The Plateau problem at infinity for horizontal ends and genus 1

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Abstract

In this paper, we study Alexandrov-embedded r-noids with genus 1 and horizontal ends. Such minimal surfaces are of two types and we build several examples of the first one. We prove that if a polygon bounds an immersed polygonal disk, it is the flux polygon of an r-noid with genus 1 of the first type. We also study the case of polygons which are invariant under a rotation. The construction of these surfaces is based on the resolution of the Dirichlet problem for the minimal surface equation on an unbounded domain.

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Introduction

The classical Plateau problem consists in finding a surface of least area bounded by a given closed curve in $\mathbb{R}^3$; it has been solved by T. Radó in 1930. A generalization of this problem is: to find a minimal surface, *i.e.* with mean curvature zero, with a given asymptotic behaviour. We first give sense to this question.

If a complete minimal surface M has finite total curvature and r embedded ends (such a surface is called an r-noid), each end of M is known to be asymptotic either to a plane or to a half-catenoid. For each end, its asymptotic behaviour is characterized by the flux vector of the end. These vectors satisfy: the sum of the flux vectors over all ends is zero. So the generalization of the Plateau problem is: given a finite number of vectors such that their sum is zero, can we find an r-noid which has these vectors as flux vectors? This problem is called the Plateau problem at infinity. Besides, we know that M is conformally equivalent to a compact Riemann surface $\overline{M}$

minus r points, the punctures of M, and what we call the genus of M is in fact the genus of $\overline{M}$.

The first examples of r-noids are due to L.P. Jorge and W.H. Meeks [JM]. These minimal surfaces are of genus 0 and mirror-symmetric with respect to an horizontal plane. Besides, these r-noids are dihedrally symmetric.

In [CR], C. Cosín and A. Ros give a description of the space of solutions of the Plateau problem at infinity with an asymptotic behaviour which is symmetric with respect to an horizontal plane (*i.e.* all the flux vectors are horizontal) for genus 0 (the Riemann surface $\overline{M}$ is the Riemann sphere $\mathbb{S}^2$). In the genus 0 case, there is a natural order on the ends. Then, since the flux vectors are horizontal and their sum is zero, the flux vectors draw a polygon in $\mathbb{R}^2$; this polygon is called the flux polygon of M. For the Jorge-Meeks r-noids, the flux polygons are regular. C. Cosín and A. Ros give a necessary and sufficient condition on this polygon for having a solution to the Plateau problem at infinity. Their proof is based on the following result: they determine the kernel of the Jacobi operator on a genus 0 r-noid. In our work, we study the case where $\overline{M}$ is of genus 1 (*i.e.* $\overline{M}$ is a torus). Now, when the topology is more complicated than a sphere minus points, the kernel of the Jacobi operator is unknown. So we shall use other technics in our study. These technics have been already used in [Ma1] to give a new proof of Cosín and Ros result.

For the genus 1 case, we need to distinguish two types of r-noids with horizontal ends; this classification depends on the place of the punctures on the torus: when M is of first type, there is a natural order on the punctures and for the second type, there is not (see Definition 6). If M is an r-noid of genus 1 and horizontal ends of first type, we define the flux polygon associated to M as for genus 0 r-noids. Then our main result can be stated as follow (see Theorem 1 and Figure 1).

Let M be an r-noid with genus 0 and horizontal ends, then there exists Σ an r-noid with genus 1 and horizontal ends of the first type which have the same flux polygon as M.

We obtain an other important existence result: it is Theorem 6 and its corollary. Corollary 3 gives us examples of polygons which are the flux polygons of r-noids with genus 1 but not the flux polygons of genus 0 r-noids.

The proofs of these existence results have two major steps. The first one is: to build some candidate minimal surfaces by solving a Dirichlet problem; the second one is: to solve a period problem.

We only consider r-noids M which are symmetric with respect to the horizontal plane $\{z = 0\}$. Then to build them, it is sufficient to build the part M^+ of M included in $\mathbb{R}^2 \times \mathbb{R}_+$. In fact, the idea is that M^+ is the conjugate of a minimal surface that is the graph of a function u over a planar "domain" which depends on the flux polygon of M. This "domain" is in fact a multi-domain with logarithmic singularity (see definitions in Section 1).

If Ω is a domain in $\mathbb{R}^2$ and u is a function on Ω, the graph of u is a minimal surface in $\mathbb{R}^3$ iff u satisfies the elliptic partial differential equation called the minimal surface equation:

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \tag{MSE}$$

Like every partial differential equation, we can associate to (MSE) the Dirichlet problem that consists in finding a function u on Ω which is a solution of the minimal surface equation and takes on assignated values on the boundary of Ω.

The first step to build a r-noid with genus 1 with V as flux polygon is then to consider a multi-domain Ω associated to V and to solve a Dirichlet problem on Ω. The boundary data is chosen such that the graph of the solution can be the conjugate of M^+. As an example if V is a triangle ABC, we can glue along the three edges of V three half-strips ($[A, B] \times \mathbb{R}_+$ along $[A, B]$, $[B, C] \times \mathbb{R}_+$ along $[B, C]$...), we get an unbounded domain D. Then the multi-domain Ω is the universal cover of $D\backslash\{Q\}$ where Q is a point in the triangle ABC. The boundary value on Ω is $\pm\infty$ alternating the sign such that for every half-strip in Ω one side has $+\infty$ and the other has $-\infty$. The Dirichlet problem is studied on unbounded domains which are similar to the ones that appear in [Ma1] and [RSE].

If Ω is a multi-domain associated to the polygon V, the conjugate surface to the graph of the solution of the Dirichlet problem is a minimal surface which is invariant under a translation by a horizontal vector. The second step of our proofs is then to solve our period problem which is to ensure that we can choose a multi-domain associated to V such that the corresponding vector is zero.

One idea to solve this question is to use the symmetry of the polygon V; this is what we do in the proof of Theorem 6. But in the general case, it is not sufficient.

In the example of the triangle, the only choice we have is the position of the point Q in ABC. For each point Q in the interior of the triangle

ABC, we get the horizontal vector that lets the conjugate surface to the graph invariant; this defines a continuous map from the triangle to $\mathbb{R}^2$ and we want to show that it vanishes. We call this map the period map and the idea to solve the period problem is to compute its degree. When the point Q moves to the boundary of the triangle, we show that the sequence of graphs converges to a graph which appears in the genus 0 case. This implies that the sequence of conjugate surfaces to the graphs converges to a symmetric r-noid with genus 0 and horizontal ends ($r = 3$ if Q moves to a vertex of the triangle and $r = 4$ if it moves to an edge). Since genus 0 r-noids are well known, we can determine the limit of the period map as Q goes to the boundary. Then we are able to compute the degree of the period map along the boundary and find a non-zero number; so the period map must vanishes at one point. The proof of our main result is based on a generalization of this argument.

The paper is organized as follows; in the first section, we define all the notions of multi-domain we use and the objects associated to a function on such domains.

Section 2 precises what kind of r-noids we study in this paper and state our main existence result.

Section 3 is devoted to the resolution of the Dirichlet problem on a multi-domain Ω associated to a polygon V. In this section, we make use of tools developped in [Ma1] and recalled in Appendix A.

In the fourth section, we study the regularity of the graph build in Section 3 near the singularity point of the multi-domain Ω.

In Section 5, we explain our period problem for the construction of a r-noid. We also generalize the notion of the period map and give the proof of our main result (Theorem 1) in using Proposition 6 which is proved in Section 6.

Section 6 is devoted to an extension of the period map and the computation of its degree. In this section, we use results on convex curves which are given in Appendix B.

In section 7, we prove Theorem 6 that gives examples of r-noids with genus 1 that are not given by Theorem 1. In particular, we consider the case where the polygon V is a regular polygon.

Let us fix some notations. In the following, when u is a function on a domain of $\mathbb{R}^2$ we shall note $W = \sqrt{1 + |\nabla u|^2}$. We shall also use the classical following notations for partial derivatives: $p = \dfrac{\partial u}{\partial x}$ and $q = \dfrac{\partial u}{\partial y}$. Besides, for the graph of u, we shall always chose the downward pointing normal to give an orientation to the graph.

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1 Graph on multi-domains

The aim of this section is to explain how a lot of minimal surfaces can be seen as graphs of a function. So, we give several generalizations of the notion of domain of $\mathbb{R}^2$. We use notions which were introduced in [CR] and [Ma1].

Let us consider a pair (D, ψ) where D is a 2-dimensionnal flat manifold with piecewise smooth boundary and $\psi : D \to \mathbb{R}^2$ is a local isometry. The map ψ is called the *developing map*. The points where the boundary ∂D is not smooth are the *vertices*. If a part of the boundary of D is linear, this part is called an *edge* of ∂D.

Definition 1. *A pair (D, ψ), where D is a simply-connected 2-dimensionnal complete flat manifold with piecewise smooth boundary and $\psi : D \to \mathbb{R}^2$ is a local isometry, is a* multi-domain *if each connected component of the smooth part of ∂D is a convex arc.*

Let D be a complete metric space and Q a point of D. D admits a *cone singularity* at Q of angle α if $D \backslash \{Q\}$ is a 2-dimensional flat manifold and if there exist $\rho_0 > 0$ such that $\{M \in D |\ d(M, Q) < \rho_0\}$ is isometric to $\{(\rho, \theta) |\ 0 \le \rho < \rho_0,\ 0 \le \theta \le \alpha\}$ with the polar metric $\mathrm{d}s^2 = \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\theta^2$ where all the points $(0, \theta)$ are identified and, for every ρ, $(\rho, 0)$ is identified with (ρ, α) (the isometry sends Q to $(0, 0)$).

Definition 2. *A triplet (D, Q, ψ) is a* multi-domain with a cone singularity at Q *if*

1. *D is a simply-connected complete metric space,*
2. *D admits a cone singularity at Q of angle $2q\pi$ ($q \in \mathbb{N}$),*
3. *D has piecewise smooth convex boundary and*
4. *$\psi : D \to \mathbb{R}^2$ is a local isometry outside Q.*

We remark that a multi-domain (D, ψ) can be seen as a multi-domain with a cone singularity at Q if Q is some point of D. The angle at the singularity is 2π.

Let (D, ψ) be a compact multi-domain with its boundary only composed by edges. The developing map allows us to see ∂D included in $\mathbb{R}^2$. Since ∂D has only edges, $\psi(\partial D)$ is a polygon in $\mathbb{R}^2$. The same thing can be done for (D, Q, ψ) a multi-domain with cone singularity. We say that a polygon V bounds a multi-domain (with perhaps a cone singularity) if there exists (D, ψ) or (D, Q, ψ) such that $V = \psi(\partial D)$. When V bounds a multi-domain (D, ψ), we also say that V *bounds an immersed polygonal disk* (see [CR]).

The last generalization we need is to give sense to an infinite angle cone singularity.

Let us consider $\mathcal{D} = \{(\rho, \theta)|\ \rho \in \mathbb{R}_+,\ \theta \in \mathbb{R}\}$ with the polar metric and where all the points $(0, \theta)$ are identified: this point is called the vertex of $\mathcal{D}$ and denoted by $\mathcal{O}$. The space $\mathcal{D}$ is a simply-connected complete metric space and $\mathcal{D}\backslash\mathcal{O}$ is a 2-dimensional flat manifold.

Definition 3. *A triplet* $(\Omega, \mathcal{Q}, \varphi)$ *is a* multi-domain with a logarithmic singularity at $\mathcal{Q}$ *if*

1. Ω *is a simply-connected complete metric space,*
2. $\mathcal{Q} \in \Omega$,
3. $\Omega\backslash\mathcal{Q}$ *is a* 2*-dimensional flat manifold with piecewise smooth convex boundary,*
4. $\varphi : \Omega \to \mathcal{D}$ *is a local isometry such that* $\varphi(\mathcal{Q}) = \mathcal{O}$ *and*
5. *there exist a neighborhood* $\mathcal{N}$ *of* $\mathcal{Q}$ *in* Ω *and* $\rho > 0$ *such that* $\varphi|_{\mathcal{N}}$ *is an isometry into* $\{M \in \mathcal{D} \mid d(\mathcal{O}, M) < \rho\}$.

Let us define $R_\alpha : \mathcal{D} \to \mathcal{D}$ by $R_\alpha(r, \theta) = (r, \theta + \alpha)$, R_α is an isometry of $\mathcal{D}$.

Definition 4. *A multi-domain with a logarithmic singularity* $(\Omega, \mathcal{Q}, \varphi)$ *is* periodic *if there exists* $f : \Omega \to \Omega$ *an isometry and* $n \in \mathbb{N}^*$ *such that*

$$\varphi \circ f = R_{2n\pi} \circ \varphi \qquad (*)$$

The period of Ω *is then* $2\pi q$ *where* q *is the smallest integer* n *such that there exists* f *making* $(*)$ *true.*

The first example of a multi-domain with a logarithmic singularity is $(\mathcal{D}, \mathcal{O}, \mathrm{id})$. This multi-domain is periodic of period 2π.

Construction 1. Let us consider (D, ψ) a multidomain and Q a point in D. We denote by $\Omega \xrightarrow{\pi} D\backslash Q$ a universal cover of $D\backslash Q$. We can pull back to Ω

the flat metric of D. The metric completion of Ω is just $\Omega \cup \{\mathcal{Q}\} = \overline{\Omega}$ where $\mathcal{Q}$ is a point "above" Q (*i.e.* if $\mathcal{A}_n \to \mathcal{Q}$, we have $\pi(\mathcal{A}_n) \to Q$). If (ρ, θ) are the polar coordinates on $\mathbb{R}^2$ of center $\psi(Q)$, on Ω, the 1-forms $(\pi \circ \psi)^* \mathrm{d}\rho$ and $(\pi \circ \psi)^* \mathrm{d}\theta$ are closed. By integration, we define a map $\varphi : \Omega \cup \{\mathcal{Q}\} \to \mathcal{D}$ such that $(\overline{\Omega}, \mathcal{Q}, \varphi)$ is a multi-domain with a logarithmic singularity. The multi-domain, we have just build, is periodic of period 2π.

In fact, the same work can be done for (D, Q, ψ) a multi-domain with a cone singularity at Q of angle $2q\pi$. We get $(\overline{\Omega}, \mathcal{Q}, \varphi)$ a periodic multi-domain with a logarithmic singularity (the period is less than $2q\pi$) and a covering map $\pi : \overline{\Omega} \to D$ with $\pi(\mathcal{Q}) = Q$.

Construction 2. The inverse construction is also possible. Let $(\Omega, \mathcal{Q}, \varphi)$ be a periodic multi-domain with a logarithmic singularity of period $2q\pi$ and isometry f. In taking the quotient of Ω by the group $\{f^n\}_{n\in\mathbb{Z}}$, we build a multi-domain with a cone singularity at Q, the image of $\mathcal{Q}$ in the quotient, and angle $2q\pi$.

Remark 1. Let $(\Omega, \mathcal{Q}, \varphi)$ be a periodic multi-domain with a logarithmic singularity of period $2q\pi$ and isometry f. Let a be in $\mathbb{N}^*$, the quotient of Ω by the group $\{f^{an}\}_{n\in\mathbb{Z}}$ is a multi-domain with a cone singularity (D, Q, ψ). The cone singularity of D has $2qa\pi$ as angle. But if we apply Construction 1 to D, we get Ω which have a period less than $2qa\pi$ if $a > 1$.

Let $V = (v_1, \dots, v_r)$ be a polygon which, for example, bounds an immersed polygonal disk (D, ψ). If $Q \in D$, we make Construction 1 and get a multi-domain with logarithmic singularity $(\overline{\Omega}, \mathcal{Q}, \varphi)$. The quotient (D', Q', φ') of $\overline{\Omega}$ by $\{f^{2n}\}_{n\in\mathbb{Z}}$ is a multi-domain with cone singularity of angle 4π; besides it bounds the polygon $V \cdot V = (v_1, \dots, v_r, v_1, \cdots, v_r)$. Since Construction 1 gives $\overline{\Omega}$ for D and D', the two polygons V and $V \cdot V$ will not be distinguished in the following.

Let $(\Omega, \mathcal{Q}, \varphi)$ be a multi-domain with logarithmic singularity and A be a point in $\mathbb{R}^2$. We then define the map $\varphi_A : \Omega \to \mathbb{R}^2$ by $\varphi_A = G \circ \varphi$ where G is defined on $\mathcal{D}$ by $G(\rho, \theta) = A + (\rho \cos\theta, \rho \sin\theta)$. When Ω is given by Construction 1, we choose $A = \psi(Q)$.

Let Ω be such that (Ω, φ) or (Ω, Q, φ) corresponds to one of the three above definitions of multi-domain. Let u be a function defined on Ω or Ω minus its singularity. The *graph* of u is the surface in $\mathbb{R}^3$ defined by $\{\varphi(x), u(x)\}_{x\in\Omega}$ or $\{\varphi_A(x), u(x)\}_{x\in\Omega\setminus\{\mathcal{Q}\}}$ with $A \in \mathbb{R}^2$.

In the following, we are interested in functions u which are solutions of the minimal surface equation (MSE). The graph of u then becomes a minimal surface of $\mathbb{R}^3$. u being a solution of (MSE), we define a closed 1-form $\mathrm{d}\Psi_u$ on Ω, $\mathrm{d}\Psi_u$ is the inner product $\frac{\nabla u}{W} \lrcorner \mathrm{d}V$ where $\mathrm{d}V$ is the volume form on Ω. Since $\mathrm{d}\Psi_u$ is closed, a function Ψ_u is locally defined (obviously Ψ_u is well defined only if we fix its value at one point). *A priori*, Ψ_u is not defined on $\partial\Omega$ and at the possible singularity of Ω. However, Ψ_u is 1-Lipschitz continuous; then it continuously extends to the singularity and $\partial\Omega$. Since Ω is simply connected, Ψ_u is then single valued on Ω. In fact Ψ_u correponds to the third coordinates of the conjugate surface to the graph of u; Ψ_u is called the *conjugate function to* u. For other properties on Ψ_u, we refer to [JS] and Appendix A.

In [JS] and [Ma1], we can find the most general answer to the Dirichlet problem on compact multi-domain (D, ψ): the Dirichlet problem consists in finding a solution u on D of (MSE) knowing its value on the boundary.

On multi-domain with cone or logarithmic singularity there is no general answer. An example of solution of (MSE) on a multi-domain with logarithmic singularity is the function u defined on $(\mathcal{D}, \mathcal{O}, \mathrm{id})$ by $u(\rho, \theta) = \theta$. It is obvious that the graph of u is the half of an helicoid; more precisely it is the surface given in isothermal coordinate by $(a, b) \mapsto (\sinh a \cos b, \sinh a \sin b, b)$ for $(a, b) \in \mathbb{R}_+^* \times \mathbb{R}$. The function u is then a solution of (MSE).

2 The r-noids

In this section we give precise definitions of r-noids, flux at one end and other objects linked to the Plateau problem at infinity. After that, we will be able to write our main result concerning Plateau problem at infinity in the genus 1 case (Theorem 1).

2.1 Definitions

Let M be a complete minimal surface with finite total curvature in $\mathbb{R}^3$; M is conformally equivalent to a compact Riemann surface $\overline{M}$ minus a finite number of points (we can refer to [Os]). Then M has a finite number of annular ends. When these ends are embedded, they are asymptotic either to a half-catenoid or to a plane [Sc]. A properly immersed minimal surface with finite total curvature and r embedded ends is called a *r-noid*. We associate to each end a vector which caracterizes the direction and the growth of the asymptotic half-catenoid (when the end is asymptotic to a plane this vector is zero); this vector is called the *flux* of the end (for a precise definition of the flux, see [HK]). If $v_1, \dots, v_r$ are the fluxes at each end, M satisfies the following balancing condition:

$$v_1 + \cdots + v_r = 0 \tag{1}$$

This condition tells us that the total flux of the system vanishes. If $v_1, \dots, v_r$ are vectors in $\mathbb{R}^3$ such that (1) is verified and g is a non-negative integer, the Plateau problem at infinity for these data is: find an r-noid of genus g which has $v_1, \dots, v_r$ as fluxes at its ends (the genus g is the genus of $\overline{M}$).

Let $X : M \longrightarrow \mathbb{R}^3$ be an r-noid. M is conformally equivalent to $\overline{M} \backslash \{p_1, \dots, p_r\}$. M is *Alexandrov-embedded* if $\overline{M}$ bounds a compact 3-manifold $\overline{\Omega}$ and the immersion X extends to a proper local diffeomorphism $f : \overline{\Omega} \backslash \{p_1, \dots, p_r\} \longrightarrow \mathbb{R}^3$. An Alexandrov-embedded surface has a canonical orientation; we choose the Gauss map to be the outward pointing normal. An Alexandrov-embedded r-noid can not have a planar end (see [CR]).

We are interested in the case where $X : M \longrightarrow \mathbb{R}^3$ is an Alexandrov-embedded r-noid of genus g and r horizontal ends (*i.e.* the flux at each end is an horizontal vector).

Let $X : M \longrightarrow \mathbb{R}^3$ be a nonflat immersion of a connected orientable surface M and Π be a plane in $\mathbb{R}^3$ (we normalize it to be $\{z = 0\}$). We denote by S the Euclidiean symmetry with respect to Π and consider:

$$\begin{aligned} M^+ &= \{p \in M | x_3(p) > 0\} \\ M^- &= \{p \in M | x_3(p) < 0\} \\ M^0 &= \{p \in M | x_3(p) = 0\} \end{aligned}$$

With these notations we have:

Definition 5. *M is* strongly symmetric with respect to Π *if*

- *There exists an isometric involution $s : M \longrightarrow M$ such that $\psi \circ s = S \circ \psi$.*
- *$\{p \in M | s(p) = p\} = M^0$.*
- *The third coordinate N_3 of the Gauss map of M takes positive (resp. negative) values on M^+ (resp. M^-).*

In [CR], C. Cosín and A. Ros have proved:

Proposition 1. *Let M be an r-noid with horizontal ends. Then M is strongly symmetric with respect to an horizontal plane if and only if M is Alexandrov-embedded.*

This proposition explains why strong symmetry is important for the study of Alexandrov-embedded r-noids.

In [CR] C. Cosín and A. Ros have studied genus 0 r-noids. They show that, on such a minimal surface, the ends are naturally ordered. Let M be an Alexandrov-embedded r-noid of genus 0 and $2v_1, \dots, 2v_r$ denote the fluxes of M ordered as the ends. Because of (1), $(v_1, \cdots, v_r)$ is a polygon: it is called the *flux polygon* of M and denoted by $F(M)$. We then have

Theorem (Cosín-Ros). *Let $v_1, \dots, v_r$ be horizontal vectors such that $v_1 + \cdots + v_r = 0$ and V the associated polygon, then there exists M an Alexandrov-embedded r-noid of genus 0 such that $F(M) = V$ if, and only if, V bounds an immersed polygonal disk. Besides there is a bijection from the set of M such that $F(M) = V$ and the set of the immersed polygonal disks bounded by V.*

In the sequel, when $(\mathcal{P}, \psi)$ is an immersed polygonal disk, $\Sigma(\mathcal{P})$ denotes the genus 0 r-noid associated to $\mathcal{P}$ by this bijection. We refer to [CR] and [Ma1] for more explanations on this theorem.

2.2 Genus 1 r-noids

In this paper, we are interested in the case of genus 1 r-noids. Let M be an Alexandrov-embedded r-noid of genus 1 with horizontal ends. M is conformally a torus $\overline{M}$ minus r points $p_1, \dots, p_r$. By Proposition 1, M is strongly symmetric with respect to an horizontal plane which is normalized to be $\{z = 0\}$. As in [CR], the punctures p_i are fixed by the antiholomorphic involution s given by Definition 5. We have the following lemma.

Lemma 1. *Let $\overline{M}$ be a conformal torus and s an antiholomorphic involution on $\overline{M}$. We assume that s has a fixed point then the set of the fixed points of s is two separated circles.*

Then we can give the following definition.

Definition 6. *Let M be an Alexandrov-embedded r-noid with genus one and horizontal ends and p_i the punctures of M. We denote by s the antiholomorphic involution associated to M and by C_1 and C_2 the two circles of fixed points of s. With this notations, we say that:*

- *M is of* type I *if all the p_i are in one of the two circles C_1 and C_2,*
- *M is of* type II *if not.*

We suppose now that M is of type I, the circles of fixed points are the boundary of $M^+ \cup M^0$, this minimal surface is oriented by the outward pointing normal and then its boundary has a natural orientation. Then the circle that contains the points p_i has a natural orientation and we suppose that the points p_i are numbered with respect to this orientation. Let $2v_i$ denote the flux vector associated to p_i. We know that $v_1+\cdots+v_r = 0$; so, as in $g = 0$ case, the list $(v_1, \ldots, v_r)$ defines a polygon which is called the *flux polygon* of M and denoted by $F(M)$. If M is of type II, there is not such an easy definition. Figure 1 shows an example of an Alexandrov-embedded 3-noid with genus one and horizontal ends of type I. The flux polygon is a triangle.

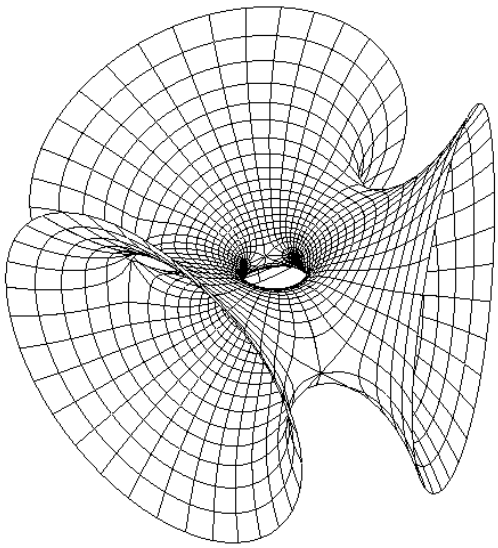

Figure 1: An example of 3-noid with genus 1 [MSL]

Now we can state the main result of the paper. It give a sufficient condition on a polygon to ensure the existence of an r-noid of genus 1 and type I with this polygon as flux polygon.

Theorem 1. *Let $v_1, \ldots, v_r$ be r non zero vectors of $\mathbb{R}^2$ such that $(v_1, \ldots, v_r)$ is a polygon that bounds an immersed polygonal disk. Then there exists M an Alexandrov-embedded r-noid with genus one and horizontal ends of type I such that $F(M) = (v_1, \ldots, v_r)$.*

This implies that every flux polygon of a genus 0 r-noid is also the flux polygon of an r-noid with genus 1. In Section 7, we give a result of existence for polygons that bound multi-domain with cone-singularity (Theorem 6); this gives an other class of polygons.

In the following paragraphes, we give some facts about r-noid of genus 1. We use arguments that C. Cosín and A. Ros have developped in [CR] for the study of the genus 0 case.

Let $X : \overline{M}\backslash\{p_1, \dots, p_r\} \to M$ be an Alexandrov-embedded r-noid with genus one and horizontal ends of type I. The surface $M^+ \cup M^0$ is topologically an annulus. So using the universal cover, we can define its conjugate surface $M^*_{0,+}$. $M^*_{0,+}$ is a periodic minimal surface and its period vector is $\int_\gamma \mathrm{d}X^*$ where γ is a path in $M^+ \cup M^0$ that generates $\pi_1(M^+ \cup M^0)$. Since M is of type I all the p_i are in one circle of fixed points, then the other circle of fixed points generates $\pi_1(M^+ \cup M^0)$. Along this circle the normal is horizontal, the first two coordinates of the period vector are then zero, *i.e.* the period vector is vertical.

The symmetry curves M^0 consists of r complete strictly convex curves in the plane $\{z = 0\}$ (they are the images by X of the arcs p_ip_{i+1}) and the image by X of the circle of fixed points that contains no p_i (see Figure 1). In the conjugate surface, these curves are transformed into vertical lines: the image of the arc p_ip_{i+1} become vertical straight-lines over the vertices of the polygon $F(M)$ in $M^*_{0,+}$.

On M^+ the third coordinate of the normal is positive, the projection map Π from M^*_+ into the plane $\{z = 0\}$ is then a local diffeomorphism. Let us pull back to M^*_+ the flat metric of this plane. Besides M^*_+ is stable by the translation t of vector $\int_\gamma \mathrm{d}X^*$. Since this vector is vertical, $\Pi \circ t = \Pi$; t is then an isometry of M^*_+ with the flat metric $\Pi^*(\mathrm{d}s^2_{\mathbb{R}^2})$.

Using arguments of C. Cosín and A. Ros, we prove that there exists $(\Omega, \mathcal{Q}, \varphi)$ a multi-domain with logarithmic singularity such that M^*_+ with its flat metric is the interior of Ω minus the singularity point $\mathcal{Q}$. Besides Ω is periodic because of the existence of the isometry t. The boundary of Ω is composed of half-lines. M^*_+ is then a graph over Ω; its boundary is composed of vertical lines over the vertices of Ω (the conjugates of the arcs p_ip_{i+1}) and a vertical line above $\mathcal{Q}$ (the conjugate of the circle of fixed points that contains no p_i).

The quotient of Ω by the group $(t^n)_{n\in\mathbb{Z}}$ is then a multi-domain with cone singularity (D', Q, ψ). D' has r vertices $P_1, \dots, P_r$ and is bounded by $2r$ half-lines; more precisely, D' is a multi-domain with cone singularity D which is bounded by the flux polygon $F(M) = (v_1, \dots, v_r)$ (we have $v_i = \overrightarrow{\psi(P_i)\psi(P_{i+1})}$) to which we have glued r half-strips (along the edge

$[P_i, P_{i+1}]$, we glue a half-strip isometric to $[P_i, P_{i+1}] \times \mathbb{R}_+$). This proves:

Proposition 2. *Let M be an Alexandrov-embedded genus 1 r-noid with horizontal ends of type I. Its flux polygon $F(M)$ bounds a multi-domain with cone singularity.*

The question is the converse of the above proposition. Our strategy to construct genus 1 r-noid is the following. Let V be a polygon that bounds a multi-domain with cone singularity. We consider the associated periodic multi-domain with logarithmic singularity Ω and solve a Dirichlet problem on $\Omega \backslash \{\mathcal{Q}\}$ that give a candidate for M_+^*. The last step is to choose a good multi-domain with cone singularity bounded by the polygon to solve a period problem. To solve this period problem, we restrict ourselves to polygons bounding an immersed polygonal disk (Theorem 1) or polygons having symmetries (Theorem 6).

In Section 3 and Section 4, we give a precise definition of the multi-domain Ω, we solve the Dirichlet problem and give properties of the obtained graph.

In Section 5, we present the period problem and give the proof of Theorem 1 subject to a proposition which Section 6 proves.

In Section 7, we solve the period problem in the case of polygons with symmetries.

3 A Dirichlet problem

In this section, we make the first step of the proofs of our existence results: solve a Dirichlet problem on a multi-domain associated to a polygon that bounds a multi-domain with cone singularity. First we define this multi-domain.

Let $V = (v_1, \dots, v_r)$ be a polygon that bounds a multi-domain with cone singularity (D, Q, ψ) ($v_i \neq 0$). Let $P_1, \dots, P_r$ denote the vertices of the polygon, we put $P_{r+1} = P_1$. In the following, we assume that the orientation on V is given as boundary of D (we remark that the vertices of V will be identified with the vertices of D). From (D, Q, ψ), Construction 1 gives a multi-domain with logarithmic singularity $(\mathcal{W}, \mathcal{Q}, \varphi)$ with a projection map $\pi : \mathcal{W} \to D$. Because of Remark 1, we assume that the period $2q\pi$ of $\mathcal{W}$ is equal to the angle of the cone singularity of D (*i.e.* we assume that D is the quotient of $\mathcal{W}$ by its isometry f). In the following, we say that D satisfies the hypothesis H.

Construction 3. Let i be in $\{1, \dots, r\}$ and consider E an edge of $\mathcal{W}$ which is send by π to the edge $[P_i, P_{i+1}]$ of D. Let us glue to $\mathcal{W}$ a half strip S_i isometric to $[P_i, P_{i+1}] \times \mathbb{R}_+^*$ along E (if $A \in E$ the point A is identified with $(\pi(A), 0)$). Making this for every i and every edges E, we get a new multi-domain with a logarithmic singularity at $\mathcal{Q}$: $(\Omega, \mathcal{Q}, \varphi)$ (we keep the same notation for the developping map since it is an extension of the original one). Since $\mathcal{W}$ is periodic, Ω is periodic and has the same period, we still denote by f the corresponding isometry.

For $i \in \{1, \dots, r\}$, $\mathcal{L}_i^+$ (resp. $\mathcal{L}_i^-$) denotes the union of the straight-lines corresponding to $\{P_i\} \times \mathbb{R}_+^*$ in the half-strips glued along the edges E such that $\pi(E) = [P_i, P_{i+1}]$ (resp. $\pi(E) = [P_{i-1}, P_i]$). $\mathcal{L}_i^+$ and $\mathcal{L}_i^-$ are a countable union of half straight-lines. $\mathcal{V}$ denotes the set of the vertices of Ω, this set is $\pi^{-1}\{P_1, \dots, P_r\}$.

We then have the following existence and uniqueness result.

Theorem 2. *Let (D, Q, ψ) be a multi-domain with cone singularity that bounds a polygon V, Constructions 1 and 3 give us a periodic multi-domain with logarithmic singularity $(\Omega, \mathcal{Q}, \varphi)$; we suppose that the period of Ω is the cone angle of D at Q. Then there exists a solution u of the minimal surface equation on Ω such that*

1. *u tends to $+\infty$ along $\mathcal{L}_i^+$ and $-\infty$ along $\mathcal{L}_i^-$ and*
2. *$\Psi_u(\mathcal{Q}) = 0 = \Psi_u(\mathcal{V})$,*

where $\mathcal{L}_i^+$, $\mathcal{L}_i^-$ and $\mathcal{V}$ are the notations given in Construction 3. Besides, the solution is unique up to an additive constant.

The data of this Dirichlet problem implies that the graph of its solution is a good candidate for the conjugate surface to a genus 1 r-noid with V as flux polygon.

Corollary 1. *If u is a solution on Ω of the Dirichlet problem asked in Theorem 2, there exists a constant $c \in \mathbb{R}$ such that $u \circ f = u + c$.*

Proof. Let u be a solution of the Dirichlet problem asked in Theorem 2, then $u \circ f$ is also a solution of this Dirichlet problem. This proves that $u \circ f - u$ is constant. $\square$

The following of this section is devoted to the proof of Theorem 2

3.1 Notations

First, the period of Ω is $2q\pi$. We assume that P_1 is such that $d(Q,P_1) = \min_i d(Q,P_i)$. Let $\mathcal{P}_1(0)$ denote a vertex in $\mathcal{V}$ which is a lift of P_1. Since $d(M,P_1)$ is minimal, the geodesic joining $\mathcal{Q}$ to $\mathcal{P}_1(0)$ is embedded, this implies that the first polar coordinate of $\varphi(\mathcal{P}_1(0))$ is $d(\mathcal{Q},\mathcal{P}_1(0)) = d(Q,P_1)$. Using $R_\alpha \circ \varphi$ instead of φ, we can assume that $\varphi(\mathcal{P}_1(0)) = (d(\mathcal{Q},\mathcal{P}_1(0),0)$. We write $\mathcal{P}_1(k) = f^k(\mathcal{P}_1(0))$ for $k \in \mathbb{Z}$; then $\varphi(\mathcal{P}_1(k)) = (d(M,P_0), 2kq\pi)$. So $\{\mathcal{P}_1(i), i \in \mathbb{Z}\}$ is the set of the vertex of Ω corresponding to P_1 (*i.e.* $\{\mathcal{P}_1(i), i \in \mathbb{Z}\} = \pi^{-1}(P_1)$).

The two geodesics $[\mathcal{Q},\mathcal{P}_1(k)]$ and $[\mathcal{Q},\mathcal{P}_1(l)]$ $(k < l)$ devide Ω into three connected components. One of them is such that its intersection with $\mathcal{N}$, the neighborhood of $\mathcal{Q}$ (see Definition 3), is isometric to $]0,r[\times]2kq\pi, 2lq\pi[$ by φ; Ω_k^l denotes this part. For $n \in \mathbb{Z}$, we have $f^n(\Omega_k^l) = \Omega_{k+n}^{l+n}$. In Ω_0^1, there is exactly one lift of every P_i $(2 \le i \le r)$, let $\mathcal{P}_i(0)$ denote the lift of P_i that is in Ω_0^1. For $k \in \mathbb{Z}$, we write $\mathcal{P}_i(k) = f^k(\mathcal{P}_i(0))$ such a way that $\mathcal{P}_i(k)$ is the only lift of P_i in Ω_k^{k+1}. A part Ω_k^{k+1} of Ω is called a *period* of Ω.

For $i \in \{1,\dots,r\}$ and $k \in \mathbb{Z}$, let $\mathcal{L}_i^+(k)$ (resp. $\mathcal{L}_i^-(k)$) denote the half straight-line included in $\mathcal{L}_i^+$ (resp. $\mathcal{L}_i^-$) and having $\mathcal{P}_i(k)$ as end-point.

Let us consider k and l in $\mathbb{Z}$ such that $k < l$, Ω_k^l is a multi-domain. Its vertices are $\mathcal{Q}$, the $\mathcal{P}_1(m)$ for $k \le m \le l$ and the $\mathcal{P}_i(m)$ for $2 \le i \le r$ and $k \le m < l$. Its boundary is composed of two segments $[\mathcal{Q},\mathcal{P}_1(k)]$ and $[\mathcal{Q},\mathcal{P}_1(l)]$ and $2r(l-k)$ half straight-lines which are the $\mathcal{L}_1^+(m)$ for $k \le m < l$, the $\mathcal{L}_1^-(m)$ for $k < m \le l$ and the $\mathcal{L}_i^\pm(m)$ for $2 \le i \le r$ and $k \le m < l$. From Ω_k^l, we build a new multi-domain: let us glue to Ω_k^l two half-strips, one is isometric to $[\mathcal{Q},\mathcal{P}_1(k)] \times \mathbb{R}_+$ and is glued along $[\mathcal{Q},\mathcal{P}_1(k)]$, the second is isometric to $[\mathcal{P}_1(l),\mathcal{Q}] \times \mathbb{R}_+$ and is glued along $[\mathcal{P}_1(l),\mathcal{Q}]$. We denote by $\widetilde{\Omega}_k^l$ this new multi-domain. Let $\widetilde{\mathcal{L}}_k^-$ (resp. $\widetilde{\mathcal{L}}_l^+$) denote the new half straight-line in the boundary of $\widetilde{\Omega}_k^l$ with $\mathcal{P}_1(k)$ (resp. $\mathcal{P}_1(l)$) as end-point. We also denote by $\widetilde{\mathcal{L}}^+$ and $\widetilde{\mathcal{L}}^-$ the two half straight-lines in the boundary with $\mathcal{Q}$ as end-point such that $\widetilde{\mathcal{L}}^+$ is in the same half-strip than $\widetilde{\mathcal{L}}_k^-$.

3.2 Proof of the existence

We begin by the proof of the existence part. First, for n in $\mathbb{N}^*$, we prove that there exists a solution u_n of (MSE) on $\widetilde{\Omega}_{-n}^n$ which:

1. tends to $+\infty$ on $\widetilde{\mathcal{L}}^+$, $\widetilde{\mathcal{L}}_n^+$ and all the $\mathcal{L}_i^+(k)$ included in $\partial\widetilde{\Omega}_{-n}^n$ and

2. tends to $-\infty$ on $\widetilde{\mathcal{L}}^-$, $\widetilde{\mathcal{L}}_{-n}^-$ and all the $\mathcal{L}_i^-(k)$ included in $\partial\widetilde{\Omega}_{-n}^n$.

Let us consider the following polygon:

$$(\overrightarrow{QP_1}, \underbrace{v_1, \dots, v_r, \dots \dots, v_1, \dots, v_r}_{2n \text{ times}}, \overrightarrow{P_1Q})$$

In removing to $\widetilde{\Omega}_{-n}^n$ all its half-strips, we get a multi-domain which is bounded by the above polygon. Then, the existence of the solution u_n is ensured by Theorem 7 in [Ma1].

Let us now only consider the restriction of u_n to Ω_{-n}^n. This gives us a sequence of solutions of (MSE) defined on an increasing sequence of domains Ω_{-n}^n and $\bigcup_{n\in\mathbb{N}^*} \Omega_{-n}^n = \Omega$. Then (u_n) can be considered as a sequence of functions on Ω. We want (u_n) to converge, so we shall prove: the sequence has no line of divergence (see Appendix A).

First we give some preliminary results on u_n.

Lemma 2. *Let u be a solution of* (MSE) *on the half-strip* $[0,a]\times\mathbb{R}_+$ *such that u tends to $-\infty$ on $\{0\}\times\mathbb{R}_+$ and $+\infty$ on $\{a\}\times\mathbb{R}_+$. Then, for $y\geq 4a$, we have:*

$$\frac{|p|}{W}(x,y) \geq 1-\frac{a^2}{y^2} \qquad\qquad \frac{|q|}{W}(x,y)\leq \sqrt{2}\frac{a}{y}$$

Proof. Let A denote the point of coordinates (x,y) and B a point in the boundary of the half-strip which realizes the distance from A to this boundary. Since $y\geq 4a$, B is $(0,y)$ or (a,y); besides the distance $|AB|$ is less than $a/2$. Because of the value of u on the boundary, the distance along the graph from the point above A to the boundary of the graph is bigger than $4a$. The ratio of this two distances is less than $1/8$, then we can apply Lemma 1 in [JS]. This gives the lemma. $\square$

Corollary 2. *Let u be a solution of* (MSE) *on the half-strip* $[0,a]\times\mathbb{R}_+$ *such that u tends to $-\infty$ on $\{0\}\times\mathbb{R}_+$ and $+\infty$ on $\{a\}\times\mathbb{R}_+$. We have $\Psi_u(0,y)=\Psi_u(a,y)$ and, if $\Psi_u(0,0)=0$, $\Psi_u\geq 0$ in the half-strip.*

Proof. Because of the value of u on the boundary, $\Psi_u(0,y)=\Psi_u(0,0)+y$ and $\Psi_u(a,y)=\Psi_u(a,0)+y$. Then for $y\geq 4a$, we have:

$$\begin{aligned}|\Psi_u(0,0)-\Psi_u(a,0)| = |\Psi_u(0,y)-\Psi_u(a,y)| &= \left|\int_0^a -\frac{q}{w}(x,y)\mathrm{d}x\right| \\ &\leq \int_0^a \sqrt{2}\frac{a}{y}\mathrm{d}x = \sqrt{2}\frac{a^2}{y}\end{aligned}$$

Then in letting y goes to $+\infty$, we obtain $\Psi_u(0,0) = \Psi_u(a,0)$ and then $\Psi_u(0,y) = \Psi_u(a,y)$.

If $\Psi_u(0,0) = 0$, $\Psi_u(0,y) = y$. If A is in the interior of the half-strip, there exists y such that A is at a distance less than y from the point $(0,y)$; since Ψ_u is 1-Lipschitz continuous, $\Psi_u(A) \geq 0$. $\square$

This corollary implies that, with Ψ_{u_n} such that $\Psi_{u_n}(\mathcal{Q}) = 0$, we have $\Psi_{u_n}(\mathcal{P}_i(k)) = 0$ for all $\mathcal{P}_i(k) \in \Omega^n_{-n}$ and $\Psi_{u_n} \geq 0$ in the half-trips included in $\widetilde{\Omega}^n_{-n}$. Since Ψ_{u_n} satisfies a maximum principle, $\Psi_{u_n} \geq 0$ in Ω^n_{-n}.

Let us assume that (u_n) has a line of divergence L. By Lemma A.1, L can not have an end point in the interior of one of the half straight-lines that compose the boundary of Ω. Then, if L has end-points, it must be $\mathcal{Q}$ or one $\mathcal{P}_i(k)$.

If L has no end point, let $\mathcal{A}$ be a point of L and let d denote the distance between $\mathcal{A}$ and $\mathcal{Q}$. Since $\Psi_{u_n}(\mathcal{Q}) = 0$ and Ψ_{u_n} is 1-Lipschitz, we have $0 \leq \Psi_{u_n}(\mathcal{A}) \leq d$. Let us fix an orientation to L such that the limit normal along this line of divergence is the right-hand unit normal. We denote by s the arc-length along L with $\mathcal{A}$ as origin. Let us consider $\mathcal{A}'$ the point on L of arc-length $s = -2d$. Then, for the subsequence that makes L appear, we have $\Psi_{u_{n'}}(\mathcal{A}) - \Psi_{u_{n'}}(\mathcal{A}') \to 2d$. Since $\Psi_{u_n}(\mathcal{Q}) \leq d$ this implies that $\Psi_{u_{n'}}(\mathcal{A}') < 0$ for big n'; this is a contradiction.

If L is a segment (L has two end-points denoted by $\mathcal{A}_1$ and $\mathcal{A}_2$), we know that for, each n, $\Psi_{u_n}(\mathcal{A}_1) = 0 = \Psi_{u_n}(\mathcal{A}_2)$. But for the subsequence that make L appear, we have $|\Psi_{u_{n'}}(\mathcal{A}_1) - \Psi_{u_{n'}}(\mathcal{A}_2)| \to d(\mathcal{A}_1, \mathcal{A}_2) > 0$; this gives us a contradiction.

We then assume that L has only one end-point $\mathcal{F}$ and goes to infinity in one half-strip. L can be parametrized isometricaly by $[0,a] \times \mathbb{R}_+$. We remark that for one half-strip in Ω the number of such lines L is finite. There exists $b \in]0,a[$ such that the part of L in the half-strip is $\{b\} \times \mathbb{R}_+$. By changing L if necessary we can assume: the part $(b,a) \times \mathbb{R}_+$ is included in $\mathcal{B}(u_n)$. Let $\mathcal{A}_1 = (b,0)$, $\mathcal{A}_2 = (b, 2(a-b))$, $\mathcal{A}_3 = (a, 2(a-b))$ and $\mathcal{A}_4 = (a,0)$ be four points in the half-strip. Since $\mathrm{d}\Psi_{u_n}$ is closed we have:

$$\begin{aligned}
\int_{[\mathcal{A}_1,\mathcal{A}_2]} \mathrm{d}\Psi_{u_n} &= -\int_{[\mathcal{A}_2,\mathcal{A}_3]} \mathrm{d}\Psi_{u_n} - \int_{[\mathcal{A}_3,\mathcal{A}_4]} \mathrm{d}\Psi_{u_n} - \int_{[\mathcal{A}_4,\mathcal{A}_1]} \mathrm{d}\Psi_{u_n} \\
&= 2(a-b) - \int_{[\mathcal{A}_2,\mathcal{A}_3]} \mathrm{d}\Psi_{u_n} - \int_{[\mathcal{A}_4,\mathcal{A}_1]} \mathrm{d}\Psi_{u_n} \\
&\geq 2(a-b) - (a-b) - (a-b) = 0
\end{aligned}$$

This implies that we have only one possibility for the limit normal . Since $]b,a[\times\mathbb{R}_+ \subset \mathcal{B}(u_n)$, there is a subsequence $(u_{n'})$ that converges to a function v on $]b,a[\times\mathbb{R}_+$ (n' is chosen such that the line of divergence L appears). v is a solution of (MSE) and by Lemma A.2, v tends to $+\infty$ along $\{b\}\times\mathbb{R}_+^*$ and $-\infty$ along $\{a\}\times\mathbb{R}_+^*$. Then by Lemma 2, $\Psi_v(\mathcal{A}_1)=\Psi_v(\mathcal{A}_4)=\lim\Psi_{u_{n'}}(\mathcal{A}_4)=0$. For the subsequence, $\Psi_{u_{n'}}(\mathcal{A}_1)-\Psi_{u_{n'}}(\mathcal{F})\to d(\mathcal{F},\mathcal{A}_1)>0$, this contradicts: $\Psi_{u_n}(\mathcal{F})=0$ and $\lim\Psi_{u_{n'}}(\mathcal{A}_1)=0$.

We have proved that $\mathcal{B}(u_n)=\Omega$, and, using a subsequence, we assume that (u_n) converges to a solution u of (MSE) on Ω. By Lemma A.2, u tends to $+\infty$ along the $\mathcal{L}_i^+(k)$ and $-\infty$ along the $\mathcal{L}_i^-(k)$. By construction, we have also that $\Psi_u(\mathcal{Q})=0=\Psi_u(\mathcal{P}_i(k))$ and $\Psi_u\geq 0$. This completes the existence part of Theorem 2 proof.

3.3 Property of the solution 1

Before the uniqueness proof, we need to know some properties of a solution of the Dirichlet problem asked in Theorem 2. We recall that there exist ρ_0 and a neighborhood $\mathcal{N}$ of $\mathcal{Q}$ such that φ is an isometry from $\mathcal{N}$ into $\{M\in\mathcal{D}|d(\mathcal{O},M)\leq\rho_0\}$. We then use the polar coordinates for v a function defined on $\mathcal{N}$.

Proposition 3. *Let ε be a positive number. There exists $d>0$ such that for every $\alpha\in\mathbb{R}$ and every solution v of* (MSE) *on $\mathcal{N}\cap\{\alpha-\pi<\theta<\alpha+\pi\}$:*

$$\sup_{0<\rho\leq\rho_0} v(\rho,\alpha) < \sup_{[d,\rho_0]\times[\alpha-3\pi/4,\alpha+3\pi/4]} v(\rho,\theta)+\varepsilon$$
$$\inf_{0<\rho\leq\rho_0} v(\rho,\alpha) > \inf_{[d,\rho_0]\times[\alpha-3\pi/4,\alpha+3\pi/4]} v(\rho,\theta)-\varepsilon$$

The proof is based on the idea used by R. Osserman to prove Theorem 10.3 in [Os].

Proof. First let us assume that $\alpha=\pi/2$. Then $\mathcal{N}\cap\{\alpha-\pi<\theta<\alpha+\pi\}$ is isometric to the disk of center the origin and radius ρ_0 minus the segment joining the origin to $(0,-\rho_0)$. We only prove the first inequality; the second one is a consequence: substitute v for $-v$.

Let $\varepsilon>0$ and d be a positive number, we denote by A the point of coordinates $(0,-d)$ and D the domain in $D(0,\rho_0)$ delimited by the segment joining $(-\sqrt{\rho_0^2-d^2},-d)$ to $(-d,-d)$, the half circle of center A and radius d: $\theta\mapsto(d\cos\theta,d(\sin\theta-1))$ for $\theta\in[0,\pi]$ and the segment joining $(d,-d)$ to $(\sqrt{\rho_0^2-d^2},-d)$ and containing $(0,\rho_0/2)$. On D, we consider the function $c: M\mapsto d\left(-\operatorname{argch}(\frac{|AM|}{d})+\operatorname{argch}(\frac{\rho_0+d}{d})\right)$. The graph of c is a piece of a

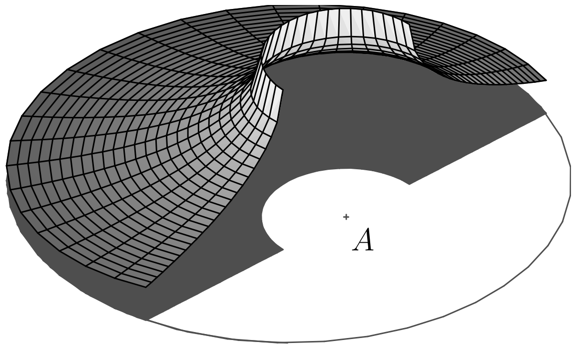

Figure 2: the graph of the function c

catenoid. c is a positive function and upper bounded by $d\,\mathrm{argch}(\frac{\rho_0+d}{d})$. Let d be small enough such that this upper-bound is less than ε.

On the part of the boundary of D which is a half circle of center A, the derivative $\nabla c \cdot n$, with n the outward pointing normal, takes the value $+\infty$. Lemma 10.2 in [Os] implies that $c + K \geq v$ where K is the supremum of v on the boundary of D minus the half circle of center A. This part of the boundary is included in the set of points of polar coordinates in $[d, \rho_0] \times [-\pi/4, 5\pi/4]$ and D contains the segment joining the origin to $(0, 1)$. The proposition is then established. $\square$

Let u be a solution of the dirichlet problem asked in Theorem 2. The above proposition implies that u is bounded on $\mathcal{N} \cap \Omega_k^l$ for every k and l.

3.4 Proof of the uniqueness

Let us now prove the uniqueness part of Theorem 2. Let u and v be two solutions of the Dirichlet problem asked in Theorem 2 and we assume that these two solutions are different: $u - v$ is not constant. Let $\Omega_0^{+\infty}$ denote $\bigcup_{n>0} \Omega_0^n$. In changing v by $v + c$ where c is a real constant we can assume:

1. $\{u - v > 0\} \cap \Omega_0^{+\infty}$ and $\{u - v < 0\} \cap \Omega_0^{+\infty}$ are non-empty and
2. the segment $[\mathcal{Q}, \mathcal{P}_1(0)]$ is included in $\{u - v > 0\}$.

The first assertion is due to the fact that $u - v$ is non-constant and that we can exchange u and v. the second one is a consequence of the fact that u and v are bounded on $[\mathcal{Q}, \mathcal{P}_1(0)]$ (Proposition 3 and Lemma 2 in [Ma1]).

We write $\Delta = \{u - v < 0\} \cap \Omega_0^{+\infty}$. Let Δ^n denote the intersection of Δ and Ω_0^n. The boundary $\partial\Delta^n$ is composed of a part included in $\partial\Omega$, a part included in the interior of Ω_0^n and one in the segment $[\mathcal{Q}, \mathcal{P}_1(n)]$. Along the first part $\mathrm{d}\widetilde{\Psi} = \mathrm{d}\Psi_u - \mathrm{d}\Psi_v = 0$ and along the second part $\mathrm{d}\widetilde{\Psi}$ is positive by Lemma 2 in [CK]; the union of this first two parts will be denoted by $\partial^n\Delta$.

Lemma 3. *We have:*

$$0 = \int_{\partial\Delta^n} \mathrm{d}\widetilde{\Psi} = \int_{\partial\Delta^n \cap [\mathcal{Q}, \mathcal{P}_1(n)]} \mathrm{d}\widetilde{\Psi} + \int_{\partial^n\Delta} \mathrm{d}\widetilde{\Psi} \tag{2}$$

Proof. Let Δ_l^n denote the part of Δ^n such that a point A is in this part if either A is not in one half-strip included in Ω_0^n or $A = (x, y)$ is in one of these half-strips which is isometricaly parametrized by $[0, a] \times \mathbb{R}_+$ and $y \le l$. The boundary of Δ_l^n is composed of three parts: the first is $\partial\Delta^n \cap [\mathcal{Q}, \mathcal{P}_1(n)]$, the second is $\partial^n\Delta \cap \overline{\Delta_l^n}$ and the third is included in the union of the segments parametrized by $[0, a] \times \{l\}$ in each half-strip; we denote by Γ_l this third part. Since $\mathrm{d}\widetilde{\Psi}$ is closed we have:

$$\int_{\partial\Delta^n \cap [\mathcal{Q}, \mathcal{P}_1(n)]} \mathrm{d}\widetilde{\Psi} + \int_{\partial^n\Delta \cap \overline{\Delta_l^n}} \mathrm{d}\widetilde{\Psi} + \int_{\Gamma_l} \mathrm{d}\widetilde{\Psi} = 0 \tag{3}$$

Then $\displaystyle\int_{\partial^n\Delta \cap \overline{\Delta_l^n}} \mathrm{d}\widetilde{\Psi} \le |\mathcal{Q}\mathcal{P}_1(n)| + \left| \int_{\Gamma_l} \mathrm{d}\widetilde{\Psi} \right|$

Since $\mathrm{d}\widetilde{\Psi} \ge 0$ along $\partial^n\Delta$, $\int_{\partial^n\Delta \cap \overline{\Delta_l^n}} \mathrm{d}\widetilde{\Psi}$ increases as l increases. By Lemma 2, $\int_{\Gamma_l} \mathrm{d}\widetilde{\Psi} \to 0$ as l goes to $+\infty$. Then the above equation implies that $\mathrm{d}\widetilde{\Psi}$ is integrable on $\partial^n\Delta$ and by passing to the limit in (3), we obtain (2). $\square$

Equation (2) implies that $\displaystyle\int_{\partial^n\Delta} \mathrm{d}\widetilde{\Psi} \le |\mathcal{Q}\mathcal{P}_1(n)| = |\mathcal{Q}\mathcal{P}_1(0)|$. Then $\mathrm{d}\widetilde{\Psi}$ is integrable on $\partial\Delta$ and $\int_{\partial\Delta} \mathrm{d}\widetilde{\Psi} > 0$ since the part of $\partial\Delta$ included in the interior of Ω is non empty. We have

$$\int_{\partial\Delta} \mathrm{d}\widetilde{\Psi} = \sum_{n=0}^{+\infty} \int_{\partial\Delta \cap \Omega_n^{n+1}} \mathrm{d}\widetilde{\Psi}$$

Then $\displaystyle\int_{\partial\Delta \cap \Omega_n^{n+1}} \mathrm{d}\widetilde{\Psi} \longrightarrow 0$. Let us study what occurs on Ω_n^{n+1} for big n.

For $n \in \mathbb{N}$, we consider $u_n = u \circ f^{-n}$ and $v_n = v \circ f^{-n}$; the restriction of u_n to Ω_0^1 is equal to the restriction of u to Ω_n^{n+1}. The same is true for v. As in the existence proof, (u_n) and (v_n) have no line of diververgence.

So there exist two real sequences (a_n) and (b_n) such that $u_{n'} - a_{n'} \to \tilde{u}$ and $v_{n'} - b_{n'} \to \tilde{v}$ on Ω (in fact $a_n = u_n(P)$ and $b_n = v_n(P)$ for a point P in Ω). The functions $\tilde{u}$ and $\tilde{v}$ are two solutions of the Dirichlet problem asked in Theorem 2. The convergence is uniform for the derivatives on compact subsets. By changing our subsequence if necessary, we assume that $b_{n'} - a_{n'} \to \pm\infty$ or $b_{n'} - a_{n'} \to c \in \mathbb{R}$. We are interested in $\{u_n - v_n < 0\} \cap \Omega_0^1$; in a certain sense, these sets "converge" to $\{\tilde{u} - \tilde{v} < \lim b_{n'} - a_{n'}\} \cap \Omega_0^1$.

Let $\gamma : [0, |\mathcal{Q}\mathcal{P}_1(1)|] \to [\mathcal{Q}, \mathcal{P}_1(1)]$ be the parametrization by arc-length of the segment $[\mathcal{Q}, \mathcal{P}_1(1)]$, with $\gamma(0) = \mathcal{Q}$. We distinguish two cases.

• We assume that $b_{n'} - a_{n'} \to \pm\infty$. Let $\varepsilon > 0$ be less than $\dfrac{1}{3}\displaystyle\int_{\partial\Delta} \mathrm{d}\widetilde{\Psi}$ and $|\mathcal{Q}, \mathcal{P}_1(1)|/2$. Since $\tilde{u}$ and $\tilde{v}$ are bounded on $[\mathcal{Q}, \mathcal{P}_1(1)]$ and there is uniform convergence on $\gamma([\varepsilon, |\mathcal{Q}\mathcal{P}_1(1)| - \varepsilon])$, we can ensure that, for n' big enough, $\gamma([\varepsilon, |\mathcal{Q}\mathcal{P}_1(1)| - \varepsilon])$ is included in $\{u_{n'} - v_{n'} < 0\}$ or does not intersect $\{u_{n'} - v_{n'} < 0\}$ following the sign of the limit of $b_{n'} - a_{n'}$. For every n, $\displaystyle\int_{[\mathcal{Q}, \mathcal{P}_1(n)]} \mathrm{d}\widetilde{\Psi} = 0$ because of the boundary values of u and v. Then for big n':

$$\left| \int_{\partial\Delta^{n'+1} \cap [\mathcal{Q}, \mathcal{P}_1(n'+1)]} \mathrm{d}\widetilde{\Psi} \right| \leq 2\varepsilon \tag{4}$$

Equality (4) implies that $\displaystyle\int_{\partial^{n'+1}\Delta} \mathrm{d}\widetilde{\Psi} \leq 2\varepsilon$ for big n'. Then by passing to the limit, we obtain a contradiction with our hypothesis on ε and (2).

• We assume that $b_{n'} - a_{n'} \to c$. Let ε be as above. If the segment $\gamma([\varepsilon, |\mathcal{Q}\mathcal{P}_1(1)| - \varepsilon])$ is included in $\{\tilde{u} - \tilde{v} < c\}$ or does not intersect $\{\tilde{u} - \tilde{v} < c\}$ then the same property is true for $\{u_{n'} - v_{n'} < 0\}$ for big n'. Then same arguments as above give us a contradiction. Then we can ensure that there is a point in $\gamma([\varepsilon, |\mathcal{Q}\mathcal{P}_1(1)| - \varepsilon])$ where $\tilde{u} - \tilde{v} = c$. If $\tilde{u} = \tilde{v} + c$ on Ω, we have $\mathrm{d}\Psi_{u_{n'}} - \mathrm{d}\Psi_{v_{n'}} \to 0$ uniformly on $\gamma([\varepsilon, |\mathcal{Q}\mathcal{P}_1(1)| - \varepsilon])$. Then for n' big enough, the inequality (4) is true, this gives a contradiction. Then $\tilde{u} \neq \tilde{v} + c$ on Ω and there is a compact part of the boundary of $\{\tilde{u} - \tilde{v} < c\}$ that crosses the segment $\gamma([\varepsilon, |\mathcal{Q}\mathcal{P}_1(1)| - \varepsilon])$; let Γ denote this part of the boundary (Γ is oriented as boundary). We have $\displaystyle\int_\Gamma \mathrm{d}\Psi_{\tilde{u}} - \mathrm{d}\Psi_{\tilde{v}} > 0$ by Lemma 2 in [CK]. Γ is included in a compact part of Ω_0^2. Since $(u_{n'} - v_{n'})$ converges to $\tilde{u} - \tilde{v} - c$, we can ensure that, for n' big enough:

$$\int_{\partial\{u_{n'} - v_{n'} < 0\} \cap \Omega_0^2} \mathrm{d}\Psi_{u_{n'}} - \mathrm{d}\Psi_{v_{n'}} \geq \frac{1}{2} \int_\Gamma \mathrm{d}\Psi_{\tilde{u}} - \mathrm{d}\Psi_{\tilde{v}}$$

This implies that $\displaystyle\int_{\partial\Delta\cap\Omega_{n'}^{n'+2}} \mathrm{d}\widetilde{\Psi} \geq \frac{1}{2}\int_\Gamma \mathrm{d}\Psi_{\tilde{u}} - \mathrm{d}\Psi_{\tilde{v}} > 0$

We have a contradiction with $\displaystyle\int_{\partial\Delta\cap\Omega_n^{n+1}} \mathrm{d}\widetilde{\Psi} \longrightarrow 0$. then, our two solutions u and v differ only by a real additive constant. This ends the proof of Theorem 2.

3.5 Property of the solution 2

Let u be the solution given by Theorem 2. The first remark is that u can be built by the way used in the existence part of the proof. As a consequence, the conjugate function Ψ_u is non-negative on Ω.

From Corollary 1, there exists a constant $c \in \mathbb{R}$ such that $u \circ f = u + c$. We have the following result for such a situation.

Proposition 4. *Let $(\Omega, \mathcal{Q}, \varphi)$ be a periodic multi-domain with logarithmic singularity, and f be the isometry associated to its periodicity. Let v be a solution of the minimal surface equation on Ω such that:*

1. *there exists a constant $c \in \mathbb{R}$ such that $v \circ f = v + c$ and*
2. *if Ψ_v is the conjugate function to v with $\Psi_v(\mathcal{Q}) = 0$, Ψ_v is non-negative.*

Under these hypotheses, the constant c is non zero.

Proof. Let us assume that $c = 0$ and consider v only on the neighborhood $\mathcal{N}$ of the singularity point $\mathcal{Q}$. Since $v \circ f = v$, by taking the quotient with respect to f, the function v can be seen as a function $\widetilde{v}$ defined on $\mathcal{C}$ a flat disk of radius ρ_0 with a cone singularity at its center of angle $2q\pi$ minus the singularity point (i.e. $\mathcal{C}$ is $\{(\rho, \theta), \rho \in (0, \rho_0), \theta \in [0, 2q\pi]\}$ where $(\rho, 0)$ and $(\rho, 2q\pi)$ are identified, the metric on $\mathcal{C}$ is the polar metric). The graph of $\widetilde{v}$ is a minimal surface of $\mathcal{C} \times \mathbb{R}$. This surface is topologicaly an annulus then it is conformally parametrized by a Riemann surface R which is an annulus: there are two harmonic maps $X : R \to \mathcal{C}$ and $x_3 : R \to \mathbb{R}$ such that $x_3 = \widetilde{v} \circ X$. R is either $\{\zeta \in \mathbb{C} | 1 < |\zeta| < a\}$ for $1 < a \leq +\infty$ or $\{\zeta \in \mathbb{C} | 0 < |\zeta| < a\}$ for $0 < a \leq +\infty$. On $\mathcal{C}$, the function $\Psi_{\widetilde{v}}$ is well defined and satisfies $\Psi_{\widetilde{v}} \geq 0$ and $\Psi_{\widetilde{v}} = 0$ at the cone singularity by hypothesis. Since x_3 is harmonic, we can define on R its harmonic conjugate x_3^*. *A priori* x_3^* is multi-valued but, for a good choice of x_3^*, we have $x_3^* = \Psi_{\widetilde{v}} \circ X$, so x_3^* is single valued. As $\Psi_{\widetilde{v}} = 0$ at the cone singularity, $x_3^*(\zeta)$ converges to zero as either $|\zeta|$ tends to 1 when $R = \{\zeta \in \mathbb{C} | 1 < |\zeta| < a\}$ or $|\zeta|$ tends to zero when $R = \{\zeta \in \mathbb{C} | 0 < |\zeta| < a\}$.

When $R = \{\zeta \in \mathbb{C} | 0 < |\zeta| < a\}$, x_3^* is a harmonic function on a pointed disk, which has a continuous extension to the whole disk. Such an harmonic function can not have an isolated singularity, then the continuous extension is harmonic. It has an extremum at the origin, since $x_3^* \geq 0$ on R; this gives a contradiction.

When $R = \{\zeta \in \mathbb{C} | 1 < |\zeta| < a\}$, $x_3^* = 0$ on the unit circle. Then x_3^* extends to an harmonic function on $\{\zeta \in \mathbb{C} | \frac{1}{a} < |\zeta| < a\}$ by Schwartz reflection principle (the extension is defined by $x_3^*(\zeta) = -x_3^*(\frac{1}{\bar{\zeta}})$; see [ABR]). By taking the harmonic conjugate, x_3 also extends to $\{\zeta \in \mathbb{C} | \frac{1}{a} < |\zeta| < a\}$. Along the unit circle $\mathbb{S}^1$, $x_3^* = 0$, then $\nabla x_3^* \cdot n = 0$ with n the unit tangent vector to $\mathbb{S}^1$. Besides by Rolle's Theorem there is a point on $\mathbb{S}^1$ where $\nabla x_3 \cdot n = 0$. At this point, $\nabla x_3^* = 0$; the local structure of a critical point of a non constant harmonic function implies that x_3^* must be negative on R in the neighborhood of this point, this contradicts $x_3^* \geq 0$ on R. □

Remark 2. Proposition 4 proves that, for a solution u of the Dirichlet problem asked in Theorem 2, the constant c such that $u \circ f = u + c$ is non-zero. Then u can not pass to the quotient by f. But since $u \circ f = u + c$ the derivatives of u are invariant by f: these derivatives are well defined on the quotient. Besides, this implies that the function Ψ_u is also invariant by f.

4 The regularity over the singularity point

In this section we understand some regularity properties of the graphs that we build in Section 3 thanks to Theorem 2.

As in the preceding section, we consider a polygon $V \in \mathbb{R}^2$ that bounds a multi-domain with cone singularity (D, Q, ψ); using Construction 1 and 3, we get a multi-domain with logarithmic singularity $(\Omega, \mathcal{Q}, \varphi)$. We assume that D satisfies the hypothesis H. Theorem 2 allows us to construct a minimal surface in $\mathbb{R}^3$ which is a graph over Ω. We want to use this surface to build an r-noid with genus 1 and horizontal ends of type I with V as flux polygon therefore the graph needs to be regular up to its boundary. In this section, we understand the behaviour of this minimal surface over the singularity point $\mathcal{Q}$: we shall prove the following result.

Theorem 3. *Let u be a solution on Ω of the Dirichlet problem asked in Theorem 2. Then the surface $\{\varphi_{\psi(Q)}(x), u(x)\}$ for $x \in \mathcal{N}$ is a minimal* *surface with boundary and its boundary is* $\{\varphi_{\psi(Q)}(x), u(x)\}$ *for* $x \in \partial\mathcal{N}$ *and the vertical straight line passing through* $\psi(Q)$.

4.1 Proof of Theorem 3

First we fix some notations : we write $\mathcal{D}_\rho = \{M \in \mathcal{D} \,|\, d(M, \mathcal{O}) < \rho\}$ and we recall that, if $G(\rho, \theta) = (\rho\cos\theta, \rho\sin\theta)$, the graph of a function u on $\mathcal{D}_\rho$ is given by $\{(G(M), u(M))\}_{M \in \mathcal{D}_\rho \setminus \{\mathcal{O}\}}$. By definition, the neighborhood $\mathcal{N}$ of the singularity point $\mathcal{Q}$ in Ω is isometric to $\mathcal{D}_{\rho_0}$ for some $\rho_0 > 0$. Theorem 3 is then a consequence of the following result :

Theorem 4. *Let* ρ *be positive and* v *be a non-constant solution of the minimal surface equation on* $\mathcal{D}_\rho$ *such that :*

1. *there exist* $c \in \mathbb{R}$ *and* $n \in \mathbb{N}^*$ *such that* $v \circ R_{2n\pi} = v + c$ *and*
2. *if* Ψ_v *is the conjugate function to* v *with* $\Psi_v(\mathcal{O}) = 0$, Ψ_v *is non-negative.*

Then, near the singularity point $\mathcal{O}$*, the graph of* v *is a minimal surface with boundary and its boundary is the vertical straight line passing through* $(0,0)$.

Proof. Let Σ denote the graph of v. From the first hypothesis, Σ is invariant by translation by the vector $(0, 0, c)$. Then the harmonic functions x_1 and x_2 are invariant by this isometry, it is also true for the harmonic conjugate function x_3^* since $x_3^*(G(M), v(M)) = \Psi_v(M)$. The quotient surface of Σ by the vertical translation is homeomorphic to the quotient of $\mathcal{D}_\rho$ by $R_{2n\pi}$ which is an annulus; this quotient is then conformal to some A_{r_1,r_2} ($A_{r_1,r_2} = \{\zeta \in \mathbb{C} \,|\, r_1 < |\zeta| < r_2\}$ with $r_1 \in \mathbb{R}_+$ and $r_2 \in \mathbb{R}_+^* \cup \{\infty\}$). There exist a conformal covering map $\pi : \Sigma \to A_{r_1,r_2}$ and three harmonic functions on A_{r_1,r_2}, σ_1, σ_2 and σ_3^*, that satisfy :

$$\pi^*(\sigma_1) = x_1, \quad \pi^*(\sigma_2) = x_2, \quad \pi^*(\sigma_3^*) = x_3^*$$

We can assume that the covering map π sends the points of Σ above neighborhoods of $\mathcal{O}$ to neighborhoods of the circle $\{|\zeta| = r_1\}$. This implies that, as $|\zeta|$ goes to r_1, $\big((\sigma_1(\zeta), \sigma_2(\zeta_2)\big)$ goes to $(0,0)$, since the singularity point $\mathcal{O}$ is send to $(0,0)$ by G and $\sigma_3^*(\zeta)$ goes to 0. By the second hypothesis, σ_3^* is a non-negative harmonic function. If $r_1 = 0$, σ_3^* is a non-negative harmonic function which is defined on a punctured disk and has a continuous extension to the whole disk. This extension is harmonic and has an extremum at the origin then it is constant. Since v is non-constant, σ_3^* can not be constant then $r_1 > 0$.

Let S_{r_1,r_2} denote the strip $\{\zeta \in \mathbb{C} \mid \ln r_1 < \Re(\zeta) < \ln r_2\}$. The exponential map $\exp : S_{r_1,r_2} \to A_{r_1,r_2}$ is an universal covering of A_{r_1,r_2}. Since Σ is simply connected, there exists a conformal isomorphism Φ making the following diagram commutative:

$$\begin{tikzcd}
\Sigma \arrow[rr, "\Phi", "\sim"'] \arrow[dr, "\pi"'] & & S_{r_1,r_2} \arrow[dl, "\exp"] \\
& A_{r_1,r_2} &
\end{tikzcd}$$

Using Φ, we assume that the harmonic functions x_i for $i = 1, 2, 3$ and x_3^* are defined on S_{r_1,r_2}. As $\Re(\zeta)$ goes to $\ln r_1$, $x_1(\zeta)$, $x_2(\zeta)$ and $x_3^*(\zeta)$ go to 0. The definition of these three harmonic functions then extends by Schwarz reflection with respect to $\{\Re(\zeta) = \ln r_1\}$ to S_{r_0,r_2} with $\ln r_0 = 2\ln r_1 - \ln r_2$. Then, by taking the conjugate of x_3^*, x_3 extends to S_{r_0,r_2}.

The map (x_1, x_2, x_3), which is defined on S_{r_0,r_2}, extends Σ across the vertical line passing by $(0, 0)$ to a minimal surface which is symmetric with respect to this axis. As in the proof of Theorem 3 in [Ma1], the fact that x_3^* is non-negative in S_{r_1,r_2} implies that this minimal surface does not have any branching point along $\{\Re(\zeta) = \ln r_1\}$. The graph of v is then regular up to its boundary near the singularity point $\mathcal{O}$ and this boundary is the vertical straight line passing by $(0, 0)$. $\square$

Let us now give the proof of Theorem 3.

Proof of Theorem 3. Let u be a solution on Ω of the Dirichlet problem asked in Theorem 2. A remark in subsection 3.5 says us that $\Psi_u \geq 0$ when $\Psi_u(\mathcal{Q}) = 0$. From Corollary 1, there exists $c \in \mathbb{R}$ such that $u \circ f = u + c$ and $c \neq 0$ by Proposition 4. Then u satisfies Theorem 4 hypotheses and Theorem 3 is proved. $\square$

4.2 A technical lemma

In the following of the paper, we shall need the following result that explains the behaviour of a graph bounded by a vertical line.

Lemma 4. *Let v be a solution of* (MSE) *on a sector of $\mathcal{D}$ $\{(r,\theta) \in \mathcal{D} \mid r \leq r_0,\ \alpha_1 \leq \theta \leq \alpha_2\}$ ($\alpha_1 < 0 < \alpha_2$). Suppose that the graph of v in $\mathbb{R}^3$ is a complete minimal surface with boundary and the part of the boundary over the origin is an interval of the vertical straight-line passing by the origin. Then if v is bounded on $\{\theta = 0\}$, $\lim_{r\to 0} v(r, 0)$ exists. Besides the normal* *to the graph of v at the point $(0,0,\lim_{r\to 0} v(r,0))$ is $\pm(0,1,0)$ and the curve $r \mapsto (r,0,v(r,0))$ is normal to the boundary at the limit point.*

Proof. We denote by Σ the graph of v. All the cluster points of $(r,0,v(r,0))$ as r goes to 0 are in the boundary of the graph and more precisely in the part of the boundary consisting in the interval of the vertical straight-line passing by the origin. Besides, the curve $r \mapsto (r,0,v(r,0))$ is in the intersection of the vertical plane $\{y=0\}$ and the graph of v, this curve is then in the intersection of two minimal surfaces. Let $(0,0,a)$ be a cluster point of $(r,0,v(r,0))$ as r goes to 0. Since the boundary of Σ is a vertical straight-line near $(0,0,a)$, the normal to Σ at this point is $(\cos\alpha, \sin\alpha, 0)$ for some $-\pi \leq \alpha < \pi$. Near $(0,0,a)$, Σ is then a graph over the vertical plane $\{x\cos\alpha + y\sin\alpha = 0\}$ and is tangent to this plane at $(0,0,a)$. Then, if $|\alpha| \neq \pi/2$, the intersection of Σ and $\{y=0\}$ is only the vertical straight-line near $(0,0,a)$: no $(r_n, 0, v(r_n,0))$ can converge to $(0,0,a)$. The normal at $(0,0,a)$ is then $\pm(0,1,0)$. Since Σ is a graph over $\{y=0\}$ the intersection of Σ with this plane near $(0,0,a)$ is the vertical straight-line and one smooth curve passing by $(0,0,a)$ such that $x \neq 0$ along them; this curve is normal to the vertical straight-line since $(0,0,a)$ is not a branching point of the graph. By continuity, this curve is $(r,0,v(r,0))$; this prove that $\lim_{r\to 0} v(r,0) = a$ and ends the proof. $\square$

4.3 Property of the solution 3

Let u be a solution of the Dirichlet problem asked in Theorem 2. Now we can understand the boundary behaviour of the graph of u in $\mathbb{R}^3$.

From Theorem 3, we know that, over the neighborhood $\mathcal{N}$ of $\mathcal{Q}$, the graph is bounded by the vertical straight-line passing by the point $\psi(Q)$.

The other points where the graph of u has boundary components are the vertices of Ω. This points statisfy the hypotheses of Theorem 3 in [Ma1]. Then the graph of u is a minimal surface with boundary; it is bounded by vertical straight-lines near the vertices and the singularity point $\mathcal{Q}$.

Besides we have $\Psi_u(\mathcal{Q}) = 0 = \Psi_u(\mathcal{V})$. This implies that the conjugate surface of the graph of u, which is bounded by the conjugate of the boundary of the graph of u, has its boundary included in the plane $\{z=0\}$.

Let Σ denote the graph of u over Ω_0^1, a period of Ω. Σ is a minimal surface with boundary according to Lemma 4. From the remark of the above paragraph, its conjugate surface can be extend by symmetry with respect to the plane $\{z=0\}$; let Σ^* denote this symmetric surface with boundary. We then have the following result.

Lemma 5. Σ^* *is of finite total curvature and its total curvature is* $4\pi r$.

Proof. Using arguments given in [Ma1], we can prove that Σ is of finite total curvature, this implies that the same is true for Σ^*. Then, as in Proposition 2.2 in [HK], each catenoidal end gives a contribution of 2π to the total curvature (see [JM], for the original arguments) and, using Gauss-Bonnet Theorem, we compute the value of the total curvature and get $4\pi r$. $\square$

5 The period problem

In this section we explain the period problem that appears in the construction of genus 1 r-noids. We then solve this problem in the case of our main existence theorem.

5.1 The general case

We now try to build a r-noid with genus 1 and horizontal ends of type I for a given polygon of flux.

Let $V = (v_1, \dots, v_r)$ be a polygon that bounds a multi-domain with cone singularity (D, Q, ψ). Using Construction 1 and Construction 3 as in Section 3, we get a multi-domain with logarithmic singularity $(\Omega, \mathcal{Q}, \varphi)$ and a solution u on Ω of the Dirichlet problem asked in Theorem 2; as in Section 3, we assume that the period $2q\pi$ of Ω is the angle at the cone singularity of D. Let us consider Ω_0^1 one period of the multi-domain Ω and Σ the graph of u over Ω_0^1. We know, because of the result of the preceding sections, that Σ is a minimal surface bounded by $r-1$ vertical lines passing by the vertices P_i $(i \neq 1)$ of the polygon V, two vertical half-lines over P_1, a vertical segment over $\psi(Q)$ and two curves over the segment $[\psi(Q), \psi(P_1)]$. From a remark in Section 3.5, the conjugate surface of Σ is included in $\{z \geq 0\}$ and the conjugates of the $r-1$ vertical lines, the two vertical half-lines and the vertical segment are exactly the intersection of the conjugate surface with the plane $\{z = 0\}$. We then can extend the conjugate surface by symmetry with respect to this plane, we get a new surface that we denote by Σ^*. Σ^* is a solution for the Plateau problem at infinity for the data V iff the two components of boundary of Σ^*, coming, by conjugation, from the two curves which are over $[\psi(Q), \psi(P_1)]$, glue together such a way that Σ^* has no boundary.

In fact this two components of boundary differ from a translation, how can we compute the vector of this translation? From the function u, we derive three closed 1-forms $\mathrm{d}X_1^*$, $\mathrm{d}X_2^*$ and $\mathrm{d}X_3^*$ on Ω which are the differential of the three coordinate functions of the conjugate surface to the graph of u (these 1-forms depend only on the first derivatives of u). For example, we have $\mathrm{d}X_3^* = \mathrm{d}\Psi_u$. In $\mathcal{N}$, the neighborhood of $\mathcal{Q}$ in Ω, we consider the path $\Gamma : \theta \mapsto (\rho, \theta)$ for some $\rho < \rho_0$ and $\theta \in [0, 2q\pi]$; Γ is a lift of a generator of $\pi_1(D \backslash \{Q\})$. Then the two components of boundary of Σ^* differ from the following vector (it is called the *period vector*):

$$\left(\int_\Gamma \mathrm{d}X_1^*,\ \int_\Gamma \mathrm{d}X_2^*,\ \int_\Gamma \mathrm{d}X_3^* \right)$$

Since $\mathrm{d}X_3^* = \mathrm{d}\Psi_u$ and as Ψ_u is invariant by f, $\displaystyle\int_\Gamma \mathrm{d}X_3^* = 0$. Obviously, the value of the integrals is the same for every Γ which is the lift of a generator of $\pi_1(D \backslash \{Q\})$; in fact, from Remark 2, since $\mathrm{d}X_i^*$ depends only on the derivatives of u, $\mathrm{d}X_i^*$ is well defined on $D \backslash Q$

Then the question of the existence of a solution to the Plateau problem at infinity for the data V becomes: to know if there exists (D, Q, ψ) bounded by V such that the associated period vector is zero; this is the *period problem.* To solve this problem, we can use different technics. One of them is to use the symmetries of the polygon V, this is what we do in Section 7 for the proof of Theorem 6. For Theorem 1, we use an other technic (see the next subsection).

Remark 3. We now give some explanations on Remark 1 and the hypothesis H on D. Let $V = (v_1, ..., v_r)$, (D, Q, ψ), $(\Omega, \mathcal{Q}, \varphi)$ and u be as above (D sastisfies the hypothesis H). We also suppose the the period vector associated to D is zero. Then Σ^* which is the conjugate surface to Σ, the graph of u over Ω_0^1, extended by symmetry is a r-noid with genus 1 and horizontal ends of type I having V as flux polygon. Let f be the isometry of Ω associated to its periodicity. Let $a \in \mathbb{N}^*$, then the quotient of $\mathcal{W}$ by the group $\{f^{an}\}_{n \in \mathbb{Z}}$ ($\mathcal{W}$ is given by Construction 1 applied to D) is a multi-domain with cone singularity that bounds the polygon

$$V_a = (\underbrace{v_1, \ldots, v_r, \ldots\ldots, v_1, \ldots, v_r}_{a \text{ times}})$$

Besides, if Σ_a is the graph of u on Ω_0^a, the conjugate surface Σ_a^* of Σ_a extended by symmetry is a ar-noid with genus 1 and horizontal ends of type I having V_a as flux polygon. In fact Σ_a^* is just Σ^* that we cover a times. Then we also can find solution to the Plateau problem at infinity for D that does not satisfy the hypothesis H.

5.2 Proof of Theorem 1: the period map

In this subsection, we give the proof of Theorem 1. We use a result which is proved in the next section.

Proof of Theorem 1. Let $V = (v_1, \ldots, v_r)$ be a polygon bounded by an immersed polygonal disk $(\mathcal{P}, \psi)$. For each A in the interior of $\mathcal{P}$, Construction 1 gives $(\mathcal{W}_A, \mathcal{A}, \varphi_A)$ a multidomain with a logarithmic singularity. Then in applying Construction 3 and Theorem 2, we get a periodic multi-domain with logarithmic singularity $(\Omega_A, \mathcal{A}, \varphi_A)$ and a function u_A defined on Ω_A. We then have the three closed 1-forms $\mathrm{d}X_i^*(A)$. To prove Theorem 1, we need to find a point $A \in \overset{\circ}{\mathcal{P}}$ such that the period vector associated to the above constuction for A is 0.

In fact, we have defined a map from the interior of $\mathcal{P}$ to $\mathbb{R}^2$ which associates to every point $A \in \overset{\circ}{\mathcal{P}}$ the vector $\left(\int_\Gamma \mathrm{d}X_1^*(A),\ \int_\Gamma \mathrm{d}X_2^*(A)\right)$, this map is the period map and will be denoted by Per. We want to prove that this map vanishes at one point of $\overset{\circ}{\mathcal{P}}$. The period map satisfies the following proposition.

Proposition 5. *The period map Per is continuous on the interior of $\mathcal{P}$.*

Proof. Let us consider a sequence (A_n) of points in $\overset{\circ}{\mathcal{P}}$ that converges to $A \in \overset{\circ}{\mathcal{P}}$. Let Γ be a closed path in $\mathcal{P} \backslash \{A, A_0, A_1, \ldots, A_n, \ldots\}$ such that for every n, Γ is a generator of $\pi_1(\mathcal{P}\backslash\{A_n\})$ and Γ is a generator of $\pi_1(\mathcal{P}\backslash\{A\})$. From Remark 2, the derivatives of u_A are well defined on $\mathcal{P}\backslash\{A\}$. Then the two closed 1-forms $\mathrm{d}X_1^*(A)$ and $\mathrm{d}X_2^*(A)$ are well defined on $\mathcal{P}\backslash\{A\}$. The same is true for A_n. We then have:

$$Per(A_n) = \left(\int_\Gamma \mathrm{d}X_1^*(A_n),\ \int_\Gamma \mathrm{d}X_2^*(A_n)\right)$$

On $\mathcal{P}$, we have the sequence of the derivatives of u_{A_n} and these derivatives converge to the derivatives of u_A if there is no line of divergence (a line of divergence is a phenomenon linked to the behaviour of the first derivatives so we can use this argument in this case). Since the arguments used in the proof of the existence part of Theorem 2 are always true, there is no line of divergence and $\mathrm{d}X_1^*(A_n) \to \mathrm{d}X_1^*(A)$ and $\mathrm{d}X_2^*(A_n) \to \mathrm{d}X_2^*(A)$, the convergence is uniform along Γ. This proves that $Per(A_n) \to Per(A)$ by integration. □

The idea to prove that the period map vanishes at one point is to compute its degree. First we make a little change in the definition of Per: let A be a point of $\overset{\circ}{\mathcal{P}}$, if $||Per(A)|| \leq 1$ we do not change the value of $Per(A)$ but if $||Per(A)|| \geq 1$ the new value of $Per(A)$ is $Per(A)/||Per(A)||$. The new period map is still continuous and is bounded in norm by 1. We have the following proposition

Proposition 6. *There exists $\widetilde{\mathcal{P}}$ a topological space which is homeomorphic to the closed unit disk such that:*

- *There exists h an homeomorphism from the interior of $\widetilde{\mathcal{P}}$ into $\overset{\circ}{\mathcal{P}}$.*
- *The map $Per \circ h : \overset{\circ}{\widetilde{\mathcal{P}}} \to \mathbb{R}^2$ continously extends to the whole $\widetilde{\mathcal{P}}$.*
- *The extended map $Per \circ h$ does not vanish on $\partial\widetilde{\mathcal{P}}$ and its degree along it is non zero.*

The proof of this proposition is long and the whole Section 6 is devoted to it. The first two points are the consequence of Propositions 7 and 8; the computation of the degree is made in Theorem 5. The proof consists in understanding the limit of the period map when we are close to $\partial\mathcal{P}$.

Now, using Proposition 3.20 in [Fu], Proposition 6 proves that there is a point in the interior of $\widetilde{\mathcal{P}}$ where $Per \circ h$ vanishes. Then there exists a point $A \in \overset{\circ}{\mathcal{P}}$ where $Per(A) = 0$. This ends the proof of Theorem 1. □

6 The period map on the boundary of $\mathcal{P}$

In this section, we end the proof of Theorem 1 in proving Proposition 6. Using notations introduced in Subsection 5.2, we study the period map near the boundary of $\mathcal{P}$ (Proposition 7 and 8) and compute its degree along the boundary (Theorem 5). As in [Ma1], $\Omega(\mathcal{P})$ denotes the multi-domain obtained when we glue to $\mathcal{P}$ along every edge $[P_i, P_{i+1}]$ a half strip isometric to $[P_i, P_{i+1}] \times \mathbb{R}_+$.

6.1 The behaviour on the edges

We begin our study by the limiting behaviour of the period map when the singularity point A moves to an edge of the polygon. The proof of the following result uses only the study of the sequence (u_n) of solutions to the Dirichlet problem.

Proposition 7. *Let (A_n) be a sequence in $\overset{\circ}{\mathcal{P}}$ such that $A_n \to A$ where A is a point in the interior of one edge of the boundary of $\mathcal{P}$. Then $Per(A_n)$ converges to $\mathrm{d}\psi|_A(N)$ where N is the outer unit normal to the edge at A, we recall that ψ is the developping map of $\mathcal{P}$.*

Proof. For every A_n, the covering map $\pi : \mathcal{W}_{A_n} \to \mathcal{P}$ extends to a covering map $\pi : \Omega_{A_n} \to \Omega(\mathcal{P})$ and the derivatives of u_{A_n} are then well defined on $\Omega(\mathcal{P})\backslash\{A_n\}$, by Remark 2. We assume that the point A is in the interior of the edge $[P_1, P_2]$ and that $|P_1P_2| = 2$. By choosing a good chart, we can suppose that $[-1,1] \times \mathbb{R}_+$ is the half-strip glued to this edge. Let D_r denote the domain in $\mathbb{R}^2$ which is the intersection of the domain $y \le 0$ and the disk of center $(0,r)$ and radius $\sqrt{r^2+1}$. By choosing $r > 0$ big enough, D_r is a neighborhood of the edge $[P_1, P_2]$ in $\mathcal{P}$.

A has $(a,0)$ as coordinates $(-1 < a < 1)$ and the points $A_n = (a_n, b_n)$ lie in D_r for big n $(b_n < 0)$. We have $a_n \to a$ and $b_n \to 0$. For every n, we denote by L_n the half straight-line $\{(a_n, b_n + t)\}_{t\ge 0}$ and L the half straight-line $\{(a,t)\}_{t\ge0}$. Using the covering map π and u_{A_n}, we define on $\Omega(\mathcal{P})\backslash L_n$ a function u_n which has the same derivatives as u_{A_n}; u_n is a solution of (MSE) and takes the values $+\infty$ on $\pi(\mathcal{L}_i^+)$ and $-\infty$ on $\pi(\mathcal{L}_i^-)$. Let us study the convergence of (u_n). By the arguments of the proof of Theorem 2, (u_n) has no line of divergence in $\Omega(\mathcal{P})\backslash L$. Then for a subsequence, (u_n) converges to a solution u of (MSE) on $\Omega(\mathcal{P})\backslash L$. By Lemma A.2, u takes the values $+\infty$ on $\pi(\mathcal{L}_i^+)$ and $-\infty$ on $\pi(\mathcal{L}_i^-)$. We then need to understand its behaviour near L to know the function u. Since in our normalization $\Psi_{u_n}(A_n) = 0$, we fix $\Psi_u(A) = 0$

Lemma 6. *With this normalization, $\Psi_u(a,t) = t$ for $t \ge 0$.*

Proof. Since Ψ_u is 1-Lipschitz continuous, $\Psi_u(a,t) \le t$. Let us suppose that for some t_0: $\Psi_u(a,t_0) = t_0 - \varepsilon$ with $\varepsilon > 0$. Then for $t > t_0$, $\Psi_u(a,t) \le t - \varepsilon$. We have $\Psi_u(1,t) = t$ for every $t \ge 0$ because u takes the value $+\infty$ along $\pi(\mathcal{L}_0^+)$. We have:

$$\varepsilon \le \Psi_u(1,t) - \Psi_u(a,t) = \lim_{n\to+\infty} \Psi_{u_n}(1,t) - \Psi_{u_n}(a,t) = \lim_{n\to+\infty} \int_{[(a,t),(1,t)]} \mathrm{d}\Psi_{u_n}$$

By Lemma 2, the integral is less than $2\sqrt{2}\frac{1-a}{t}$ for big t. So it tends to 0; this gives us a contradiction. □

Lemma 6 tells us, by Lemma A.2, that u takes the value $+\infty$ on one side of L and $-\infty$ on the other side; more precisely, $u(a+\eta,t)$ tends to $+\infty$

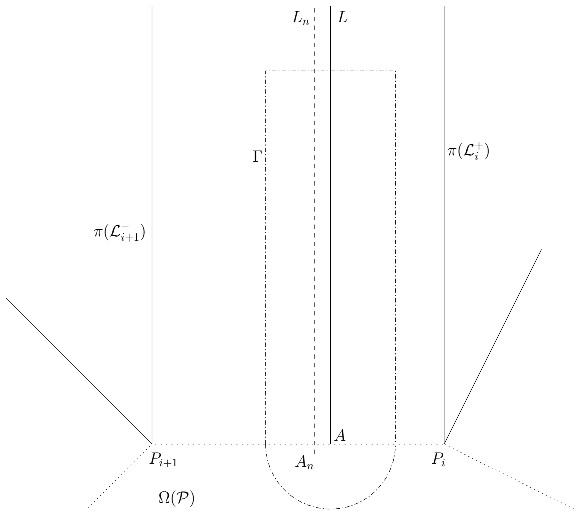

Figure 3:

(resp. $-\infty$) if η tends to 0 by negative value (resp. positive value). There is only one solution for the Dirichlet problem for such boundary condition (we apply Theorem 7 in [Ma1] with the polygon $(\overrightarrow{P_1A}, \overrightarrow{AP_2}, v_2, \cdots, v_r)$). The limit for a subsequences of (u_n) is then unique. Then the sequence (u_n) converges to the function u.

In $[-1,1] \times \mathbb{R}_+ \cup D_r$, the 1-form $\mathrm{d}X_1^*(A_n)$ and $\mathrm{d}X_2^*(A_n)$ are:

$$\mathrm{d}X_1^*(A_n) = \frac{q_n p_n}{W_n}\mathrm{d}x + \frac{1+q_n^2}{W_n}\mathrm{d}y \qquad \mathrm{d}X_2^*(A_n) = -\frac{1+p_n^2}{W_n}\mathrm{d}x - \frac{p_n q_n}{W_n}\mathrm{d}y$$

with p_n and q_n the first derivatives of u_n (see [Os]). Using these expressions, we also define the 1-forms $\mathrm{d}X_1^*(A)$ and $\mathrm{d}X_2^*(A)$. Let η_1 be a small positive number, $\eta_2 < \eta_1$ and l positive numbers. Let Γ be the closed path which consists in the segment $[(a+\eta_1, l), (a-\eta_1, l)]$, the segment $[(a-\eta_1, l), (a-\eta_1, 0)]$, the half circle in D_r of center A and radius η_1 and the segment $[(a+\eta_1, 0), (a+\eta_1, l)]$. Besides we call Γ_1 the part of Γ consisting in the two vertical segments and the half circle, Γ_2 the union of the two segments $[(a+\eta_1, l), (a+\eta_2, l)]$ and $[(a-\eta_2, l), (a-\eta_1, l)]$ and Γ_3 the segment $[(a+\eta_2, l), (a-\eta_2, l)]$ (see Figure 3). For every η_1, η_2 and l, there exists $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$, $Per(A_n)$ is computed by:

$$\left(\int_\Gamma \mathrm{d}X_1^*(A_n), \int_\Gamma \mathrm{d}X_2^*(A_n)\right)$$

Let $0 < \alpha < 1$. By Lemma 2, we have $\dfrac{|q_n|}{|p_n|} \le \sqrt{2}\dfrac{2}{l}(1 - \dfrac{4}{l^2})^{-1}$ on $\Gamma_2 \cup \Gamma_3$. Then in choosing l big enough, we can ensure that:

$$\left| \int_{\Gamma_2+\Gamma_3} \mathrm{d}X_1^*(A_n) \right| < \frac{\alpha}{2} \int_{\Gamma_2+\Gamma_3} \mathrm{d}X_2^*(A_n)$$

For $i \in \{1, 2\}$, we have:

$$\lim_{n\to+\infty} \int_{\Gamma_i} \mathrm{d}X_1^*(A_n) = \int_{\Gamma_i} \mathrm{d}X_1^*(A) \qquad \lim_{n\to+\infty} \int_{\Gamma_i} \mathrm{d}X_2^*(A_n) = \int_{\Gamma_i} \mathrm{d}X_2^*(A)$$

We have $\lim\limits_{\eta_2\to 0} \displaystyle\int_{\Gamma_2} \mathrm{d}X_2^*(A) \to +\infty$; this assertion is due to Lemma 1 in [JS] which implies that, as η goes to zero $\dfrac{p}{W}(a+\eta, l) \longrightarrow 1$ and $p(a+\eta, l) \ge C/\eta$ for some constant C. We then choose η_2 such that for big n, we have

$$\frac{\left| \int_{\Gamma_1} \mathrm{d}X_1^*(A_n) \right|}{\int_{\Gamma_2} \mathrm{d}X_2^*(A_n)} < \frac{\alpha}{2} \quad \text{and} \quad \frac{\left| \int_{\Gamma_1} \mathrm{d}X_2^*(A_n) \right|}{\int_{\Gamma_2} \mathrm{d}X_2^*(A_n)} < \frac{\alpha}{8}$$

This implies first that $\lim\limits_{n\to+\infty} \displaystyle\int_{\Gamma} \mathrm{d}X_2^*(A_n) = +\infty$, since $\displaystyle\int_{\Gamma_3} \mathrm{d}X_2^*(A_n) \ge 0$; then the period at A_n, for big n, is renormalized and must have a non negative second coordinate. Secondly, for big n, we have:

$$\begin{aligned}
\frac{\left| \int_{\Gamma} \mathrm{d}X_1^*(A_n) \right|}{\left| \int_{\Gamma} \mathrm{d}X_2^*(A_n) \right|} &\le \frac{\left| \int_{\Gamma_1} \mathrm{d}X_1^*(A_n) \right| + \left| \int_{\Gamma_2} \mathrm{d}X_1^*(A_n) \right| + \left| \int_{\Gamma_3} \mathrm{d}X_1^*(A_n) \right|}{-\left| \int_{\Gamma_1} \mathrm{d}X_2^*(A_n) \right| + \int_{\Gamma_2} \mathrm{d}X_2^*(A_n) + \int_{\Gamma_3} \mathrm{d}X_2^*(A_n)} \\
&\le \frac{\frac{\alpha}{2} \int_{\Gamma_2} \mathrm{d}X_2^*(A_n) + \frac{\alpha}{2} \int_{\Gamma_2} \mathrm{d}X_2^*(A_n) + \alpha \int_{\Gamma_3} \mathrm{d}X_2^*(A_n)}{-\frac{\alpha}{8} \int_{\Gamma_2} \mathrm{d}X_2^*(A_n) + \int_{\Gamma_2} \mathrm{d}X_2^*(A_n) + (1 - \frac{\alpha}{8}) \int_{\Gamma_3} \mathrm{d}X_2^*(A_n)} \\
&\le \frac{\alpha \int_{\Gamma_2+\Gamma_3} \mathrm{d}X_2^*(A_n)}{(1 - \frac{\alpha}{8}) \int_{\Gamma_2+\Gamma_3} \mathrm{d}X_2^*(A_n)} \le \frac{\alpha}{1 - \frac{\alpha}{8}}
\end{aligned}$$

This proves that the renormalized period converges to the vector $(0, 1)$; this is what we want to prove. □

6.2 The behaviour at the vertices

Because of Proposition 7, it is clear that we can not extend the period map to the vertices and obtain a continuous map on the boundary. This explain why Proposition 6 introduces $\widetilde{\mathcal{P}}$. This new topological space is obtained by making a blow-up of $\mathcal{P}$ at its vertices.

Let P_i be a vertex of $\mathcal{P}$, there exists $\alpha > 0$ such that a neighborhood of P_i is isometric to $\{(\rho, \theta),\ 0 \le \rho < \mu, 0 \le \theta \le \alpha\}$ with the polar metric $\mathrm{d}\rho^2 + \rho^2 \mathrm{d}\theta^2$. A blow-up at P_i consists in replacing the point P_i with the segment of all the points $(P_i, \theta)_{0 \le \theta \le \alpha}$. The topology near these new points is: if (A_n) is a sequence in $\overset{\circ}{\mathcal{P}}$ that converges to P_i in the original topology, we say that (A_n) converges to (P_i, θ) if $\theta_n \to \theta$ where (ρ_n, θ_n) are the coordinates of A_n near P_i. We then define $\widetilde{\mathcal{P}}$ in making this blow-up at all the vertices, we get a new topological space which is still homeomorphic to the closed unit disk. Besides its interior is equal to the interior of $\mathcal{P}$. We have proved the first item of Proposition 6. The question is now to understand what is the limit of $Per(A_n)$ when (A_n) tends to a boundary point (P_i, θ).

Let us consider the case where $i = 1$; a neighborhood of P_1 in $\mathcal{P}$ is $\{(\rho, \theta),\ 0 \le \rho < \mu, 0 \le \theta \le \alpha\}$. We know that there is a bijection between the Alexandrov-embedded r-noid with genus 0 and horizontal ends and the polygonal immersed disk (see [CR]). This bijection sends $\mathcal{P}$ to a r-noid $\Sigma(\mathcal{P})$ where $\Sigma(\mathcal{P})^+$ is the conjugate surface to the graph of the function $u_{\mathcal{P}}$ over the multi-domain $\Omega(\mathcal{P})$. $u_{\mathcal{P}}$ is the solution of (MSE) on $\Omega(\mathcal{P})$ which takes the value $+\infty$ on $\pi(\mathcal{L}_i^+)$ and $-\infty$ on $\pi(\mathcal{L}_i^-)$; $u_{\mathcal{P}}$ exists and is unique by Theorem 7 in [Ma1]. The graph of $u_{\mathcal{P}}$ is bounded by r vertical straight-lines passing by the vertices of $\mathcal{P}$ (see [CR] and [Ma1]). Let us consider $\mathcal{C}$ the conjugate of the straight-line passing by P_1. $\mathcal{C}$ is a strictly convex curve and there exists $\gamma : (-\pi/2, \alpha + \pi/2) \to \{z = 0\}$ a parametrization of $\mathcal{C}$ by its normal (see Appendix B). The end of this subsection is devoted to the proof of the following result.

Proposition 8. *Let (A_n) be a sequence of points in the interior of $\mathcal{P}$ converging to the point (P_1, θ) of $\partial\widetilde{\mathcal{P}}$. Then $Per(A_n)$ converges to:*

- $(0, -1)$ *if* $\theta = 0$,
- $(-\sin\alpha, \cos\alpha)$ *if* $\theta = \alpha$,
- $\overrightarrow{\gamma(\theta - \pi/2)\gamma(\theta + \pi/2)}$ *or* $\dfrac{\overrightarrow{\gamma(\theta - \pi/2)\gamma(\theta + \pi/2)}}{||\overrightarrow{\gamma(\theta - \pi/2)\gamma(\theta + \pi/2)}||}$, *following the sign of* $||\overrightarrow{\gamma(\theta - \pi/2)\gamma(\theta + \pi/2)}|| - 1$, *if* $\theta \in (0, \alpha)$.

6.2.1 Preliminaries

Let (A_n) be a sequence in $\overset{\circ}{\mathcal{P}}$ that converges to (P_1, θ). A neighborhood of P_1 in $\mathcal{P}$ is $T(0, \alpha, \mu) = \{(\rho, \theta),\ 0 \le \rho < \mu, 0 \le \theta \le \alpha\}$. In $\Omega(\mathcal{P})$, a neighborhood of P_1 is $T(-\frac{\pi}{2}, \alpha + \frac{\pi}{2}, \mu)\}$. Let u_n be the restriction of the solution u_{A_n} to the period $\Omega_{A_n}{}_0^1$; we remark that the period $\Omega_{A_n}{}_0^1$ can be identified with $\Omega(\mathcal{P})\backslash[A_n, P_1]$ in using the covering map π.

Let Σ_n denote the graph in $\mathbb{R}^3$ of u_n and Σ_n^* the minimal surface consisting in the union of the conjugate surface of Σ_n with its symmetric with respect to $\{z = 0\}$ (the conjugate surface to Σ_n is normalized such that the conjugates of the vertical lines satisfy $z = 0$). We also denote by $\widetilde{\Sigma}_n^*$ the periodic minimal surface consisting in the union of the conjugate surface of the graph of u_{A_n} with its symmetric with respect to $\{z = 0\}$. In a certain way, Σ_n^* is a period of $\widetilde{\Sigma}_n^*$, and the vector that lets $\widetilde{\Sigma}_n^*$ invariant is the non renormalized $Per(A_n)$. Proposition 8 will be a consequence of the understanding of the convergence of (Σ_n^*).

From Lemma 4 and Lemma 5, Σ_n^* is a minimal surface with two components of boundary and its total curvature $4\pi r$ does not depend on n.

The proof of Proposition 8 begins by the study of the convergence of the sequence (u_n) and sequences of rescaled functions. After, we understand the asymptotic behaviour of (Σ_n^*). The last step consist in following the convex curves which are $\Sigma_n^* \cap \{z = 0\}$ up to their limit.

6.2.2 The convergence of the graphs

The first step of the proof is to understand the asymptotic behaviour of the sequence (u_n). We consider u_n as a function on $\Omega(\mathcal{P})\backslash[A_n, P_1]$; then the study of the convergence is on the limit multi-domain $\Omega(\mathcal{P})$.

Lemma 7. *The sequence (u_n) converges to $u_{\mathcal{P}}$ on $\Omega(\mathcal{P})$.*

Proof. Using the arguments of the proof of Theorem 2, we see that there is no line of divergence so a subsequence $(u_{n'})$ converges to a function u solution of (MSE) on $\Omega(\mathcal{P})$. By Lemma A.2, u takes the value $+\infty$ on $\pi(\mathcal{L}_i^+)$ and $-\infty$ on $\pi(\mathcal{L}_i^-)$ (more precisely, if B is a point in $\mathcal{P}$ we have $u_{n'} - u_{n'}(B) \to u$). By Theorem 7 in [Ma1], $u = u_{\mathcal{P}}$; the limit is then unique and the sequence (u_n) converges to $u_{\mathcal{P}}$. The graph of this function is the conjugate surface to $\Sigma(\mathcal{P})^+$. $\square$

We also need to understand the convergence of (u_n) near the point P_1: to do this we renormalize a neighborhood of P_1. u_n is defined on the neighborhood $T(-\frac{\pi}{2}, \alpha + \frac{\pi}{2}, \mu)\backslash[A_n, P_1]$ of P_1; more precisely, the derivatives of

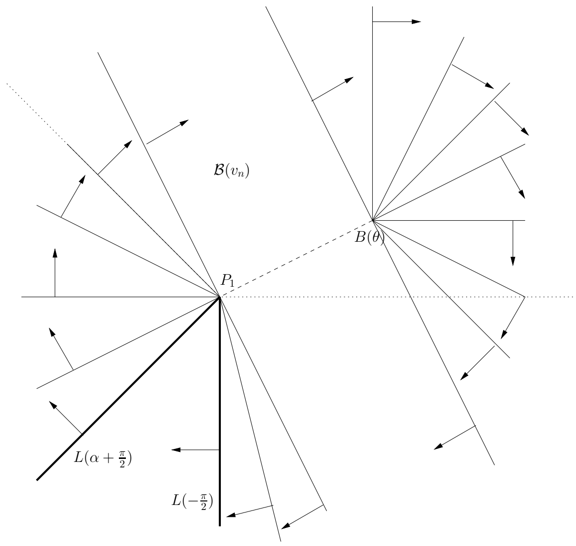

Figure 4: the asymptotic behaviour of v_n

u_n are well defined on $T(-\frac{\pi}{2}, \alpha+\frac{\pi}{2}, \mu)\backslash\{A_n\}$. We have $A_n = (\rho_n, \theta_n)$ with $\theta_n \to \theta$ and $\rho_n \to 0$. Let us renormalize by $\frac{1}{\rho_n}$: we get a function v_n defined on $T(-\frac{\pi}{2}, \alpha+\frac{\pi}{2}, \frac{\mu}{\rho_n})\backslash\{(\rho, \theta_n), \rho \in [0,1]\}$ by $v_n(\rho, \beta) = \frac{1}{\rho_n} u_n(\rho_n \rho, \beta)$. v_n is a solution of (MSE). We want to understand the asymptotic behaviour of v_n. In the following, let $B(\beta)$ denote the point of polar coordinates $(1, \beta)$ and $L(\beta)$ the half straight-line $\{(\rho, \beta)\}_{\rho>0}$.Since $\frac{\mu}{\rho_n} \to +\infty$ the limit multi-domain is $T(-\frac{\pi}{2}, \alpha+\frac{\pi}{2}, +\infty)\backslash\{B(\theta)\}$ for the derivatives of v_n. We want to prove that the asymptotic behaviour is given by Figure 4

First, we study the lines of divergence. We know that v_n takes the value $+\infty$ on $L(-\frac{\pi}{2})$ and the value $-\infty$ on $L(\alpha+\frac{\pi}{2})$; we have $\Psi_{v_n}(P_1) = 0$, $\Psi_{v_n}(B(\theta_n)) = 0$ and $\Psi_{v_n} \geq 0$. Let L be a line of divergence, L must have an end-point, otherwise we can apply the argument of the proof of Theorem 2 with the point P_1. This end point can not be on $L(\alpha+\frac{\pi}{2})$ or $L(-\frac{\pi}{2})$ because of Lemma A.1. Then the end point must be P_1 or $B(\theta)$. If the line of divergence has two end-points, it is the segment $[P_0, B(\theta)]$ then we have

$$0 = |\Psi_{v_n}(B(\theta_n)) - \Psi_{v_n}(P_1)| = \left|\int_{[P_1, B(\theta_n)]} \mathrm{d}\Psi_{v_n}\right| \longrightarrow \left|\int_{[P_1, B(\theta)]} \lim \mathrm{d}\Psi_{v_n}\right| \longrightarrow 1$$

This is a contradiction.

We then can ensure that L is a half straight-line with P_1 or $B(\theta)$ as end-point. Let us suppose that the end-point is P_1, L is one $L(\beta)$.

Lemma 8. *Let L be a line of divergence with P_1 as end-point: $L = L(\beta)$. Then $\beta \notin (\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2})$*

Proof. Let us suppose that $\beta \in (\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2})$. $C(\rho)$ denotes the point of $L(\beta)$ with coordinates (ρ, β). Since $\mathrm{d}\Psi_{v_n}$ is closed,

$$\int_{[P_1,B(\theta_n)]} \mathrm{d}\Psi_{v_n} + \int_{[B(\theta_n),C(\rho)]} \mathrm{d}\Psi_{v_n} + \int_{[C(\rho),P_1]} \mathrm{d}\Psi_{v_n} = 0$$

The first integral is always zero, then $\left| \int_{[C(\rho),P_1]} \mathrm{d}\Psi_{v_n} \right| \le |B(\theta_n)C(\rho)|$.

Then taking the limit for a subsequence making L appear, we get $\rho \le |B(\theta)C(\rho)|$. But $|B(\theta)C(\rho)| = \sqrt{\rho^2 + 1 - 2\rho\cos(\beta - \theta)}$ then for big ρ the inequality is not true. □

We suppose now that the end-point of L is $B(\theta)$. We use (ρ', γ') the polar coordinates on $T(-\frac{\pi}{2}, \alpha + \frac{\pi}{2}, +\infty)$ with $B(\theta)$ as origin point; γ' is chosen such that the coordinates of P_0 in this new coordinates are $(1, \pi)$. In this polar coordinates, the line of divergence L is one $L'(\beta) = \{\gamma' = \beta,\ \rho' > 0\}$.

Lemma 9. *Let L be a line of divergence with $B(\theta)$ as end point, $L = L'(\beta)$. Then $\beta \notin (-\pi, -\frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$.*

Proof. The proof is the same as the one of Lemma 8 in exchanging P_1 and $B(\theta)$. □

In the following, we prove that, in fact, all the lines of divergence that we have not excluded by Lemma 8 and 9 yet appear. We first observe that the allowed lines of divergence do not intersect themselves. Since, for every n, $\Psi_{v_n}(P_1) = 0 = \Psi_{v_n}(B(\theta_n))$ and $\Psi_{v_n} \ge 0$, there is only one possibility for the limiting normal on each line of divergence. Besides, we know that $\mathcal{B}(v_n) = \{P \in T(-\frac{\pi}{2}, \alpha + \frac{\pi}{2}, +\infty) |\ |\nabla v_n(P)| \text{ is bounded}\}$ contains a strip which is delimited by the two half straight-lines with P_1 as end-point $L(\theta - \frac{\pi}{2})$ and $L(\theta + \frac{\pi}{2})$ and the two half straight-lines with $B(\theta)$ as end-point $L'(-\frac{\pi}{2})$ and $L'(\frac{\pi}{2})$ for the polar coordinates centred on $B(\theta)$. If all the lines of divergence appear, $\mathcal{B}(v_n)$ is exactly this strip (see Figure 4).

Let L denote one allowed line of divergence and assume L does not appear, *i.e.* $L \subset \mathcal{B}(v_n)$. Then the connected component $\mathcal{B}$ of $\mathcal{B}(v_n)$ that contains L is a multi-domain which contains a subset K such that $\mathcal{B} \backslash K$ is isometric to an angular sector (let us observe that the angle at the vertex can be greater than 2π) minus the set of the points at a distance less than d from the vertex of the angular sector (d is a positive number). On $\mathcal{B}$, a subsequence $v_{n'}$ converges to a function v. Since $\mathcal{B}$ is bounded by lines of divergence or by the boundary of $T(-\frac{\pi}{2}, \alpha + \frac{\pi}{2}, +\infty)$ the value of v is $+\infty$ on one side of the angular sector and $-\infty$ on the other side (Lemma A.2). Besides $\Psi_v \geq 0$ since $\Psi_{v_n} \geq 0$ for every n. Then, v satisfies many conditions that contradict Theorem 2 in [Ma2]; this proves there is no sub-sequence such that one of the allowed lines of divergence does not appear. We then know the limit of the normal to the graph of v_n for all the points outside the strip.

On the strip, there is a sub-sequence $(v_{n'})$ that converges to some function v (in fact only the derivatives of $v_{n'}$ are well defined and converge to the derivatives of some function v). The function v takes the value $-\infty$ on $L(\theta + \frac{\pi}{2})$ and $L'(-\frac{\pi}{2})$ and the value $+\infty$ on $L(\theta - \frac{\pi}{2})$ and $L'(\frac{\pi}{2})$ by Lemma A.2; such a solution v is unique and is a peace of helicoid. More precisely, if the strip is isometrically parametrized by $\mathbb{R} \times [-1/2, 1/2]$ with $P_1 = (0, 1/2)$ and $B(\theta) = (0, -1/2)$ then we have $v(x, y) = x \tan(\pi y)$.

This ends our study of the convergence of renormalized graphs near P_1.

Remark 4. Let P be a point in the strip, the curvature of the helicoid at this point is non zero, then the curvature of the graph of v_n over the point P goes to the curvature of the helicoid at this point. This implies that there exists a sequence of point $p_n \in \Sigma_n$ such that the curvature of Σ_n at p_n goes to $+\infty$.

Remark 5. Besides, we know that there exists, for each n, a constant c_n such that $u_{A_n} \circ f_n = u_{A_n} + c_n$, then the above result proves that $\frac{c_n}{\rho_n} \to +\infty$.

We also want to know what occurs when the sequence (u_n) is homothetically expanded in a general way. Let (M_n) be a sequence of point in $\Omega(\mathcal{P})$ such that, for every n, $M_n \neq A_n$ and (λ_n) a sequence of positive number such that λ_n goes to $+\infty$. Let h_n be the homothety of center M_n and ratio λ_n. In applying the dilatation h_n, we get a function v_n on $h_n(\Omega(\mathcal{P}) \backslash \{A_n\})$ defined by $v_n(M) = \lambda_n u_n(h_n^{-1}(M))$. The question we ask is: which asymptotic behaviours are possible near M_n? By taking a subsequence, we can assume that we are in one of the five below cases. In particular, we proof:

Lemma 10. *If the domain of convergence is non-empty and the limit v of (v_n) is not a linear function, we are in Case 5: we have $\lambda_n \sim \frac{c}{\rho_n}$.*

Here are the five cases:

Case 1. For each i, $d(M_n, h_n(P_i)) \to +\infty$ and $d(M_n, h_n(A_n)) \to +\infty$. The limit multi-domain for the sequence v_n is $\mathbb{R}^2$. If there is no line of divergence, a sub-sequence $(v_{n'})$ must converge to a linear function by Bernstein Theorem. If there is a line of divergence L, each connected component of the domain of convergence of a subsequence $(v_{n'})$ is a strip or a half-plane and $(v_{n'})$ converges to a function v with the value $+\infty$ on one side and $-\infty$ on the other side; but such solution of (MSE) does not exist by Proposition 1 in [Ma2]. So the domain of convergence is empty and we only have lines of divergence which are all parallel to L and the limit normal is constant on $\mathbb{R}^2$.

Case 2. For each i, $d(M_n, h_n(P_i)) \to +\infty$ and $d(M_n, h_n(A_n)) \to d \geq 0$. The limit multi-domain is then $\mathbb{R}^2$ minus the limit point A' of $h_n(A_n)$. Since $d(M_n, h_n(P_1)) \to +\infty$, we have $d(h_n(A_n), h_n(P_1)) \to +\infty$ then $\lambda_n \rho_n \to +\infty$; by Remark 5, this implies that $\lambda_n c_n$, which is the vertical jump over $[h_n(A_n), h_n(P_1)]$ for the function v_n, goes to $+\infty$. Then the derivatives of v_n can not converge on $\mathbb{R}^2 \backslash \{A'\}$ and there are lines of divergence. Since $\Psi_{v_n}(h_n(A_n)) = 0$ and $\Psi_{v_n} \geq 0$, using arguments that we have already seen, we can ensure that a line of divergence must be a half straight-line with A' as end-point. Besides, only one limit normal is possible along the line of divergence. If the domain of convergence $\mathcal{B}(v_n)$ is non empty, each connected component of it is an angular sector of $\mathbb{R}^2$; then on one component, a subsequence $(v_{n'})$ converges to a solution v of (MSE) with the value $+\infty$ on one side and $-\infty$ on the other side. Proposition 2 in [Ma2] forbids this and we have only lines of divergence as asymptotic behaviour.

Case 3. There exists $j \neq 1$ such that $d(M_n, h_n(P_j)) \to c \geq 0$. This implies that, for $i \neq j$, $d(M_n, h_n(P_i)) \to +\infty$ and $d(M_n, h_n(A_n)) \to +\infty$; then the limit multi-domain is an angular sector isometric to some $T(0, \beta, +\infty)$ with $P'_j = \lim h_n(P_j)$ as vertex. As above, the lines of divergence must be half straight-lines with P'_j as end-point and the limit normal on a line of divergence is given by the condition $\Psi_{v_n} \geq 0$. As in Case 2, $\mathcal{B}(v_n)$ is empty and the asymptotic behaviour is given by the lines of divergence.

Case 4. $d(M_n, h_n(P_1)) \to c \geq 0$ and $d(h_n(P_1), h_n(A_n)) \to 0$ or $+\infty$. The limit multi-domain is then an angular sector with $P'_1 = \lim h_n(P_1)$ as vertex and the asymptotic behaviour is the same as in Case 3.

Case 5. $d(M_n, h_n(P_1)) \to c \geq 0$ and $d(h_n(P_1), h_n(A_n)) \to c' > 0$. The limit-multi-domain is then an angular sector, with P'_1 as vertex, minus the point $A' = \lim h_n(A_n)$. We are in the situation studied above and we know that the asymptotic behaviour is a domain of convergence which is a strip where (v_n) converges to a piece of helicoid and, outside the strip, lines of divergence with A' or P'_1 as end-point (see Figure 4).

6.2.3 The convergence of Σ_n^*

Now, we translate the convergence for the graphs into the behaviour of the sequence (Σ_n^*).

We first fix notations. Let Q be a point in the interior of $\mathcal{P}$ and q_n the corresponding point in Σ_n. For every n, the value $\Psi_{u_n}(Q)$ is well defined and we normalize Σ_n^* such that q_n^*, the corresponding point to q_n, has coordinates $(0,0,\Psi_{u_n}(Q))$; Σ_n^* is then still symmetric with respect to the plane $\{z=0\}$. We want to determine the limit of the sequence of minimal surfaces (Σ_n^*).

Let M be a surface in $\mathbb{R}^3$. In the following, when we talk about a geodesical disk $D(m,\mu)$ of center $m\in M$ and radius μ, we consider the disk of radius μ in the tangent plane to M at m with the exponential map $\exp_m : T_mM\to\mathbb{R}^3$. Besides, we sometimes identify a point in the tangent plane with its image by $\exp_m$.

By periodicity, $\max|K_{\widetilde{\Sigma}_n^*}(\cdot)| = \max|K_{\Sigma_n^*}(\cdot)|$. Remark 4 tells us that $\lim\max|K_{\Sigma_n^*}(\cdot)|=+\infty$. We need to control the curvature growth.

Lemma 11. *We have* $\max|K_{\Sigma_n^*}(\cdot)| = O(\left(\frac{1}{\rho_n}\right)^2)$ *(we recall that* $\rho_n=|P_1A_n|$*).*

Proof. Since Σ_n^* is symmetric, there exists $m_n^*\in\Sigma_n^{*+,0}$ such that $|K_{\widetilde{\Sigma}_n^*}(m_n^*)| = \max|K_{\Sigma_n^*}|=\lambda_n^2$. Let μ be a positive number, and, for each n, $D(m_n^*,\mu)$ denotes the closed geodesical disk of center m_n^* and radius μ in $\widetilde{\Sigma}_n^*$. By translating m_n^* to the origin and homothetically expanding the disk $D(m_n^*,\mu)$ by the factor λ_n, we obtain a new geodesical disk $D_n'=D(0,\lambda_n\mu)$, (D_n') is a sequence of minimal surfaces. We have $|K_{D_n'}(0)|=1$ and $|K_{D_n'}(\cdot)|\le 1$ since the curvature is maximum at m_n^*. Thus the curvature is uniformly bounded. Then there exists a subsequence $(D_{n'}')$ that converges to D_∞' a complete minimal surface; this surface is complete since $\lambda_n\to+\infty$.

D_∞' is non flat since at the origin its curvature is -1. Then, there is a point $\tilde{a}$ where the normal has a negative third coordinate; there exists a neighborhood U of $\tilde{a}$ in the tangent plane to D_∞' at $\tilde{a}$ such that D_∞' and $D_{n'}'$, for big n', are graphs over U and $D_{n'}'\to D_\infty'$ as graphs. Let $\tilde{a}_{n'}$ be the sequence of points in $D_{n'}'$ over $\tilde{a}$ as graphs. The normal at $\tilde{a}_{n'}$ to $D_{n'}'$ has a negative third coordinate then before the rescaling $\tilde{a}_{n'}$ correponds to a point $a_{n'}$ which lies in the conjugate of $\Sigma_{n'}$. Let $b_{n'}$ be the point in $\Sigma_{n'}$ corresponding to $a_{n'}$ and $B_{n'}$ its projection on $\Omega(\mathcal{P})$. The convergence of $D_{n'}$ to D_∞' near $\tilde{a}$ says us that $\mathcal{B}(v_{n'})$ is non empty where $v_{n'}$ is the rescaled function of $u_{n'}$ with the factor $\lambda_{n'}$ and $B_{n'}$ as origin points. Besides the limit of $v_{n'}$ is not linear since D_∞' is non flat. Lemma 10 then says that we are in Case 5 and $\lambda_{n'}\sim\frac{c}{\rho_{n'}}$. This gives the upper-bound for the curvature. □

Let us now observe the asymptotic behaviour Σ_n^* near its boundary. This boundary is composed of two closed paths Γ_n^1 and Γ_n^2. The boundary of the graph Σ_n is composed of $r-1$ straight-lines, they are over the points P_i for $i \neq 1$, and, by Lemma 4, a curve which consists in:

- a half straight-line over P_1 that goes down from the infinity to some point called t_n^1,
- a curve that is a graph over $[P_1, A_n]$ joining t_n^1 to a point t_n^2,
- a vertical segment $[t_n^2, t_n^3]$ over A_n,
- a curve which is a graph over $[P_1, A_n]$ joining t_n^3 to a point t_n^4 (it is the vertical translation of the curve joining t_n^1 to t_n^2) and
- a half straight-line over P_1 with t_n^4 as end-point and going down to the infinity.

The path Γ_n^1 (resp. Γ_n^2) consists in the conjugate of the curve joining t_n^1 to t_n^2 (resp. t_n^3 to t_n^4) with its symmetric with respect to the plane $\{z = 0\}$.

As in Subsection 6.2.2, let v_n be the rescaled function of u_n with factor $\frac{1}{\rho_n}$ on $T(-\frac{\pi}{2}, \alpha + \frac{\pi}{2}, \frac{r}{\rho_n}) \backslash [P_1, B(\theta_n)]$. By Lemma 11, the curvature is uniformly bounded on the graph of v_n for every n. Let p_n be a point in the graph of v_n which is above the middle of the segment $[P_1, B(\theta_n)]$ and $\mathcal{Y}_n$ the conjugate of the graph of v_n that we extend by symmetry and periodicity such that the conjugate point of p_n is the origin of $\mathbb{R}^3$. The curvature is uniformly bounded on the surfaces. Then, using a Cantor diagonal process, there exists a subsequence $(\mathcal{Y}_{n'})$ which converges to some minimal surface $\mathcal{Y}$. Near p_n, the graphs of v_n converge to a piece of an helicoid, $\mathcal{Y}$ is then a catenoid whose flux is $2\overrightarrow{P_1 B(\theta)}$. There is only one possible limit for subsequences of $(\mathcal{Y}_n)$ so the sequence $(\mathcal{Y}_n)$ converges to $\mathcal{Y}$. Let us consider the catenoid given in cylindrical coordinates by $(u, v) \mapsto (\frac{1}{\pi} u, v, \frac{1}{\pi} \operatorname{argch}(u))$. Thus $\mathcal{Y}$ is the translated by $(0, 0, -\frac{1}{\pi})$ of the image by a rotation of axis $\{z = 0, x \cos\theta + y \sin\theta = 0\}$ and angle $\frac{\pi}{2}$ of this catenoid. Let p_n be the point in the graph of v_n which is the limit of the points $(1/2, \beta, v_n(1/2, \beta))$ when $\beta \to \theta_n$ with $\beta < \theta_n$. Let $\mathcal{Y}_n'$ denote the conjugate of the graph of v_n that we extend by symmetry but not by periodicity with the conjugate of the point p_n at the origin. What we have proved just above implies that $(\mathcal{Y}_n')$ converges to the part of the catenoid $\mathcal{Y}$ included in $\{x \cos\theta + y \sin\theta \leq 0\}$; this part is denoted by $\mathcal{Y}_-$. If p_n is build with $\beta > \theta_n$ we get the half catenoid included in $\{x \cos\theta + y \sin\theta \geq 0\}$.

This proves that the rescaled paths $\frac{1}{\rho_n}\Gamma_n^1$ and $\frac{1}{\rho_n}\Gamma_n^2$ converge to two circles of the same radius. Let us call s_n^i the point in Σ_n^* which is the conjugate of t_n^i for $1 \leq i \leq 4$. The above arguement shows that, for every $\varepsilon > 0$, the ball of center s_n^1 (resp. s_n^4) and radius ε contains Γ_n^1 (resp. Γ_n^2) and even a bigger and bigger part of a surface near a half catenoid. We consider the case where p_n is built with $\beta < \theta_n$. If $B(R)$ is the ball of center the origin and radius R, we have $B(R)\cap\mathcal{Y}'_n$ is near from $B(R)\cap\mathcal{Y}_-$ for big n. Besides for big n, $\rho_n R < \varepsilon$ then the ball $B(s_n^1, \varepsilon)$ contains the homothetic by ρ_n of a surface near $B(R)\cap\mathcal{Y}_-$. The same is true for p_n build with $\beta > \theta_n$. This implies that the sequence of total curvatures of the part of Σ_n^* included in this ball has a lower limit bigger than 2π, since the total curvature of an half catenoid is 2π. In fact, we have to remember that, near s_n^1 and s_n^4, the surfaces Σ_n^* behave like smaller and smaller half-catenoids.

We can now give the limit of (Σ_n^*). We know that the Σ_n^* are symmetric minimal surfaces with finite total curvature $4\pi r$. Each surface Σ_n^* has two connected components of boundary Γ_n^1 and Γ_n^2. We have just seen that the diameter of each component goes to zero and, if B_n is a sequence of balls of diameter ε centred at s_n^2, the sequence of total curvatures of $\Sigma_n^* \cap B_n$ has a lower limit bigger than 2π. By taking a subsequence we can assume that (s_n^2) and (s_n^3) diverge to the infinity or converge to s_∞^2 and s_∞^3. Then by results explained in [CR], there exist a finite number of distinct properly and simply immersed branched minimal surfaces $M_1, \ldots, M_k \subset \mathbb{R}^3$ with finite total curvature, a finite subset $X \in \mathbb{R}^3$ contained in $M = M_1 \cup \cdots \cup M_k$ and a subsequence of (Σ_n^*), that we still call (Σ_n^*), such that:

1. (Σ_n^*) converges to M (with finite multiplicity) on compact subsets of $\mathbb{R}^3\backslash(X \cup \{s_\infty^2, s_\infty^3\})$ in the C^m-topology for any positive integer m;
2. on each M_i the multiplicity m_i is well defined and is such that

$$m_1 C(M_1) + \cdots + m_k C(M_k) \leq C(\Sigma_n^*);$$

3. X is the singular set of the limit $\Sigma_n^* \to M$. Given a point $p \in X$, the amount of total curvature of the sequence (Σ_n^*) which disappears through the point p is a positive multiple of 4π.

In fact we have:

Lemma 12. *We have $X = \emptyset$ and $k = 1$ with $m_1 = 1$ and $M_1 = \Sigma(\mathcal{P})$. In fact $\Sigma_n^* \to \Sigma(\mathcal{P})$.*

Proof. Since at s_∞^2 and s_∞^3, there is 2π of total curvature that disappear, we have $m_1C(M_1)+\cdots+m_kC(M_k)\leq C(\Sigma_n^*)-4\pi=4\pi(r-1)$ even if s_n^2 or s_n^3 diverges. The sequence (u_n) on $\Omega(\mathcal{P})\backslash[P_1,A_n]$ converges on $\Omega(\mathcal{P})$ to $u_\mathcal{P}$ (Lemma 7). Then (q_n) converges to q_∞ the point in the graph of u which is above Q. The conjugate surface to the graph of $u_\mathcal{P}$, after extension by horizontal symmetry, is the genus 0 r-noid $\Sigma(\mathcal{P})$. Let q_∞^* be the point in $\Sigma(\mathcal{P})$ that correspond to q_∞. We then have $q_n^*\to q_\infty^*$ and, in a neighborhood of q_∞^*, Σ_n^* converges to $\Sigma(\mathcal{P})$, then $\Sigma(\mathcal{P})$ is one M_i. We assume that $M_1=\Sigma(\mathcal{P})$. Since $C(\Sigma(\mathcal{P}))=4\pi(r-1)$, $m_1=1$, for $i\neq 1$, M_i is a plane and X is empty. In fact, in [CR], C. Cosín and A. Ros prove that, in such a convergence, no plane can appear. This ends the proof. □

Remark 6. Since the problems of convergence appear only near the points s_∞^2 and s_∞^3, the curves in $\Sigma(\mathcal{P})$ which are the conjugates of the $r-1$ straight-lines that are over the points P_i, for $i\neq 1$, are the respective limits of the curves in Σ_n^* which are the conjugates of the $r-1$ straight-lines in Σ_n that are over the points P_i, for $i\neq 1$.

6.2.4 The convergence of $Per(A_n)$

Since we know the convergence of (Σ_n^*), in this subsection, we end the proof of Proposition 8 by computing the limit of $Per(A_n)$.

$Per(A_n)$ corresponds to the vector that lets $\widetilde{\Sigma}_n^*$ invariant. Then $Per(A_n)$ is $\overrightarrow{s_n^2s_n^3}$ or $\frac{\overrightarrow{s_n^2s_n^3}}{||\overrightarrow{s_n^2s_n^3}||}$, following the value of $||\overrightarrow{s_n^2s_n^3}||$.

We recall that $\mathcal{C}$ denotes the curve in $\Sigma(\mathcal{P})$ which is the conjugate of the vertical straight-line which is over P_1 in the graph of $u_\mathcal{P}$. This curve is strictly convex and is parametrized by its normal by $\gamma:(-\frac{\pi}{2},\alpha+\frac{\pi}{2})\to\{z=0\}$ (the normal to $\gamma(\beta)$ is $(\sin\beta,-\cos\beta,0)$). We cover $\mathcal{C}$ as we cover the straight-line over P_1 by going down. From Remark 6, outside s_∞^2 and s_∞^3, $\mathcal{C}$ is the limit, in the Hausdorff topology, of the part of $\Sigma_n^*\cap\{z=0\}$ that do not correspond to the $r-1$ straight-line that are above the vertices P_i for $i\neq 1$. For each n, this set is composed of three strictly convex arcs, denoted by $\mathcal{C}_n^1$, $\mathcal{C}_n^2$ and $\mathcal{C}_n^3$: $\mathcal{C}_n^1$ is the conjugate of the vertical half straight-line that have t_n^1 as end-point ($\mathcal{C}_n^1$ has s_n^1 as end-point), $\mathcal{C}_n^2$ is the conjugate of the vertical segment $[t_n^2,t_n^3]$ ($\mathcal{C}_n^2$ joins s_n^2 to s_n^3) and $\mathcal{C}_n^3$ is the conjugate of the vertical half straight-line that have t_n^4 as end-point ($\mathcal{C}_n^3$ has s_n^4 as end-point). As $\mathcal{C}$, this three strictly convex arcs can be parametrized by their normal, *i.e.* there exist $\gamma_n^1:(-\frac{\pi}{2},\theta_n]\to\{z=0\}$, $\gamma_n^2:[\theta_n-\pi,\theta_n+\pi]\to\{z=0\}$ and $\gamma_n^3:[\theta_n,\alpha+\frac{\pi}{2})\to\{z=0\}$ such that γ_n^i parametrized $\mathcal{C}_n^i$ and the normal at the point $\gamma_n^i(\beta)$ is $(\sin\beta,-\cos\beta,0)$. We have $\gamma_n^1(\theta_n)=s_n^1$, $\gamma_n^2(\theta_n-\pi)=s_n^2$,

$\gamma_n^2(\theta_n+\pi)=s_n^3$ and $\gamma_n^3(\theta_n)=s_n^4$. I_n^i denotes the definition set of γ_n^i. We then have $I_n^i\to I^i$ where $I^1=(\frac{\pi}{2},\theta]$, $I^2=[\theta-\pi,\theta+\pi]$ and $I^3=[\theta,\alpha+\frac{\pi}{2})$. The question is: does (γ_n^i) converge on I^i?

Since there is a half catenoid that appears near the points s_n^i for big n, if $\beta\in(\theta-\frac{\pi}{2},\theta]\subset I^1$ the curvature at $\gamma_n^1(\beta)$ becomes infinite, the same is true for the sequence of point $(\gamma_n^2(\beta))$ for $\beta\in[\theta-\pi,\theta-\frac{\pi}{2})\cup(\theta+\frac{\pi}{2},\theta+\pi]\subset I^2$ and the sequence of points $(\gamma_n^3(\beta))$ for $\beta\in[\theta,\theta+\frac{\pi}{2})\subset I^3$. We have the following convergence.

Lemma 13. *There exist*

1. *k_3 open intervals in $I=(-\frac{\pi}{2},\alpha+\frac{\pi}{2})$: $J_1,\dots,J_{k_1}\subset I^1\backslash[\theta-\frac{\pi}{2},\theta]$, $J_{k_1+1},\dots,J_{k_2}\subset I^2\backslash\big([\theta-\pi,\theta-\frac{\pi}{2}]\cup[\theta+\frac{\pi}{2},\theta+\pi]\big)$ and $J_{k_2+1},\dots,J_{k_3}\subset I^3\backslash[\theta,\theta+\frac{\pi}{2}]$, these intervals satisfy $J_j\cap J_l=\emptyset$ if $j\neq l$, and*
2. *a map $\widetilde{\gamma}$ with value in $\{z=0\}$, defined on the union of the k_3 intervals, $\widetilde{\gamma}$ parametrizes $\mathcal{C}\backslash\{s_\infty^2,s_\infty^3\}$ by its normal,*
3. *a subsequence n',*

such that, on J_j, the sequence $(\gamma_{n'}^i)$ (for the corresponding i) converges to $\widetilde{\gamma}$; besides, the J_j are maximal in the sense that $\widetilde{\gamma}$ restricted to J_j parametrized one of the convex arcs that composed $\mathcal{C}\backslash\{s_\infty^2,s_\infty^3\}$.

Proof. The set $\mathcal{C}\backslash\{s_\infty^2,s_\infty^3\}$ is composed of a finite number of convex arcs. Let a be a point in $\mathcal{C}\backslash\{s_\infty^2,s_\infty^3\}$, at this point the convergence of Σ_n^* to $\Sigma(\mathcal{P})$ well behaves so there exists a neighborhood U of a in the tangent plane to $\Sigma(\mathcal{P})$ at a (this plane is vertical) such that, over U, $\Sigma(\mathcal{P})$ and Σ_n^*, for big n, are graphs and, as graphs, (Σ_n^*) converge to $\Sigma(\mathcal{P})$. Over U, there is a neighborhood of a in $\mathcal{C}\backslash\{s_\infty^2,s_\infty^3\}$ and this neighborhood is the limit of the part of Σ_n^* which is included in $\{z=0\}$. Since each unit vector is reached a finite number of times on $\mathcal{C}_n^1\cup\mathcal{C}_n^2\cup\mathcal{C}_n^3$, by taking a subsequence n', there exist β, ε and $i\in\{1,2,3\}$ such that, for every n', the part of $\Sigma_{n'}^*\cap\{z=0\}$ which is over U contains $\gamma_{n'}^i(\beta-\varepsilon,\beta+\varepsilon)$. Then the convergence as graphs of Σ_n^* implies that on $(\beta-\varepsilon,\beta+\varepsilon)$ the sequence $(\gamma_{n'}^i)$ converges to some map $\widetilde{\gamma}$ that parametrized a neighborhood of a in $\mathcal{C}$ such that the normal to the point $\widetilde{\gamma}(\omega)$ is $(\sin\omega,-\cos\omega,0)$ (the convergence is in the C^m topology for every m). Since $\gamma_{n'}^i\to\widetilde{\gamma}$, for every $\omega\in(\beta-\varepsilon,\beta+\varepsilon)$ the curvature at the point $\gamma_{n'}^i(\omega)$ remains bounded. Then, if $i=1$, $(\beta-\varepsilon,\beta+\varepsilon)\cap[\theta-\frac{\pi}{2},\theta]=\emptyset$, if $i=2$, $(\beta-\varepsilon,\beta+\varepsilon)\cap\big([\theta-\pi,\theta-\frac{\pi}{2}]\cup[\theta+\frac{\pi}{2},\theta+\pi]\big)=\emptyset$ and, if $i=3$, $(\beta-\varepsilon,\beta+\varepsilon)\cap[\theta,\theta+\frac{\pi}{2}]=\emptyset$.

Now in applying the above argument to a countable number of points and constructing a subsequence by diagonal Cantor process, we proof the lemma. $\square$

The curve $\mathcal{C}$ has total curvature $\alpha + \pi$ and since $\widetilde{\gamma}$ parametrizes by the normal $\mathcal{C}\backslash\{s_\infty^2, s_\infty^3\}$, which has the same total curvature, we have

$$\alpha + \pi = \sum_{j=1}^{k_3} l(J_j) \leq l(I) = \alpha + \pi \tag{5}$$

where $l(J_j)$ is the length of the interval J_j; here we use $\bigcup J_j \subset I$. In distinguishing two cases, we can finish the proof of Proposition 8.

• We assume that $\theta \notin \{0, \alpha\}$. Computation (5) then implies that there exist points:

$$-\frac{\pi}{2} = \beta_0 < \beta_1 < \cdots < \beta_{k_1} = \theta - \frac{\pi}{2} < \beta_{k_1+1} < \cdots$$
$$\cdots < \beta_{k_2} = \theta + \frac{\pi}{2} < \beta_{k_2+1} < \cdots < \beta_{k_3} = \alpha + \frac{\pi}{2}$$

such that, for every j, $J_j = (\beta_{j-1}, \beta_j)$.

Lemma 14. *for $j \notin \{0, k_1, k_2, k_3\}$, the sequence $(\gamma_{n'}^i)$ (for the corresponding i) converges in a neighborhood of β_j in the C^1 topology, then we can extend the definition of $\widetilde{\gamma}$ at β_j.*

Proof. We apply Proposition B.3, if there is no C^1 convergence near β_j, since $\Sigma_n^* \to \Sigma(\mathcal{P})$ in the Hausdorff topology, $\Sigma(\mathcal{P})$ contains a segment. But this is not true and we can extend the definition of $\widetilde{\gamma}$ at β_j. $\square$

We then have $\gamma_{n'}^1 \to \widetilde{\gamma}$ on $(-\frac{\pi}{2}, \theta - \frac{\pi}{2})$, $\gamma_{n'}^2 \to \widetilde{\gamma}$ on $(\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2})$ and $\gamma_{n'}^3 \to \widetilde{\gamma}$ on $(\theta + \frac{\pi}{2}, \alpha + \frac{\pi}{2})$.

Let us study the behaviour near $\theta - \frac{\pi}{2} = \theta'$.

Let $\beta < \frac{\pi}{4}$ be a small positive angle and $\varepsilon > 0$. We apply Proposition B.2 to $\gamma_{n'}^1$ on $(\theta' - \beta, \theta' + \beta)$: it builds S an angular sector of vertex $\gamma_{n'}^1(\theta' - \beta)$ and angle 2β that contains $\gamma_{n'}^1(\theta' - \beta, \theta' + \beta)$. We have $\gamma_{n'}^1(\theta' + \beta) \in S$. Then, for big n', the behaviour of the sequence $(\Sigma_{n'}^*)$ near the point $s_{n'}^2 = \gamma_{n'}^2(\theta)$ implies that the distances $d\big(\gamma_{n'}^1(\theta' + \beta), s_{n'}^2\big)$ and $d\big(\gamma_{n'}^1(\theta' + \beta), \gamma_{n'}^2(\theta' - \beta)\big)$ are less than ε. Now we apply Proposition B.2 to $\gamma_{n'}^2$ on $(\theta' - \beta, \theta' + \beta)$: it builds S' an angular sector of vertex $\gamma_{n'}^2(\theta' - \beta)$ and angle 2β that contains $\gamma_{n'}^2(\theta' - \beta, \theta' + \beta)$. At $\gamma_{n'}^1(\theta' - \beta)$ and $\gamma_{n'}^2(\theta' - \beta)$, the two strictly convex curves $\mathcal{C}_{n'}^1$ and $\mathcal{C}_{n'}^2$ have the same normal and their curvature have the same

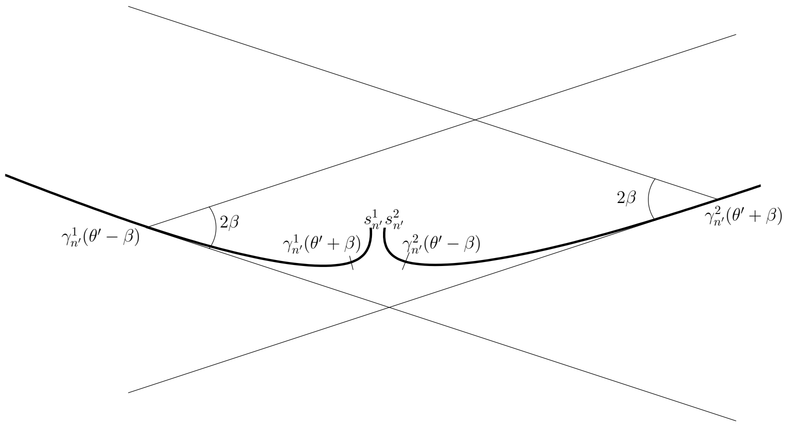

Figure 5: the local behaviour near $s^2_{n'}$

sign, the angular sector S' is then just the translation of S by the vector $\overrightarrow{\gamma^1_{n'}(\theta'-\beta)\gamma^2_{n'}(\theta'-\beta)}$. Since $\gamma^2_{n'}(\theta'-\beta)$ is at distance less than ε from S, the angular sector S' is included in S_ε the ε- neighborhood of S. In particular $\gamma^1_{n'}(\theta'-\beta,\theta'+\beta)\bigcup\gamma^2_{n'}(\theta'-\beta,\theta'+\beta)\subset S_\varepsilon$. Then passing to the limit $n'\to+\infty$, we get that $\widetilde{\gamma}\left((\theta'-\beta,\theta'+\beta)\backslash\{\theta'\}\right)$ is included in the ε-neighborhood of an angular sector which have $\widetilde{\gamma}(\theta'-\beta)$ as vertex and 2β as angle. The points $s^2_{n'}$ are also in this set for big n' (see Figure 5).

We can do the same work in starting from the point $\gamma^2_{n'}(\theta'+\beta)$ and in covering the curves $\mathcal{C}^1_{n'}$ and $\mathcal{C}^2_{n'}$ in the opposite sense. We get that $\widetilde{\gamma}\left((\theta'-\beta,\theta'+\beta)\backslash\{\theta'\}\right)$ and $s^2_{n'}$, for big n', are included in the ε-neighborhood of an angular sector which have $\widetilde{\gamma}(\theta'+\beta)$ as vertex and 2β as angle. Since β is less than $\frac{\pi}{4}$, the intersection of the two sets we have just build is a compact subset of $\mathbb{R}^2$ which contains $s^2_{n'}$, then s^2_∞ exists. Besides $\widetilde{\gamma}(t)$ admits two limits: g_1 when $t\to\theta'$, $t<\theta'$ and g_2 when $t\to\theta'$, $t>\theta'$. This two limit points are in $\mathcal{C}$ and $\widetilde{\gamma}$ parametrized $\mathcal{C}$ except for two points where the normals are different. g_1 and g_2 need to be one of these two points. But the normals at g_1 and g_2 are the same, then $g_1=g_2$. The definition of $\widetilde{\gamma}$ then extends at θ' in a differentiable way. Now, letting β and ε goes to zero, we see that the limit compact is just $\widetilde{\gamma}(\theta')$ then $\widetilde{\gamma}(\theta')=s^2_\infty$.

In the same way, we prove that s^3_∞ exists and $\widetilde{\gamma}(\theta+\frac{\pi}{2})=s^3_\infty$. In fact this proves that $\widetilde{\gamma}$ parametrizes $\mathcal{C}$ by its normal on I and then $\widetilde{\gamma}=\gamma$. Then the sequence $(Per(A_n))$ has only one possible cluster point: the sequence converges to the limit given in Proposition 8.

• We assume that $\theta = 0$ (the case $\theta = \alpha$ can be done in the same way). In this case, $k_1 = 0$ and, using the same arguments as above, we prove: s^3_∞ exists, $\gamma = \widetilde{\gamma}$ and $\gamma(\frac{\pi}{2}) = s^3_\infty$. For t near 0, $\gamma(t)$ is in a catenoidal end of $\Sigma(\mathcal{P})$ with flux vector $V_1 = ||V_1||(1,0)$. This asymptotic behaviour of $\mathcal{C}$ implies that there exists $0 < \beta < \frac{\pi}{2}$ such that for every $t \in (-\frac{\pi}{2}, -\frac{\pi}{2} + \beta)$, $|\gamma(t)s^3_\infty| > 2$ and $\frac{\overrightarrow{\gamma(t)s^3_\infty}}{|\gamma(t)s^3_\infty|}$ is at a distance less than ε from the vector $(0,-1)$. Let $0 < \beta' < \beta$, we apply Proposition B.2 to γ on $(-\frac{\pi}{2}, -\frac{\pi}{2} + \beta')$ covered in the opposite sense: it builds S an angular sector of vertex $\gamma(-\frac{\pi}{2} + \beta')$ and angle 2β that contains $\gamma(-\frac{\pi}{2}, -\frac{\pi}{2} + \beta')$. Then, because of the asymptotic behaviour of $\mathcal{C}$, there exist $0 < \beta' < \beta$ and ε' such that, for every point s in the ε'-neighborhood of S, we have: $|ss^3_\infty| > 2$ and $\frac{\overrightarrow{ss^3_\infty}}{|ss^3_\infty|}$ is at a distance less than ε from $(0,-1)$.

Now we apply Proposition B.2 to $\gamma^2_{n'}$ on $(-\frac{\pi}{2} - \beta', -\frac{\pi}{2} + \beta')$ covered in the opposite sense: we obtain that $\gamma^2_{n'}(-\frac{\pi}{2} - \beta', -\frac{\pi}{2} + \beta')$ is included in an angular sector $S_{n'}$ with vertex $\gamma^2_{n'}(-\frac{\pi}{2} + \beta')$ and angle 2β. Since $\gamma^2_{n'} \to \gamma$ on $(-\frac{\pi}{2}, -\frac{\pi}{2} + \beta)$, $(S_{n'})$ converges to S: in fact, $S_{n'}$ is just the translation of S by the vector $\overrightarrow{\gamma(-\frac{\pi}{2} + \beta')\gamma^2_{n'}(-\frac{\pi}{2} + \beta')}$ which goes to zero. Besides, the distance between $\gamma^2_{n'}(-\frac{\pi}{2} - \beta')$ and $s^2_{n'}$ goes to zero. Then for big n', $s^2_{n'}$ is in the ε'-neighborhood of S. Then, using that $s^3_{n'} \to s^3_\infty$, this proves that for big n', $|s^2_{n'}s^3_{n'}| > 2$ and $\frac{\overrightarrow{s^2_{n'}s^3_{n'}}}{|s^2_{n'}s^3_{n'}|} \to (0,-1)$.

Since in each case, there is only one possible limit for $(Per(A_{n'}))$ this proves that $(Per(A_n))$ converges to this limit. The second case of Proposition 8 is then established.

6.3 Conclusion

In this subsection, we finish the proof of Proposition 6. In Subsection 6.2, $\widetilde{\mathcal{P}}$ is defined. Propositions 7 and 8 allow us to continuously extend Per to $\partial\widetilde{\mathcal{P}}$. This extension satisfies:

Proposition 9. *The period map does not vanish on the boundary of* $\widetilde{\mathcal{P}}$

Proof. The only points where Per can vanish are points in the vertices and if $Per(A) = 0$ we have $\overrightarrow{\gamma(\theta - \frac{\pi}{2})\gamma(\theta + \frac{\pi}{2})} = 0$ for some strictly convex curve γ. But γ on $(\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2})$ is a graph over a straight-line then the above vector can not be 0. □

This proves that we can compute the degree of the period map along the boundary of $\widetilde{\mathcal{P}}$.

Theorem 5. *The degree of the period map along the boundary of $\widetilde{\mathcal{P}}$ is $-(r-1)$*

Proof. The edges of $\mathcal{P}$ does not contribute toward the degree. So only the behaviour at the vertices is interesting for the degree. Let us compute the contribution of the vertex P_1; we use the notation of Proposition 8. Since the curve γ is strictly convex, the map $\theta \mapsto \frac{\overrightarrow{\gamma(\theta-\frac{\pi}{2})\gamma(\theta+\frac{\pi}{2})}}{|\gamma(\theta-\frac{\pi}{2})\gamma(\theta+\frac{\pi}{2})|}$ is a monotone map. Let θ be in $[0,\alpha]$, the unit vector tangent to γ at $\gamma(\theta)$ is $(\cos\theta,\sin\theta)$ which turns in the clockwise sense, when θ increases. Besides, for $0<\theta<\alpha$, $\gamma([\theta-\pi/2,\theta+\pi/2])$ is a graph over a straight-line generated by $(\cos\theta,\sin\theta)$. This implies that $Per(P_1,\theta)\cdot(\cos\theta,\sin\theta)$ never vanishes ((P_1,θ) is a point of the boundary of $\widetilde{\mathcal{P}}$); by looking at the behaviour for small θ we have $Per(P_1,\theta)\cdot(\cos\theta,\sin\theta)\geq 0$. Besides, for $\theta=0$, the basis composed of $(1,0)$ and $Per(P_1,0)$ is an indirect one and, for $\theta=\alpha$, the basis $((\cos\alpha,\sin\alpha),Per(P_1,\alpha)$ is a direct one. Then, when θ increases from 0 to α, $Per(P_1,\theta)$ describes an angle α (since the unit tangent describes this angle) plus π (since the basis composed of the unit tangent at $\gamma(\theta)$ and $Per(P_1,\theta)$ is an orthonormal indirect one for $\theta=0$ and an orthonormal direct one for $\theta=\alpha$). Since, when we describe $\partial\widetilde{\mathcal{P}}$ in the clockwise sense, θ decreases, the contribution of the vertex P_1 towards the degree is $-\frac{1}{2\pi}(\alpha+\pi)$.

The degree is then $-(r\pi+\alpha_1+\cdots+\alpha_r)/(2\pi)$ where α_i is the inner angle at the vertex P_i. In applying Gauss-Bonnet Theorem to $\mathcal{P}$, $\alpha_1+\cdots+\alpha_r=r\pi-2\pi$ and then the degree is $-(r-1)$. □

Theorem 5 says that the degree of the period map is non zero and the last item of Proposition 6 is proved.

7 Other examples of genus 1 r-noids

Theorem 1 gives a wide class of solutions of the Plateau problem at infinity with genus 1. But it gives no example of a polygon which is the flux polygon of an r-noid of genus 1 but not the flux polygon of an r-noid of genus 0. Corollary 3 gives such polygons. In fact we study the case where the multi-domain with cone singularity bounded by the polygon is invariant under a "rotation".

Theorem 6. *Let V be a polygon that bounds a multi-domain with cone singularity (D,Q,ψ). We suppose that D satisfies the hypothesis H and that there exists an isometry h of D such that $\psi\circ h=R\circ\psi$ where R is the rotation in $\mathbb{R}^2$ with center $\psi(Q)$ and angle $\alpha\in(0,2\pi)$. Then the period* *vector associated to D vanishes and there exists an Alexandrov-embedded r-noid with genus 1 and horizontal ends having V as flux polygon.*

Proof. From Construction 1 and 3, we have a multi-domain with logarithmic singularity $(\Omega, \mathcal{Q}, \varphi)$ and Theorem 2 gives us a function u on Ω which is unique up to an additive constant. By construction, the map h can be lifted to Ω to a map $\tilde{h}$ which is an isometry of Ω such that $\varphi_{\psi(Q)} \circ \tilde{h} = R \circ \varphi_{\psi(Q)}$. Since $\tilde{h}$ is an isometry of Ω, $u \circ \tilde{h}$ is a solution of the same Dirichlet problem as u then $u \circ \tilde{h} = u + c$ with $c \in \mathbb{R}$. Then the two equations $\varphi_{\psi(Q)} \circ \tilde{h} = R \circ \varphi_{\psi(Q)}$ and $u \circ \tilde{h} = u + c$ imply that:

$$\tilde{h}^*(\mathrm{d}X_1^*, \mathrm{d}X_2^*) = R(\mathrm{d}X_1^*, \mathrm{d}X_2^*)$$

with $\mathrm{d}X_1^*$ and $\mathrm{d}X_2^*$ the 1-forms associated to u as in Subsection 5.1. Then, if Γ is a lift of a generator of $\pi_1(D \backslash \{Q\})$, we have:

$$\begin{aligned}\int_\Gamma (\mathrm{d}X_1^*, \mathrm{d}X_2^*) = \int_{\tilde{h}(\Gamma)} (\mathrm{d}X_1^*, \mathrm{d}X_2^*) &= \int_\Gamma \tilde{h}^*(\mathrm{d}X_1^*, \mathrm{d}X_2^*) \\ &= \int_\Gamma R(\mathrm{d}X_1^*, \mathrm{d}X_2^*) \\ &= R\left(\int_\Gamma (\mathrm{d}X_1^*, \mathrm{d}X_2^*)\right)\end{aligned}$$

The first equality is due to the fact that Γ and $\tilde{h}(\Gamma)$ are lifts of two closed pathes that give the same generator of $\pi_1(D \backslash \{Q\})$. Besides R has a unique fixed point, since $\alpha \in (0, 2\pi)$, which is 0: the period vector vanishes. □

To give some examples of polygons V that satify this condition, we consider the case where V is a regular convex polygon or a regular star polygon (see [Cox]).

Corollary 3. *Let q and r be in $\mathbb{N}^*$ such that $\gcd(q, r) = 1$ and $2q < r$. For $i = 1, \dots, r+1$, we write $P_i = e^{2(i-1)\sqrt{-1}\frac{q}{r}\pi} \in \mathbb{C} = \mathbb{R}^2$ ($P_1 = P_{r+1}$) (see Figure 6). Then there exists an Alexandrov-embedded r-noid of genus 1 and horizontal ends with $(\overrightarrow{P_1P_2}, \overrightarrow{P_2P_3}, \dots, \overrightarrow{P_rP_{r+1}})$ as flux polygon.*

Proof. The idea of the proof is to build a multi-domain with cone singularity bounded by the above polygon and satisfying the hypothesis of Theorem 6. Let $\mathbb{R}^2$ be identified with $\mathbb{C}$. T denotes the set of $(\rho, \theta) \in \mathbb{R}_+ \times [0, \pi]$ such that $\rho e^{i\theta}$ is in the triangle P_1P_2O, where O is the origin. We denote by D the set of (ρ, θ) in $\mathbb{R}^+ \times [0, 2q\pi]$ such that (ρ, θ) is in D if (ρ, θ') is in T

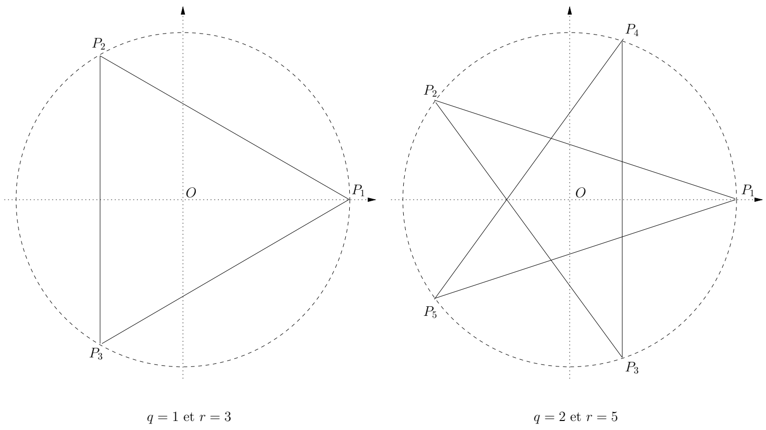

Figure 6: Examples of polygons studied in Corollary 3

where $\theta = n\left(2\frac{q}{r}\pi\right) + \theta'$ is the only writing with $n \in \mathbb{Z}$ and $\theta' \in [0, 2\frac{q}{r}\pi)$ (like an euclidean division of θ by $2\frac{q}{r}\pi$). We remark $\{0\} \times [0, 2q\pi] \subset D$ and $(\rho, 0) \in D$ iff $(\rho, 2q\pi) \in D$ iff $\rho \in [0, 1]$. Because of these remarks, we can consider the polar metric on D (we identify all the points $(0, \theta)$ and call the new point Q) and indentify the point $(\rho, 0)$ with $(\rho, 2q\pi)$ for $0 \leq \rho \leq 1$. Then if we consider on D the map $\psi : (\rho, \theta) \mapsto (\rho \cos\theta, \rho \sin\theta)$, the triplet (D, Q, ψ) is a multi-domain with cone singularity which is bounded by the polygon $(\overrightarrow{P_1P_2}, \overrightarrow{P_2P_3}, \ldots, \overrightarrow{P_rP_{r+1}})$. The angle of D is $2q\pi$. Besides on D the map

$$h : (\rho, \theta) \longmapsto \begin{cases} (\rho, \theta + 2\frac{q}{r}\pi) & \text{if } \theta \leq 2q\pi - 2\frac{q}{r}\pi \\ (\rho, \theta - 2\frac{r-1}{r}q\pi) & \text{if } \theta \geq 2q\pi - 2\frac{q}{r}\pi \end{cases}$$

is well defined and is an isometry of D. Besides $\psi \circ h = R \circ \psi$ with R the rotation of center O and angle $2\frac{q}{r}\pi < \pi$. Then we can apply Theorem 6. $\square$

When $q = 1$, the polygon is a convex regular polygon and the multi-domain D that we build is in fact an immersed polygonal disk. Theorem 1 then gives also the result but what we know is that, in the proof of Theorem 1, the period map vanishes at the isobarycenter of the polygon.

When $q > 1$, the polygon $(\overrightarrow{P_1P_2}, \overrightarrow{P_2P_3}, \ldots, \overrightarrow{P_rP_{r+1}})$ does not bound an immersed polygonal disk. Then we get new examples of polygons which are flux polygons of an Alexandrov-embedded r-noid with genus 1.

Remark 7. In [JM], L. P. Jorge and W. H. Meeks give Weierstrass data for r-noid with genus 0 and horizontal ends having as flux polygon the polygon studied in Corollary 3 with $q = 1$. Then Corollary 3 gives r-noids similar to the Jorge-Meeks examples for the genus 1 case. In fact this examples are known by H. Karcher which gave in [Ka] Weierstrass data for some r-noids which correspond to the ones we have just built. These Weierstrass data are expressed in terms of the Weierstrass $\mathfrak{p}$-function.

A Convergence of sequences of solution of (MSE) and line of divergence

The aim of this appendix is to explain some results on the convergence of sequences of solutions of the minimal surface equation.

Let (u_n) be a sequence of solution of (MSE) on a multi-domain D (D has possibly a cone or logarithmic singularity), we have a classical convergence result.

Proposition A.1. *If (u_n) is an uniformly bounded sequence on D, there exists a subsequence which converges to a solution of* (MSE) *on D. The convergence is uniform on every compact subset of D.*

If the sequence $(||\nabla u_n||)$ is uniformly bounded on every compact subset of D, this result proves that a subsequence $(u_{n'} - u_{n'}(P))$ (P is a point in D) converges to a solution of (MSE) on D (D is connected). So we define the domain of convergence of the sequence (u_n) as the set of points $P \in D$ where the sequence $(||\nabla u_n(P)||)$ is bounded, we denote by $\mathcal{B}(u_n)$ this set. $\mathcal{B}(u_n)$ is an open subset of D such that $(||\nabla u_n||)$ is uniformly bounded on every compact subset of $\mathcal{B}(u_n)$ (see [Ma1]). Then, on every connected component of the domain of convergence, a subsequence $(u_{n'} - u_{n'}(P))$ converges. In fact, in our paper, we often write that a subsequence $(u_{n'})$ converges instead of $(u_{n'} - u_{n'}(P))$; but, since the value on the boundary are often infinite, this does not matter. When it is necessary we use $(u_{n'} - u_{n'}(P))$. The question is to understand the domain of convergence or its complementary $D\backslash\mathcal{B}(u_n)$.

Let P be in $D\backslash\mathcal{B}(u_n)$, then for a subsequence $W_{n'} \to +\infty$. Since the normal to the graph at the point over P is

$$N_{n'}(P) = \left(\frac{p_{n'}}{W_{n'}}(P), \frac{q_{n'}}{W_{n'}}(P), -\frac{1}{W_{n'}}(P)\right)$$

(we use euclidean coordinates in a neighborhood of P), we can assume that $N_{n'}(P) \to N$ where N is a unit horizontal vector. We have the following result.

Theorem A.1. *Let (D, ψ) be a multi-domain. Let (u_n) be a sequence of solutions of* (MSE) *on D. Let $P \in D$ and N be a unit horizontal vector and*

L the geodesic of D passing by P and normal to N. If the sequence $(N_n(P))$ converges to N, then $N_n(A)$ converges to N at every point A of L

Since D is locally isometric to $\mathbb{R}^2$, N can be seen as a vector in $\mathbb{R}^2$; so N is well defined at all the points of D (in fact this definition coincides with the parallel transport of N). The proof of this theorem is in [Ma1].

In our situation $N_{n'}(P)$ is converging to a unit horizontal vector then $N_{n'}$ converges to this vector along a straight-line L. Such a line L is called a *line of divergence* of the sequence since L must be included in $D\backslash\mathcal{B}(u_n)$. The set $D\backslash\mathcal{B}(u_n)$ is then an union of geodesics of D and the question is: what are the possible lines of divergence? The following lemmas give some tools to answer to this discussion.

First, we observe that the existence of a line of divergence has a consequence on the sequence of 1-forms $\mathrm{d}\Psi_{u_n}$. If $N_n \to N$ along a line of divergence L and T is a segment included in L then $\int_T \mathrm{d}\Psi_{u_n}$ converges to $|T|$ the length of T (the orientation of T is such that N is the right-hand normal). Since (Ψ_{u_n}) is a sequence of 1-Lipschitz continuous function, we can always take a subsequence and assume that it has a limit Ψ; the above remark on $\mathrm{d}\Psi_{u_n}$ allows us to make some calculations on Ψ.

As in Figure 4, when we make a picture to explain the convergence of a sequence of solutions of (MSE), we draw the limit normal along the lines of divergence to explain the asymptotical behaviour.

We then have the two following results concerning the lines of divergence and the convergence of sequences of solutions of (MSE)

Lemma A.1. *Let (u_n) be a sequence of solutions of* (MSE) *on $[0,1]^2$ such that, for all n, the function u_n tends to $+\infty$ on $\{1\}\times]0,1[$. Then, no line of divergence of the sequence (u_n) has $(1,\frac{1}{2})$ as end-point.*

Proof. Let us assume that such a line of divergence L exists. We write $A=(1,0)$, $B=(1,\frac{1}{2})$ and $C=(1,1)$. Let us consider a point $M\in L$. We suppose that the limit normal along L is such that the basis composed of $\overrightarrow{MB}$ and the limit normal is direct. Since $\mathrm{d}\Psi_{u_n}$ is closed, we then have:

$$\int_{[A,B]} \mathrm{d}\Psi_{u_n} + \int_{[B,M]} \mathrm{d}\Psi_{u_n} + \int_{[M,A]} \mathrm{d}\Psi_{u_n} = 0$$

Since u_n takes the value $+\infty$ on $[A,B]$, this equality proves that:

$$|AB| + \int_{[B,M]} \mathrm{d}\Psi_{u_n} \leq |MA|$$

But, by our choice of limit normal, we have $\int_{[B,M]} \mathrm{d}\Psi_{u_n} \to |BM|$ for a subsequence, then $|AB| + |BM| \leq |MA|$ which contradicts the triangle inequality. If the limit normal is the opposite of the one we consider, we write the same arguments with the triangle BCM. $\square$

Lemma A.2. *Let (u_n) be a sequence of solutions of* (MSE) *on* $[0,1]^2$ *such that (u_n) converges in the interior of the square to a solution u and such that we are in one of the following two cases*

- *for all n, the function u_n tends to $+\infty$ on $\{1\}\times]0,1[$ or,*
- *for all n, u_n is the restriction to $[0,1]^2$ of a solution v_n of* (MSE) *defined on $[0,2]\times[0,1]$ and for $y \in]0,1[$ we have $N_n(1,y) \to (1,0,0)$.*

Then, the limit function u tends to $+\infty$ on $\{1\}\times]0,1[$.

Proof. let $0 < \varepsilon < \frac{1}{2}$. For $x \in [0,1]$, we write $A_x = (x,\varepsilon)$ and $B_x = (x, 1-\varepsilon)$. Since $\mathrm{d}\Psi_{u_n}$ is closed, for $x < 1$ we have:

$$\int_{[A_x,B_x]} \mathrm{d}\Psi_{u_n} - \int_{[A_1,B_1]} \mathrm{d}\Psi_{u_n} = \int_{[B_x,B_1]} \mathrm{d}\Psi_{u_n} + \int_{[A_1,A_x]} \mathrm{d}\Psi_{u_n}$$

Then, in taking the limit and using that $\int_{[A_1,B_1]} \mathrm{d}\Psi_{u_n} \to 1-2\varepsilon$ in the two cases, we obtain: $\left| \int_{[A_x,B_x]} \mathrm{d}\Psi_u - (1-2\varepsilon) \right| \leq 2(1-x)$

This proves that $\mathrm{d}\Psi_u = \mathrm{d}y$ on $[A_1,B_1]$. Let $x \in]0,1[$ and v be the solution of (MSE) on $A_xB_xB_1A_1$ such that v tends to $+\infty$ on $[A_1,B_1]$ and $v = u$ on the rest of the boundary; we shall prove that $u = v$.

If $u \neq v$ there exist $\eta \neq 0$ such that $\Omega = \{u - v > \eta\}$ is non-empty. The boundary of Ω is composed of one part included in the interior of $A_xB_xB_1A_1$ and one part included in $[A_1,B_1]$. Since $\mathrm{d}\Psi_u$ and $\mathrm{d}\Psi_v$ are closed, $\int_{\partial\Omega} \mathrm{d}\Psi_u - \mathrm{d}\Psi_v = 0$.

On the part of the boundary included in $[A_1,B_1]$, the integral is 0 since $\mathrm{d}\Psi_u = \mathrm{d}y = \mathrm{d}\Psi_v$. On the part included in the interior of $A_xB_xB_1A_1$, the integral is negative by Lemma 2 in [CK]. We have our contradiction and $u = v$; then u goes to $+\infty$ on $[A_1,B_1]$. $\square$

B Convex curves in $\mathbb{R}^2$

A convex curve in $\mathbb{R}^2$ is a curve such that its geodesical curvature has always the same sign. A curve is strictly convex if this geodesical curvature never vanishes.

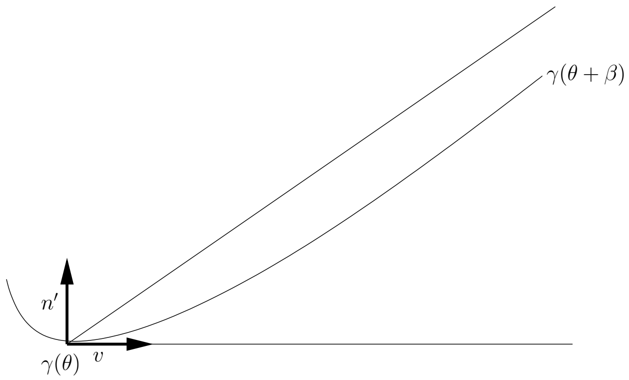

Figure 7: the situation of Proposition B.2

If a curve $c : s \mapsto c(s) \in \mathbb{R}^2$ is convex, the map with value in $\mathbb{S}^1$ that associates to the parameter s the normal to c at $c(s)$ is a monotone map and, if c is strictly convex, the above map is strictly monotone. Then if $c : I \to \mathbb{R}^2$ is a strictly convex curve, there exists $h : J \to I$ a diffeomorphism such that for $\beta \in J$ the normal to c at the point $c \circ h(\beta)$ is $(\sin\beta, -\cos\beta)$. We say that c is parametrized by its normal.

Proposition B.1. *Let $c : I \to \mathbb{R}^2$ be a strictly convex curve parametrized by its normal. If $I \subset (\beta_0 + \frac{\pi}{2}, \beta_0 + \frac{3\pi}{2})$, the curve c is a graph over the straight line $y\cos\beta_0 - x\sin\beta_0 = 0$.*

Proof. We choose $\beta_0 = 0$. If $c(\beta) = (x(\beta), y(\beta))$, the map $\beta \mapsto x(\beta)$ is a local diffeomorphism since the second coordinate of the normal is positive. Besides $\beta \mapsto x(\beta)$ is injective: if $x(\beta') = x(\beta'')$ there would exist, by Rolle's Theorem, $\beta \in (\beta', \beta'')$ such that the normal at $c(\beta)$ is horizontal. $\beta \mapsto x(\beta)$ is then a global diffeomorphism and c is a graph over $y = 0$. $\square$

Proposition B.2. *Let c be a strictly convex curve parametrized by its normal on $(\theta - \varepsilon, \theta + \beta]$ with $\beta \le \frac{\pi}{2}$. Let v be the unit tangent vector to c at $c(\theta)$ and n' the unit vector which is the normal to c at $c(\theta)$ if the curvature is positive or the opposite of the normal if the curvature is negative. Let w be the unit vector $\cos\beta v + \sin\beta n'$. Then the curve $c([\theta, \theta+\beta])$ is in the angular sector delimited by the two half straight-lines with $c(\theta)$ as end-point and respectively generated by v and w (see Figure 7).*

Proof. We assume that $\theta = 0$, the curvature is negative and $c(0) = (0, 0)$. $c([0, \beta])$ is then the graph of a function f over a segment $[0, a]$ by Proposition

B.1. By hypotheses, f is a convex function and $f'(0) = 0$. This implies first that $f \geq 0$. If $\beta = \frac{\pi}{2}$ the proposition is proved. If $\beta < \frac{\pi}{2}$ and $c([0, \beta])$ is not in $\{(x, y)|\ y \geq 0,\ y \leq x \tan \beta\}$ there is a parameter $b \in [0, a]$ such that $f(b) = b \tan \beta$. Then it exists $d < b$ such that $f'(d) = \tan \beta$. Since the normal map is injective on $[0, \beta]$, we have $(d, f(d)) = c(\beta) = (a, f(a))$ which is impossible since $d < a$. $\square$

Lemma B.1. *Let c be a strictly convex curve parametrized by its normal on $[\beta - \varepsilon, \beta)$. We assume that the curve c is included in a compact of $\mathbb{R}^2$. Then $c(t)$ converges when $t \to \beta$.*

Proof. Since we are in a compact part of $\mathbb{R}^2$, it is enough to prove that there is only one cluster point for $c(t)$. Let us assume that $c(t_n) \to p_1$ and $c(s_n) \to p_2$ and, for every n, $t_n < s_n < t_{n+1}$. We apply Proposition B.2 on $[t_n, \beta)$ and $[s_n, \beta)$. This proves that p_1 is in an angular sector with vertex $c(s_n)$ and $\beta - s_n$ as angle and p_2 is in an angular sector with vertex $c(t_n)$ and $\beta - t_n$ as angle. Letting n goes to $+\infty$, we obtain that p_1 is in the half straight-line with p_2 as end-point and generated by v (where v is the limit unit tangent vector) and p_2 is in the half straight-line with p_1 as end-point and generated by v. It is possible only if $p_1 = p_2$. $\square$

Proposition B.3. *Let (c_n) be a sequence of stricly convex curves parametrized by their normal on $(\theta - \varepsilon, \theta + \varepsilon)$. We assume that (c_n) converges to $\widetilde{c}$ on $(\theta - \varepsilon, \theta) \cup (\theta, \theta + \varepsilon)$ in the C^1 topology. Then:*

- *we have $\widetilde{c}(t) \to p_1$ when $t \to \theta$, $t < \theta$ and $\widetilde{c}(t) \to p_2$ when $t \to \theta$, $t > \theta$,*
- *as sets, $c_n(\theta - \varepsilon, \theta + \varepsilon)$ converges to $\widetilde{c}(\theta - \varepsilon, \theta) \cup \widetilde{c}(\theta, \theta + \varepsilon) \cup [p_1, p_2]$.*

Moreover, if $p_1 = p_2$, the sequence (c_n) converges to $\widetilde{c}$ (that we extend by $\widetilde{c}(\theta) = p_1$) on $(\theta - \varepsilon, \theta + \varepsilon)$ in the C^1 topology.

Proof. ε is supposed to be small and we choose $\varepsilon' < \varepsilon$. We apply Proposition B.2 to $c_n(\theta - \varepsilon', \theta + \varepsilon')$ at the points $c_n(\theta - \varepsilon')$ and $c_n(\theta + \varepsilon')$. We get that, for every n, $c_n(\theta - \varepsilon', \theta + \varepsilon')$ is included in an angular sector of vertex $c_n(\theta - \varepsilon')$ and angle $2\varepsilon'$ and an angular sector of vertex $c_n(\theta + \varepsilon')$ and angle $2\varepsilon'$ (here, we apply Proposition B.2 to the curve c that we cover in the opposite sense). Letting n goes to $+\infty$, we get that $\widetilde{c}((\theta - \varepsilon', \theta) \cup (\theta, \theta + \varepsilon'))$ is included in the intersection of two angular sectors of angle $2\varepsilon'$, one has $\widetilde{c}(\theta - \varepsilon')$ as vertex the other has $\widetilde{c}(\theta + \varepsilon')$ as vertex. The intersection of this two angular sector is a compact; then, by Lemma B.1, p_1 and p_2 exists. We have also proved that the cluster points of the sequence of curves (c_n) are $\widetilde{c}((\theta - \varepsilon, \theta) \cup (\theta, \theta + \varepsilon))$

and points included in the intersection of the two angular sectors. If we let ε' goes to 0 the intersections of the two sectors converge to the segment $[p_1, p_2]$ which have $(\sin\theta, -\cos\theta)$ as normal. We must show that each point of the segment $[p_1, p_2]$ is the limit of a sequence $(c_n(t_n))$.

We suppose now that $\theta = 0$. By Proposition B.1, the curves c_n are graphs over $\{y = 0\}$. Let a and b be the respective first coordinates of $\widetilde{c}(-\varepsilon/2)$ and $\widetilde{c}(\varepsilon/2)$; we assume $a < b$. Since $c_n \to \widetilde{c}$, we can ensure that, over the segment $[a, b]$, the curves c_n are graphs for big n. We have $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$; since the segment $[p_1, p_2]$ is horizontal, $y_1 = y_2$. Besides by convergence of (c_n), $a < x_1 \leq x_2 < b$. Let $x \in [x_1, x_2]$; since c_n is a graph over $[a, b]$ for big n, there exist a parameter t_n such that $c_n(t_n)$ has x as first coordinate. The only possible cluster point for the sequence $(c_n(t_n))$ is (x, y_1) since every cluster point must have x as first coordinate. This proves that all the points of the segment $[p_1, p_2]$ is in the limit set.

If $p_1 = p_2$, (c_n) converges in the C^1 topology since the normal at the point $\widetilde{c}(\theta) = p_1$ is $(\sin\theta, -\cos\theta)$ by continuity. □

References

[ABR] S. Axler, P. Bourdon and W. Ramey, *Harmonic Function Theory*, Graduate Texts in Mathematics **137** (Springer-Verlag, 2001).

[CK] P. Collin and R. Krust, Le problème de Dirichlet pour l'équation des surfaces minimales sur des domaines non bornés, Bul. Soc. Math. France. **119** (1991), 443–462.

[CR] C. Cosín and A. Ros, A Plateau problem at infinity for properly immersed sufaces with finite total curvature, Indiana Univ. Math. J. **50** (2001), 847–879.

[Cox] H. S. M. Coxeter, *Introduction to Geometry*, Wiley Classics Library (John Wiley & Sons Inc., 1989).

[Fu] W. Fulton, *Algebraic Topology, A First Course*, Graduate Texts in Mathematics **153** (Springer-Verlag, 1995).

[HK] D. Hoffman and H. Karcher, Complete embedded minimal surfaces of finite total curvature, *Geometry V*, Encylopaedia Math. Sci. **90** (1997), 5–93.

[JS] H. Jenkins and J. Serrin, Variational problems of minimal surface type II, Arch. Rational Mech. Anal. **21** (1966), 321–342.

[JM] L. P. Jorge and W. H. Meeks III, The topology of complete minimal surfaces of finite totale Gaussian curvature, Topology **22** (1983), 203–221.

[Ka] H. Karcher, Embedded minimal surfaces derived from Scherk's examples, Manuscripta Math. **62** (1988), 83–114.

[Ma1] L. Mazet, The Dirichlet problem for minimal surfaces equation and Plateau problem at infinity, J. Inst. Math. Jussieu. **3** (2004), 397–420.

[Ma2] L. Mazet, Quelques résultats de non-existence pour l'équation des surfaces minimales, Bull. Sci. Math. **128** (2004), 577–586.

[MSL] Minimal Surface Library, `http://www-sbf256.iam.uni-bonn.de/grape/EXAMPLES/AMANDUS/amandus.html`

[Os] R. Osserman, *A Survey On Minimal Surfaces*, Van Nostrand Math. Studies. (1969).

[RSE] H. Rosenberg and R. Sa-Earp, The Dirichlet problem for the minimal surface equation on unbounded planar domains, J. Math. Pures Appl. **68** (1989), 163–183.

[Sc] R. Schoen, Uniqueness, symmetry, and embeddedness of minimal surfaces. J. Differential Geom. **18** (1983), 791–809.

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