

A HALF-SPACE THEOREM FOR GRAPHS OF CONSTANT MEAN CURVATURE $0 < H < \frac{1}{2}$ IN $\mathbb{H}^2 \times \mathbb{R}$

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ABSTRACT. We study a half-space problem related to graphs in $\mathbb{H}^2 \times \mathbb{R}$, where \mathbb{H}^2 is the hyperbolic plane, having constant mean curvature H defined over unbounded domains in \mathbb{H}^2 .

1. INTRODUCTION

The half-space theorem by Hoffman and Meeks [10] states that if a properly immersed minimal surface S in \mathbb{R}^3 lies on one side of some plane P , then S is a plane parallel to P . As a consequence, they proved the strong half-space theorem which says that two properly immersed minimal surfaces in \mathbb{R}^3 that do not intersect must be parallel planes.

These theorems have been generalized to some other ambient simply connected homogeneous manifolds with dimension 3. For example, we have half-space theorems with respect to horospheres in \mathbb{H}^3 [17], vertical minimal planes in Nil_3 and Sol_3 [3],[4] and entire minimal graph in Nil_3 [4]. It is known that there is no half-space theorem for horizontal slices in $\mathbb{H}^2 \times \mathbb{R}$, since rotational minimal surfaces (catenoids) are contained in a slab [14]-[15], but one has half-space theorems for constant mean curvature $\frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$ [9].

In [11], the first author proved a general half-space theorem for constant mean curvature surfaces. Under certain hypothesis, he proved that in a Riemannian 3-manifold of bounded geometry, a constant mean curvature H surface on one side of a parabolic constant mean curvature H surface Σ is an equidistant surface to Σ .

In Euclidian spaces of dimension higher than 4, there is no strong half-space theorem, since there exist rotational proper minimal hypersurfaces contained in a slab.

In [13], Menezes proves a half-space theorem for some complete vertical minimal graphs, more precisely, she looks at some particular graphs $\Sigma \subset M \times \mathbb{R}$ over a domain $D \subset M$, where M is a Hadamard surface with bounded curvature, these graphs are called ideal Scherk graphs and their existence was proved by Collin and Rosenberg in [2] for \mathbb{H}^2 and by Galvez and Rosenberg in [5] in the general case.

Theorem 1. *Let M denote a Hadamard surface with bounded curvature and let $\Sigma = \text{Graph}(u)$ be an ideal Scherk graph over an admissible polygonal domain $D \subset M$. If S is a properly immersed minimal surface contained in $D \times \mathbb{R}$ and disjoint from Σ , then S is a translate of Σ .*

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In this paper, we are interested in the case where the graph has constant mean curvature. More precisely, we consider graphs over \mathbb{H}^2 with constant mean curvature $0 < H < \frac{1}{2}$. In that case, we prove a result similar to the one of Menezes. We notice that the value $H = \frac{1}{2}$ is critical in this setting (see [12, 16] for the $H = \frac{1}{2}$ case).

The graphs that we will work with are graphs of functions u defined in a domain $D \subset \mathbb{H}^2$ whose boundary ∂D is composed of complete arcs $\{A_i\}$ and $\{B_j\}$, such that the curvatures of the arcs with respect to the domain are $\kappa(A_i) = 2H$ and $\kappa(B_j) = -2H$. These graphs will have constant mean curvature and u will assume the value $+\infty$ on each A_i and $-\infty$ on each B_j . These domains D will be called *Scherk type domains* and the functions u *Scherk type solutions*. The existence of these graphs is assured by A. Folha and S. Melo in [6]. There, the authors give necessary and sufficient conditions on the geometry of the domain D to prove the existence of such a solution. In this context, we prove the following result.

Theorem 2. *Let $D \subset \mathbb{H}^2$ be a Scherk type domain and u be a Scherk type solution over D (for some value $0 < H < \frac{1}{2}$). Denote by $\Sigma = \text{Graph}(u)$. If S is a properly immersed CMC H surface contained in $D \times \mathbb{R}$ and above Σ , then S is a vertical translate of Σ .*

The original idea of Hoffman and Meeks is to use the 1-parameter family of catenoids as *a priori* barriers to control minimal surfaces on one side of a plane (here *a priori* means that the choice of catenoids is independent of the particular minimal surface you want to control). In more general situations, it is not easy to construct such a continuous family of barriers so some authors use a discrete family (see for example [4, 18]). Menezes works also with such a discrete family. In our case, it does not seem possible to construct such a family in an easy way. Our approach is based on the existence of only one barrier whose construction depends on the particular surface S .

This paper is organized as follows. In Section 2, we will give a brief presentation of the Scherk type graphs and the result of Folha and Melo. Section 3 contains the proof of Theorem 2, so one of the main step is the existence of the barriers which uses the Perron method. We also prove a uniqueness result for the constant mean curvature equation.

2. CONSTANT MEAN CURVATURE SCHERK TYPE GRAPHS

In this section we present the theorem by A. Folha and S. Melo in [6] that assure the existence of constant mean curvature graphs which take the boundary value $+\infty$ on certain arcs A_i and $-\infty$ on arcs B_j . All along this section H will be a real constant in $(0, \frac{1}{2})$.

First, let us fix some notations. Let \mathbb{H}^2 be the hyperbolic plane, and $\mathbb{H}^2 \times \mathbb{R}$ be endowed with the product metric. Let D be a simply connected domain in \mathbb{H}^2 and $u : D \rightarrow \mathbb{R}$ a function. Denote by

$$\Sigma = \text{Graph}(u) = \{(x, u(x)), x \in D\}.$$

The upward unit normal to Σ is given by

$$(1) \quad N = \frac{1}{W}(\partial_t - \nabla u),$$

where

$$(2) \quad W = \sqrt{1 + |\nabla u|^2}.$$

The graph Σ has mean curvature H if u satisfies the equation

$$(3) \quad \mathcal{L}u := \operatorname{div} \frac{\nabla u}{W} - 2H = 0,$$

where the divergence and the gradient are taken with respect to the metric on \mathbb{H}^2 . Let us now give some definitions.

Definition 1. We say that the boundary of a domain D in \mathbb{H}^2 is a *2H-polygon* if its boundary is made of a finite number of complete arcs with constant curvature $2H$ and the cluster points of D in $\partial_\infty \mathbb{H}^2$ are the end-points of these arcs. The arcs are called the edges of D and the cluster points are the vertices of D .

We notice that a complete curve in \mathbb{H}^2 with constant curvature $2H$ is proper.

If Ω is a domain whose boundary is a $2H$ -polygon, we will denote by A_i (resp. B_i) the arcs of the boundary whose curvature is $2H$ (resp. $-2H$) with respect to the inward pointing unit normal.

Definition 2. We say that a domain D in \mathbb{H}^2 is a *Scherk type domain* if its boundary is a $2H$ -polygon and if each vertex is the end point of one arc A_i and one arc B_j .

Such a domain D is drawn in Figure 1.

Definition 3. Let Ω be a Scherk type domain. We say that P is an *admissible inscribed polygon* if $P \subset \Omega$ is a domain whose boundary is a $2H$ -polygon and its vertices are among the ones of Ω .

Let D be a Scherk type domain, in [6], Folha and Melo study the following Dirichlet problem

$$(4) \quad \begin{cases} \mathcal{L}(u) = 0 & \text{in } D \\ u = +\infty & \text{on } A_i \\ u = -\infty & \text{on } B_i \end{cases}$$

In order to state the result of Folha and Melo, let us introduce some notations. Let P be an admissible inscribed polygon in D and let $\{d_i\}_{i \in I}$ denote the vertices of P . Consider the set

$$\Theta = \{(\mathcal{H}_i)_{i \in I} \mid \mathcal{H}_i \text{ is a horodisk at } d_i \text{ and } \mathcal{H}_i \cap \mathcal{H}_j = \emptyset \text{ if } i \neq j\}.$$

we notice that, by choosing sufficiently small horodisks, Θ is not empty.

Let $(\mathcal{H}_i)_{i \in I}$ be in Θ such that the following is true : each arc A_i and B_j meets exactly two of these horodisks. Denote by \tilde{A}_i the compact arc of A_i which is the part of A_i outside these two horodisks. Let $|A_i|$ denote the length of \tilde{A}_i . We introduce the same notations of the B_j . For each arc $\eta_j \in \partial P$, we also define $\tilde{\eta}_j$ and $|\eta_j|$ in the same way.

We define

$$\alpha(\partial P) = \sum_{A_i \in \partial P} |A_i|, \quad \beta(\partial P) = \sum_{B_i \in \partial P} |B_i| \quad \text{and} \quad l(\partial P) = \sum_j |\eta_j|,$$

where $\partial P = \cup_j \eta_j$. We remark that a Scherk type domain has finite area. So we can introduce $\mathcal{A}(D)$ the area of D and $\mathcal{A}(P)$ the area of P .

With these definitions we can state the main theorem of [6].

Theorem 3. *Let D be a Scherk type domain. Then there exists a solution u for the Dirichlet problem (4) in D if and only if for some choice of the horodisks (in Θ) at the vertices,*

$$\alpha(\partial D) = \beta(\partial D) + 2HA(D)$$

and for any admissible inscribed polygons $P \neq D$,

$$2\alpha(\partial P) < l(\partial P) + 2HA(\Omega) \quad \text{and} \quad 2\beta(\partial P) < l(\partial P) - 2HA(\Omega).$$

It could seem that the conditions depend on the choice of the horodisks in Θ , actually they are independent of that choice if the horodisks are small enough. The details and the proof of this theorem can be found in [6].

3. THE MAIN RESULT

In this section we will prove the following result

Theorem 2. *Let $D \subset \mathbb{H}^2$ be a Scherk type domain and u be a Scherk type solution over D (for some value $0 < H < \frac{1}{2}$). Denote by $\Sigma = \text{Graph}(u)$. If S is a properly immersed CMC H surface contained in $D \times \mathbb{R}$ and above Σ , then S is a vertical translate of Σ .*

The proof of the theorem consists in constructing barriers to control the surface S . Before starting the proof, let us give some notations and preliminary results that we will use.

So we fix a value of $H \in (0, \frac{1}{2})$, a Scherk type domain D and a Scherk type solution u . Let $y \in D$ and B_y and B'_y be open balls centered in y such that $B_y \subsetneq B'_y \subsetneq D$ (see Figure 1). The following result consists in constructing a first barrier to control S .

Lemma 1. *There exists a constant $\epsilon > 0$ such that for all $t \in [0, \epsilon)$ there exists $v \in C^2(B'_y \setminus B_y) \cap C^0(\partial(B'_y \setminus B_y))$ such that v solves (3) and $v = u$ in $\partial B'_y$ and $v = u + t$ over ∂B_y .*

Proof. Consider the operator $F : C^{2,\alpha}(\overline{B'_y \setminus B_y}) \times C^{2,\alpha}(\partial(B'_y \setminus B_y)) \longrightarrow C^{0,\alpha}(\overline{B'_y \setminus B_y}) \times C^{2,\alpha}(\partial(B'_y \setminus B_y))$ given by

$$F(v, \phi) = (\mathcal{L}v, v - \phi).$$

Observe that

$$F(u, u) = 0.$$

Moreover, consider the operator

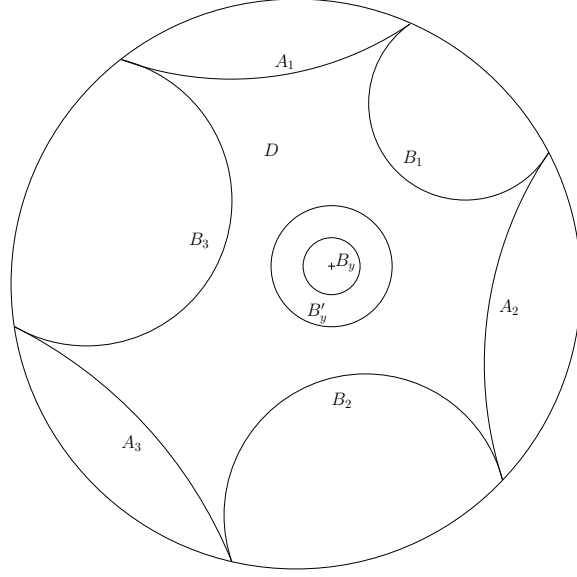


FIGURE 1. The Scherk type domain D and the balls B_y and B'_y

$$T := D_1 F(u, u) : C^{2,\alpha}(\overline{B'_y \setminus B_y}) \longrightarrow C^{0,\alpha}(\overline{B'_y \setminus B_y}) \times C^{2,\alpha}(\partial(B'_y \setminus B_y))$$

$$h \longmapsto \lim_{t \rightarrow 0} \frac{F(u + th, u) - F(u, u)}{t}$$

We have that

$$T(h) = \left(\operatorname{Div} \left(\frac{\nabla h - \frac{\nabla u}{W} \langle \frac{\nabla u}{W}, \nabla h \rangle}{W} \right), h \right).$$

Observe that T is a linear operator, of the form $T = (T_1, T_2)$ where T_1 is an elliptic operator of the form

$$T_1(v) = a^{ij}(x)D_{ij}v + b^i(x)D_iv; \quad a^{ij} = a^{ji}.$$

Moreover, since $|\nabla u| \leq C$, we have that $\frac{|\nabla u|}{W} \leq C' < 1$, this implies that T_1 is uniformly elliptic. We also have that the coefficients of T_1 belong to $C^{0,\alpha}(\overline{B'_y \setminus B_y})$. It follows by Theorem 6.14 in [7] that if $g \in C^{0,\alpha}(\overline{B'_y \setminus B_y})$ and $\phi \in C^{2,\alpha}(\partial(B'_y \setminus B_y))$, then there exists a unique $w \in C^{2,\alpha}(\overline{B'_y \setminus B_y})$ such that $T_1(w) = g$ in $B'_y \setminus B_y$ and $w = \phi$ in $\partial(B'_y \setminus B_y)$. Then, T_1 is invertible.

We conclude that T is invertible. It follows by the implicit function theorem that for all ϕ close to u there exists a solution of $\mathcal{L}v = 0$ in $B'_y \setminus \overline{B_y}$ with $v = \phi$ in $\partial(B'_y \setminus B_y)$. In other words, it exists $\epsilon > 0$ such that for all $t \in [0, \epsilon)$ there exists v such that v solves (3) and $v = u$ over $\partial B'_y$ and $v = u + t$ over ∂B_y . \square

Let S be as in Theorem 2. Give $p \in D$, define $g(p)$ by

$$g(p) = \inf_{t \in \mathbb{R}} \{(p, t) \in S\} \in \mathbb{R} \cup \{+\infty\}.$$

Observe that g is a lower semicontinuous functions and $g \geq u$. From now on, we will assume that $g > u$ (the case where $g(p) = u(p)$ for a point $p \in D$ will be considered in the proof of the theorem). Then for $\epsilon' > 0$ sufficiently small, we have that

$$(5) \quad g > u + \epsilon' \quad \text{on } \partial B_y.$$

Now, let ϵ be as in Lemma 1, fix $\epsilon' < \epsilon$ where ϵ' satisfies (5) and v given by Lemma 1 associated to ϵ' . We will construct a second barrier to control the surface S . More precisely, we will prove the existence of a function β that satisfies

$$(6) \quad \mathcal{L}\beta = 0 \quad \text{in } D \setminus \overline{B_y},$$

$$(7) \quad \beta = u + \epsilon' \quad \text{in } \partial B_y.$$

Proposition 1. *There is a solution $\beta \in C^2(D \setminus \overline{B_y}) \cap C^0(D \setminus B_y)$ for the Dirichlet problem (6)-(7) such that $\max(u, v) \leq \beta \leq \min(u + \epsilon', g)$.*

Proof. To prove this proposition we will use the Perron method. Let us recall the framework of this method. A function $w \in C^0(D \setminus B_y)$ is called a *subsolution* for \mathcal{L} if for any compact subdomain $U \subset D \setminus B_y$ and any solution h of (3) with $w \leq h$ on the boundary of U , we have $w \leq h$ on U .

First, observe that u is a subsolution for (3). Moreover, if w' and w are subsolutions, the continuous function $\max(w', w)$ is also a subsolution.

Let $\Delta \subset D \setminus B_y$ be a geodesic disk of small radius such that $\kappa(\partial\Delta) \geq 2H$. Theorem 3.2 in [8] implies that the Dirichlet problem for Equation (3) can be solved in Δ . So, for any such disk Δ and subsolution w , we can define a continuous function $M_\Delta(w)$ as

$$M_\Delta(w)(x) = \begin{cases} w(x) & \text{if } x \in D \setminus \Delta \\ \nu(x) & \text{if } x \in \Delta \end{cases}$$

where ν is the solution of $\mathcal{L}\nu = 0$ in Δ , with $\nu = w$ in $\partial\Delta$.

Also, define $u_+ = \min(u + \epsilon', g)$ and $u_- = \max(u, v)$. Denote by Γ the set of all subsolutions w such that $w \leq u_+$ on $D \setminus B_y$,

Claim 1. *If $w \in \Gamma$, then $M_\Delta(w) \in \Gamma$.*

Proof of Claim 1. First we have to prove that $M_\Delta(w)$ is a subsolution. So, take a arbitrary compact subdomain $U \subset D \setminus B_y$, and let h a solution of (3) in U with $M_\Delta(w) \leq h$ on ∂U . Since $w = M_\Delta(w)$ in $U \setminus \Delta$ we have that $M_\Delta(w) \leq h$ in $U \setminus \Delta$.

Moreover, $M_\Delta(w)$ is a solution of (3) in Δ . Then, by the maximum principle, we have that $M_\Delta(w) \leq h$ in $U \cap \Delta$. So, $M_\Delta(w) \leq h$ in U . Since U is arbitrary, it follows that $M_\Delta(w)$ is a subsolution in $D \setminus B_y$.

Now we have to prove that $M_\Delta(w) \leq u_+$. Observe that in $D \setminus \Delta$, $M_\Delta(w) = w$, since $w \in \Gamma$, then $w \leq u_+$, and so, $M_\Delta(w) \leq u_+$ in $D \setminus \Delta$.

On the other hand, $M_\Delta(w) = \nu$ in Δ , where ν is a solution of $\mathcal{L}(\nu) = 0$ in Δ and $\nu = w$ on $\partial\Delta$. Thus $\nu \leq u_+ = \min(u + \epsilon', g)$ on $\partial\Delta$. It follows by the maximum principle that $\nu \leq u + \epsilon'$ in Δ . So, we have to prove that $\nu \leq g$ in Δ .

Suppose that there exists $q \in \Delta$ such that $(\nu - g)(q) < 0$. Then, there exists $p \in \Delta$ such that $(\nu - g)(p) = \min(\nu - g) = C < 0$.

Now, observe that the graph of g in Δ is a piece of the surface S , let us denote it by S_g . Since $g \geq \nu + C$, then the graph $\Sigma_{\nu+C}$ of $\nu + C$ is a CMC $2H$ surface which is below the surface S_g . Moreover, $(p, g(p))$ is a point of contact of $\Sigma_{\nu+C}$ and S_g , and by the maximum principle, $S_g = \Sigma_{\nu+C}$. It follows that $g = \nu + C$ in Δ , since $\nu \leq g$ on $\partial\Delta$ then $C \geq 0$, and this contradicts $C < 0$. Then $\nu \leq g$ in Δ . \square

We define our solution by the following formula

$$\beta(q) = \sup_{w \in \Gamma} w(q).$$

Observe that u_- is a subsolution, since u and v are subsolutions. Also, $u \leq u_+ = \min(u + \epsilon', g)$. Moreover, in the proof of Claim 1 we see that $v \leq u_+$. Then $u_- = \max(u, v) \leq u_+$. We conclude that $u_- \in \Gamma$, then Γ is non empty, and u_+ is an upper bound for any w in Γ , thus β is well defined.

The method of Perron claims that β is a solution of Equation (3).

Claim 2. *The function β is a solution of (3) in $D \setminus \overline{B}_y$.*

Proof of Claim 2. Let $p \in D \setminus B_y$ and $\Delta \subset D \setminus B_y$ a geodesic disk of small radius centered at p as above. By definition of β there exists a sequence of subsolutions (w_n) such that $w_n(p) \rightarrow \beta(p)$. Then, consider the sequence of subsolutions $M_\Delta(w_n)$, we have that $M_\Delta(w_n)(p) \rightarrow \beta(p)$. Also, we have that $M_\Delta(w_n)$ is a bounded sequence of solutions of (3) in Δ , so, by considering a subsequence if necessary, we can assume that it converges to a solution \bar{w} on Δ with $\beta \geq \bar{w}$ and $\bar{w}(p) = \beta(p)$. Let us prove that $\beta = \bar{w}$ on Δ , then β will be a solution of (3).

We have that $\beta \geq \bar{w}$. Suppose that there is a point $q \in \Delta$ where $\beta(q) > \bar{w}(q)$. So, there is a subsolution s such that $s(q) > \bar{w}(q)$. Now consider the sequence of subsolutions $M_\Delta(\max(s, w_n))$. We have that $M_\Delta(\max(s, w_n))$ is a sequence of solutions of (3) in Δ . Thus, considering a subsequence, it converges to a solution $\bar{s} \geq \bar{w}$ of (3) in Δ .

So, we have \bar{w} and \bar{s} solutions of (3) in Δ , with $\bar{w}(p) = \beta(p) = \bar{s}(p)$, thus by the maximum principle we have that $\bar{w} = \bar{s}$ in Δ .

But, since $M_\Delta(\max(s, w_n)) \geq s$, we have that $\bar{s} \geq s$. This implies that $\bar{s}(q) \geq s(q) > \bar{w}(q)$, which contradicts $\bar{w} = \bar{s}$ in Δ . \square

Until now we have a function $\beta \in C^2(D \setminus \overline{B}_y)$ solution of (3) in $D \setminus \overline{B}_y$ such that $u \leq \beta \leq \min(u + \epsilon', g)$. But we don't have any information about β on the boundary ∂B_y . So, the next step is prove that β is continuous up to ∂B_y and $\beta = u + \epsilon'$ in ∂B_y .

Claim 3. *The function β is continuous up to the boundary ∂B_y and takes the value $u + \epsilon'$ on it.*

Proof of Claim 3. We have that $\beta(q) = \sup_{w \in \Gamma} w(q)$, then, $\beta(q) \leq u_+(q) \leq u + \epsilon'$. On the other hand, $u_-(q) = \max(u, v) \in \Gamma$, then $u_-(q) \leq \beta(q)$. Moreover, in ∂B_y we have that $u_-(q) = u + \epsilon'$. Thus, let $p \in \partial B_y$, and $\{x_n\}$ a sequence such that $x_n \rightarrow p$. We have

$$u_-(x_n) \leq \beta(x_n) \leq u_+(x_n),$$

since

$$\lim_{x_n \rightarrow p} u_-(x_n) = \lim_{x_n \rightarrow p} u_+(x_n) = u(p) + \epsilon',$$

we have that

$$\lim_{x_n \rightarrow p} \beta(x_n) = u(p) + \epsilon'.$$

Then we define $\beta(p)$ to be continuous on $D \setminus B_y$ and $\beta = u + \epsilon'$ in ∂B_y . □

This concludes the proof of Proposition 1. □

We have proved the existence of a function β defined on $D \setminus B_y$ such that $\max(u, v) \leq \beta \leq \min(u + \epsilon', g)$ and $\beta \in C^2(D \setminus \overline{B_y}) \cap C^0(D \setminus B_y)$. Now we observe that, since $v \leq \beta \leq u + \epsilon'$ we have by Theorem 1.1 in [19] that $\nabla \beta$ is uniformly bounded close to ∂B_y . This implies that β is a solution of a linear uniformly elliptic equation. It follows by Corollary 8.36 in [7] that $\beta \in C^{1,\alpha}(D \setminus \overline{B_y})$, and then $\beta \in C^1(D \setminus \overline{B_y})$.

In the next result we will prove a uniqueness result for CMC graphs in $\mathbb{H}^2 \times \mathbb{R}$ defined over unbounded domain in \mathbb{H}^2 whose the existence was proved by A. Folha and S. Melo in [6].

Proposition 2. *Let $\beta \in C^1(D \setminus \overline{B_y})$ solution of the Dirichlet problem (6)-(7) such that $u \leq \beta \leq u + \epsilon'$. Then, $\beta = u + \epsilon'$.*

Proof. Let us first analyze the boundary ∂D . As in section 2, let $A_1, B_1, \dots, A_k, B_k$ be the edges of the ∂D with $u(A_i) = +\infty = \beta(A_i)$ and $u(B_i) = -\infty = \beta(B_i)$.

For each i , let $\mathcal{H}_i(n)$ be a horodisk asymptotic to the vertex d_i of D such that $\mathcal{H}_i(n+1) \subset \mathcal{H}_i(n)$ and $\cap_n \mathcal{H}_i(n) = \emptyset$. For each side A_i , let us denote by A_i^n the compact subarc of A_i which is the part of A_i outside the two horodisks, and by $|A_i^n|$ the length of A_i^n . Analogously, we define B_i^n for each side B_i . Denote by C_i^n the compact arc of $\partial \mathcal{H}_i(n)$ contained in the domain D and let P^n be the subdomain of D bounded by the closed curve formed by the arcs A_i^n , B_i^n and C_i^n and let us denote $\Delta^n = \partial(P^n \setminus \overline{B_y})$.

We have by the theorem of Stokes that

$$0 = \int_{P^n \setminus B_y} \text{Div} \left(\frac{\nabla u}{W_u} - \frac{\nabla \beta}{W_\beta} \right) = \int_{\Delta^n} \left(\frac{\nabla u}{W_u} - \frac{\nabla \beta}{W_\beta} \right) \cdot \eta$$

where $W_u^2 = 1 + |\nabla u|^2$ and $W_\beta^2 = 1 + |\nabla \beta|^2$.

Thus

$$0 = \int_{A_i^n} \left(\frac{\nabla u}{W_u} - \frac{\nabla \beta}{W_\beta} \right) \cdot \eta + \int_{B_i^n} \left(\frac{\nabla u}{W_u} - \frac{\nabla \beta}{W_\beta} \right) \cdot \eta \\ + \int_{C_i^n} \left(\frac{\nabla u}{W_u} - \frac{\nabla \beta}{W_\beta} \right) \cdot \eta + \int_{\partial B_y} \left(\frac{\nabla u}{W_u} - \frac{\nabla \beta}{W_\beta} \right) \cdot \eta.$$

By theorem 5.1 in [6], these integrals can be estimated. We have

$$0 < |A_i^n| - |A_i^n| + |B_i^n| - |B_i^n| + 2|C_i^n| + \int_{\partial B_y} \left(\frac{\nabla u}{W_u} - \frac{\nabla \beta}{W_\beta} \right) \cdot \eta.$$

Then

$$\int_{\partial B_y} \left(\frac{\nabla u}{W_u} - \frac{\nabla \beta}{W_\beta} \right) \cdot \eta > -2|C_i^n|.$$

Now, we have that $\beta \leq u + \epsilon'$, suppose that $\beta < u + \epsilon'$, it follows by Lemmas 1 and 2 in [1] that $\int_{\partial B_y} \left(\frac{\nabla u}{W_u} - \frac{\nabla \beta}{W_\beta} \right) \cdot \eta < 0$. This yields a contradiction, since $|C_i^n| \rightarrow 0$ when $n \rightarrow \infty$, so we conclude that $\beta = u + \epsilon'$. \square

Now we are able to prove our main theorem.

Proof of Theorem 2. We know that S is a properly immersed CMC surface contained in $D \times \mathbb{R}$ above Σ . Then, let $y \in D$, $B_y \subset D$ and ϵ' as above. We have three cases to analyze

1) Suppose that there exists $p \in D$ such that $g(p) = u(p)$ (is this the case we had let aside before Proposition 1). In this case, by the maximum principle, we conclude that $u = \beta$.

2) Suppose that $g > u$ and $\inf(g - u) = 0$. In this case, by the Proposition 1 there exists β solution of (6)-(7) defined over $D \setminus B_y$ such that $u \leq \beta \leq g$. Moreover, Proposition 2 assures that $\beta = u + \epsilon'$. This yields a contradiction, since we assume that $\inf(g - u) = 0$.

3) Finally, suppose that $g > u$ and $\inf(g - u) = \alpha > 0$. Then, pushing up Σ by a vertical translations, that is, by considering $u + \alpha$ instead of u , we have now that $g \geq u + \alpha$ and $\inf(g - \alpha - u) = 0$, this case reduces to cases (1) and (2) and we conclude that $\beta = u + \alpha$, where α is a constant. \square

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