A general halfspace theorem for constant mean curvature surfaces

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Abstract

In this paper, we prove a general halfspace theorem for constant mean curvature surfaces. Under certain hypotheses, we prove that, in an ambient space M^3, any constant mean curvature H_0 surface on one side of a constant mean curvature H_0 surface Σ_0 is an equidistant surface to Σ_0. The main hypotheses of the theorem are that Σ_0 is parabolic and the mean curvature of the equidistant surfaces to Σ_0 evolves in a certain way.

1 Introduction

One problem in the theory of constant mean curvature surfaces (cmc surfaces) is to know when two surfaces with the same constant mean curvature can coexist in the same ambient space M^3. More precisely, if Σ_1 and Σ_2 are two properly immersed constant mean curvature H_0 surfaces in a Riemannian 3 manifold M^3 (these surfaces are called H_0 surfaces), is the intersection $\Sigma_1 \cap \Sigma_2$ empty?

If we consider two spheres in $\mathbb{R}^3$ with the same radius, we can put them such a way that they do not meet. But inside a sphere of radius one, there is no compact constant mean curvature one surface.

If we consider non intersecting properly immersed minimal surfaces in $\mathbb{R}^3$, D. Hoffman and W. Meeks [11] proved that these minimal surfaces are parallel planes. For example, any minimal surface on one side of a plane is a plane. This result is called a halfspace theorem.

This result can also be stated in an other way. Let us consider a properly immersed minimal surface Σ in $\mathbb{R}^3$ with compact boundary and P a plane. We assume that Σ lies on one side of P, then the distance between Σ and P satisfies $d(P, \Sigma) = d(P, \partial\Sigma)$ *i.e.* the distance is achieved along the boundary.

Such a result is called a maximum principle at infinity. A very general maximum principle at infinity was proved by W. Meeks and H. Rosenberg in [14].

In a general setting, if Σ_0 is a properly embedded constant mean curvature H_0 surface in M^3, a halfspace theorem with respect to Σ_0 says that H_0 surfaces Σ that lies on one side of Σ_0 are "classified". Often, the classification implies that Σ has to be an equidistant surface to Σ_0. In this case, the halfspace theorem can be interpreted as a maximum principle at infinity.

For example, A. Ros and H. Rosenberg [21] proved in $\mathbb{R}^3$ that no H_0 surface can lie in the mean convex side of a properly embedded H_0 surface Σ_0 ($H_0 > 0$). We notice that this result says that any H_0 surface in the mean convex side of Σ_0 is an equidistant surface to Σ_0 but, since the equidistant surface to Σ_0 do not have constant mean curvature H_0, no such surface can exist.

Other halfspace theorems were proved by several authors. We have halfspace theorems with respect to horospheres in $\mathbb{H}^3$ [20], horocylinders in $\mathbb{H}^2 \times \mathbb{R}$ [10], vertical minimal planes in Nil_3 and Sol_3 [5, 6], rotationnal cmc $1/2$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$ [18] and entire minimal graph in Nil_3 [6]. We notice that, in [6], B. Daniel, W. Meeks and H. Rosenberg prove that the only minimal surfaces on one side of an entire minimal graph in Nil_3 are the vertical translate of the entire graph. Since the distance between an entire graph and one of its translate is not constant, the classification is of a different nature.

The aim of this paper is to give a general situation where a halfspace theorem is true. More precisely, we prove that, under some hypotheses, a H_0 surfaces that lies on one side of a given H_0 surface is necessarily an equidistant surface.

Let M^3 be a complete Riemannian 3 manifold which is geometrically bounded and Σ_0 a properly embedded constant mean curvature H_0 surface. Our main theorem says principally the following (see Theorem 7, for a precise statement)

Theorem. *Let $\Sigma_0 \hookrightarrow M^3$ be as above. We assume that Σ_0 is parabolic.*

1. *Assume that the equidistant surfaces to Σ_0 has mean curvature less than H_0 in the non mean convex side of Σ_0. Then any H_0 surface that lies in the non mean convex side of Σ_0 and is well oriented is an equidistant surface to Σ_0.*

2. *Assume that the equidistant surfaces to Σ_0 has mean curvature larger than H_0 in the mean convex side of Σ. Then any H_0 surface that lies*

in the mean convex side of Σ_0 is an equidistant surface to Σ_0.

In this result, the two important hypotheses are the parabolicity of Σ_0 and the value of the mean curvature of the equidistant surfaces. In fact, Σ_0 will be assumed to satisfy some other technical hypotheses (see Theorem 7). The "well oriented" hypothesis means that, along the surface, the mean curvature vector points to Σ_0. When Σ_0 is a minimal surface ($H_0 = 0$), the hypothesis on the mean curvature of the equidistant surfaces is that the mean curvature vector does not point to Σ_0. In fact the hypothesis about the mean curvature of the equidistant surface says that the mean curvature evolves like the one of concentric spheres: inside the sphere of radius 1 the mean curvature is larger than 1 outside it is less than 1.

If we consider $M^3 = \mathbb{R}^3$ and Σ_0 is a plane. Σ_0 is parabolic and the equidistant surface are also planes, thus the mean curvature hypothesis is satisfied. The theorem then applies and we recover the classical halfspace theorem.

Let us see why the hypotheses are important. We consider $M^3 = \mathbb{H}^2 \times \mathbb{R}$ and the upper halfspace model for $\mathbb{H}^2$ *i.e.* $\mathbb{H}^2 = \{(x, y) \in \mathbb{R} \times \mathbb{R}_+^*\}$ with the metric $\frac{1}{y^2}(\mathrm{d}x^2 + \mathrm{d}y^2)$. In $\Omega = \{(x, y) \in \mathbb{R} \times \mathbb{R}^*,\ x > 0\}$, we consider the function $u(x, y) = \ln \frac{\sqrt{x^2+y^2}+y}{x}$. This function is a solution to the minimal surface equation (its graph in $\mathbb{H}^2 \times \mathbb{R}$ is a minimal surface). As $x \to 0$, $u(x, y) \to +\infty$ and, as $y \to 0$, $u(x, y) \to 0$. Let Σ be the graph of u. The minimal surface Σ lies on one side of the minimal surface $\Sigma_0 = \mathbb{H}^2 \times \{0\}$ and is asymptotic to it; so there is no halfspace theorem for Σ_0. In fact the mean curvature of the equidistant surfaces to Σ_0 is 0. So the mean curvature hypothesis of the theorem is satisfied but Σ_0 is not parabolic. The surface Σ lies also on one side of the minimal surface $\Sigma_1 = \{x = 0\} \times \mathbb{R}$. This times, Σ_1 is parabolic (it is a flat $\mathbb{R}^2$) but the hypothesis for the mean curvature of the equidistant surfaces is not satisfied. Thus, both hypotheses are important in our statement.

Let us make a remark about the halfspace theorem of B. Daniel, W. Meeks and H. Rosenberg with respect to entire minimal graph in Nil_3 (Theorem 1.4 in [6]). Among all entire minimal graphs, certain are not parabolic, so their result is really of a different nature from the one we prove.

The paper is divided as follows. In the first section, we recall some definition about constant mean curvature surfaces and we write the Stokes formula in a general framework that we need.

In Section 3, we explain what is a parabolic manifold and we give a result that explain when the parabolicity is preserved by quasi-isometry.

In Section 4, we explain what kind of ambient space we consider in our halfspace theorem.

Section 5 is devoted to the proof of the first step of our main theorem. It consist in proving that, if a H_0 surface lies on one side of an other one, we can assume it is stable. In section 6, we state our main theorem and finish its proof.

In the last section, we apply our main theorem to some ambient spaces that have a Lie group structure. In this way, we recover known halfspace theorems [11, 20, 10, 5, 6] and prove new results.

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2 Preliminaries

In this section we recall some facts about cmc surfaces: what is the stability and what can be said about self-intersection. We also explain what is the Stokes formula in the setting of rectifiable boundary. Finally we define the area estimate we will use in the following sections.

2.1 Stability

Let S be a cmc surface in a Riemannian 3-manifold M. On S, the stability operator L acts on smooth functions with compact support on S by

$$Lu = -\Delta u - (2Ric(\vec{n}, \vec{n}) + |A|^2)u,$$

where $Ric(\vec{n}, \vec{n})$ is the Ricci curvature of the ambient manifold, $\vec{n}$ is the unit normal vector to the surface and $|A|$ the norm of the second fundamental form of S. L is also called the Jacobi operator of S.

The cmc surface S is said to be *stable* if the stability operator is non-negative on the set of smooth functions with compact support *i.e.*, for any smooth function u with compact support,

$$0 \leq \int_S uLu = \int_S \|\nabla u\|^2 - (2Ric(\vec{n}, \vec{n}) + |A|^2)u^2.$$

The stability operator appears as the second derivative of the area for normal variations of the surface S or as the first derivative of the mean curvature (see [2]).

2.2 Stokes formula

Let Ω be a domain in $\mathbb{R}^n$ with a rectifiable boundary of finite $\mathcal{H}^{n-1}$ measure. This is the same as saying that the current $[\Omega]$ associated to Ω has a rectifiable boundary $\partial[\Omega]$. By Theorem 4.1.28 and Theorem 4.5.6 in [7] (see also 4.5.12 in [7] and 12.2 in [17]), for any smooth vector field X with compact support in $\mathbb{R}^n$, the Stokes formula can be written:

$$\int_\Omega \operatorname{div} X(x) \mathrm{d}\mathcal{L}^n x = \int_{\partial\Omega} X(x) \cdot \vec{n}(\Omega, x) \mathrm{d}\mathcal{H}^{n-1} x \tag{1}$$

where $\vec{n}(\Omega, x)$ is a unit vector called the exterior normal of Ω at x (see 4.5.5 in [7] for a definition in this situation). This exterior normal is defined $\mathcal{H}^{n-1}$ almost everywhere along $\partial\Omega$. We notice that the definition of $\vec{n}(\Omega, x)$ is local and coincides with the classical unit outgoing normal vector for smooth boundaries.

Moreover, the $\mathcal{H}^{n-1}$ measure of $\partial\Omega$ is equal to the mass of the $n-1$ current $\partial[\Omega]$.

2.3 Self intersection

Now let us consider D_2, an open disk in $\mathbb{R}^2$, and D_1, the open disk with the same center and half radius. On $D_2 \times \mathbb{R}$, we consider a Riemannian metric g. Let $f_1, \cdots, f_n$ be smooth pairwise different functions on D_2 such that their graphs have constant mean curvature H_0 with respect to the metric g and the mean curvature vector points downward. Let p in D_2 such that $f_i(p) = f_j(p)$ and $\nabla(f_i - f_j)(p) = 0$, p is a singular intersection point. The structure of the set $\{f_i = f_j\}$ near p is then described by Theorem 5.3 in [4]: it is the union of $2d$ embedded arcs meeting at p. Moreover such points are isolated.

Let f_0 be a smooth function on D_2 such that $\nabla(f_0 - f_i)$ does not vanish at any point where $f_0 = f_i$. We notice that if p satisfies $f_i(p) = f_j(p)$ and $\nabla(f_i - f_j)(p) \neq 0$, the level set $\{f_i = f_j\}$ is locally an embedded arc. This implies that $I_{i,j} = \{f_i = f_j\}$ is locally either a smooth arc or the union of embedded arcs meeting at a point. Thus $I_{i,j} \cap \overline{D_1}$ is compact.

We define the function f by $f(p) = \min_i f_i(p)$. Let Ω_i be the open subset $\Omega_i = \{p \in D_1 | \, f(p) = f_i(p) \text{ and } \forall j \neq i \; f(p) < f_j(p)\}$. The question is: what is the Stokes formula for such a domain Ω_i?

First we see that $\partial\Omega_i$ is included in the sets $I_{i,j}$ and the boundary of D_1. This implies that $\partial\Omega_i$ is a 1-rectifiable subset of finite $\mathcal{H}^1$ measure. Thus the above formula (1) can be applied. But we need to understand what is $\vec{n}(\Omega_i, x)$ and where it is defined.

Let p be a point in $\partial\Omega_i \setminus \partial D_1$. First, since the singular intersection points form a discrete set, this set is finite in D_1 and has vanishing $\mathcal{H}^1$ measure. So we assume that p is not such a point. We denote by $\Lambda(p)$ the set of indices j such that $p \in \partial\Omega_j$. Then $i \in \Lambda(p)$ and, for any $j \in \Lambda(p)$, we have $f(p) = f_j(p)$. There are two situations.

First, the vectors $\nabla(f_j - f_l)(p)$ for $j \neq l$ and $j, l \in \Lambda(p)$ are not all linearly dependent (this implies that the intersection of the tangent planes to the graphs of the f_j, $j \in \Lambda(p)$, is a point). In this case, near p the domain Ω_i is included in an angular sector of angle strictly less than π. This implies that the exterior normal $\vec{n}(\Omega_i, p)$ does not exist. This is the same for $\vec{n}(\Omega_j, p)$, $j \in \Lambda(p)$.

Let us assume now that the vectors $\nabla(f_j - f_l)(p)$ for $j \neq l$ and $j, l \in \Lambda(p)$ are all linearly dependent: the intersection of the tangent planes to the graphs is now a line (this is the case when $\Lambda(p)$ has only two elements). In this case, all the curves $I_{j,l}$, $j, l \in \Lambda(p)$, are tangent at p. Let L_j be the differential of f_j at p. For any $x \in \mathbb{R}^2$, we define $L(x) = \min_{j \in A(p)} L_j(x)$. Since the L_j are linear and different, there exists, in fact, a unique subset $\{j_1, j_2\} \subset \Lambda(p)$ such that $L(x) = \min(L_{j_1}(x), L_{j_2}(x))$. $\{L = L_{j_1}\}$ and $\{L = L_{j_2}\}$ are half-planes and we denote by $\vec{\eta}$ the unit vector normal to $\{L_{j_1} = L_{j_2}\}$ and pointing in $\{L = L_{j_2}\}$. Thus, for any $\lambda < 1$, the affine angular sector $\{p + x, \lambda\|x\| < \vec{\eta} \cdot x\}$ is included in Ω_{j_2} near p and the affine angular sector $\{p + x, \lambda\|x\| < -\vec{\eta} \cdot x\}$ is included in Ω_{j_1} near p. This implies that $\vec{n}(\Omega_{j_1}, p) = \vec{\eta}$, $\vec{n}(\Omega_{j_2}, p) = -\vec{\eta}$ and $\vec{n}(\Omega_j, p)$ is not defined for any $j \in \Lambda(p) \setminus \{j_1, j_2\}$. In this case we say that p is in the set $\Gamma_{j_1,j_2} = \Gamma_{j_2,j_1}$.

Finally, with this definition, if X is a smooth vector field with compact support in D_1, we get the following Stokes formula:

$$\int_{\Omega_i} \operatorname{div} X(x) \mathrm{d}\mathcal{L}^2 x = \sum_{j \neq i} \int_{\Gamma_{i,j}} X(x) \cdot \vec{n}(\Omega_i, x) \mathrm{d}\mathcal{H}^1 x.$$

2.4 Area bounds

In this subsection, we define a notion of area bound. Let M be a Riemannian 3-manifold. Let p be a point in M and P be a plane in T_pM. Let (e_1, e_2, e_3) be an orthonormal basis of T_pM such that P is the plane generated by e_1 and e_2. We denote by $O_{p,P}(t)$ the image by the exponential map at p of the ellipsoid $\{(x, y, z) \in T_pM | x^2 + y^2 + 4z^2 \leq t^2\}$ where (x, y, z) are the coordinates in T_pM with respect to (e_1, e_2, e_3).

Let $V(O_{p,P}(t))$ be the volume of $O_{p,P}(t)$ and $A(\partial O_{p,P}(t))$ be the area of its boundary. We notice that, for t small, $V(O_{p,P}(t)) \sim (2/3)\pi t^3$ and

$A(\partial O_{p,P}(t)) \sim 4\pi\alpha t^2$ with $\alpha < 1$.

Definition 1. *Let $(S_i)_{i\in\mathcal{I}}$ be a family of immersed surfaces in a Riemannian 3-manifold M. We say that the family satisfies a uniform area estimate of at most one leaf if for any $p \in M$ and P a plane in T_pM there exists $t_0 > 0$ and $\beta < 1$ such that, for any $t < t_0$ and $i \in \mathcal{I}$,*

$$A(S_i \cap O_{p,P}(t)) \leq 2\beta\pi t^2.$$

Remark 1. If S is an immersed surface and p is a point in S, we have

$$\liminf_{t\to 0} \frac{A(S \cap O_{p,T_pS}(t))}{t^2} \geq \pi.$$

Thus the area estimate of at most one leaf prevents S to pass at p more than one time with the same tangent plane.

3 Parabolic manifolds

In this section, we recall some definitions about the conformal type of Riemannian manifolds and we explain when the conformal type is preserved by quasi-isometries. We refer to [8] for a general presentation of conformal types.

Let (M, g) be a Riemannian manifold. A continuous function u on a domain $\Omega \in M$ is *superharmonic* if, for any precompact domain $U \subset\subset \Omega$ and any harmonic function $v \in C^2(U) \cap C^0(\overline{U})$, $v \leq u$ on ∂U implies $v \leq u$ on U. If $u_1, \cdots, u_n$ are superharmonic functions, we remark that $u = \inf_i u_i$ is also a superharmonic function.

Definition 2. *Let (M, g) be a Riemannian manifold.*

1. *If $\partial M = \emptyset$, M is* parabolic *if any bounded superharmonic function on M is constant.*
2. *If $\partial M \neq \emptyset$, M is* parabolic at infinity *if any bounded non-positive superharmonic function on M with $u = 0$ on ∂M is constant.*

When $\partial M \neq \emptyset$, M is often said to be "parabolic" instead of "parabolic at infinity", but we prefer to use different terminologies. In fact, there are a lot of equivalent characterizations of parabolicity (see [8]) and we will use certain of them below. As an example, a Riemannian manifold M without boundary is parabolic if and only if there exists a sequence $(\varphi_n)_n$ of smooth functions with compact support in M such that $0 \leq \varphi_n \leq 1$, $(\varphi_n^{-1}(1))_n$ is an increasing exhaustion by compact subsets of M and

$$\lim_{n\to+\infty} \int_M \|\nabla\varphi_n\|^2 = 0.$$

We remark that a subdomain of a parabolic manifold, viewed as a manifold with boundary, is parabolic at infinity.

Let (M, g) and (N, h) be two n-dimensional Riemannian manifold and let F be a map from M to N. If $k \geq 1$, we say that F is k quasi-isometric or a local k quasi-isometry if, for any $p \in M$ and $v \in T_pM$, we have $\frac{1}{k}\|v\|_g \leq \|T_pF(v)\|_h \leq k\|v\|_g$. If M and N has no boundary and $F : M \to N$ is a k quasi-isometric diffeomorphism, M is parabolic if an only if N is parabolic. For parabolicity at infinity, we do not have such a result. In fact we have the following proposition:

Proposition 1. *Let (M, g) and (N, h) be two n-dimensional Riemannian manifold such that $\partial M \neq \emptyset$ and N has no boundary. We assume that (N, h) is parabolic and that there exists $F : M \to N$ an injective local k quasi-isometry. Then M is parabolic at infinity.*

Proof. Let us assume that M is not parabolic at infinity, then it exists a harmonic function u_M such that $0 < u_M \leq 1$, $u_M = 1$ on ∂M and $\inf_M u_M = 0$ (see [8], $u_M(x)$ is the probability that a Brownian motion from x hits the boundary of M). Let $\eta \in (0, 1)$ be a regular value of u_M and $\varphi \in C^\infty(\mathbb{R}, [0, 1])$ be a function such that $\varphi = 0$ on $[(1 + \eta)/2, +\infty)$ and $\varphi = 1$ on $(-\infty, \eta]$.

Since F is a quasi-isometry, DF is invertible and $\tilde{g} = F_*(g)$ is well defined. $\tilde{g}$ is a section over $F(M)$ of the symmetric 2-tensor bundle. Moreover $\tilde{g}$ is positive definite and we have $\frac{1}{k^2}h \leq \tilde{g} \leq k^2 h$.

We denote by v the function $u_M \circ F^{-1}$ on $F(M)$ and we consider $\mu = \varphi \circ v$. The function μ is C^∞ on $F(M)$ and vanishes on $v^{-1}([(1 + \eta)/2, 1])$. This domain contains a neighborhood of $F(\partial M) = \partial F(M)$. So we can extend the definition of the function μ by 0 to the complement of $F(M)$. The function μ is then C^∞ on N with $\mu = 1$ on $v^{-1}([0, \eta])$.

On N, we define $\tilde{h} = (1 - \mu)h + \mu\tilde{g}$ ($\mu\tilde{g}$ is well defined on N since μ vanishes outside $F(M)$). $\tilde{h}$ is a global section of the symmetric 2-tensor bundle and we have:

$$\frac{1}{k^2}h \leq ((1 - \mu) + \frac{1}{k^2}\mu)h \leq \tilde{h} \leq ((1 - \mu) + k^2\mu)h \leq k^2 h$$

So $\tilde{h}$ defines a Riemannian metric on N and $\mathrm{id}_N : (N, h) \to (N, \tilde{h})$ is a local quasi-isometry.

Since (N, h) is parabolic, so is $(N, \tilde{h})$. Let $\tilde{v}$ be the function defined by η outside $F(M)$ and by $\min(\eta, v)$ on $F(M)$; $\tilde{v}$ is continuous on M. On $v^{-1}([0, \eta])$, $\tilde{h} = \tilde{g}$ so $v^{-1}([0, \eta])$ with the metric $\tilde{h}$ is isometric by F with $u_M([0, \eta]) \subset M$. Thus, on $v^{-1}([0, \eta])$, $\Delta_{\tilde{h}}\tilde{v} = \Delta_{\tilde{g}}v = (\Delta_g u_M) \circ F^{-1} = 0$ since u_M is harmonic. On the complement to $v^{-1}([0, \eta])$, $\tilde{v}$ is constant so $\Delta_{\tilde{h}}\tilde{v} = 0$. Therefore, $\tilde{v}$ is a positive superharmonic function on $(N, \tilde{h})$ (it is locally the infimum of two harmonic function) and it is bounded from above by η. $\tilde{v}$ is then constant and equal to η. This implies that $u_M = \eta$ on $u_M^{-1}([0, \eta])$ which contradicts $\inf_M u_M = 0$. This ends the proof of the proposition. □

4 Regular ε-neighborhood

In this section, we explain what kind of ambient space we will consider in our main theorem. Let Σ be a properly embedded constant mean curvature H_0 surface in an ambient 3-manifold M. The ε-tubular neighborhood of Σ is the set of points in M at distance less than ε from Σ. We can define the map $F : \Sigma \times [-\varepsilon, \varepsilon] \to M$, $(x, t) \mapsto \exp_x(t\vec{n}(x))$ where $\vec{n}(p)$ is the unit normal vector such that the mean curvature vector of Σ at p is $-H_0\vec{n}(p)$. The image of F is the ε-tubular neighborhood of Σ. When F is a diffeomorphism, it gives a global parametrization of the neighborhood. Besides if $H_0 > 0$, the image of $F(\Sigma \times [-\varepsilon, 0])$ is the mean convex side of the tubular neighborhood and $F(\Sigma \times [0, \varepsilon])$ is the non-mean convex side. When $H_0 = 0$, no such distinction can be done.

We want to take this situation as a model for our ambient spaces.

Definition 3. *Let $(\Sigma, \mathrm{d}\sigma_0^2)$ be a 2-dimensional complete Riemannian manifold. An* outside ε-half neighborhood *of Σ is the 3-manifold with boundary $M_+(\varepsilon) = \Sigma \times [0, \varepsilon]$ with a Riemannian metric $\mathrm{d}s^2 = \mathrm{d}\sigma_t^2 + \mathrm{d}t^2$ where $t \mapsto \mathrm{d}\sigma_t^2$ is a smooth family of Riemannian metric on Σ such that $\mathrm{d}s^2$ is complete.*

An inside ε-half neighborhood *of Σ is the 3-manifold with boundary $M_-(\varepsilon) = \Sigma \times [-\varepsilon, 0]$ with a Riemannian metric $\mathrm{d}s^2 = \mathrm{d}\sigma_t^2 + \mathrm{d}t^2$ where $t \mapsto \mathrm{d}\sigma_t^2$ is a smooth family of Riemannian metric on Σ such that $\mathrm{d}s^2$ is complete.*

It seems that we define twice the same object but we prefer to use two different terms for the model of the mean convex side (the inside ε-half neighborhood) and the non-mean convex side (the outside ε-half neighborhood).

Let $M_\pm(\varepsilon)$ be a ε-half neighborhood of Σ. If $\varepsilon' \leq \varepsilon$, the submanifold $\Sigma \times [0, \varepsilon'] \subset M_+(\varepsilon)$ is denoted by $M_+(\varepsilon')$ and is an outside ε'-half neighborhood. $M_-(\varepsilon') = \Sigma \times [-\varepsilon', 0] \subset M_-(\varepsilon)$ is an inside ε'-half neighborhood.

We denote by Σ_t the submanifold $\Sigma \times \{t\}$, Σ_0 with its induced metric is then isometric to $(\Sigma, \mathrm{d}\sigma_0^2)$. Σ is then isometrically embedded in $M_\pm(\varepsilon)$. Σ and Σ_0 will be often viewed as the same object. We denote $M^*_\pm(\varepsilon) = M_\pm(\varepsilon) \setminus \Sigma_0$. We also define the distance function $\mathbf{d}$ as $\mathbf{d}(x,t) = |t|$, $\mathbf{d}$ is then the distance from Σ_0. Σ_t is the equidistant surface from Σ_0 at distance $|t|$. On $M_\pm(\varepsilon)$, we define the projection map $\pi : M_\pm(\varepsilon) \to \Sigma_0$ by $\pi(x,t) = (x, 0)$. We denote by π_t the restriction of π to Σ_t.

Let $\vec{\xi}$ denote the unit vectorfield $\dfrac{\partial}{\partial t}$. In the following, we always consider $-\vec{\xi}$ as the unit normal vector to the surface Σ_t. This is the normal vector field w.r.t. we will compute the mean curvature. For $(x,t) \in M_\pm(\varepsilon)$, this implies that $\operatorname{div} \vec{\xi}(x,t) = 2H(x,t)$ where $H(x,t)$ is the mean curvature of Σ_t at (x,t).

Definition 4. *Let $H_0 \geq 0$ be a constant.*

- *We say that $M_+(\varepsilon)$ satisfies the $H \leq H_0$* hypothesis *if, for any $t \in [0, \varepsilon]$, the mean curvature of Σ_t is less than or equals to H_0 at any point.*
- *We say that $M_-(\varepsilon)$ satisfies the $H \geq H_0$* hypothesis *if, for any $t \in [-\varepsilon, 0]$, the mean curvature of Σ_t is larger than or equals to H_0 at any point.*

Definition 5. *Let $(\Sigma, \mathrm{d}\sigma_0^2)$ be a complete 2-dimensional Riemannian manifold. Let $M_\pm(\varepsilon)$ be an outside or inside ε-half neighborhood. We say that $M_\pm(\varepsilon)$ is* regular *if*

1. *there is $k > 0$ such that π_t is a k quasi-isometric map for any t with $|t| \leq \varepsilon$.*
2. *there is C such that the norm of the second fundamental form of Σ_t is bounded by C for any t with $|t| \leq \varepsilon$.*
3. *$M_\pm(\varepsilon)$ is geometrically bounded.*

Let S be a properly immersed cmc H_0 surface ($H_0 \geq 0$) in $M_\pm(\varepsilon)$ with $S \subset M^*_\pm(\varepsilon)$ and possibly nonempty boundary in $\Sigma_{\pm\varepsilon}$. Along S, we always consider the unit normal vector $\vec{N}$ such that the mean curvature vector is $H_0\vec{N}$. We denote by D the connected component of $M_\pm(\varepsilon) \setminus S$ that contains

Σ_0. Consider a point p in $S \cap \partial D$. Let $\Delta \subset S$ an embedded neighborhood of p. Let us consider the map $F : \Delta \times (-\eta, \eta) \to M_\pm(\varepsilon), (q,t) \mapsto \exp_q(t\vec{N}(q))$. F is an embedding if η is small enough and its image is a neighborhood of p in $M_\pm(\varepsilon)$. We say that the mean curvature vector of S at p points into D (resp. not into D) if, for any sequence $(p_n)_n$ in D with $p_n \to p$, $p_n \in F(\Delta \times [0, \eta))$ (resp $p_n \in F(\Delta \times (-\eta, 0]))$ for large n.

We say that S is *well oriented* if, for any point in $S \cap \partial D$, the mean curvature vector of S points into D (resp. not into D) when $S \looparrowright M_+(\varepsilon)$ (resp. $S \looparrowright M_-(\varepsilon)$) (see Figure 1). We notice that, when S is minimal ($H_0 = 0$), S can be assumed to be orientable by considering a covering space. Moreover it can always be considered as well oriented.

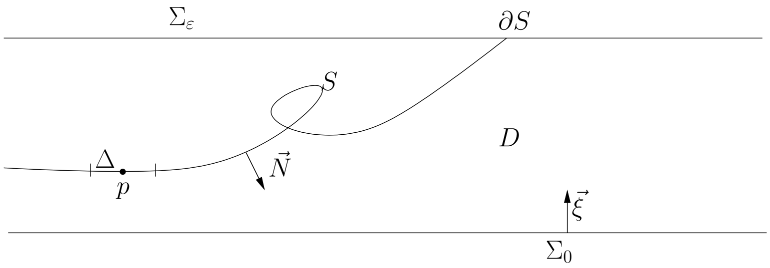

Figure 1: A well oriented surface S

5 Construction of stable constant mean curvature surface

Let $M_\pm(\varepsilon)$ be an ε-half neighborhood of a surface $(\Sigma, \mathrm{d}\sigma_0^2)$ such that Σ_0 has constant mean curvature H_0. The main result of our paper says under which hypotheses we have a halfspace theorem for Σ_0 : Any properly immersed constant mean curvature H_0 surface in $M_\pm(\varepsilon)$ is an equidistant surface to Σ_0. In this section, we explain that, if such a constant mean curvature H_0 surface exists, we can assume that it is stable.

Theorem 2. *Let $(\Sigma_0, \mathrm{d}\sigma_0^2)$ be a complete orientable Riemannian surface, ε be a positive constant and H_0 be a non-negative constant. Let $M_\pm(\varepsilon)$ be an inside or outside ε-half neighborhood of Σ. We consider a properly immersed constant mean curvature H_0 surface S in $M_\pm(\varepsilon)$ with possibly nonempty boundary in Σ_ε and $S \subset M_\pm^*(\varepsilon)$. We assume that the lower bound of the distance function $\mathbf{d}$ on S is 0.*

1. *If $S \looparrowright M_+(\varepsilon)$ is well oriented and $M_+(\varepsilon)$ satisfies the $H \leq H_0$ hypothesis, there exist $\varepsilon' > 0$ and a properly immersed constant mean curvature H_0 surface S' in $M_+(\varepsilon')$ with non empty boundary in $\Sigma_{\varepsilon'}$ such that $S' \subset M_+^*(\varepsilon')$, S' is stable, well oriented and the distance function $\mathbf{d}$ on S' is not constant.*

2. *If $S \looparrowright M_-(\varepsilon)$ and $M_-(\varepsilon)$ satisfies the $H \geq H_0$ hypothesis, there exist $\varepsilon' > 0$ and a properly immersed constant mean curvature H_0 surface S' in $M_-(\varepsilon')$ with non empty boundary in $\Sigma_{\varepsilon'}$ such that $S' \subset M_-^*(\ \varepsilon')$, S' is stable and the distance function $\mathbf{d}$ on S' is not constant.*

The remaining part of the section is devoted to the proof of this result. But let us begin by some remarks on the proof and the result.

The first remark is that, for $H_0 = 0$, both cases are in fact the same since the good orientation hypothesis has no meaning and the outside and inside half neighborhoods play the same role for minimal surfaces.

Besides the proof of both cases are very similar. So we write a detailed proof only for the first case with $H_0 > 0$. Then we explain what are the important changes to do in the other cases.

One more remark about this result is that, if S is stable or if a surface $S \cap M_\pm(\varepsilon')$, for $\varepsilon' < \varepsilon$, is stable then this surface gives the surface S' we look for. If no such surface is stable, the surface S' produced by the proof is, in fact, embedded and well oriented in both cases.

A large part of the proof is inspired by the work of A. Ros and H. Rosenberg in [21] and L. Hauswirth, P. Roitman and H. Rosenberg in [9].

5.1 $S \looparrowright M_+(\varepsilon)$ and $H_0 > 0$

Let us consider S in $M_+^*(\varepsilon)$ with $H_0 > 0$. First we need to introduce objects that will be used in the proof.

5.1.1 Definition of the barriers

Let x_0 be a point in Σ and $\eta_0 > 0$ such that the exponential map $\exp_{x_0}$ for the metric $d\sigma_0^2$ is a diffeomorphism from the disk of radius η_0 in $T_{x_0}\Sigma$ into a neighborhood D_{η_0} of x_0. Since S is properly immersed, there is ε_0 such that $D_{\eta_0} \times [0, \varepsilon_0] \cap S = \emptyset$. In $D_{\eta_0} \times [0, \varepsilon_0]$ we consider the chart $\exp_{x_0} \times \mathrm{id}$ defined on $\Delta_{\eta_0} \times [0, \varepsilon_0]$ where Δ_{η_0} is the Euclidean disk in $T_{x_0}\Sigma$ of radius η_0. Let η be small and consider in $\Delta_{\eta_0} \times [0, \varepsilon_0]$ the surfaces of revolution $C_{\eta,t}$ parametrized by

$$X_{\eta,t}(u, v) = \Big((t - \frac{\eta}{6}\cos v)\cos u, (t - \frac{\eta}{6}\cos v)\sin u, \frac{\eta}{6}(1 + \sin v)\Big)$$

where $(u, v) \in [0, 2\pi] \times [-\pi/2, \pi/2]$, $t \le \eta_0$ and $\eta \le \min(\eta_0, \varepsilon_0)$ (see Appendix A).

Let η be sufficiently small so that the surfaces $C_{\eta,t}$ are well defined for $t \in [\eta/2, \eta]$. We denote by K the compact domain of $\Delta_{\eta_0} \times [0, \eta/3]$ bounded by $C_{\eta,\eta/2}$ and containing the origin. For $t \in [\eta/2, \eta]$, we denote by Q_t the domain of $\Delta_{\eta_0} \times [0, \eta/3]$ bounded by $C_{\eta,\eta/2}$ and $C_{\eta,t}$ ($Q_t \subset Q_\eta$). Q_η is foliated by the surfaces $C_{\eta,t}$ for $t \in [\eta/2, \eta]$. As explained in Appendix A, on these surfaces, the mean curvature vector does not point to K and its norm is larger than $1/\eta$. We denote by $\mathcal{K}_{bar}$, $\mathcal{Q}_{bar,t}$ and $\mathcal{C}_{\eta,t}$ the images of K, Q_t and $C_{\eta,t}$ in $D_{\eta_0} \times [0, \varepsilon_0] \subset M_+(\varepsilon_0)$ (see Figure 2), we also denote $\overset{\circ}{\mathcal{Q}}_{bar,t} = \mathcal{Q}_{bar,t} \setminus \mathcal{C}_{\eta,t}$. In fact, $\mathcal{K}_{bar} \cup \mathcal{Q}_{bar,\eta} \subset M_+(\eta/3)$ and these subsets do not meet S. Let $\vec{\xi}_{bar}$ be the unit vector normal to $\mathcal{C}_{\eta,t}$ which does not point to $\mathcal{K}_{bar}$. $\vec{\xi}_{bar}$ is a unit vector field on $\mathcal{Q}_{bar,\eta}$. Since $\operatorname{div} \vec{\xi}_{bar}$ is the opposite of the mean curvature of $\mathcal{C}_{\eta,t}$, the value of the Euclidean mean curvature implies that η can be chosen sufficiently small such that $\operatorname{div} \vec{\xi}_{bar}$ is as small as we want. So we choose η such that $\operatorname{div} \vec{\xi}_{bar} \le 2H_0$ and $\eta/3$ is a regular value of the function $\mathbf{d}$ on S.

We write $\varepsilon_1 = \eta/3$. From now on, we work in $M_+(\varepsilon_1)$ and we consider the restriction of S to $M_+(\varepsilon_1)$ which we still call S.

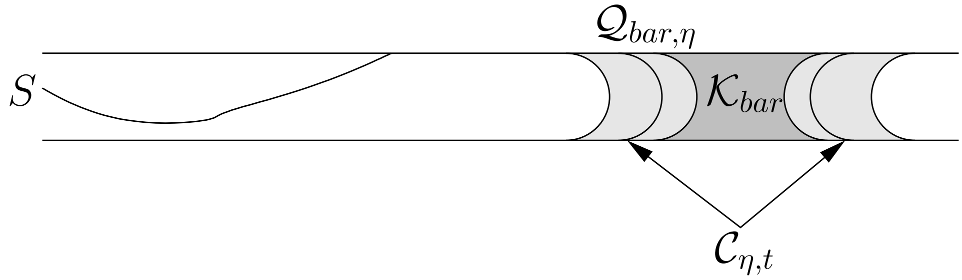

Figure 2: The domains $\mathcal{K}_{bar}$ and $\mathcal{Q}_{bar,\eta}$

As explained above, if S is stable Theorem 2 is already proved so we can assume S is not stable. Hence there exists an exhaustion $(K_n)_n$ of Σ by compact subsets such that, for any n, $S_n = S \cap (K_n \times [0, \varepsilon_1])$ is not stable.

We denote by D the connected component of $M_+(\varepsilon_1) \backslash S$ that contains Σ_0 and $D_n = D \cap (K_n \times [0, \varepsilon_1])$. We notice that $\mathcal{K}_{bar} \cup \mathcal{Q}_{bar,\eta} \subset D_n$ for large n. Since S is well oriented, the mean curvature vector of S points into D_n along $S \cap \partial D_n$. Let φ_n be the first eigenfunction of the Jacobi operator of S_n; φ_n vanishes on ∂S_n, is positive in the interior of S_n and satisfies

$-L\varphi_n + \lambda_{1,n}\varphi_n = 0$ where L is the stability operator and $\lambda_{1,n}$ is a negative constant. Perturbing K_n, we can assume that 0 is not an eigenvalue of $-L$, hence there is a smooth function v_n on S_n, vanishing on the boundary such that $-Lv_n = 1$ in S_n. By the boundary maximum principle, the outing derivative $\dfrac{\partial \varphi_n}{\partial \nu}$ is negative along ∂S_n. Thus for a_n small enough, the function $u_n = \varphi_n + a_n v_n$ is positive in the interior of S_n.

Let $\vec{N}(x)$ be the unit normal to S_n such the mean curvature vector is $H_0\vec{N}(x)$. For $t_{0,n} > 0$, we define, on $S_n \times [0, t_{0,n}]$, the map $F(x,t) = \exp_x(tu_n(x)\vec{N}(x))$, we assume $t_{0,n}$ small such that F is an immersion. We then denote $\widetilde{\mathcal{Q}}^n_{uns} = S_n \times [0, t_{0,n}]$ with the induced metric $F^*\mathrm{d}s^2$. In $\widetilde{\mathcal{Q}}^n_{uns}$ we consider the surfaces $\mathcal{S}^n_t = S_n \times \{t\}$ which foliates $\widetilde{\mathcal{Q}}^n_{uns}$. Let $\vec{\xi}_{uns}$ be the unit vector field defined on $\widetilde{\mathcal{Q}}^n_{uns}$ normal to $\mathcal{S}^n_t$. We have $\operatorname{div}\vec{\xi}_{uns} = -2H_t$ where H_t is the mean curvature of $\mathcal{S}^n_t$. Moreover, we have:

$$\frac{d}{dt}_{|t=0} 2H_t = -L'u = -\lambda_{1,n}\varphi_n + a_n > 0$$

so, choosing $t_{0,n}$ small enough, we get $H_t > H_0$. We define $\mathcal{Q}^{n,0}_{uns} = F(\widetilde{\mathcal{Q}}^n_{uns}) \cap D_n$ and $\mathcal{D}_n = D_n \setminus \mathcal{Q}^{n,0}_{uns}$ (see Figure 3). In fact, $t_{0,n}$ is also chosen such that $\mathcal{D}_{n-1} \cap (K_{n-2} \times [0, \varepsilon_1]) \subset \mathcal{D}_n \cap (K_{n-2} \times [0, \varepsilon_1])$. This implies that the sequence $(\mathcal{D}_n)_n$ is increasing with respect to compact subsets. We can also assume that $\cup_n \mathcal{D}_n = D$.

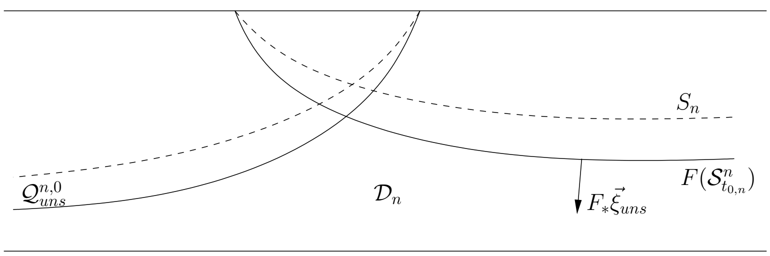

Figure 3:

The surface $\mathcal{S}^n_{t_{0,n}}$ is immersed by F in $M_+(\varepsilon_1)$ and the normal vector $F_*(\vec{\xi}_{uns})$ points to $\mathcal{D}_n$ along $\mathcal{S}_n$ where $\mathcal{S}_n = F(\mathcal{S}^n_{t_{0,n}}) \cap \partial \mathcal{D}_n$ (see Figure 3). Let x be a point in $\mathcal{S}^n_{t_{0,n}}$ and consider $D_x \subset \mathcal{S}^n_{t_{0,n}}$ a small open geodesic disk which is embedded in $M_+(\varepsilon_1)$ by F. Let ψ be a smooth function on $\mathcal{S}^n_{t_{0,n}}$ vanishing outside D_x and positive in D_x. We then define on $\mathcal{S}^n_{t_{0,n}} \times [0, 2t_x]$

$$G(p,t) = \exp_{F(p)}(t\psi(p)F_*(\vec{\xi}_{uns}(p)))$$

If we choose t_x small enough, we can assume that G is an embedding on $D_x \times [0, 2t_x]$. In $G(D_x \times [0, 2t_x])$, we define $\vec{\xi}_x$ the unit vector field normal to the embedded surfaces $\mathcal{S}_t^x = G(D_x \times \{t\})$ with $\vec{\xi}_x = F_*(\vec{\xi}_{uns})$ along $\mathcal{S}_0^x$. Since the mean curvature of $\mathcal{S}_0^x$ is larger than H_0, if t_x is small enough, we can assume that $\operatorname{div} \vec{\xi}_x < -2H_0$.

For $\delta \in [1, 2]$, we denote $\mathcal{Q}_{x,\delta} = G(D_x \times [0, \delta t_x])$ (see Figure 4) and $\overset{\circ}{\mathcal{Q}}_{x,\delta} = G(D_x \times [0, \delta t_x))$. Now we define

$$\mathcal{Q}_{uns}^{n,1} = \mathcal{D}_n \cap \bigcup_{x \in \mathcal{S}_{t_0}^n} \mathcal{Q}_{x,1}.$$

Since $F_*(\vec{\xi}_{uns})$ points to $\mathcal{D}_n$ along $\mathcal{S}_n$, any point at $\mathcal{S}_n$ is at a positive distance from $\mathcal{D}_n \setminus \mathcal{Q}_{uns}^{n,1}$.

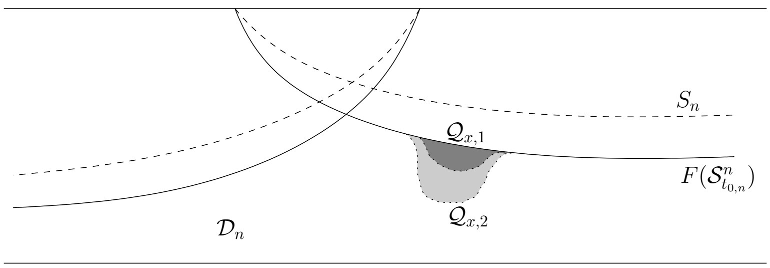

Figure 4:

Let $t_{1,n} > 0$ be small and, for $\mu \in (0, 1]$, let us define $\mathcal{Q}_{par,\mu}^n = K_n \times [0, \mu t_{1,n}]$. $t_{1,n}$ is chosen such that $\mathcal{Q}_{par,1}^n \subset \mathcal{D}_n$ and $\mathcal{Q}_{par,1}^n \cap \mathcal{Q}_{uns}^{n,1} = \emptyset$. $\mathcal{Q}_{par,1}^n$ is foliated by the equidistant surfaces to Σ_0 and we have $\operatorname{div} \vec{\xi} \leq 2H_0$ since the $H \leq H_0$ hypothesis is satisfied.

5.1.2 Construction of compact stable constant mean curvature surfaces

With the notations of the preceding subsection, we have the following lemma.

Lemma 3. *There exists $\varepsilon_2 \in (0, \varepsilon_1)$ and $p_0 \in S$ such that, for large n, there exists a stable constant mean curvature H_0 embedded surface S_n' in $(\mathcal{D}_{n+1} \cap (K_n \times \mathbb{R})) \setminus (\mathcal{K}_{bar} \cup \mathcal{Q}_{par,1/2}^{n+1})$ with boundary on $\partial K_n \times \mathbb{R}$ and $K_n \times \{\varepsilon_2\}$ and $S_n' \cap [\pi(p_0), p_0] \neq \emptyset$. Moreover the surfaces S_n' are well oriented i.e. the mean curvature vector points into the connected component of $(\mathcal{D}_{n+1} \cap (K_n \times$*

$\mathbb{R}))\setminus S'_n$ *which contains* $K_n \times 0$ *and the surfaces* S'_n*,* n *large, satisfy a uniform local area estimate of at most one leaf.*

Before the proof of Lemma 3, let us explain why we introduced the subsets $\mathcal{K}_{bar}$, $\mathcal{Q}_{bar,t}$, $\mathcal{Q}^{n,1}_{uns}$, $\mathcal{Q}_{x,\delta}$ and $\mathcal{Q}^n_{par,\mu}$. In fact the subsets $\mathcal{Q}_{bar,t}$, $\mathcal{Q}^{n,1}_{uns}$, $\mathcal{Q}_{x,\delta}$ and $\mathcal{Q}^n_{par,\mu}$ are used as barriers to prevent the surface S'_n from touching $\mathcal{K}_{bar}$, $\mathcal{S}_n$ and Σ_0. So $\mathcal{Q}^{n,1}_{uns}$ and $\mathcal{Q}^n_{par,\mu}$ are used to prescribe the boundary of S'_n. Once we have the sequence S'_n, we construct S' as the limit of this sequence. We then use $\mathcal{K}_{bar}$ as a barrier to control the possible limits of the sequence.

Let us come back to the proof.

Proof of Lemma 3. We begin by fixing $n \in \mathbb{N}$. Let $\mathcal{F}$ be the family of open domains $\mathcal{Q}$ in $\mathcal{D}_{n+1} \setminus \mathcal{K}_{bar}$ with rectifiable boundary such that $\mathcal{S}_{n+1} \subset \partial\mathcal{Q}$. In the following, $\partial_c\mathcal{Q}$ will denote the complement of $\mathcal{S}_{n+1}$ in $\partial\mathcal{Q}$. On $\mathcal{F}$, we define the functional:

$$F(\mathcal{Q}) = A(\partial\mathcal{Q}) + 2H_0V(\mathcal{Q})$$

where $V(\mathcal{Q})$ is the volume of $\mathcal{Q}$ and $A(\partial\mathcal{Q})$ is the $\mathcal{H}^2$ measure of $\partial\mathcal{Q}$. We recall that $A(\partial\mathcal{Q})$ is also the mass of the current $\partial[\mathcal{Q}]$, it is interpreted as the area of the boundary of $\mathcal{Q}$. The idea is to find $\mathcal{Q}_0 \in \mathcal{F}$ which minimizes F in $\mathcal{F}$ then the part of the boundary of $\mathcal{Q}_0$ in $\mathcal{D}_{n+1}$ will be the surface S'_n we look for.

Claim 4. *Let* $\mathcal{Q}$ *be in* $\mathcal{F}$*.*

1. *If* $\mathcal{Q} \cap \mathcal{Q}_{bar,2\eta/3} \neq \emptyset$*, there exists* $t \in [2\eta/3, \eta]$ *such that* $\mathcal{Q} \setminus \mathcal{Q}_{bar,t} \in \mathcal{F}$ *and* $F(\mathcal{Q} \setminus \mathcal{Q}_{bar,t}) \leq F(\mathcal{Q})$*.*
2. *If* $\mathcal{Q} \cap \mathcal{Q}^{n+1}_{par,1/2} \neq \emptyset$*, there exists* $\mu \in [1/2, 1]$ *such that* $\mathcal{Q} \setminus \mathcal{Q}^{n+1}_{par,\mu} \in \mathcal{F}$ *and* $F(\mathcal{Q} \setminus \mathcal{Q}^{n+1}_{par,\mu}) \leq F(\mathcal{Q})$*.*

Proof of Claim 4. Let $\mathcal{Q}$ be in $\mathcal{F}$ and assume that $\mathcal{Q} \cap \mathcal{Q}_{bar,2\eta/3} \neq \emptyset$ as in Assertion *1* (see Figure 5).

Since $\partial\mathcal{Q}$ has finite $\mathcal{H}^2$ measure, the coarea formula implies that there exists $t \in [2\eta/3, \eta]$ such that $\mathcal{H}^1(\partial_c\mathcal{Q} \cap \mathcal{C}_{\eta,t}) < +\infty$. Thus $\mathcal{H}^2(\partial_c\mathcal{Q} \cap \mathcal{C}_{\eta,t}) = 0$; this set is negligible in the following computations.

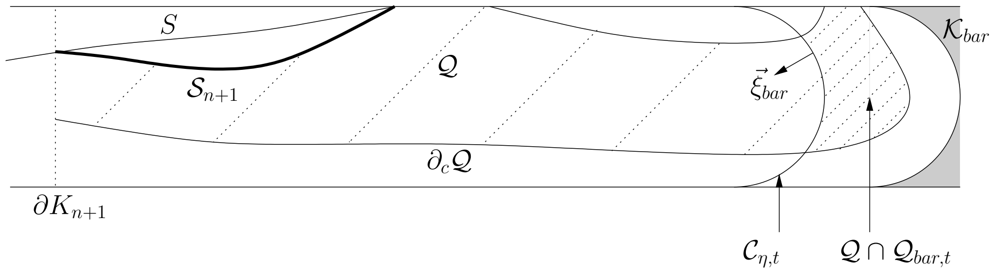

Figure 5:

First $\mathcal{Q} \cap \mathcal{Q}_{bar,t} \neq \emptyset$ has a rectifiable boundary, thus applying Equation (1) of Subsection 2.2 with $\operatorname{div} \vec{\xi}_{bar} \leq 2H_0$, we have:

$$\begin{aligned}
2H_0 V(\mathcal{Q} \cap \mathcal{Q}_{bar,t}) &\geq \int_{\mathcal{Q} \cap \mathcal{Q}_{bar,t}} \operatorname{div} \vec{\xi}_{bar} \mathrm{d}\mathcal{L}_{\mathrm{d}s^2} \\
&\geq \int_{\partial(\mathcal{Q} \cap \mathcal{Q}_{bar,t})} \langle \vec{\xi}_{bar}(x), \vec{n}(\mathcal{Q} \cap \mathcal{Q}_{bar,t}, x) \rangle \mathrm{d}\mathcal{H}^2_{\mathrm{d}s^2} \\
&\geq \int_{\mathcal{Q} \cap \mathcal{C}_{\eta,t}} \langle \vec{\xi}_{bar}(x), \vec{n}(\mathcal{Q} \cap \mathcal{Q}_{bar,t}, x) \rangle \mathrm{d}\mathcal{H}^2_{\mathrm{d}s^2} \\
&\qquad + \int_{\overset{\circ}{\mathcal{Q}}_{bar,t} \cap \partial_c \mathcal{Q}} \langle \vec{\xi}_{bar}, \vec{n}(\mathcal{Q} \cap \mathcal{Q}_{bar,t}, x) \rangle \mathrm{d}\mathcal{H}^2_{\mathrm{d}s^2}.
\end{aligned}$$

We notice that the computation are made with respect to the metric $\mathrm{d}s^2$ and results of Subsection 2.2 are still valid in this setting. On $\mathcal{C}_{\eta,t} \cap \mathcal{Q}$, we have $\vec{\xi}_{bar}(x) = \vec{n}(\mathcal{Q} \cap \mathcal{Q}_{bar}, x)$ everywhere, thus:

$$\begin{aligned}
A(\mathcal{Q} \cap \mathcal{C}_{\eta,t}) &= \int_{\mathcal{Q} \cap \mathcal{C}_{\eta,t}} \langle \vec{\xi}_{bar}, \vec{n}(\mathcal{Q} \cap \mathcal{Q}_{bar}, x) \rangle \mathrm{d}\mathcal{H}^2_{\mathrm{d}s^2} \\
&\leq - \int_{\overset{\circ}{\mathcal{Q}}_{bar,t} \cap \partial_c \mathcal{Q}} \langle \vec{\xi}_{bar}, \vec{n}(\mathcal{Q} \cap \mathcal{Q}_{bar}, x) \rangle \mathrm{d}\mathcal{H}^2_{\mathrm{d}s^2} + 2H_0 V(\mathcal{Q} \cap \mathcal{Q}_{bar}) \\
&\leq A(\overset{\circ}{\mathcal{Q}}_{bar,t} \cap \partial_c \mathcal{Q}) + 2H_0 V(\mathcal{Q} \cap \mathcal{Q}_{bar}).
\end{aligned}$$

This implies that

$$\begin{aligned}
F(\mathcal{Q} \setminus \mathcal{Q}_{bar,t}) &= A(\partial \mathcal{Q}) - A(\overset{\circ}{\mathcal{Q}}_{par,t} \cap \partial_c \mathcal{Q}) + A(\mathcal{Q} \cap \mathcal{C}_{\eta,t}) + 2H_0(V(\mathcal{Q}) - V(\mathcal{Q} \cap \mathcal{Q}_{bar})) \\
&\leq F(\mathcal{Q}).
\end{aligned}$$

Assertion *1* is then proved. Assertion *2* follows from the same arguments; Figure 6 shows the situation.

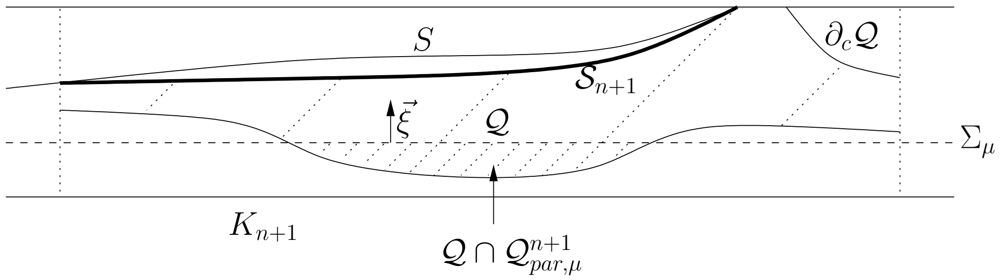

Figure 6:

□

Let $K_{n+1/2}$ be a compact subset of Σ such that

$$K_n \subset \overset{\circ}{K}_{n+1/2} \subset K_{n+1/2} \subset \overset{\circ}{K}_{n+1}.$$

Claim 5. *Let $\mathcal{Q}$ be in $\mathcal{F}$. If $\mathcal{Q}^{n+1,1}_{uns} \cap (K_{n+1/2} \times [0, \varepsilon_1]) \not\subset \mathcal{Q}$, there exists $\mathcal{Q}' \in \mathcal{F}$ such that $\mathcal{Q}^{n+1,1}_{uns} \cap (K_{n+1/2} \times [0, \varepsilon_1]) \subset \mathcal{Q}'$ and $F(\mathcal{Q}') \leq F(\mathcal{Q})$.*

Proof of Claim 5. Let $\mathcal{Q}$ be in $\mathcal{F}$ as in the claim. The subset $\mathcal{Q}^{n+1,1}_{uns} \cap (K_{n+1/2} \times [0, \varepsilon_1])$ is compact so there exists a finite number of points $x_i \in \mathcal{S}^{n+1}_{t_{0,n+1}}$ such that

$$\mathcal{Q}^{n+1,1}_{uns} \cap (K_{n+1/2} \times [0, \varepsilon_1]) \subset \bigcup_i \mathcal{Q}_{x_i,3/2}$$

As in proof of Claim 4, there is $\delta_1 \in [3/2, 2]$ such that $\mathcal{H}^2(\partial_c \mathcal{Q} \cap \mathcal{S}^{x_1}_{\delta_1 t_{x_1}}) = 0$. We denote $\mathcal{O}_1 = (\mathcal{Q}_{x_1,\delta_1} \cap \mathcal{D}_{n+1}) \setminus \mathcal{Q}$. The boundary of $\mathcal{O}_1$ is composed of a part $\partial_1 \mathcal{O}_1 = \partial_c \mathcal{Q} \cap \overset{\circ}{\mathcal{Q}}_{x_1,\delta_1}$, a second part $\partial_2 \mathcal{O}_1 \subset \mathcal{S}^{x_1}_{\delta_1 t_{x_1}}$ in the complement of $\overline{\mathcal{Q}}$ and a third one of vanishing $\mathcal{H}^2$ measure (see Figure 7). In $\mathcal{Q}_{x_1,\delta_1}$, we have the unit vector field $\vec{\xi}_{x_1}$ which satisfies $\operatorname{div} \vec{\xi}_{x_1} < -2H_0$. Then:

$$\begin{aligned} 2H_0 V(\mathcal{O}_1) &\leq -\int_{\mathcal{O}_1} \operatorname{div} \vec{\xi}_{x_1} \\ &\leq -\int_{\partial \mathcal{O}_1} \langle \vec{\xi}_{x_1}(x), \vec{n}(\mathcal{O}_1, x) \rangle \\ &\leq -\int_{\partial_2 \mathcal{O}_1} \langle \vec{\xi}_{x_1}(x), \vec{n}(\mathcal{O}_1, x) \rangle - \int_{\partial_1 \mathcal{O}_1} \langle \vec{\xi}_{x_1}(x), \vec{n}(\mathcal{O}_1, x) \rangle. \end{aligned}$$

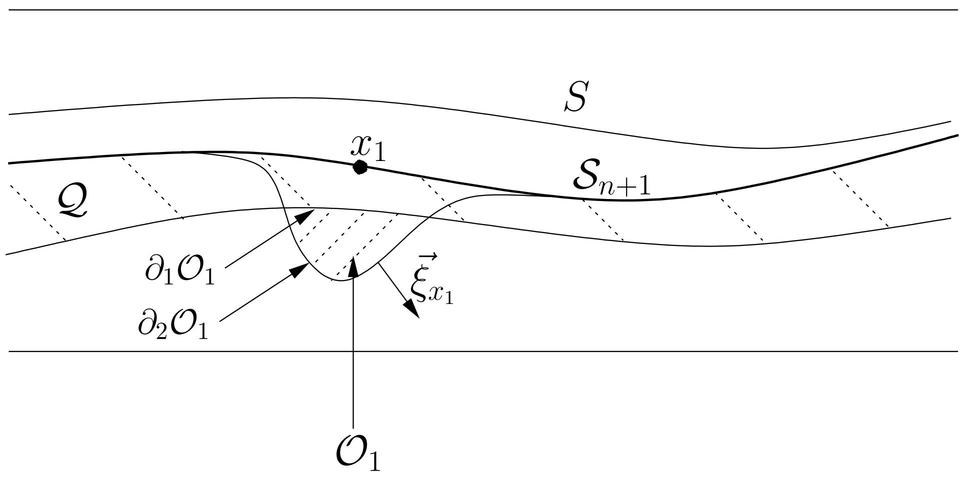

Figure 7:

where, for simplicity, we have omitted to write the measures. On $\partial_2\mathcal{O}_1$,$\vec{\xi}_{x_1} = \vec{n}(\mathcal{O}_1, x)$ thus

$$\begin{aligned} 2H_0V(\mathcal{O}_1) + A(\partial_2\mathcal{O}_1) &= 2H_0V(\mathcal{O}_1) + \int_{\partial_2\mathcal{O}_1} \langle \vec{\xi}_{x_1}(x), \vec{n}(\mathcal{O}_1, x)\rangle \\ &\leq -\int_{\partial_1\mathcal{O}_1} \langle \vec{\xi}_{x_1}(x), \vec{n}(\mathcal{O}_1, x)\rangle \\ &\leq A(\partial_1\mathcal{O}_1) \end{aligned}$$

The interior $\mathcal{Q}_1$ of $\mathcal{Q}\cup\mathcal{O}_1$ is an element of $\mathcal{F}$ (the boundary is still rectifiable) and

$$\begin{aligned} F(\mathcal{Q}_1) &= 2H_0(V(\mathcal{Q}) + V(\mathcal{O}_1) + A(\partial\mathcal{Q}) + A(\partial_2\mathcal{O}_1) - A(\partial_1\mathcal{O}_1) \\ &\leq F(\mathcal{Q}) \end{aligned}$$

Now considering $\mathcal{O}_2 = (\mathcal{Q}_{x_2,\delta_2} \cap \mathcal{D}_{n+1}) \setminus \mathcal{Q}_1$ and $\mathcal{Q}_2$ the interior of $\mathcal{Q}_1 \cup \mathcal{O}_2$, we prove by the same argument that $\mathcal{Q}_2 \in \mathcal{F}$ and $F(\mathcal{Q}_2) \leq F(\mathcal{Q}_1)$. Doing this a finite number of times, we construct the subset $\mathcal{Q}'$. □

Let us now consider $(\mathcal{Q}_k)_k$ a minimizing sequence for F. Because of the claims, we can assume that the sequence satisfies $\mathcal{Q}_k \cap \mathcal{Q}_{bar,2\eta/3} = \emptyset$, $\mathcal{Q}_k \cap \mathcal{Q}_{par,1/2}^{n+1} = \emptyset$ and $\mathcal{Q}_{uns}^{n+1,1} \cap (K_{n+1/2} \times [0, \varepsilon_1]) \subset \mathcal{Q}_k$. By the compactness theorem for integral currents (see Theorem 5.5 in [17]), there is $\mathcal{Q}_\infty$ a cluster point of the sequence for the flat topology. As a limit of a subsequence of $(\mathcal{Q}_k)_k$, $\mathcal{Q}_\infty$ is a domain in $\mathcal{D}_{n+1}$ with a rectifiable boundary such that $\mathcal{Q}_\infty \cap \mathcal{Q}_{bar,2\eta/3} = \emptyset$, $\mathcal{Q}_\infty \cap \mathcal{Q}_{par,1/2}^{n+1} = \emptyset$ and $\mathcal{Q}_{uns}^{n+1,1} \cap (K_{n+1/2} \times [0, \varepsilon_1]) \subset \mathcal{Q}_\infty$. Moreover $\mathcal{Q}_\infty$ minimizes F since the area functional $A(\partial\mathcal{Q})$ is lower semi-continuous for the flat convergence and $V(\mathcal{Q})$ is the integral over $\mathcal{Q}$ of the volume differential form. Since $\mathcal{Q}_\infty$ minimizes F, the part of $\partial\mathcal{Q}_\infty$ inside the interior of $\mathcal{D}_{n+1}$ is a local isoperimetric surface in the sense of [16], by regularity theory (see Corollary 3.7 in [16]) we obtain that this part of $\partial\mathcal{Q}_\infty$ is a smooth surface which we denote by $\mathcal{S}'_{n+1}$. Since $\mathcal{Q}_\infty$ minimizes F, $\mathcal{S}'_{n+1}$ has constant mean curvature H_0 with mean curvature vector pointing outside of $\mathcal{Q}_\infty$ and it is stable (see computations in [2]). Since $\mathcal{Q}_\infty \cap \mathcal{Q}_{bar,2\eta/3} = \emptyset$, $\mathcal{Q}_\infty \cap \mathcal{Q}_{par,1/2}^{n+1} = \emptyset$ and $\mathcal{Q}_{uns}^{n+1,1} \cap (K_{n+1/2} \times [0, \varepsilon_1]) \subset \mathcal{Q}_\infty$, the part of the boundary of $\mathcal{S}'_{n+1}$ in $K_{n+1/2} \times [0, \varepsilon_1]$ is only in $K_{n+1/2} \times \{\varepsilon_1\}$ (here we speak about a non necessarily regular boundary).

Once all the surfaces $\mathcal{S}'_{n+1}$ are constructed, we choose $\varepsilon_2 < \varepsilon_1$ a constant which is a regular value of the distance function for all the $\mathcal{S}'_{n+1}$ (such a ε_2 exists since for each n the set of critical value of the distance function along $\mathcal{S}'_{n+1}$ has vanishing Lebegues measure) and we define $S'_n = \mathcal{S}'_{n+1} \cap (K_n \times [0, \varepsilon_2])$. We notice that S'_n may be empty for small n if ε_2 is too small; but, for large n, $S'_n \neq \emptyset$. Let $p_0 \in S \cap M_+(\varepsilon_2)$ be a point such the geodesic arc $[p_0, \pi(p_0)]$ does not meet the surface S. For n large enough $\pi(p_0) \in \overline{\mathcal{D}_n}$ and $p_0 \notin \mathcal{D}_n$, this implies that $S'_n \cap [p_0, \pi(p_0)] \neq \emptyset$. These surfaces S'_n are in fact the ones we want to construct. First the surface is well oriented since it is a part of the boundary of $\mathcal{Q}_\infty$. For the area estimate, let us consider a point p in D and P a plane in the tangent space. Since $\cup_n \mathcal{D}_n = D$ and the sequence $(\mathcal{D}_n)_n$ is increasing with respect to compact subsets, there is t_0 and n_0 such that, for $t \le t_0$ and $n \ge n_0$, $O_{p,P}(t)$ is a subset of $\mathcal{D}_n$. Since $\mathcal{Q}_\infty$ minimizes F we have $F(\mathcal{Q}_\infty) \le F(\mathcal{Q}_\infty \setminus O_{p,P}(t))$ and $F(\mathcal{Q}_\infty) \le F(\mathcal{Q}_\infty \cup O_{p,P}(t))$, this implies that:

$$A(S'_n \cap O_{p,P}(t)) + 2H_0 V(\mathcal{Q}_\infty \cap O_{p,P}(t)) \le A(\partial O_{p,P}(t) \cap \mathcal{Q}_\infty)$$
$$A(S'_n \cap O_{p,P}(t)) \le A(\partial O_{p,P}(t) \setminus \mathcal{Q}_\infty) + 2H_0 V(O_{p,P}(t) \setminus \mathcal{Q}_\infty)$$

Thus, taking the sum and dividing by two,

$$A(S'_n \cap O_{p,P}(t)) \le A(\partial O_{p,P}(t))/2 + H_0 V(O_{p,P}(t))$$

which is uniformly less that $2\beta\pi t^2$ for some $\beta < 1$ and t small because of the asymptotic behaviour of $A(\partial O_{p,P}(t))$ and $V(O_{p,P}(t))$ (see Subsection 2.4). □

5.1.3 Construction of the surface S'

The last step of the proof of Theorem 2 is to obtain a limit to the sequence $(S'_n)_n$. We choose ε_3 less than ε_2 and we consider $k \in \mathbb{N}$. For every $n \geq k+1$ and $p \in S'_n \cap (K_k \times [0, \varepsilon_3])$ the distance from p to the boundary of S_n is bounded from below by a constant depending only on k and ε_3. From the stability of S_n, this implies that the norm of second fundamental form of S'_n is bounded in $K_k \times [0, \varepsilon_3]$. Besides the sequence $(S'_n)_n$ satisfies a uniform local area estimate. The curvature and area estimates imply that the sequence of surface has a subsequence that converge to a stable cmc H_0 surface in $K_k \times [0, \varepsilon_3]$. Because of the area estimate, the convergence has multiplicity one and the limit surface is embedded. Since the surfaces S'_n cut the geodesic arc $[\pi(p_0), p_0]$ we can assume that this is also the case for this limit surface. Then by a diagonal process, we obtain a stable cmc H_0 surface S_∞ in $\Sigma \times [0, \varepsilon_3]$. We have $S_\infty \cap [\pi(p_0), p_0] \neq \emptyset$ thus $S_\infty \not\subset \Sigma_{\varepsilon_3}$. Moreover S_∞ is well oriented as limit of well oriented surfaces.

One thing we have to check is that S_∞ is, in fact, in $\Sigma \times (0, \varepsilon_3]$. If it is not the case, S_∞ touches Σ_0 and by the maximum principle we have $S_\infty = \Sigma_0$. By construction, the sequence S'_n never enters in $\mathcal{K}_{bar}$ so it is the same for S_∞ and we obtain $S_\infty \neq \Sigma_0$.

Moreover S_∞ is not included in an equidistant surface Σ_t. By construction, S_∞ is between Σ_0 and S and $\inf_S \mathbf{d} = 0$, this implies that $\mathbf{d}$ can not be constant along S_∞.

Now choosing ε' a regular value of the distance function $\mathbf{d}$ on S_∞ (we assume that ε' is part of the image of $\mathbf{d}$ along S_∞), we can consider $S' = S_\infty \cap \Sigma \times [0, \varepsilon']$: S' then has its non empty boundary in $\Sigma \times \{\varepsilon'\}$. S' is then a complete stable cmc H_0 surface which is properly embedded in $\Sigma \times [0, \varepsilon']$. Moreover S' is well oriented and $\mathbf{d}$ is not constant along S'.

5.2 $H_0 = 0$

In this case, the cases *1* and *2* of Theorem 2 are the same, so assume that $S \looparrowright M_+(\varepsilon)$. The proof is essentially the same, the only difference comes from the fact that the "well oriented" hypothesis has no more meaning.

So as above we define, $\mathcal{K}_{bar}$, $\mathcal{Q}_{bar,t}$ and ξ_{bar} such that $\operatorname{div} \xi_{bar} \leq 0$. This gives a ε_1.

We introduce the compact K_n and the domain D_n. As above we assume the instability of S_n and consider φ_n such that $L\varphi_n = \lambda_{1,n}\varphi_n$, v_n such that $-Lv_n = 1$ and $u_n = \varphi_n + a_n v_n$. Let $\vec{N}(x)$ be the unit normal to S_n. For $t_{0,n} > 0$ we define, on $S_n \times [-t_{0,n}, t_{0,n}]$, the map $F(x,t) = \exp_x(t u_n(x) \vec{N}(x))$

and assume that $t_{0,n}$ is small enough to ensure that F is an immersion. $S_n \times [-t_{0,n}, t_{0,n}]$ with the metric $F^*\mathrm{d}s^2$ is foliated by $\mathcal{S}_t^n = S_n \times \{t\}$. Because of $-Lu_n = -\lambda_{1,n}\varphi_n + a_n > 0$, if $t_{0,n}$ is chosen small enough, the mean curvature vector of $F(\mathcal{S}_{t_{0,n}}^n)$ and $F(\mathcal{S}_{-t_{0,n}}^n)$ is non vanishing and points "outside" $F(S_n \times [-t_{0,n}, t_{0,n}])$.

Thus for any $x \in \mathcal{S}_{t_{0,n}}^n \cup \mathcal{S}_{-t_{0,n}}^n$ we can define as above $\mathcal{Q}_{x,\delta}$ and $\vec{\xi}_x$ with $\operatorname{div} \vec{\xi}_x < 0$. Then we define $\mathcal{Q}_{uns}^{n,0} = F(S_n \times [-t_{0,n}, t_{0,n}]) \cap D_n$, $\mathcal{D}_n = D_n \setminus \mathcal{Q}_{uns}^{n,0}$ and

$$\mathcal{Q}_{uns}^{n,1} = \mathcal{D}_n \cap \bigcup_{x \in \mathcal{S}_{t_{0,n}}^n \cup \mathcal{S}_{-t_{0,n}}^n} \mathcal{Q}_{x,1}$$

With these notations, the end of the proof is the same.

5.3 $S \looparrowright M_-(\varepsilon)$ and $H_0 > 0$

When $S \looparrowright M_-(\varepsilon)$, the differences comes from the fact that the surface is not assumed to be well oriented.

As above, we define, $\mathcal{K}_{bar}$, $\mathcal{Q}_{bar,t}$ and $\vec{\xi}_{bar}$ such that $\operatorname{div} \vec{\xi}_{bar} \leq -2H_0$. This gives ε_1. We introduce the compact subsets K_n and the domain D_n.

We use the instability of S to define φ_n such that $L\varphi_n = \lambda_{1,n}\varphi_n$, v_n such that $-Lv_n = 1$ and $u_n = \varphi_n + a_n v_n$. If $\vec{N}(x)$ is the unit normal to S_n such that the mean curvature vector is $2H_0\vec{N}(x)$, we define $F(x,t) = \exp_x(tu_n(x)\vec{N}(x))$ on $S_n \times [-t_{0,n}, 0]$ with $t_{0,n} > 0$ small so that F is an immersion. $S_n \times [-t_{0,n}, 0]$ with the metric $F^*\mathrm{d}s^2$ is foliated by $\mathcal{S}_t^n = S_n \times \{t\}$ and we extend to $S_n \times [-t_{0,n}, 0]$ the definition of $\vec{N}$ as the unit normal vectorfield to the surfaces $\mathcal{S}_t^n$.

Since $-Lu_n = -\lambda_{1,n}\varphi_n + a_n > 0$, if $t_{0,n}$ is small enough, the mean curvature of $F(\mathcal{S}_{-t_{0,n}}^n)$ computed with respect to $\vec{N}$ is less than H_0. We define $\mathcal{Q}_{uns}^{n,0} = F(S_n \times [-t_{0,n}, 0]) \cap D_n$ and $\mathcal{D}_n = D_n \setminus \mathcal{Q}_{uns}^{n,0}$.

As above for any $x \in \mathcal{S}_{-t_{0,n}}^n$, we consider $D_x \subset \mathcal{S}_{-t_{0,n}}^n$ a small open geodesic disk which is embedded in $M_+(\varepsilon_1)$ by F. Let ψ be a smooth function on $\mathcal{S}_{t_{0,n}}^n$ vanishing outside D_x and positive in D_x. We then define on $\mathcal{S}_{-t_{0,n}}^n \times [0, 2t_x]$

$$G(p,t) = \exp_{F(p)}(-t\psi(p)F_*(\vec{N}(p)))$$

If we choose t_x small, we can assume that G is an embedding on $D_x \times [0, 2t_x]$. In $G(D_x \times [0, 2t_x])$, we define $\vec{\xi}_x$ the unit vector field normal to the embedded surfaces $\mathcal{S}_t^x = G(D_x \times \{t\})$ with $\vec{\xi}_x = -F_*(\vec{N})$ along $\mathcal{S}_0^x$. Since the mean curvature of $\mathcal{S}_0^x$ is less than $2H_0$, if t_x is small, we can assume that $\operatorname{div}\vec{\xi}_x < 2H_0$.

We denote $\mathcal{Q}_{x,\delta} = G(D_x \times [0, \delta t_x])$ for $\delta \in [1,2]$. We also define

$$\mathcal{Q}_{uns}^{n,1} = \mathcal{D}_n \cap \bigcup_{x \in \mathcal{S}_{t_0}^n} \mathcal{Q}_{x,1}.$$

Since the surface S_n can be not well oriented, we need to introduce new barriers. For any $x \in S_n$, we consider $\Delta_x \subset S_n$ a small open geodesic disk which is embedded in $M_+(\varepsilon_1)$. Let η be a smooth function on S_n vanishing outside Δ_x and positive in Δ_x. We define on $S_n \times [0, 2t_x]$

$$Z(p,t) = \exp_p(t\eta(p)\vec{N}(p))$$

with t_x small enough such that Z is an embedding on $\Delta_x \times [0, 2t_x]$. In $Z(\Delta_x \times [0, 2t_x])$, we define $\vec{\xi}_{ori}^x$ the unit vector field normal to the surfaces $\mathcal{S}_{ori,t}^x = Z(\Delta_x \times \{t\})$ with $\vec{\xi}_{ori}^x = \vec{N}$ along Δ_x. Since the mean curvature vector of S_n is $H_0\vec{N}$, if t_x is small enough we have $\operatorname{div}\vec{\xi}_{ori}^x < 2H_0$.

We denote $\mathcal{Q}_{ori}^{x,\nu} = Z(\Delta_x \times [0, \nu t_x])$ and $\overset{\circ}{\mathcal{Q}}{}_{ori}^{x,\nu} = Z(\Delta_x \times [0, \nu t_x))$, for $\nu \in [1,2]$ and :

$$\mathcal{Q}_{ori}^{n,1} = \mathcal{D}_n \cap \bigcup_{x \in S_n} \mathcal{Q}_{ori}^{x,1}$$

Let $\mathcal{S}_n$ be part of the boundary of $\mathcal{D}_n$ in $F(\mathcal{S}_{-t_0}^n) \cup S_n$; this the part of $\partial\mathcal{D}_n$ is not in Σ_0 and $\partial K_n \times \mathbb{R}$. Any point in $\mathcal{S}_n$ is at positive distance from $\mathcal{D}_n \setminus (\mathcal{Q}_{uns}^{n,1} \cup \mathcal{Q}_{ori}^{n,1})$.

Then we define $\mathcal{Q}_{par,\mu}^n = K_n \times [-\mu t_1, 0]$ and introduce $\vec{\xi}_{par} = -\vec{\xi}$. We have $\operatorname{div}\vec{\xi}_{par} = -\operatorname{div}\vec{\xi} \le -2H_0$ because of the $H \ge H_0$ hypothesis in $M_-(\varepsilon)$.

An equivalent of Lemma 3 can be proved. The idea is now to minimize the functional $F(\mathcal{Q}) = A(\partial\mathcal{Q}) - 2H_0V(\mathcal{Q})$ where $\mathcal{Q} \in \mathcal{F}$ and $\mathcal{F}$ is the same family of domains in $\mathcal{D}_{n+1}$. We first remark that Claim 4 is still true. Claim 5 is replaced by

Claim 6. *Let $\mathcal{Q}$ be in $\mathcal{F}$.*

1. *If $\mathcal{Q}_{uns}^{n+1,1} \cap (K_{n+1/2} \times [-\varepsilon_1, 0]) \not\subset \mathcal{Q}$, there exists $\mathcal{Q}' \in \mathcal{F}$ such that $\mathcal{Q}_{uns}^{n+1,1} \cap (K_{n+1/2} \times [-\varepsilon_1, 0]) \subset \mathcal{Q}'$ and $F(\mathcal{Q}') \le F(\mathcal{Q})$.*

2. *If $\mathcal{Q}_{ori}^{n+1,1} \cap (K_{n+1/2} \times [-\varepsilon_1, 0]) \not\subset \mathcal{Q}$, there exists $\mathcal{Q}' \in \mathcal{F}$ such that $\mathcal{Q}_{ori}^{n+1,1} \cap (K_{n+1/2} \times [-\varepsilon_1, 0]) \subset \mathcal{Q}'$ and $F(\mathcal{Q}') \le F(\mathcal{Q})$.*

Proof of Claim 6. The proof of both items are the same so let us prove the second one (the situation is very similar to the one of Claim 5 so we refer to Figure 7). Let $\mathcal{Q}$ be in $\mathcal{F}$ and $\mathcal{Q}_{ori}^{n+1,1} \cap (K_{n+1/2} \times [-\varepsilon_1, 0]) \not\subset \mathcal{Q}$. As in the proof of Claim 5, the subset $\mathcal{Q}_{ori}^{n+1,1} \cap (K_{n+1/2} \times [-\varepsilon_1, 0])$ is compact so there exists a finite number of points $x_i \in S_{n+1}$ such that

$$\mathcal{Q}_{ori}^{n+1,1} \cap (K_{n+1/2} \times [-\varepsilon_1, 0]) \subset \bigcup_i \mathcal{Q}_{ori}^{x_i,3/2}.$$

As in claims 4 and 5, there is $\nu_1 \in [3/2, 2]$ such that $\mathcal{H}^2(\mathcal{S} \cap \mathcal{S}_{ori,\nu_1 t_{x_1}}^{x_1}) = 0$. Then we denote $\mathcal{O}_1 = (\mathcal{Q}_{ori}^{x_1,\nu_1} \cap \mathcal{D}_{n+1}) \setminus \mathcal{Q}$. The boundary of $\mathcal{O}_1$ is composed of a part $\partial_1 \mathcal{O}_1 = \partial_c \mathcal{Q} \cap \overset{\circ}{\mathcal{Q}}{}_{ori}^{x_1,\nu_1}$ and a second part $\partial_2 \mathcal{O}_1 \subset \mathcal{S}_{ori,\nu_1 t_{x_1}}^{x_1}$ in the complement of $\overline{\mathcal{Q}}$ and a third part of vanishing $\mathcal{H}^2$ measure. In $\mathcal{Q}_{ori}^{x_1,\nu_1}$, we have the unit vector field $\vec{\xi}_{ori}^{x_1}$ which satisfies $\operatorname{div} \vec{\xi}_{ori}^{x_1} < 2H_0$. Then:

$$\begin{aligned}
2H_0 V(\mathcal{O}_1) &\geq \int_{\mathcal{O}_1} \operatorname{div} \vec{\xi}_{ori}^{x_1} \\
&\geq \int_{\partial \mathcal{O}_1} \langle \vec{\xi}_{ori}^{x_1}, \vec{n}(\mathcal{O}_1, x) \rangle \\
&\geq \int_{\partial_2 \mathcal{O}_1} \langle \vec{\xi}_{ori}^{x_1}, \vec{n}(\mathcal{O}_1, x) \rangle + \int_{\partial_1 \mathcal{O}_1} \langle \vec{\xi}_{ori}^{x_1}, \vec{n}(\mathcal{O}_1, x) \rangle.
\end{aligned}$$

On $\partial_2 \mathcal{O}_1$, $\vec{\xi}_{ori}^{x_1} = \vec{n}(\mathcal{O}_1, x)$ thus

$$\begin{aligned}
A(\partial_2 \mathcal{O}_1) - 2H_0 V(\mathcal{O}_1) &= \int_{\partial_2 \mathcal{O}_1} \langle \vec{\xi}_{ori}^{x_1}, \vec{n}(\mathcal{O}_1, x) \rangle - 2H_0 V(\mathcal{O}_1) \\
&\leq - \int_{\partial_1 \mathcal{O}_1} \langle \vec{\xi}_{ori}^{x_1}, \vec{n}(\mathcal{O}_1, x) \rangle \\
&\leq A(\partial_1 \mathcal{O}_1).
\end{aligned}$$

$\mathcal{Q}_1 = \mathcal{Q} \cup \mathcal{O}_1$ is an element of $\mathcal{F}$ since the boundary is still rectifiable and

$$\begin{aligned}
F(\mathcal{Q} \cup \mathcal{O}_1) &= -2H_0(V(\mathcal{Q}) + V(\mathcal{O}_1)) + A(\partial \mathcal{Q}) + A(\partial_2 \mathcal{O}_1) - A(\partial_1 \mathcal{O}_1) \\
&\leq F(\mathcal{Q}).
\end{aligned}$$

Repeating this a finite number of times we construct the subset $\mathcal{Q}'$. ☐

As in proof of Lemma 3, we obtain a minimizer $\mathcal{Q}_\infty$ and a smooth surface $\mathcal{S}'_{n+1}$ which gives us S'_n. The uniform area estimate is also proved by the same way.

Once the sequence S'_n is constructed, the end of the proof of Theorem 2 is the same as in the first case.

6 The halfspace theorem

In this section we prove our main theorem.

6.1 Some preliminary computations

We begin by some computations. Let Σ be a Riemannian surface and $M_{\pm}(\varepsilon)$ be an ε-half neighborhood of Σ. Let S be a constant mean curvature H_0 surface in $M_{\pm}(\varepsilon)$. We denote by ∇ the connection on $M_{\pm}(\varepsilon)$ and we denote by $\widetilde{\nabla}$ and $\widetilde{\Delta}$ the connection on S and its associated Laplace operator.

Let f be a function on $\mathbb{R}$, we want to compute $\widetilde{\Delta} f(\mathbf{d})$. Along S, we denote by $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ an orthonormal basis of $TM_{\pm}(\varepsilon)$ such that $\vec{e}_3$ is normal to S and the mean curvature vector to S is $H_0\vec{e}_3$. For any function g defined in $M_{\pm}(\varepsilon)$ we have :

$$\widetilde{\Delta} g = \sum_{i=1}^{2} \langle \nabla_{\vec{e}_i} \nabla g, \vec{e}_i \rangle + 2\langle \nabla g, H_0 \vec{e}_3 \rangle$$

Thus if $g = f \circ \mathbf{d}$, we get:

$$\widetilde{\Delta} f \circ \mathbf{d} = f''(\mathbf{d}) \sum_{i=1}^{2} \langle \nabla \mathbf{d}, \vec{e}_i \rangle^2 + f'(\mathbf{d}) \left(\sum_{i=1}^{2} \langle \nabla_{\vec{e}_i} \nabla \mathbf{d}, \vec{e}_i \rangle + 2\langle \nabla \mathbf{d}, H_0 \vec{e}_3 \rangle \right)$$

For any point in $M_{\pm}(\varepsilon)$, we denote by $(\vec{a}_1, \vec{a}_2)$ an orthonormal basis of $T\Sigma_t$ which diagonalized the shape operator of Σ_t. Let κ_1 and κ_2 the associated principal curvature such that $\nabla_{\vec{a}_i} \vec{\xi} = \kappa_i \vec{a}_i$ for $i = 1, 2$. $(\vec{a}_1, \vec{a}_2, \vec{\xi})$ is then an orthonormal basis of $TM_{\pm}(\varepsilon)$, we write $\vec{a}_3 = \vec{\xi}$ and we have $\nabla_{\vec{a}_3} \vec{a}_3 = 0$. Moreover, we define $(\lambda_i^j)_{1 \le i,j \le 3}$ such that

$$\vec{e}_i = \sum_{j=1}^{3} \lambda_i^j \vec{a}_j$$

Using these expressions and working in $M_{+}(\varepsilon)$ where $\nabla \mathbf{d} = \vec{a}_3$, we have:

$$\begin{aligned}
\widetilde{\Delta} f \circ \mathbf{d} &= f''(\mathbf{d})({\lambda_1^3}^2 + {\lambda_2^3}^2) + f'(\mathbf{d}) \left(\sum_{i=1}^{2} \langle \lambda_i^1 \kappa_1 \vec{a}_1 + \lambda_i^2 \kappa_2 \vec{a}_2, \lambda_i^1 \vec{a}_1 + \lambda_i^2 \vec{a}_2 + \lambda_i^3 \vec{a}_3 \rangle + 2H_0 \lambda_3^3 \right) \\
&= f''(\mathbf{d})(1 - {\lambda_3^3}^2) + f'(\mathbf{d}) \left(\kappa_1({\lambda_1^1}^2 + {\lambda_2^1}^2) + \kappa_2({\lambda_1^2}^2 + {\lambda_2^2}^2) + 2H_0 \lambda_3^3 \right) \\
&= f''(\mathbf{d})(1 - {\lambda_3^3}^2) + f'(\mathbf{d}) \left((\kappa_1 + \kappa_2) + 2H_0 \lambda_3^3 - \kappa_1 {\lambda_3^1}^2 - \kappa_2 {\lambda_3^2}^2 \right)
\end{aligned}$$

Since the vector $(\lambda_3^1, \lambda_3^2, \lambda_3^3)$ has norm 1, there exists $(\varphi, \theta) \in [0, \pi] \times [0, 2\pi]$ such that

$$(\lambda_3^1, \lambda_3^2, \lambda_3^3) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, -\cos\varphi)$$

The "$-$" sign in the last coordinate is there in order to make φ close to 0 in the proof below. Besides, if $M_+(\varepsilon)$ satisfies the $H \leq H_0$ hypothesis and f is an increasing function, we obtain:

$$\widetilde{\Delta} f \circ \mathbf{d} \leq f''(\mathbf{d})(1-\cos^2\varphi) + f'(\mathbf{d})(2H_0(1-\cos\varphi) - (\kappa_1\cos^2\theta + \kappa_2\sin^2\theta)\sin^2\varphi)$$

$$\widetilde{\Delta} f \circ \mathbf{d} \leq f''(\mathbf{d})\sin^2\varphi + f'(\mathbf{d})(2H_0(1-\cos\varphi) - (\kappa_1\cos^2\theta + \kappa_2\sin^2\theta)\sin^2\varphi) \tag{2}$$

If we work in $M_-(\varepsilon)$, we have $\nabla\mathbf{d} = -\vec{a}_3$ thus:

$$\begin{aligned}
\widetilde{\Delta} f \circ \mathbf{d} &= f''(\mathbf{d})({\lambda_1^3}^2 + {\lambda_2^3}^2) - f'(\mathbf{d})\left(\sum_{i=1}^{2}\langle\lambda_i^1\kappa_1 a_1 + \lambda_i^2\kappa_2 a_2, \lambda_i^1 a_1 + \lambda_i^2 a_2 + \lambda_i^3 a_3\rangle + 2H_0\lambda_3^3\right) \\
&= f''(\mathbf{d})(1 - {\lambda_3^3}^2) - f'(\mathbf{d})\left(\kappa_1({\lambda_1^1}^2 + {\lambda_2^1}^2) + \kappa_2({\lambda_1^2}^2 + {\lambda_2^2}^2) + 2H_0\lambda_3^3\right) \\
&= f''(\mathbf{d})(1 - {\lambda_3^3}^2) - f'(\mathbf{d})\left((\kappa_1 + \kappa_2) + 2H_0\lambda_3^3 - \kappa_1{\lambda_3^1}^2 - \kappa_2{\lambda_3^2}^2\right)
\end{aligned}$$

The vector $(\lambda_3^1, \lambda_3^2, \lambda_3^3)$ has still norm 1, so there exists $(\varphi, \theta) \in [0, \pi] \times [0, 2\pi]$ such that

$$(\lambda_3^1, \lambda_3^2, \lambda_3^3) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, -\cos\varphi)$$

If $M_-(\varepsilon)$ satisfies the $H \geq H_0$ hypothesis and f is an increasing function, we obtain:

$$\begin{aligned}
\widetilde{\Delta} f \circ \mathbf{d} &\leq f''(\mathbf{d})(1-\cos^2\varphi) + f'(\mathbf{d})(2H_0(\cos\varphi - 1) + (\kappa_1\cos^2\theta + \kappa_2\sin^2\theta)\sin^2\varphi) \\
&\leq f''(\mathbf{d})\sin^2\varphi + f'(\mathbf{d})(2H_0(\cos\varphi - 1) + (\kappa_1\cos^2\theta + \kappa_2\sin^2\theta)\sin^2\varphi)
\end{aligned}$$

Since $\cos\varphi - 1 \leq 0$, we get:

$$\widetilde{\Delta} f \circ \mathbf{d} \leq \left(f''(\mathbf{d}) + f'(\mathbf{d})(\kappa_1\cos^2\theta + \kappa_2\sin^2\theta)\right)\sin^2\varphi \tag{3}$$

6.2 The main theorem

Let us now state and prove our main result.

Theorem 7. *Let $(\Sigma, \mathrm{d}\sigma_0^2)$ be a complete orientable Riemannian surface, ε be positive and H_0 non-negative. Let $M_\pm(\varepsilon)$ be an inside or outside ε-half neighborhood of Σ. We consider a properly immersed constant mean* *curvature* H_0 *surface* S *in* $M_\pm(\varepsilon)$ *with possibly non-empty boundary in* Σ_ε *and* $S \subset M^*_\pm(\varepsilon)$.

We assume that $(\Sigma, \mathrm{d}\sigma_0^2)$ *is parabolic. We also assume that* $M_\pm(\varepsilon)$ *is regular.*

1. *If* $S \looparrowright M_+(\varepsilon)$ *is well oriented and* $M_+(\varepsilon)$ *satisfies the* $H \leq H_0$ *hypothesis, the distance function* $\mathbf{d}$ *is constant on* S.
2. *If* $S \looparrowright M_-(\varepsilon)$ *and* $M_-(\varepsilon)$ *satisfies the* $H \geq H_0$ *hypothesis, the distance function* $\mathbf{d}$ *is constant on* S.

Theorem 7 says that the equidistant surfaces are the only possible constant mean curvature H_0 surfaces in $M_\pm(\varepsilon)$ (with good orientation in $M_+(\varepsilon)$). If no equidistant surface has mean curvature H_0, no cmc H_0 surface exists in $M_\pm(\varepsilon)$.

As for Theorem 2, the proof of both cases are very similar so we will mainly focus on the first one.

6.2.1 $S \looparrowright M_+(\varepsilon)$ and $H_0 > 0$

Let us consider S in $M_+(\varepsilon)$ and assume that S is not in one equidistant surface Σ_t ($\mathbf{d}$ is not constant along S). Let $\mu < \varepsilon$ be the lower bound of $\mathbf{d}$ on S, we notice that this lower bound is never reached because of the $H \leq H_0$ hypothesis and the maximum principle.

The space $N_+(\varepsilon - \mu) = \Sigma \times [\mu, \varepsilon]$ with the Riemannian metric $\mathrm{d}s^2$ can be seen as an outside $(\varepsilon - \mu)$-half neighborhood of $(\Sigma, \mathrm{d}\sigma_\mu^2)$. Since $M_+(\varepsilon)$ is regular, $(\Sigma, \mathrm{d}\sigma_\mu^2)$ is parabolic ($\pi_\mu : \Sigma_\mu \to \Sigma_0$ is quasi-isometric) and $N_+(\varepsilon - \mu)$ is regular.

S can be viewed as properly immersed in $N_+(\varepsilon - \mu)$; thus we can assume that $\inf_S \mathbf{d} = 0$ in the statement of Theorem 7.

Thus $S \looparrowright M_+(\varepsilon)$ satisfies all the hypotheses of Theorem 2. So there are $\varepsilon' > 0$ and a surface S' properly immersed in $M_+(\varepsilon')$ with nonempty boundary in $\Sigma_{\varepsilon'}$. S' is well oriented, has cmc H_0 and is stable, moreover the distance function $\mathbf{d}$ on S' is not constant.

Let ε_1 be less that ε'; for any point in $S' \cap M_+(\varepsilon_1)$, the geodesic distance to $\partial S'$ is lower bounded by $\varepsilon' - \varepsilon_1$. Since S' is stable and $M_+(\varepsilon')$ is geometrically bounded, the norm of the second fundamental form of S' is bounded in $M_+(\varepsilon_1)$. Choosing ε_1 sufficiently close to $\inf_{S'} \mathbf{d}$, there is a constant $c > 0$ such that, along $S' \cap M_+(\varepsilon_1)$, $|\langle \vec{N}, \vec{\xi} \rangle| > c$ where $\vec{N}$ is the normal to S'. Thus π is a local quasi-isometry from S' to Σ_0.

Let D by the connected component of $M_+(\varepsilon_1) \setminus S'$ which contains Σ_0. Let p be in $S' \cap M_+(\varepsilon_1)$ and $q = \pi(p)$, along the geodesic segment $[q, p]$ there is a point $p' \in S'$ which is the closest to q. p' is in ∂D and we denote by S'' the connected component of $S' \cap M_+(\varepsilon_1)$ which contains p'. S'' is not in an equidistant surface to Σ_0. We notice that since S' is well oriented the mean curvature vector at p' points into D. Thus $\langle \vec{N}, \vec{\xi} \rangle \le 0$ at p' which gives $\langle \vec{N}, \vec{\xi} \rangle \le -c$ at p'. Since S'' is connected, we get $\langle \vec{N}, \vec{\xi} \rangle \le -c$ along S''.

Let us construct on S'' a non constant bounded superharmonic function which does not reach its lower bound on the boundary. Let K be a real constant and consider the function:

$$f_K : \mathbb{R}_+ \to \mathbb{R},\ x \mapsto \frac{1}{K}(1 - \exp(-Kx))$$

We have $f_K'(x) = \exp(-Kx) \ge 0$ so f_K is increasing and $f_K''(x) + K f_K'(x) = 0$.

Now, we use the computation (2) with $f = f_K$. On S'' we have $|\langle \vec{N}, \vec{\xi} \rangle| < -c$, this means that $\cos\varphi \ge c$ in (2). But there exists $A \ge 0$ such that $1 - \cos\varphi \le A \sin^2\varphi$ when $\cos\varphi \ge c$. Then, from (2), we get:

$$\begin{aligned}\Delta_{S''} f_K \circ \mathbf{d} &\le f_K''(\mathbf{d}) \sin^2\varphi + f_K'(\mathbf{d})(2H_0 A \sin^2\varphi - (\kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta)\sin^2\varphi) \\ &\le \big(f_K''(\mathbf{d}) + f_K'(\mathbf{d})(2H_0 A - (\kappa_1 \cos^2\theta + \kappa_2 \cos^2\theta))\big)\sin^2\varphi\end{aligned}$$

Since $M_+(\varepsilon)$ is assumed to be regular there is a constant C such that $\max(|\kappa_1|, |\kappa_2|) \le C$. Then considering $K = 2H_0 A + C$ we get

$$\begin{aligned}\Delta_{S''} f_K \circ \mathbf{d} &\le \big(f_K''(\mathbf{d}) + f_K'(\mathbf{d})(2H_0 A + (C \cos^2\theta + C \sin^2\theta))\big)\sin^2\varphi \\ &\le \big(f_K''(\mathbf{d}) + f_K'(\mathbf{d})(2H_0 A + C)\big)\sin^2\varphi \\ &\le 0\end{aligned}$$

$f_K \circ \mathbf{d}$ is then superharmonic on S'', bounded since $\mathbf{d}$ is bounded and $f_K \circ \mathbf{d} \le f_K(\varepsilon_1) = (f_K \circ \mathbf{d})_{|\partial S''}$. If we prove that S'' is parabolic at infinity we could conclude that $f_K \circ \mathbf{d}$ is constant and $S'' \subset \Sigma_{\varepsilon_1}$; this will give the contradiction we look for and the first case of Theorem 7 will be proved.

First we deal with a special case: S'' is embedded. This case is not necessary for the general one but it explains some ideas. We have the following claim

Claim 8. *π is injective on S''.*

Proof. Let us assume that there is p_0 and p_1 in S'' such that $\pi(p_0) = \pi(p_1)$ and $\mathbf{d}(p_0) > \mathbf{d}(p_1)$. Let $\gamma : [0,1] \to S''$ be a curve such that $\gamma(0) = p_0$ and

$\gamma(1) = p_1$. We denote $\pi \circ \gamma$ by $\tilde{\gamma}$. $\tilde{\gamma}$ is a closed curve in Σ_0, so we can extend the definition of $\tilde{\gamma}$ by periodicity to $\mathbb{R}_+$. Since $\pi : S'' \to \Sigma_0$ is a local diffeomorphism, we can extend the definition of γ as a lift of $\tilde{\gamma}$ to $[0, t_0]$ where $\gamma(t_0) \in \partial S''$ or to $\mathbb{R}_+$.

We have $\mathbf{d}(\gamma(0)) - \mathbf{d}(\gamma(1)) > 0$ then, for any $t \in [0, t_0 - 1]$, $\mathbf{d}(\gamma(t)) - \mathbf{d}(\gamma(t+1)) > 0$ since this quantity never vanishes. Since $\mathbf{d}(\gamma(t)) \leq \varepsilon'$, we get $\mathbf{d}(\gamma(t)) < \varepsilon'$ for any $t \geq 1$. Hence $\gamma(t) \notin \partial S''$ for $t \geq 1$ and γ is then defined on $\mathbb{R}_+$. Thus $\gamma(n)$ is a sequence of distinct points in S'' with $\pi(\gamma(n)) = \pi(p_0)$. This contradicts the fact that S'' is properly embedded and $|\langle \vec{N}, \vec{\xi} \rangle| > c$. The map π is then injective on S''. □

Since $\pi : S'' \to \Sigma_0$ is an injective quasi-isometry and Σ_0 is parabolic, S'' is parabolic at infinity by Proposition 1; Theorem 7 would then be proved.

Let us now write the general case: S'' is only immersed.

We recall that D_0 is the connected component of $M_+(\varepsilon_1) \setminus S''$ that contains Σ_0. The boundary of D_0 is composed by Σ_0 and a set S_0 made of points in S'' and Σ_{ε_1} (see Figure 8). For any x in Σ_0, we define $v(x) = \min\{\mathbf{d}(p), p \in \pi^{-1}(x) \cap (S'' \cup \Sigma_{\varepsilon_1})\}$. It is clear that the graph of v, $\{(x, v(x)) \in \Sigma \times [0, \varepsilon_1]\}$, is included in S_0. In fact we have equality because of the following claim.

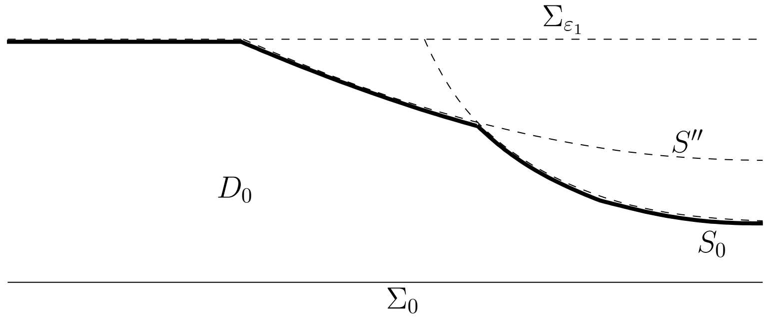

Figure 8:

Claim 9. *The function v is continuous.*

Proof. If v is not continuous there is a sequence of points (x_n) converging to x in Σ_0 such that $\lim v(x_n) = v_0 \neq v(x)$. Since $S'' \cup \Sigma_{\varepsilon_1}$ is closed, $(x, v_0) \subset S'' \cup \Sigma_{\varepsilon_1}$ thus $v_0 > v(x)$. $(x, v(x))$ is in $S'' \cup \Sigma_{\varepsilon_1}$ thus there is a smooth function f defined in a neighborhood of x in Σ_0 such that the graph of f is included in $S'' \cup \Sigma_{\varepsilon_1}$ and $f(x) = v(x)$ (we used the fact that

$|\langle \vec{N}, \vec{\xi} \rangle| > c$ along S''). Then $f < v_0$ near x and $v(x_n) \leq f(x_n)$ for n large. We get a contradiction. □

In fact near a point $p \in S_0$, S'' and Σ_{ε_1} can be viewed as a finite union of graphs above a small disk in Σ_0 around $\pi(p)$. Let us denote the associated functions by f_i, then $v = \min_i f_i$ (in view of Subsection 2.3, $f_0 = \varepsilon_1$ and $f_1, \cdots, f_p$ have constant mean curvature graphs). The projection map $\pi : S_0 \to \Sigma_0$ is then a homeomorphism.

Let us denote by O_i the connected component of S_0 minus the set of self-intersection points in S'' and the set $S'' \cap \Sigma_{\varepsilon_1}$ (these are the points where v is given by only one f_j).

We denote $\Omega_i = \pi(O_i) \subset \Sigma_0$. By the description made in Subsection 2.3, the boundary of O_i can be decomposed as the union of part $\Gamma_{i,j}$ and a set of vanishing $\mathcal{H}^1$ measure. The set $\Gamma_{i,j}$ is the part of $\partial\Omega_i \cap \partial\Omega_j$ where Ω_i "touches" Ω_j. On $\overline{\Omega_i}$, we consider the metric $g_i = \pi_*(\mathrm{d}s^2_{|O_i})$, this metric is well defined since π is smooth on S'' and Σ_{ε_1}. Moreover since π is quasi-isometric along S'' and Σ_{ε_1} there is $k > 0$ such that $\frac{1}{k^2}\mathrm{d}\sigma_0^2 \leq g_i \leq k^2 \mathrm{d}\sigma_0^2$.

On Σ_0 we consider the function u defined by $u(p) = f_K \circ \mathbf{d}(\pi^{-1}_{|S_0}(p)) - \varepsilon_1 = f_K \circ v - \varepsilon_1$. u is non-positive, smooth on each Ω_i and $\Delta_{g_i} u \leq 0$. In fact, in view of its definition and the definition of S_0, u can be interpreted as the minimum of several superharmonic functions so, in some sense, u is a superharmonic function. Let us explain how this idea can be used. The following computations are inspired by [12] (see also [1, 3]).

Since $(\Sigma_0, \mathrm{d}\sigma_0^2)$ is parabolic there exists a sequence of compactly supported smooth functions $(\varphi_n)_n$ such that $0 \leq \varphi_n \leq 1$, $(\varphi_n^{-1}(1))_n$ is a compact exhaustion of Σ_0 and

$$\lim_n \int_{\Sigma_0} \|\nabla_0 \varphi_n\|_0 \mathrm{d}v_0 = 0$$

The subscript 0 means that the computation are made with respect to the metric $\mathrm{d}\sigma_0^2$.

We use the subscript i when the computation are made with respect to g_i in $\overline{\Omega_i}$. Let us define the following quantity :

$$I_n = \sum_i \int_{\Omega_i} \mathrm{div}_i(\varphi_n^2 u \nabla_i u) \mathrm{d}v_i$$

We notice that, since φ_n is compactly supported, I_n is well defined.

In fact, because $u\Delta_i u \geq 0$, we have:

$$\begin{aligned} I_n &= \sum_i \Big(\int_{\Omega_i} 2\varphi_n u \langle \nabla_i \varphi_n, \nabla_i u \rangle_i \mathrm{d}v_i + \int_{\Omega_i} \varphi_n^2 \|\nabla_i u\|_i^2 \mathrm{d}v_i + \int_{\Omega_i} \varphi_n^2 u \Delta_i u \mathrm{d}v_i \Big) \\ &\geq \sum_i \Big(\int_{\Omega_i} 2\varphi_n u \langle \nabla_i \varphi_n, \nabla_i u \rangle_i \mathrm{d}v_i + \int_{\Omega_i} \varphi_n^2 \|\nabla_i u\|_i^2 \mathrm{d}v_i \Big) \end{aligned}$$

Because of Section 2, we also have :

$$\begin{aligned} I_n &= \sum_i \int_{\partial\Omega_i} \varphi_n^2 u \langle \nabla_i u, \vec{\nu}_i \rangle_i \mathrm{d}\mathcal{H}_i^1 \\ &= \frac{1}{2} \sum_{(i,j)} \left(\int_{\Gamma_{i,j}} \varphi_n^2 u \langle \nabla_i u, \vec{\nu}_i \rangle_i \mathrm{d}\mathcal{H}_i^1 + \int_{\Gamma_{i,j}} \varphi_n^2 u \langle \nabla_j u, \vec{\nu}_j \rangle_j \mathrm{d}\mathcal{H}_j^1 \right) \end{aligned}$$

where $\vec{\nu}_i$ is the outgoing normal from Ω_i along $\Gamma_{i,j}$. We notice that the results of Subsection 2.3 are applied for a Riemannian metric however this Stokes formula can be easily deduced from the Euclidean one. Let $C_{i,j}$ be the part of $\partial O_i \cap \partial O_j$ such that $\pi(C_{i,j}) = \Gamma_{i,j}$. Let $\vec{n}_i$ be the unit outgoing normal from O_i in S'' or Σ_{ε_1}. We then have:

$$\begin{aligned} \int_{\Gamma_{i,j}} \varphi_n^2 u \langle \nabla_i u, \vec{\nu}_i \rangle_i \mathrm{d}\mathcal{H}_i^1 &+ \int_{\Gamma_{i,j}} \varphi_n^2 u \langle \nabla_j u, \vec{\nu}_j \rangle_j \mathrm{d}\mathcal{H}_j^1 \\ &= \int_{C_{i,j}} \varphi_n^2 (f_K \circ \mathbf{d} - \varepsilon_1) \langle \nabla (f_K \circ \mathbf{d}), \vec{n}_i \rangle \mathrm{d}\mathcal{H}_{\mathrm{d}s^2} \\ &\quad + \int_{C_{i,j}} \varphi_n^2 (f_K \circ \mathbf{d} - \varepsilon_1) \langle \nabla (f_K \circ \mathbf{d}), \vec{n}_j \rangle \mathrm{d}\mathcal{H}_{\mathrm{d}s^2} \\ &= \int_{C_{i,j}} \varphi_n^2 (f_K \circ \mathbf{d} - \varepsilon_1)(f_K' \circ \mathbf{d}) \langle \nabla \mathbf{d}, \vec{n}_i + \vec{n}_j \rangle \mathrm{d}\mathcal{H}_{\mathrm{d}s^2} \end{aligned}$$

where φ_n is extended to $M_+(\varepsilon_1)$ by $\varphi_n(p) = \varphi_n(\pi(p))$.

By construction, a point $p \in S_0$ is such that $\mathbf{d}(p) \leq \mathbf{d}(q)$ for any $q \in \pi^{-1}(\pi(p)) \cap (S'' \cup \Sigma_{\varepsilon_1})$. And it implies that along $C_{i,j}$, $\langle \nabla \mathbf{d}, \vec{n}_i + \vec{n}_j \rangle \geq 0$. Hence since $\varphi_n^2 (f_K \circ \mathbf{d} - \varepsilon_1)(f_K' \circ \mathbf{d}) \leq 0$, we obtain $I_n \leq 0$. This proves that

$$\sum_i \int_{\Omega_i} 2\varphi_n u \langle \nabla_i \varphi_n, \nabla_i u \rangle_i \mathrm{d}v_i + \int_{\Omega_i} \varphi_n^2 \|\nabla_i u\|_i^2 \mathrm{d}v_i \leq 0.$$

Thus:

$$\sum_i \int_{\Omega_i} \varphi_n^2 \|\nabla_i u\|_i^2 \mathrm{d}v_i \leq -2 \sum_i \int_{\Omega_i} \varphi_n u \langle \nabla_i \varphi_n, \nabla_i u \rangle_i \mathrm{d}v_i$$

$$\leq 2 \left(\sum_i \int_{\Omega_i} \varphi_n^2 \|\nabla_i u\|_i^2 \mathrm{d}v_i \right)^{\frac{1}{2}} \left(\sum_i \int_{\Omega_i} u^2 \|\nabla_i \varphi_n\|_i^2 \mathrm{d}v_i \right)^{\frac{1}{2}}.$$

Thus

$$\sum_i \int_{\Omega_i} \varphi_n^2 \|\nabla_i u\|_i^2 \mathrm{d}v_i \leq 4 \sum_i \int_{\Omega_i} u^2 \|\nabla_i \varphi_n\|_i^2 \mathrm{d}v_i.$$

The function u is bounded and the metric g_i and $\mathrm{d}\sigma_0^2$ are k-quasi-isometric so there exists a constant C which does not depend on i and n such that

$$\int_{\Omega_i} u^2 \|\nabla_i \varphi_n\|_i^2 \mathrm{d}v_i \leq C \int_{\Omega_i} \|\nabla_0 \varphi_n\|_0^2 \mathrm{d}v_0.$$

Hence :

$$\sum_i \int_{\Omega_i \cap \varphi_n^{-1}(1)} \|\nabla_i u\|_i^2 \mathrm{d}v_i \leq 4C \int_{\Sigma_0} \|\nabla_0 \varphi_n\|_0^2 \mathrm{d}v_0.$$

Taking the limit $n \to +\infty$ we obtain :

$$\sum_i \int_{\Omega_i} \|\nabla_i u\|_i^2 \mathrm{d}v_i = 0.$$

This implies that u is constant so $S_0 \subset \Sigma_{\varepsilon_1}$, this gives the contradiction we look for and Theorem 7 is proved.

6.2.2 $S \looparrowright M_+(\varepsilon)$ and $H_0 \geq 0$

In the second case, the only difference is the construction of the superharmonic function. It is in fact simpler since we do not have to control $\langle \vec{N}, \vec{\xi} \rangle$. From (3), we have

$$\Delta_{S''} f_K \circ \mathbf{d} \leq \big(f_K''(\mathbf{d}) + f_K'(\mathbf{d})(\kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta) \big) \sin^2\varphi$$

There is still a constant C such that $\max(|\kappa_1|, |\kappa_2|) \leq C$. Then considering $K = C$ we get

$$\begin{aligned} \Delta_{S''} f_K \circ \mathbf{d} &\leq \big(f_K''(\mathbf{d}) + f_K'(\mathbf{d}) C \big) \sin^2\varphi \\ &\leq 0 \end{aligned}$$

$f_K \circ \mathbf{d}$ is then superharmonic and this gives also a contradiction.

6.3 Hypotheses and stable surfaces

In this subsection, we want to make a remark about the hypothesis of Theorem 7.

Let Σ_0 be as in the theorem and assume that Σ_0 has constant mean curvature H_0. Applying the Jacobi operator to the constant function 1, the hypothesis about the mean curvature of the equidistant surfaces implies that $0 \geq L(1) = -(2Ric(\vec{n}, \vec{n}) + |A|^2)$ along Σ_0.

Now assume that Σ_0 is stable, since Σ_0 is parabolic there exists a sequence of compactly supported smooth functions $(\varphi_k)_k$ such that $0 \leq \varphi_k \leq 1$, $(\varphi_k^{-1}(1))_k$ is a compact exhaustion of Σ_0 and

$$\lim_k \int_{\Sigma_0} \|\nabla \varphi_k\|^2 = 0$$

Then by stability we get:

$$\begin{aligned} 0 \geq \int_{\varphi_k^{-1}(1)} -(2Ric(\vec{n}, \vec{n}) + |A|^2) &\geq \int_{\Sigma_0} -(2Ric(\vec{n}, \vec{n}) + |A|^2)\varphi_k^2 \\ &\geq \int_{\Sigma_0} \varphi_k L\varphi_k - \int_{\Sigma_0} \|\nabla \varphi_k\|^2 \\ &\geq -\int_{\Sigma_0} \|\nabla \varphi_k\|^2. \end{aligned}$$

Taking the limit as k goes to $+\infty$, we obtain $2Ric(\vec{n}, \vec{n}) + |A|^2 = 0$ along Σ_0. This implies that, at first order, the equidistant surfaces to Σ_0 have constant mean curvature H_0.

Now if the equidistant surfaces have constant mean curvature H_0, we get $0 = L(1)$ and $2Ric(\vec{n}, \vec{n}) + |A|^2 = 0$. Σ_0 is then a stable cmc H_0 surface.

If Σ_0 is not stable, we see that there exists $\varepsilon' > 0$ such that no Σ_t, $0 < t < \varepsilon'$, has constant mean curvature H_0. Thus Theorem 7 says that there is no constant mean curvature H_0 surface in $M_\pm(\varepsilon')$ (with good orientation in $M_+(\varepsilon')$).

7 Halfspace theorems in certain ambient spaces

In this section, we prove a halfspace result when the ambient space is a Lie group with a left invariant Riemannian metric. For a complete study of 3 dimensional metric Lie groups we refer to [15, 13].

Let G be a 3-dimensional connected Lie group and F be a normal properly embedded 2-dimensional Lie subgroup. We denote by $\mathfrak{g}$ and $\mathfrak{f}$ the associated Lie algebras.

Let $\mathrm{d}s^2$ be a left invariant metric on G. F is then a constant mean curvature surface in G. Do we have a halfspace theorem with respect to F? In fact for any $g \in G$, the coset gF is also a constant mean curvature surface in G. Since the left multiplication by g is an isometry, the halfspace problem is the same as the one for F.

Let $X \in \mathfrak{g}$ be the left invariant unit vector field which is normal to F at e. Let Y be a left invariant vector field, we have

$$\langle \nabla_X X, Y \rangle = -\langle [X, Y], X \rangle$$

Since F is normal, for any $Y \in \mathfrak{f}$, $[X, Y] \in \mathfrak{f}$. Then X normal to $\mathfrak{f}$ implies that $\nabla_X X = 0$. Then $t \mapsto \exp(tX)$ is the geodesic from e with speed X at e.

The map $F \times \mathbb{R} \to G,\ (f, t) \mapsto f \exp(tX)$ is onto. Let $t_0 > 0$ be the infimum of $\{t > 0 | \exp(tX) \in F\}$. If t_0 exists, F does not separate G and the above map is bijective on $F \times [0, t_0)$. If $t_0 = +\infty$, G is diffeomorphic to $F \times \mathbb{R}$ and F separates G.

We have the following halfspace result.

Proposition 10. *Let G be a 3-dimensional connected Lie group with a left invariant metric $\mathrm{d}s^2$. Let F be a normal properly embedded 2-dimensional Lie subgroup of G which is parabolic for the left invariant metric. We denote by H_0 the mean curvature of F. Let S be a properly immersed constant mean curvature H_0 surface in G with no boundary.*

- *If F does not separate G and S is included in $G \setminus F$, S is a coset gF.*
- *If F separates G and S is included in the mean convex side of F, S is a coset gF.*
- *If F separates G and S is included in the non mean convex side of F and is well oriented with respect to F, S is a coset gF.*

Let us just explain what is well oriented with respect to F. If G_+ is the non mean convex side of F and D is the connected component of $G_+ \setminus S$ containing F, we ask that along $S \cap \partial D$ the mean curvature vector of S points into D.

Proof. Let $X \in \mathfrak{g}$ still denote the left invariant unit vector field which is normal to F at e and points into the mean convex side. Let $s \mapsto g(s) = \exp(sX)$ be the geodesic curve from the unit element $e \in G$ normal to $\mathfrak{f} = T_e F$. For any $f \in F$, $s \mapsto fg(s)$ is the geodesic curve from $f \in F$ normal to $T_f F$. So the equidistant to F at distance t is $Fg(t)$. Since F is normal $Fg(t) = g(t)F$, thus the equidistant to t has the same mean curvature as F and the norm of its second fundamental form is constant. We denote by F_t this equidistant. Depending on the case, G can be parametrized by $F \times [0, t_0)$ or $F \times \mathbb{R}$ such that $F \times \{s\}$ is an equidistant surface to F. The mean convex side is the part included in $F \times \mathbb{R}_+$ (there is a change of sign with respect to the preceding section). The projection map π_s from F_s to F_0 is given by the right multiplication by $g(s)^{-1}$.

Let $s_0 \in \mathbb{R}$ be such that $F \times [0, s_0] \cap S$ is non empty. $F \times [0, s_0]$ is then a outside or inside regular s_0-neighborhood that satisfies the hypothesis about the mean curvature of the equidistant ($F \times [0, s_0]$ is regular because the right multiplication by $g(s)^{-1}$ is quasi-isometric). Moreover, F is parabolic, so Theorem 7 applies and S is an equidistant surface to F *i.e.* a coset gF. $\square$

Actually, when G is simply connected, the situation described in Proposition 10 can be classified (see [15, 13]). Since F is normal, $\mathfrak{f}$ is an ideal of $\mathfrak{g}$. Besides, F being parabolic, $\mathfrak{f}$ is then Abelian. This implies that G is isomorphic as metric Lie group to $\mathbb{R}^2 \rtimes_A \mathbb{R}$.

When A is in $\mathcal{M}_2(\mathbb{R})$, $\mathbb{R}^2 \rtimes_A \mathbb{R}$ is $\mathbb{R}^2 \times \mathbb{R}$ with the Lie group structure:

$$(p, z) * (p', z') = (p + e^{zA} p', z + z')$$

and with the canonical left invariant metric making $(\partial_x, \partial_y, \partial_z)$ an orthonormal basis at the origin. Actually, $\operatorname{tr} A$ can always be assumed non negative. So we can assume that $G = \mathbb{R}^2 \rtimes_A \mathbb{R}$ and F is the $\mathbb{R}^2$ subgroup $\mathbb{R}^2 \rtimes_A \{0\}$. The mean curvature of F is then $\operatorname{tr} A/2$ with respect to ∂_z. We then have the following consequence of Proposition 10.

Proposition 11. *Let S be a properly immersed constant mean curvature* $\operatorname{tr} A/2$ *surface in $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with no boundary.*

1. *If S is included in the mean convex side of one $\{z = t\}$, S is equal to one $\{z = t'\}$.*
2. *If S is included in the non mean convex side of one $\{z = t\}$ and S is well oriented with respect to it, S is equal to one $\{z = t'\}$.*

When $\operatorname{tr} A = 0$, $\mathbb{R}^2 \rtimes_A \mathbb{R}$ is unimodular and F is minimal. We have four possibilities for the Lie group structure

(a) $A = 0$ and $G = \mathbb{R}^3$, we recover the classical halfspace theorem for minimal surfaces with respect to planes [11].

(b) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $G = \mathrm{Nil}_3$ with its classical left invariant metric. We recover the halfspace theorem for "vertical minimal planes" in Nil_3 by Daniel and Hauswirth [5].

(c) $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $G = \mathrm{Sol}_3$. With its canonical left invariant metric, we get the halfspace result of Daniel, Meeks and Rosenberg [6] with respect to minimal planes. For other left invariant metrics we find new results.

(d) $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $G = \widetilde{E}(2)$. This gives new halfspace results.

When $\mathrm{tr}\, A \neq 0$, the group $\mathbb{R}^2 \rtimes_A \mathbb{R}$ is non unimodular. If $A \neq \lambda I_2$, the group structure is classified by the value of the Milnor invariant $D = 4 \det A / (\mathrm{tr}\, A)^2$.

(e) $A = I_2$ and $G = \mathbb{R}^2 \rtimes_{I_2} \mathbb{R} = \mathbb{H}^3$. We recover the halfspace theorem of Rodriguez and Rosenberg [20] with respect to horospheres in $\mathbb{H}^3$.

(f) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. This gives a new halfspace result.

(g) A has Milnor invariant $D < 1$. G has the same group structure has $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with $A = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$ with $b \in \mathbb{R} \backslash \{\pm 1\}$. If $b = 0$, $G = \mathbb{H}^2 \times \mathbb{R}$ and we recover the halfspace result of Hauswirth, Rosenberg and Spruck [10] with respect to vertical horocylinders in $\mathbb{H}^2 \times \mathbb{R}$.

(h) A has Milnor invariant $D > 1$. This case carries also a new halfspace theorem.

The above list gives all the simply connected metric Lie groups that satisfies hypothesis of Proposition 10. $\widetilde{SL}_2(\mathbb{R})$ is a unimodular Lie group that does not appear in the above classification but a halfspace result can be derived from this list.

In the case (g), let A be the matrix $\begin{pmatrix} 1 & 0 \\ c & 0 \end{pmatrix}$ and let us consider $G = \mathbb{R}^2 \rtimes_A \mathbb{R}$ with its canonical left invariant metric. The Lie group structure is:

$$(x, y, z) * (x', y', z') = (x + e^z x', y + y' + b(e^z - 1)x', z + z')$$

and the left invariant metric can be written

$$\mathrm{d}s^2 = e^{-2z}\mathrm{d}x^2 + (\mathrm{d}y + c(1 - e^{-z})\mathrm{d}x)^2 + \mathrm{d}z^2$$

Thus if we define $X = x$, $Y = e^z$ and $Z = y + cx$, the metric becomes

$$\mathrm{d}s^2 = \frac{1}{Y^2}(\mathrm{d}X^2 + \mathrm{d}Y^2) + (\mathrm{d}Z - \frac{c}{Y}\mathrm{d}X)^2$$

So G is isometric to the standard $\mathbb{E}(\kappa, \tau)$ space with $\kappa = -1$ and $\tau = c/2$. In fact, this metric space is isometric to $\widetilde{SL}_2(\mathbb{R})$ with a certain left invariant metric.

In this case, Proposition 11 gives Proposition 12. In fact, the Abelian subgroup $\mathbb{R}^2 \rtimes \{0\}$ is $\{Y = 1\}$ and its cosets are the surfaces $\Sigma_t = \{Y = e^t\}$. They have mean curvature $1/2$. It is important to notice that $\{Y = 1\}$ is not a subgroup of $\widetilde{SL}_2(\mathbb{R})$ but Proposition 12 is a halfspace result in $\widetilde{SL}_2(\mathbb{R})$ viewed as a metric space.

Proposition 12. *Let S be a properly immersed constant mean curvature $\frac{1}{2}$ surface in $\widetilde{SL}_2(\mathbb{R}) = \mathbb{E}(-1, c/2)$ with no boundary.*

1. *If S is included in the mean convex side of one Σ_t, S is equal to one $\Sigma_{t'}$.*

2. *If S is included in the non mean convex side of one Σ_t and S is well oriented with respect to it, S is equal to one $\Sigma_{t'}$.*

In fact, the projection map $(X, Y, Z) \mapsto (X, Y)$ is a Riemannian submersion from $\widetilde{SL}_2(\mathbb{R})$ to $\mathbb{H}^2$. So the surfaces Σ_t that foliate $\widetilde{SL}_2(\mathbb{R})$ are called "vertical horocylinders" since they are the fiber over horocycles in $\mathbb{H}^2$. Proposition 12 is then a halfspace result with respect to the vertical horocylinders in $\widetilde{SL}_2(\mathbb{R})$.

The author recently learns that this result is also proved by Carlos Peñafiel in [19]

A The surfaces $C_{\eta,t}$

In Section 5, we consider the surface $C_{\eta,t} \in \mathbb{R}^3$ which is parametrized by :

$$X_{\eta,t}(u, v) = \Big((t - \frac{\eta}{6}\cos v)\cos u, (t - \frac{\eta}{6}\cos v)\sin u, \frac{\eta}{6}(1 + \sin v)\Big)$$

with $(u, v) \in [0, 2\pi] \times [-\pi/2, \pi/2]$. This surface is drawn in Figure 9, it is a part of a rotationnel torus in $\mathbb{R}^3$. A computation gives that the mean curvature vector along $C_{\eta,t}$ is

$$\frac{3(t - \frac{\eta}{3}\cos v)}{\eta(t - \frac{\eta}{6}\cos v)} (\cos u \cos v, \sin u \cos v, -\sin v) .$$

So when $t \in [\eta/2, \eta]$, the mean curvature is always larger than $3/(2\eta)$.

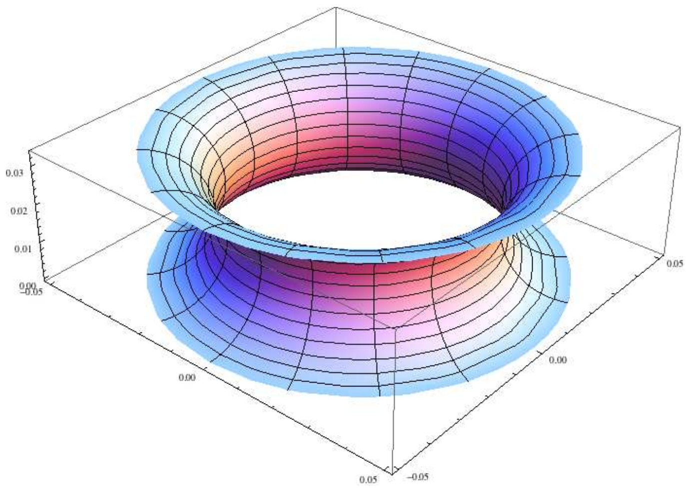

Figure 9: The surface $C_{\eta,t}$

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