Optimal length estimates for stable CMC surfaces in 3-space-forms

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Abstract

In this paper, we study stable constant mean curvature H surfaces in $\mathbb{R}^3$. We prove that, in such a surface, the distance from a point to the boundary is less that $\pi/(2H)$. This upper-bound is optimal and is extended to stable constant mean curvature surfaces in space forms.

1 Introduction

A constant mean curvature (cmc) surface Σ in a Riemannian 3-manifold $\mathbb{M}^3$ is stable, if its stability operator $L = -\Delta - \mathrm{Ric}(n,n) - |A|^2$ is nonnegative. The nonnegativity of this operator means that Σ is a local minimizer of the area functional on surfaces regard to the infinitesimal deformations fixing its boundary.

The stability hypothesis was studied by several authors and has many consequences (see [6] for an overview). For example, D. Fischer-Colbrie and R. Schoen [4] studied the case of complete stable minimal surfaces when $\mathbb{M}^3$ has non-negative scalar curvature. They obtain that the universal cover of Σ is not conformally equivalent to the disk and, as a consequence, prove that the plane is the only complete stable minimal surface in $\mathbb{R}^3$. From this, R. Schoen [8] has derived a curvature estimate for stable cmc surfaces.

In [2], T. H. Colding and W. P. Minicozzi introduced new technics and obtained area and curvature estimates for stable cmc surfaces. Afterward, these technics were used by P. Castillon [1] to answer a question asked in [4] about the consequences of the positivity of certain elliptic operators. Recently, the same ideas have been used by J. Espinar and H. Rosenberg [3] to obtain similar results.

In [7], A. Ros and H. Rosenberg study constant mean curvature H surfaces in $\mathbb{R}^3$ with $H \neq 0$: they prove a maximum principle at infinity. One of their tools is a length estimate for stable cmc surface. In fact, they prove that the intrinsic distance from a point p in a stable cmc surface Σ to the boundary of Σ is less than π/H. The aim of this paper is to improve this result. In fact, applying the ideas of [2], we prove that the distance is less than $\pi/(2H)$. This estimate is optimal since, for a hemisphere of radius $1/H$, the distance from the pole to the boundary is $\pi/(2H)$. Actually we prove that the hemisphere of radius $1/H$ is the only stable cmc H surface where the distance $\pi/(2H)$ is reached. We can generalized this result to stable cmc H surfaces in $\mathbb{M}^3(\kappa)$, where $\mathbb{M}^3(\kappa)$ is the 3-space form of sectional curvature κ. We prove that when $H^2+\kappa>0$ such an optimal estimate exists. In fact, it is already known that, when $\kappa\le 0$ and $H^2+\kappa\le 0$, there is no such estimate since there exist complete stable cmc H surfaces. But, in some sense, our results is an extension of the fact that the planes (resp. the horospheres) are the only stable complete constant mean curvature H surfaces in $\mathbb{R}^3$ (resp. $M^3(\kappa)$, $\kappa<0$) when $H=0$ (resp. $H^2+\kappa=0$).

2 Definitions

On a constant mean curvature surface Σ in a Riemannian 3-manifold $\mathbb{M}^3$, the stability operator is defined by $L=-\Delta-\mathrm{Ric}(n,n)-|A|^2$, where Δ is the Laplace operator on Σ, Ric is the Ricci tensor on $\mathbb{M}^3$, n is the normal to Σ and A is the second fundamental form on Σ. When it is necessary, we will denote the stability operator by L_f to refer to the immersion f of Σ in $\mathbb{M}^3$.

The surface Σ is called *stable* if the operator L is nonnegative *i.e.*, for every compactly supported function u, we have

$$0\le\int_\Sigma uL(u)\mathrm{d}\sigma=\int_\Sigma\|\nabla u\|^2-(\mathrm{Ric}(n,n)+|A|^2)u^2\mathrm{d}\sigma$$

We remark that this property is sometimes called strong stablility since it means that the second derivatives of the area functional is nonnegative with respect to any compactly supported infinitesimal deformations u whereas Σ is critical for this functional only for compactly supported infinitesimal deformations with vanishing mean value *i.e.* $\int_\Sigma u\mathrm{d}\sigma=0$.

In the following, on a cmc surface, the normal n is always chosen such that H is non-negative.

We will denote by d_Σ the intrinsic distance on Σ and by K the sectional curvature of the surface.

3 Results

The main result of this paper is the following theorem.

Theorem 1. *Let H be positive. Let Σ be a stable constant mean curvature H surface in $\mathbb{R}^3$. Then, for $p \in \Sigma$, we have :*

$$d_\Sigma(p, \partial\Sigma) \leq \frac{\pi}{2H} \tag{1}$$

Moreover, if the equality is satisfied, Σ is a hemisphere.

In $\mathbb{R}^3$, the stability operator can be written $L = -\Delta - 4H^2 + 2K$.

Proof. We denote by R_0 the distance $d_\Sigma(p, \partial\Sigma)$ and assume that $R_0 \geq \pi/(2H)$. If $R_0 < \pi/H$ we denote by I the segment $[\pi/(2H), R_0]$, otherwise $I = [\pi/(2H), \pi/H)$. In fact, because of the work of Ros and Rosenberg [7], we already know that $R_0 \leq \pi/H$. Let R be in I.

The surface Σ has constant mean curvature H thus its sectional curvature is less than H^2. So the exponential map $\exp_p$ is a local diffeomorphism on the disk $D(0, R) \subset T_p\Sigma$ of center 0 and radius R. On this disk, we consider the induced metric and the operator $\mathcal{L} = -\Delta - 4H^2 + 2K$. The surface Σ is stable so it exists a positive function g on Σ such that $L(g) = 0$ (see Theorem 1 in [4]). On $D(0, R)$, the function $\tilde{g} = g \circ \exp_p$ is then positive and satisfies $\mathcal{L}(\tilde{g}) = 0$ since $D(0, R)$ and Σ are locally isometric. The operator $\mathcal{L}$ is thus nonnegative on $D(0, R)$ [4].

For $r \in [0, R]$, we define $l(r)$ as the length of the circle $\{v,\ |v| = r\} \subset D(0, R)$ and $\mathcal{K}(r) = \int_{D(0,r)} K \mathrm{d}\sigma$. Since $D(0, R)$ and Σ are locally isometric, the sectional curvature K of $D(0, R)$ is less than H^2. Then

$$l(r) \geq \frac{2\pi}{H} \sin Hr \tag{2}$$

By Gauss-Bonnet, we have:

$$\mathcal{K}(r) = 2\pi - l'(r) \tag{3}$$

Let us consider a function $\eta : [0, R] \to [0, 1]$ with $\eta(0) = 1$ and $\eta(R) = 0$. Let us write the nonnegativity of $\mathcal{L}$ for the radial function $u = \eta(r)$.

$$0 \leq \int_0^R (\eta'(r))^2 l(r) \mathrm{d}r - 4H^2 \int_0^R \eta^2(r) l(r) \mathrm{d}r + 2 \int_0^R \mathcal{K}'(r) \eta^2(r) \mathrm{d}r$$

Hence, following the ideas in [2] and using (3) and the boundary values of η, we have:

$$
\begin{aligned}
\int_0^R (4H^2\eta^2 - \eta'^2) l \mathrm{d}r &\leq 2\left(\left[\mathcal{K}(r)\eta^2(r)\right]_0^R - \int_0^R \mathcal{K}(r)(\eta^2(r))'\mathrm{d}r \right) \\
&\leq -2\int_0^R \mathcal{K}(r)(\eta^2(r))'\mathrm{d}r \\
&\leq -2\int_0^R (2\pi - l'(r))(\eta^2(r))'\mathrm{d}r \\
&\leq 4\pi + 2\int_0^R (\eta^2(r))' l'(r)\mathrm{d}r \\
&\leq 4\pi + \left[2(\eta^2(r))' l(r)\right]_0^R - 2\int_0^R (\eta^2(r))'' l(r)\mathrm{d}r \\
&\leq 4\pi - 2\int_0^R (\eta^2(r))'' l(r)\mathrm{d}r
\end{aligned}
$$

Thus we obtain

$$
\int_0^R \left(4H^2\eta^2 - \eta'^2 + 2(\eta^2)''\right) l \mathrm{d}r \leq 4\pi \tag{4}
$$

We shall apply this equation to the function $\eta(r) = \cos\dfrac{\pi r}{2R}$. In this case we have

$$
\begin{aligned}
\eta'^2 &= \frac{\pi^2}{4R^2}\sin^2\frac{\pi r}{2R} \\
(\eta^2)'' &= -\frac{\pi^2}{2R^2}\left(\cos^2\frac{\pi r}{2R} - \sin^2\frac{\pi r}{2R}\right)
\end{aligned}
$$

Thus

$$
4H^2\eta^2 - \eta'^2 + 2(\eta^2)'' = (4H^2 - \frac{\pi^2}{R^2})\cos^2\frac{\pi r}{2R} + \frac{3\pi^2}{4R^2}\sin^2\frac{\pi r}{2R}
$$

As $R \geq \frac{\pi}{2H}$, $4H^2\eta^2 - \eta'^2 + 2(\eta^2)''$ is non-negative and, by (2),

$$
\begin{aligned}
\left(4H^2\eta^2 - \eta'^2 + 2(\eta^2)''\right) l &\geq \left((4H^2 - \frac{\pi^2}{R^2})\cos^2\frac{\pi r}{2R} + \frac{3\pi^2}{4R^2}\sin^2\frac{\pi r}{2R}\right)\frac{2\pi}{H}\sin Hr \\
&\geq \frac{\pi}{H}\left((4H^2 - \frac{\pi^2}{4R^2})\sin Hr + (4H^2 - \frac{7\pi^2}{4R^2})\frac{1}{2}\left(\sin(\frac{\pi}{R} + H)r - \sin(\frac{\pi}{R} - H)r\right)\right)
\end{aligned}
$$

Thus integrating in (4), we obtain (we recall that $R < \pi/H$)

$$4\pi \geq \frac{\pi}{H}\left((4H^2 - \frac{\pi^2}{4R^2})\frac{1}{H}(1-\cos HR) \right.$$
$$\left. +(4H^2 - \frac{7\pi^2}{4R^2})\frac{1}{2}\left(\frac{R}{\pi+HR}(1-\cos(\pi+HR)) - \frac{R}{\pi-HR}(1-\cos(\pi-HR))\right)\right)$$

After some simplifications in the above expression, we obtain

$$4\pi \geq \pi\frac{(-32H^2R^4 + 24\pi^2H^2R^2 - \pi^4) - (10\pi^2H^2R^2 - \pi^4)\cos HR}{4H^2R^2(\pi^2 - H^2R^2)}$$

Now, passing 4π on the right-hand side of the above inequality and simplifying by π, we obtain:

$$0 \geq \frac{-(4H^2R^2 - \pi^2)^2 - (10\pi^2H^2R^2 - \pi^4)\cos HR}{4H^2R^2(\pi^2 - H^2R^2)}$$

We denote by $F(R)$ the right-hand term of the above inequality. Hence we have proved that, for every R in I, $F(R) \leq 0$. If we write $R = \pi/(2H) + x$, we compute the Taylor expansion of F and obtain

$$F(\frac{\pi}{2H} + x) = 2Hx + o(x)$$

which is positive if $x > 0$. Thus, if $R_0 > \pi/(2H)$, we get a contradiction and the inequality (1) is proved.

Now if $R_0 = \pi/(2H)$, we have in fact equality all along the computation, so $l(r) = 2\pi/H \sin Hr$ and $\mathcal{K}(r) = 2\pi - l'(r) = 2\pi(1-\cos Hr)$. But we also know that the sectional curvature is less than H^2 thus $\mathcal{K}(r) \leq H^2\int_0^r l(u)\mathrm{d}u = 2\pi(1-\cos Hr)$. Since this inequality is in fact an equality, the sectional curvature is in fact H^2 at every point. Thus the principal curvatures of a point in Σ are H and H *i.e.* there are only umbilical points. Hence Σ is a piece of a sphere of radius $1/H$ and, since $d_\Sigma(p, \partial\Sigma) = \frac{\pi}{2H}$, it contains the hemisphere of pole p. A hemisphere can not be strictly contained in a stable subdomain of the sphere, so Σ is a hemisphere. $\square$

With this result we have an important corollary.

Corollary 2. *Let $H \geq 0$ and $\kappa \in \mathbb{R}$ such that $H^2 + \kappa > 0$. Let Σ be a stable contant mean curvature H surface in $\mathbb{M}^3(\kappa)$. Then for $p \in \Sigma$, we have :*

$$d_\Sigma(p, \partial\Sigma) \leq \frac{\pi}{2\sqrt{H^2+\kappa}}$$

Moreover, if the equality is satisfied, Σ is a geodesical hemisphere of $\mathbb{M}^3(\kappa)$.

The proof is based on the Lawson's correspondence between constant mean curvature surfaces in space forms (see [5]).

Proof. First, the case $\kappa = 0$ is Theorem 1.

Let $\Pi : \widetilde{\Sigma} \to \Sigma$ be the universal cover of Σ. We then have a constant mean curvature immersion of $\widetilde{\Sigma}$ in $\mathbb{M}^3(\kappa)$, let $\mathcal{L} = -\Delta - 2\kappa - |A|^2$ be the stability operator on $\widetilde{\Sigma}$. Σ is stable, so there exists a positive function g on Σ such that $L(g) = -\Delta g - (2\kappa + |A|^2)g = 0$. Thus the function $\tilde{g} = g \circ \Pi$ is a positive function on $\widetilde{\Sigma}$ satisfying $\mathcal{L}(\tilde{g}) = 0$. Hence $\widetilde{\Sigma}$ is stable. Let I and S be respectively the first fundamental form and the shape operator on $\widetilde{\Sigma}$. They satisfy the Gauss and Codazzi equations for $\mathbb{M}^3(\kappa)$.

We define $S' = S + (-H + \sqrt{H^2+\kappa})\mathrm{id}$ on $\widetilde{\Sigma}$. Then I and S' satisfy the Gauss and Codazzi equations for $\mathbb{M}^3(0) = \mathbb{R}^3$ (see [5]). Hence there exists an immersion f of $\widetilde{\Sigma}$ in $\mathbb{R}^3$ with first fundamental form I and shape operator S' (we notice that the induced metric is the same). Its mean curvature is then $H + (-H + \sqrt{H^2+\kappa}) = \sqrt{H^2+\kappa}$ *i.e.* the immersion has constant mean curvature. The stability operator is

$$\begin{aligned}
L_f &= -\Delta - \|S'\|^2 \\
&= -\Delta - (\|S\|^2 + 4H(-H + \sqrt{H^2+\kappa}) + 2(-H + \sqrt{H^2+\kappa})^2) \\
&= -\Delta - (\|S\|^2 + 2\kappa) \\
&= \mathcal{L}
\end{aligned}$$

Hence the surface $f(\widetilde{\Sigma})$ is stable. So, from Theorem 1, we have

$$d_\Sigma(p, \partial\Sigma) = d_{\widetilde{\Sigma}}(\tilde{p}, \partial\widetilde{\Sigma}) \le \frac{\pi}{2\sqrt{H^2+\kappa}}$$

where $\Pi(\tilde{p}) = p$.

The equality case comes from the equality case in Theorem 1 and since the Lawson's correspondence sends spheres into spheres. $\square$

References

[1] Philippe Castillon. An inverse spectral problem on surfaces. *Comment. Math. Helv.*, 81:271–286, 2006.

[2] Tobias H. Colding and William P. Minicozzi, II. Estimates for parametric elliptic integrands. *Int. Math. Res. Not.*, pages 291–297, 2002.

[3] Jose M. Espinar and Harold Rosenberg. A Colding-Minicozzi stability inequality and its applications. preprint.

[4] Doris Fischer-Colbrie and Richard Schoen. The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature. *Comm. Pure Appl. Math.*, 33:199–211, 1980.

[5] H. Blaine Lawson, Jr. Complete minimal surfaces in $\mathbb{S}^3$. *Ann. of Math. (2)*, 92:335–374, 1970.

[6] William H. Meeks, III, Joaquín Pérez, and Antonio Ros. Stable constant mean curvature surfaces. preprint.

[7] Antonio Ros and Harold Rosenberg. Properly embedded surfaces with constant mean curvature. preprint.

[8] Richard Schoen. Estimates for stable minimal surfaces in three-dimensional manifolds. In *Seminar on minimal submanifolds*, volume 103 of *Ann. of Math. Stud.*, pages 111–126. Princeton Univ. Press, 1983.