, we define the variation of f around the point x_0 by:

$$V_t(x_0, f) = \sup_{[x_0 - t, x_0 + t]} f - \inf_{[x_0 - t, x_0 + t]} f$$

Let f and g be two continuous functions $I \to \mathbb{R}$; we define the variation of the pair (f, g) around x_0 by:

$$V_t(x_0, f, g) = \max\left(V_t(x_0, f), V_t(x_0, g)\right)$$

The two preceding propositions give us the following result:

Theorem 5. *Let b_- and b_+ be continuous functions on $\mathbb{R}_+$ with $b_-(x) < b_+(x)$. We define $\Omega = \{(x,y) \in \mathbb{R}_+ \times \mathbb{R} \,|\, b_-(x) < y < b_+(x)\}$. Let u a solution of* (CMC) *on Ω and continuous on $\overline{\Omega}$. We consider $x_0 > \frac{2}{H}$ and M such that:*

$$V_{\frac{2}{H}}(x_0, f_-, f_+) \leq M$$

Then there exists M' which depends only on M and H such that:

$$\max_{I_{x_0}} u - \min_{I_{x_0}} u \leq M'$$

For example, $M' = 4M + \frac{5}{H}$ works.

Proof. We have for $x \in [x_0 - \frac{2}{H}, x_0 + \frac{2}{H}]$, $|f_-(x) - f_-(x_0)| \leq M$ and $|f_+(x) - f_+(x_0)| \leq M$. Then if we apply Theorem 2 to $\Omega \bigcap [x_0 - \frac{2}{H}, x_0 + \frac{2}{H}] \times \mathbb{R}$, we get that, over this segment, the graph of f_- is at a distance less than $\frac{2}{H}$ from the one of f_+. Since, for $\alpha \in \{-,+\}$, the graph of f_α is in the horizontal strip $f_\alpha(x_0) - M \leq z \leq f_\alpha(x_0) + M$, we have $|f_-(x_0) - f_+(x_0)| \leq 2(M + \frac{1}{H})$. This implies that, for every $x, x' \in [x_0 - \frac{2}{H}, x_0 + \frac{2}{H}]$, $|f_\alpha(x) - f_\beta(x')| \leq 4M + \frac{2}{H}$ with $\alpha, \beta \in \{-,+\}$. Then there exists $A \in \mathbb{R}$ such that, for every $x \in [x_0 - \frac{2}{H}, x_0 + \frac{2}{H}]$, we have:

$$|f_-(x) - A| \leq 2M + \frac{1}{H} \tag{4}$$

$$|f_+(x) - A| \leq 2M + \frac{1}{H} \tag{5}$$

These two equations with Proposition 3 implies that:

$$\min_{I_{x_0}} u \geq A - 2M - \frac{4}{H} \tag{6}$$

With Proposition 4, we get:

$$\max_{I_{x_0}} u \leq A + 2M + \frac{1}{H} \tag{7}$$

Then, in bringing together (6) and (7), we obtain:

$$\max_{I_{x_0}} u - \min_{I_{x_0}} u \leq 4M + \frac{5}{H}$$

□

Theorem 5 has an easy corollary.

Corollary 6. *Let b_- and b_+ be continuous functions on $\mathbb{R}_+$ with $b_-(x) < b_+(x)$. We define $\Omega = \{(x,y) \in \mathbb{R}_+ \times \mathbb{R} \,|\, b_-(x) < y < b_+(x)\}$. Let u be a solution of* (CMC) *on Ω and continuous on $\overline{\Omega}$. We consider $x_0 > \frac{2}{H}$ and M such that:*

$$V_{\frac{2}{H}}(x_0, f_-, f_+) \leq M$$

Then there exists M' which depends only on M and H such that, for every $p \in I_{x_0}$ and $\alpha \in \{-,+\}$, we have:

$$f_\alpha(x_0) - M' \leq u(p) \leq f_\alpha(x_0) + M'$$

For example, $M' = 4M + \frac{5}{H}$ works.

Proof. It is just the fact that $(x_0, b_-(x_0))$ and $(x_0, b_+(x_0))$ are in I_{x_0}. □

3 Two uniqueness results

In this section, we use Corollary 6 to prove uniqueness theorems for the Dirichlet problem associated to (CMC).

Theorem 7. *Let b_-, b_+ be two continuous functions on $\mathbb{R}_+$ such that $b_-(0) = b_+(0)$ and $b_-(x) < b_+(x)$ for every $x > 0$. We define $\Omega = \{(x,y) \in \mathbb{R}_+ \times \mathbb{R} \,|\, b_-(x) < y < b_+(x)\}$. Let f_-, f_+ be two continuous functions on $\mathbb{R}_+$ such that $f_-(0) = f_+(0)$. We suppose that there exist an increasing positive sequence $(x_n)_{n\in\mathbb{N}}$ with $\lim x_n = +\infty$ and a sequence $(M_n)_{n\in\mathbb{N}}$ with $M_n = o(\ln x_n)$ such that, for every $n \in \mathbb{N}$, we have:*

$$V_{\frac{2}{H}}(x_n, f_-, f_+) \leq M_n$$

Then, if there exists a solution u of (CMC) *on Ω with value f_- and f_+ on the boundary, this solution is unique.*

Proof. Let u_1 and u_2 be two different solutions of the Dirichlet problem with f_- and f_+ as boundary data. A result of P. Collin and R. Krust [CK] says that:

$$\liminf_{x\to+\infty} \frac{\max_{I_x} |u_1 - u_2|}{\ln x} > 0$$

But by Corollary 6, we know that:

$$\max_{I_{x_n}} |u_1 - f_-(x_n)| \leq 4M_n + \frac{5}{H}$$

$$\max_{I_{x_n}} |u_2 - f_-(x_n)| \leq 4M_n + \frac{5}{H}$$

So, we get:

$$\max_{I_{x_n}} |u_1 - u_2| \leq 8M_n + \frac{10}{H}$$

By the hypothesis on M_n, we have:

$$\lim_{n\to\infty} \frac{\max_{I_{x_n}} |u_1 - u_2|}{\ln x_n} = 0$$

This gives us a contradiction since $x_n \to +\infty$. ☐

We also have a second theorem.

Theorem 8. *Let b_-, b_+ be two continuous functions on $\mathbb{R}$ such that $b_-(x) < b_+(x)$ for every $x \in \mathbb{R}$. We define $\Omega = \{(x,y) \in \mathbb{R}^2 \,|\, b_-(x) < y < b_+(x)\}$. Let f_-, f_+ be two continuous functions on $\mathbb{R}$. We suppose that there exist one increasing sequence $(x_n)_{n\in\mathbb{N}}$ with $\lim x_n = +\infty$ and one decreasing sequence $(x'_n)_{n\in\mathbb{N}}$ with $\lim x'_n = -\infty$ and two sequences $(M_n)_{n\in\mathbb{N}}$ and $(M'_n)_{n\in\mathbb{N}}$ such that $M_n = o(\ln|x_n|)$, $M'_n = o(\ln|x'_n|)$ and, for every $n \in \mathbb{N}$, we have:*

$$V_{\frac{2}{H}}(x_n, f_-, f_+) \leq M_n$$
$$V_{\frac{2}{H}}(x'_n, f_-, f_+) \leq M'_n$$

Then, if there exists a solution u of (CMC) *on Ω with value f_- and f_+ on the boundary, this solution is unique.*

Proof. Let u_1 and u_2 be two different solutions of the Dirichlet problem with f_- and f_+ as boundary data. We know ([CK]) that:

$$\max_{I_x \cup I_{-x}} |u_1 - u_2| \xrightarrow[x\to+\infty]{} +\infty \tag{8}$$

We define $C = \max_{I_0} |u_1 - u_2|$. Then we have $u_2 - C - 1 < u_1 < u_2 + C + 1$ on I_0 and because of (8) the set $\{|u_1 - u_2| > M + 1\}$ is non-empty. Then we can assume that there exists a subdomain $\Omega^* \subset \Omega \cap \mathbb{R}^- \times \mathbb{R}$ which is a connected component of $\{u_1 > u_2 + M + 1\}$. By Theorem 2 in [CK], we have:

$$\liminf_{x\to-\infty} \frac{\max_{I_x\cap\Omega^*} |u_1 - u_2 - C - 1|}{\ln|x|} > 0$$

As in the preceding proof, Corollary 6 says us that, for every n, we have:

$$\max_{I_{x'_n}} |u_1 - u_2 - C - 1| \leq 8M'_n + \frac{10}{H} + C + 1$$

By the hypothesis on M'_n, we have:

$$\lim_{n\to\infty} \frac{\max_{I_{x'_n}} |u_1 - u_2 - C - 1|}{\ln |x'_n|} = 0$$

This gives us a contradiction since $x'_n \to -\infty$ and ends the proof. □

This theorem can be used to study the uniqueness of the solutions which were constructed by P. Collin in [Co] and by R. Lopéz in [Lo1]. For a complete proof of uniqueness in these examples, we refer to [Ma].

There are others results of uniqueness which we can prove with the same arguments. For example, if we assume that $b_+ - b_-$ is bounded in Theorems 7 and 8, we need only to assume that $M_n = o(x_n)$ and $M'_n = o(|x'_n|)$ to get the uniqueness.

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